Miscellaneous FOL Examples

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Contents

1 Natural numbers

theory *Natural-Numbers*
imports *FOL*
begin

Theory of the natural numbers: Peano's axioms, primitive recursion. (Modernized version of Larry Paulson's theory "Nat".)

typedecl *nat*
instance *nat* :: ‹*term*› ⟨*proof*⟩

axiomatization
  Zero :: ‹*nat*›   (‹*0*›) **and**
  Suc :: ‹*nat* => *nat*› **and**
  rec :: ‹[*nat*, *'a*, [*nat*, *'a*] => *'a*] => *'a*›
where
  induct [*case-names 0 Suc*, *induct type*: *nat*]:
    ‹*P*(*0*) ==> (!!*x*. *P*(*x*) ==> *P*(*Suc*(*x*))) ==> *P*(*n*)› **and**
  Suc-inject: ‹*Suc*(*m*) = *Suc*(*n*) ==> *m* = *n*› **and**
  Suc-neq-0: ‹*Suc*(*m*) = *0* ==> *R*› **and**
  rec-0: ‹*rec*(*0*, *a*, *f*) = *a*› **and**

rec-Suc: ‹$rec(Suc(m),\ a,\ f) = f(m,\ rec(m,\ a,\ f))$›

lemma *Suc-n-not-n*: ‹$Suc(k) \neq k$›
⟨*proof*⟩

definition *add* :: ‹*nat* => *nat* => *nat*› (**infixl** ‹+› *60*)
where ‹$m + n = rec(m,\ n,\ \lambda x\ y.\ Suc(y))$›

lemma *add-0* [*simp*]: ‹$0 + n = n$›
⟨*proof*⟩

lemma *add-Suc* [*simp*]: ‹$Suc(m) + n = Suc(m + n)$›
⟨*proof*⟩

lemma *add-assoc*: ‹$(k + m) + n = k + (m + n)$›
⟨*proof*⟩

lemma *add-0-right*: ‹$m + 0 = m$›
⟨*proof*⟩

lemma *add-Suc-right*: ‹$m + Suc(n) = Suc(m + n)$›
⟨*proof*⟩

lemma
assumes ‹!!$n.\ f(Suc(n)) = Suc(f(n))$›
shows ‹$f(i + j) = i + f(j)$›
⟨*proof*⟩

end

2 Examples for the manual "Introduction to Isabelle"

theory *Intro*
imports *FOL*
begin

2.0.1 Some simple backward proofs

lemma *mythm*: ‹$P \lor P \longrightarrow P$›
⟨*proof*⟩

lemma ‹$(P \land Q) \lor R \longrightarrow (P \lor R)$›
⟨*proof*⟩

Correct version, delaying use of *spec* until last.

lemma ‹$(\forall x\ y.\ P(x,y)) \longrightarrow (\forall z\ w.\ P(w,z))$›

$\langle proof \rangle$

2.0.2 Demonstration of *fast*

lemma ‹$(\exists\, y.\ \forall\, x.\ J(y,x) \longleftrightarrow \neg\ J(x,x)) \longrightarrow \neg\ (\forall\, x.\ \exists\, y.\ \forall\, z.\ J(z,y) \longleftrightarrow \neg\ J(z,x))$›
$\langle proof \rangle$

lemma ‹$\forall\, x.\ P(x,f(x)) \longleftrightarrow (\exists\, y.\ (\forall\, z.\ P(z,y) \longrightarrow P(z,f(x))) \wedge P(x,y))$›
$\langle proof \rangle$

2.0.3 Derivation of conjunction elimination rule

lemma
  assumes *major*: ‹$P \wedge Q$›
    and *minor*: ‹$[\![P;\ Q]\!] \Longrightarrow R$›
  shows ‹R›
$\langle proof \rangle$

2.1 Derived rules involving definitions

Derivation of negation introduction

lemma
  assumes ‹$P \Longrightarrow False$›
  shows ‹$\neg\ P$›
$\langle proof \rangle$

lemma
  assumes *major*: ‹$\neg\ P$›
    and *minor*: ‹P›
  shows ‹R›
$\langle proof \rangle$

Alternative proof of the result above

lemma
  assumes *major*: ‹$\neg\ P$›
    and *minor*: ‹P›
  shows ‹R›
$\langle proof \rangle$

end

3 Theory of the natural numbers: Peano's axioms, primitive recursion

theory *Nat*
  imports *FOL*
begin

typedecl *nat*
instance *nat* :: ‹*term*› ⟨*proof*⟩

axiomatization
  Zero :: ‹*nat*› (‹*0*›) **and**
  Suc :: ‹*nat* ⇒ *nat*› **and**
  rec :: ‹[*nat*, ′*a*, [*nat*, ′*a*] ⇒ ′*a*] ⇒ ′*a*›
where
  induct: ‹$[\![P(0); \bigwedge x.\ P(x) \Longrightarrow P(Suc(x))]\!] \Longrightarrow P(n)$› **and**
  Suc-inject: ‹$Suc(m)=Suc(n) \Longrightarrow m=n$› **and**
  Suc-neq-0: ‹$Suc(m)=0 \Longrightarrow R$› **and**
  rec-0: ‹$rec(0,a,f) = a$› **and**
  rec-Suc: ‹$rec(Suc(m),\ a,\ f) = f(m,\ rec(m,a,f))$›

definition *add* :: ‹[*nat*, *nat*] ⇒ *nat*› (**infixl** ‹+› *60*)
  where ‹$m + n \equiv rec(m,\ n,\ \lambda x\ y.\ Suc(y))$›

3.1 Proofs about the natural numbers

lemma *Suc-n-not-n*: ‹$Suc(k) \neq k$›
⟨*proof*⟩

lemma ‹$(k+m)+n = k+(m+n)$›
⟨*proof*⟩

lemma *add-0* [*simp*]: ‹$0+n = n$›
⟨*proof*⟩

lemma *add-Suc* [*simp*]: ‹$Suc(m)+n = Suc(m+n)$›
⟨*proof*⟩

lemma *add-assoc*: ‹$(k+m)+n = k+(m+n)$›
⟨*proof*⟩

lemma *add-0-right*: ‹$m+0 = m$›
⟨*proof*⟩

lemma *add-Suc-right*: ‹$m+Suc(n) = Suc(m+n)$›
⟨*proof*⟩

lemma
  assumes *prem*: ‹$\bigwedge n.\ f(Suc(n)) = Suc(f(n))$›
  shows ‹$f(i+j) = i+f(j)$›
⟨*proof*⟩

end

4 Theory of the natural numbers: Peano's axioms, primitive recursion

theory *Nat-Class*
  imports *FOL*
begin

This is an abstract version of `Nat.thy`. Instead of axiomatizing a single type *nat*, it defines the class of all these types (up to isomorphism).

Note: The *rec* operator has been made *monomorphic*, because class axioms cannot contain more than one type variable.

class *nat* =
  fixes *Zero* :: ‹$'a$› (‹0›)
    and *Suc* :: ‹$'a \Rightarrow 'a$›
    and *rec* :: ‹$'a \Rightarrow 'a \Rightarrow ('a \Rightarrow 'a \Rightarrow 'a) \Rightarrow 'a$›
  assumes *induct*: ‹$P(0) \Longrightarrow (\bigwedge x.\ P(x) \Longrightarrow P(Suc(x))) \Longrightarrow P(n)$›
    and *Suc-inject*: ‹$Suc(m) = Suc(n) \Longrightarrow m = n$›
    and *Suc-neq-Zero*: ‹$Suc(m) = 0 \Longrightarrow R$›
    and *rec-Zero*: ‹$rec(0,\ a,\ f) = a$›
    and *rec-Suc*: ‹$rec(Suc(m),\ a,\ f) = f(m,\ rec(m,\ a,\ f))$›
begin

definition *add* :: ‹$'a \Rightarrow 'a \Rightarrow 'a$› (**infixl** ‹+› *60*)
  where ‹$m + n = rec(m,\ n,\ \lambda x\ y.\ Suc(y))$›

lemma *Suc-n-not-n*: ‹$Suc(k) \neq (k::'a)$›
  ⟨*proof*⟩

lemma ‹$(k + m) + n = k + (m + n)$›
  ⟨*proof*⟩

lemma *add-Zero* [*simp*]: ‹$0 + n = n$›
  ⟨*proof*⟩

lemma *add-Suc* [*simp*]: ‹$Suc(m) + n = Suc(m + n)$›
  ⟨*proof*⟩

lemma *add-assoc*: ‹$(k + m) + n = k + (m + n)$›
  ⟨*proof*⟩

lemma *add-Zero-right*: ‹$m + 0 = m$›
  ⟨*proof*⟩

lemma *add-Suc-right*: ‹$m + Suc(n) = Suc(m + n)$›
  ⟨*proof*⟩

lemma
  assumes *prem*: ‹$\bigwedge n.\ f(Suc(n)) = Suc(f(n))$›

shows ‹$f(i + j) = i + f(j)$›
⟨*proof*⟩

end

end

5 Intuitionistic FOL: Examples from The Foundation of a Generic Theorem Prover

theory *Foundation*
imports *IFOL*
begin

lemma ‹$A \wedge B \longrightarrow (C \longrightarrow A \wedge C)$›
⟨*proof*⟩

A form of conj-elimination

lemma
assumes ‹$A \wedge B$›
and ‹$A \Longrightarrow B \Longrightarrow C$›
shows ‹C›
⟨*proof*⟩

lemma
assumes ‹$\bigwedge A.\ \neg\ \neg\ A \Longrightarrow A$›
shows ‹$B \vee \neg\ B$›
⟨*proof*⟩

lemma
assumes ‹$\bigwedge A.\ \neg\ \neg\ A \Longrightarrow A$›
shows ‹$B \vee \neg\ B$›
⟨*proof*⟩

lemma
assumes ‹$A \vee \neg\ A$›
and ‹$\neg\ \neg\ A$›
shows ‹A›
⟨*proof*⟩

5.1 Examples with quantifiers

lemma
assumes ‹$\forall z.\ G(z)$›
shows ‹$\forall z.\ G(z) \vee H(z)$›
⟨*proof*⟩

lemma ‹$\forall x.\ \exists y.\ x = y$›
⟨*proof*⟩

lemma ‹$\exists y.\ \forall x.\ x = y$›
⟨*proof*⟩

Parallel lifting example.

lemma ‹$\exists u.\ \forall x.\ \exists v.\ \forall y.\ \exists w.\ P(u,x,v,y,w)$›
⟨*proof*⟩

lemma
  assumes ‹$(\exists z.\ F(z)) \wedge B$›
  shows ‹$\exists z.\ F(z) \wedge B$›
⟨*proof*⟩

A bigger demonstration of quantifiers – not in the paper.

lemma ‹$(\exists y.\ \forall x.\ Q(x,y)) \longrightarrow (\forall x.\ \exists y.\ Q(x,y))$›
⟨*proof*⟩

end

6 First-Order Logic: PROLOG examples

theory *Prolog*
imports *FOL*
begin

typedecl *'a list*
instance *list* :: (‹*term*›) ‹*term*› ⟨*proof*⟩

axiomatization
  Nil :: ‹*'a list*› **and**
  Cons :: ‹[*'a*, *'a list*]=> *'a list*› (**infixr** ‹:› *60*) **and**
  app :: ‹[*'a list*, *'a list*, *'a list*] => *o*› **and**
  rev :: ‹[*'a list*, *'a list*] => *o*›
where
  appNil: ‹*app*(*Nil*,*ys*,*ys*)› **and**
  appCons: ‹*app*(*xs*,*ys*,*zs*) ==> *app*(*x*:*xs*, *ys*, *x*:*zs*)› **and**
  revNil: ‹*rev*(*Nil*,*Nil*)› **and**
  revCons: ‹[| *rev*(*xs*,*ys*); *app*(*ys*, *x*:*Nil*, *zs*) |] ==> *rev*(*x*:*xs*, *zs*)›

schematic-goal ‹*app*(*a*:*b*:*c*:*Nil*, *d*:*e*:*Nil*, *?x*)›
⟨*proof*⟩

schematic-goal ‹*app*(*?x*, *c*:*d*:*Nil*, *a*:*b*:*c*:*d*:*Nil*)›
⟨*proof*⟩

schematic-goal ‹*app*(*?x*, *?y*, *a*:*b*:*c*:*d*:*Nil*)›

⟨*proof*⟩

lemmas *rules* = *appNil appCons revNil revCons*

schematic-goal ‹$rev(a{:}b{:}c{:}d{:}Nil,\ ?x)$›
⟨*proof*⟩

schematic-goal ‹$rev(a{:}b{:}c{:}d{:}e{:}f{:}g{:}h{:}i{:}j{:}k{:}l{:}m{:}n{:}Nil,\ ?w)$›
⟨*proof*⟩

schematic-goal ‹$rev(?x,\ a{:}b{:}c{:}Nil)$›
⟨*proof*⟩

⟨*ML*⟩

schematic-goal ‹$rev(?x,\ a{:}b{:}c{:}Nil)$›
⟨*proof*⟩

schematic-goal ‹$rev(a{:}?x{:}c{:}?y{:}Nil,\ d{:}?z{:}b{:}?u)$›
⟨*proof*⟩

schematic-goal ‹$rev(a{:}b{:}c{:}d{:}e{:}f{:}g{:}h{:}i{:}j{:}k{:}l{:}m{:}n{:}o{:}p{:}Nil,\ ?w)$›
⟨*proof*⟩

schematic-goal ‹$a{:}b{:}c{:}d{:}e{:}f{:}g{:}h{:}i{:}j{:}k{:}l{:}m{:}n{:}o{:}p{:}Nil = ?x \wedge app(?x,?x,?y) \wedge rev(?y,?w)$›
⟨*proof*⟩

end

7 Intuitionistic First-Order Logic

theory *Intuitionistic*
imports *IFOL*
begin

Metatheorem (for *propositional* formulae): P is classically provable iff $\neg\neg P$ is intuitionistically provable. Therefore $\neg P$ is classically provable iff it is intuitionistically provable.

Proof: Let Q be the conjunction of the propositions $A \vee \neg A$, one for each atom A in P. Now $\neg\neg Q$ is intuitionistically provable because $\neg\neg(A \vee \neg A)$ is and because double-negation distributes over conjunction. If P is provable classically, then clearly $Q \to P$ is provable intuitionistically, so $\neg\neg(Q \to P)$

is also provable intuitionistically. The latter is intuitionistically equivalent to $\neg\neg Q \to \neg\neg P$, hence to $\neg\neg P$, since $\neg\neg Q$ is intuitionistically provable. Finally, if P is a negation then $\neg\neg P$ is intuitionstically equivalent to P. [Andy Pitts]

lemma ‹$\neg \neg (P \wedge Q) \longleftrightarrow \neg \neg P \wedge \neg \neg Q$›
⟨*proof*⟩

lemma ‹$\neg \neg ((\neg P \longrightarrow Q) \longrightarrow (\neg P \longrightarrow \neg Q) \longrightarrow P)$›
⟨*proof*⟩

Double-negation does NOT distribute over disjunction.

lemma ‹$\neg \neg (P \longrightarrow Q) \longleftrightarrow (\neg \neg P \longrightarrow \neg \neg Q)$›
⟨*proof*⟩

lemma ‹$\neg \neg \neg P \longleftrightarrow \neg P$›
⟨*proof*⟩

lemma ‹$\neg \neg ((P \longrightarrow Q \vee R) \longrightarrow (P \longrightarrow Q) \vee (P \longrightarrow R))$›
⟨*proof*⟩

lemma ‹$(P \longleftrightarrow Q) \longleftrightarrow (Q \longleftrightarrow P)$›
⟨*proof*⟩

lemma ‹$((P \longrightarrow (Q \vee (Q \longrightarrow R))) \longrightarrow R) \longrightarrow R$›
⟨*proof*⟩

lemma
‹$(((G \longrightarrow A) \longrightarrow J) \longrightarrow D \longrightarrow E) \longrightarrow (((H \longrightarrow B) \longrightarrow I) \longrightarrow C \longrightarrow J) \longrightarrow (A \longrightarrow H) \longrightarrow F \longrightarrow G \longrightarrow (((C \longrightarrow B) \longrightarrow I) \longrightarrow D) \longrightarrow (A \longrightarrow C) \longrightarrow (((F \longrightarrow A) \longrightarrow B) \longrightarrow I) \longrightarrow E$›
⟨*proof*⟩

Admissibility of the excluded middle for negated formulae

lemma ‹$(P \vee \neg P \longrightarrow \neg Q) \longrightarrow \neg Q$›
⟨*proof*⟩

The same in a more general form, no ex falso quodlibet

lemma ‹$(P \vee (P\longrightarrow R) \longrightarrow Q \longrightarrow R) \longrightarrow Q \longrightarrow R$›
⟨*proof*⟩

7.1 Lemmas for the propositional double-negation translation

lemma ‹$P \longrightarrow \neg \neg P$›
⟨*proof*⟩

lemma ‹$\neg \neg (\neg \neg P \longrightarrow P)$›
⟨*proof*⟩

lemma ‹$\neg\neg P \wedge \neg\neg(P \longrightarrow Q) \longrightarrow \neg\neg Q$›
⟨*proof*⟩

The following are classically but not constructively valid. The attempt to prove them terminates quickly!

lemma ‹$((P \longrightarrow Q) \longrightarrow P) \longrightarrow P$›
⟨*proof*⟩

lemma ‹$(P \wedge Q \longrightarrow R) \longrightarrow (P \longrightarrow R) \vee (Q \longrightarrow R)$›
⟨*proof*⟩

7.2 de Bruijn formulae

de Bruijn formula with three predicates

lemma
‹$((P \longleftrightarrow Q) \longrightarrow P \wedge Q \wedge R) \wedge$
$((Q \longleftrightarrow R) \longrightarrow P \wedge Q \wedge R) \wedge$
$((R \longleftrightarrow P) \longrightarrow P \wedge Q \wedge R) \longrightarrow P \wedge Q \wedge R$›
⟨*proof*⟩

de Bruijn formula with five predicates

lemma
‹$((P \longleftrightarrow Q) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T) \wedge$
$((Q \longleftrightarrow R) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T) \wedge$
$((R \longleftrightarrow S) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T) \wedge$
$((S \longleftrightarrow T) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T) \wedge$
$((T \longleftrightarrow P) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T$›
⟨*proof*⟩

Problems from of Sahlin, Franzen and Haridi, An Intuitionistic Predicate Logic Theorem Prover. J. Logic and Comp. 2 (5), October 1992, 619-656.

Problem 1.1

lemma
‹$(\forall x.\ \exists y.\ \forall z.\ p(x) \wedge q(y) \wedge r(z)) \longleftrightarrow$
$(\forall z.\ \exists y.\ \forall x.\ p(x) \wedge q(y) \wedge r(z))$›
⟨*proof*⟩

Problem 3.1

lemma ‹$\neg(\exists x.\ \forall y.\ mem(y,x) \longleftrightarrow \neg mem(x,x))$›
⟨*proof*⟩

Problem 4.1: hopeless!

lemma
‹$(\forall x.\ p(x) \longrightarrow p(h(x)) \vee p(g(x))) \wedge (\exists x.\ p(x)) \wedge (\forall x.\ \neg p(h(x)))$
$\longrightarrow (\exists x.\ p(g(g(g(g(g(x)))))))$›
⟨*proof*⟩

7.3 Intuitionistic FOL: propositional problems based on Pelletier.

¬¬1

lemma ‹$\neg\ \neg\ ((P \longrightarrow Q) \longleftrightarrow (\neg\ Q \longrightarrow \neg\ P))$›
⟨*proof*⟩

¬¬2

lemma ‹$\neg\ \neg\ (\neg\ \neg\ P \longleftrightarrow P)$›
⟨*proof*⟩

3

lemma ‹$\neg\ (P \longrightarrow Q) \longrightarrow (Q \longrightarrow P)$›
⟨*proof*⟩

¬¬4

lemma ‹$\neg\ \neg\ ((\neg\ P \longrightarrow Q) \longleftrightarrow (\neg\ Q \longrightarrow P))$›
⟨*proof*⟩

¬¬5

lemma ‹$\neg\ \neg\ ((P \lor Q \longrightarrow P \lor R) \longrightarrow P \lor (Q \longrightarrow R))$›
⟨*proof*⟩

¬¬6

lemma ‹$\neg\ \neg\ (P \lor \neg\ P)$›
⟨*proof*⟩

¬¬7

lemma ‹$\neg\ \neg\ (P \lor \neg\ \neg\ \neg\ P)$›
⟨*proof*⟩

¬¬8. Peirce's law

lemma ‹$\neg\ \neg\ (((P \longrightarrow Q) \longrightarrow P) \longrightarrow P)$›
⟨*proof*⟩

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lemma ‹$((P \lor Q) \land (\neg\ P \lor Q) \land (P \lor \neg\ Q)) \longrightarrow \neg\ (\neg\ P \lor \neg\ Q)$›
⟨*proof*⟩

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lemma ‹$(Q \longrightarrow R) \longrightarrow (R \longrightarrow P \land Q) \longrightarrow (P \longrightarrow (Q \lor R)) \longrightarrow (P \longleftrightarrow Q)$›
⟨*proof*⟩

7.4 11. Proved in each direction (incorrectly, says Pelletier!!)

lemma ‹$P \longleftrightarrow P$›
  ⟨*proof*⟩

¬¬12. Dijkstra's law

lemma ‹$\neg\neg (((P \longleftrightarrow Q) \longleftrightarrow R) \longleftrightarrow (P \longleftrightarrow (Q \longleftrightarrow R)))$›
  ⟨*proof*⟩

lemma ‹$((P \longleftrightarrow Q) \longleftrightarrow R) \longrightarrow \neg\neg (P \longleftrightarrow (Q \longleftrightarrow R))$›
  ⟨*proof*⟩

13. Distributive law

lemma ‹$P \vee (Q \wedge R) \longleftrightarrow (P \vee Q) \wedge (P \vee R)$›
  ⟨*proof*⟩

¬¬14

lemma ‹$\neg\neg ((P \longleftrightarrow Q) \longleftrightarrow ((Q \vee \neg P) \wedge (\neg Q \vee P)))$›
  ⟨*proof*⟩

¬¬15

lemma ‹$\neg\neg ((P \longrightarrow Q) \longleftrightarrow (\neg P \vee Q))$›
  ⟨*proof*⟩

¬¬16

lemma ‹$\neg\neg ((P \longrightarrow Q) \vee (Q \longrightarrow P))$›
  ⟨*proof*⟩

¬¬17

lemma ‹$\neg\neg (((P \wedge (Q \longrightarrow R)) \longrightarrow S) \longleftrightarrow ((\neg P \vee Q \vee S) \wedge (\neg P \vee \neg R \vee S)))$›
  ⟨*proof*⟩

Dijkstra's "Golden Rule"

lemma ‹$(P \wedge Q) \longleftrightarrow P \longleftrightarrow Q \longleftrightarrow (P \vee Q)$›
  ⟨*proof*⟩

8 Examples with quantifiers

8.1 The converse is classical in the following implications . . .

lemma ‹$(\exists x.\ P(x) \longrightarrow Q) \longrightarrow (\forall x.\ P(x)) \longrightarrow Q$›
  ⟨*proof*⟩

lemma ‹$((\forall x.\ P(x)) \longrightarrow Q) \longrightarrow \neg (\forall x.\ P(x) \wedge \neg Q)$›
  ⟨*proof*⟩

lemma ‹$((\forall x.\ \neg P(x)) \longrightarrow Q) \longrightarrow \neg (\forall x.\ \neg (P(x) \vee Q))$›

⟨*proof*⟩

lemma ‹$(\forall x.\ P(x)) \lor Q \longrightarrow (\forall x.\ P(x) \lor Q)$›
⟨*proof*⟩

lemma ‹$(\exists x.\ P \longrightarrow Q(x)) \longrightarrow (P \longrightarrow (\exists x.\ Q(x)))$›
⟨*proof*⟩

8.2 The following are not constructively valid!

The attempt to prove them terminates quickly!

lemma ‹$((\forall x.\ P(x)) \longrightarrow Q) \longrightarrow (\exists x.\ P(x) \longrightarrow Q)$›
⟨*proof*⟩

lemma ‹$(P \longrightarrow (\exists x.\ Q(x))) \longrightarrow (\exists x.\ P \longrightarrow Q(x))$›
⟨*proof*⟩

lemma ‹$(\forall x.\ P(x) \lor Q) \longrightarrow ((\forall x.\ P(x)) \lor Q)$›
⟨*proof*⟩

lemma ‹$(\forall x.\ \neg\ \neg\ P(x)) \longrightarrow \neg\ \neg\ (\forall x.\ P(x))$›
⟨*proof*⟩

Classically but not intuitionistically valid. Proved by a bug in 1986!

lemma ‹$\exists x.\ Q(x) \longrightarrow (\forall x.\ Q(x))$›
⟨*proof*⟩

8.3 Hard examples with quantifiers

The ones that have not been proved are not known to be valid! Some will require quantifier duplication – not currently available.

¬¬18

lemma ‹$\neg\ \neg\ (\exists y.\ \forall x.\ P(y) \longrightarrow P(x))$›
⟨*proof*⟩

¬¬19

lemma ‹$\neg\ \neg\ (\exists x.\ \forall y\ z.\ (P(y) \longrightarrow Q(z)) \longrightarrow (P(x) \longrightarrow Q(x)))$›
⟨*proof*⟩

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lemma
‹$(\forall x\ y.\ \exists z.\ \forall w.\ (P(x) \land Q(y) \longrightarrow R(z) \land S(w)))$
$\longrightarrow (\exists x\ y.\ P(x) \land Q(y)) \longrightarrow (\exists z.\ R(z))$›
⟨*proof*⟩

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lemma ‹$(\exists x.\ P \longrightarrow Q(x)) \wedge (\exists x.\ Q(x) \longrightarrow P) \longrightarrow \neg\ \neg\ (\exists x.\ P \longleftrightarrow Q(x))$›
$\langle proof \rangle$

22

lemma ‹$(\forall x.\ P \longleftrightarrow Q(x)) \longrightarrow (P \longleftrightarrow (\forall x.\ Q(x)))$›
$\langle proof \rangle$

¬¬23

lemma ‹$\neg\ \neg\ ((\forall x.\ P \vee Q(x)) \longleftrightarrow (P \vee (\forall x.\ Q(x))))$›
$\langle proof \rangle$

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lemma
‹$\neg\ (\exists x.\ S(x) \wedge Q(x)) \wedge (\forall x.\ P(x) \longrightarrow Q(x) \vee R(x)) \wedge$
$(\neg\ (\exists x.\ P(x)) \longrightarrow (\exists x.\ Q(x))) \wedge (\forall x.\ Q(x) \vee R(x) \longrightarrow S(x))$
$\longrightarrow \neg\ \neg\ (\exists x.\ P(x) \wedge R(x))$›

Not clear why *fast-tac*, *best-tac*, *ASTAR* and *ITER-DEEPEN* all take forever.

$\langle proof \rangle$

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lemma
‹$(\exists x.\ P(x)) \wedge$
$(\forall x.\ L(x) \longrightarrow \neg\ (M(x) \wedge R(x))) \wedge$
$(\forall x.\ P(x) \longrightarrow (M(x) \wedge L(x))) \wedge$
$((\forall x.\ P(x) \longrightarrow Q(x)) \vee (\exists x.\ P(x) \wedge R(x)))$
$\longrightarrow (\exists x.\ Q(x) \wedge P(x))$›
$\langle proof \rangle$

¬¬26

lemma
‹$(\neg\ \neg\ (\exists x.\ p(x)) \longleftrightarrow \neg\ \neg\ (\exists x.\ q(x))) \wedge$
$(\forall x.\ \forall y.\ p(x) \wedge q(y) \longrightarrow (r(x) \longleftrightarrow s(y)))$
$\longrightarrow ((\forall x.\ p(x) \longrightarrow r(x)) \longleftrightarrow (\forall x.\ q(x) \longrightarrow s(x)))$›
$\langle proof \rangle$

27

lemma
‹$(\exists x.\ P(x) \wedge \neg\ Q(x)) \wedge$
$(\forall x.\ P(x) \longrightarrow R(x)) \wedge$
$(\forall x.\ M(x) \wedge L(x) \longrightarrow P(x)) \wedge$
$((\exists x.\ R(x) \wedge \neg\ Q(x)) \longrightarrow (\forall x.\ L(x) \longrightarrow \neg\ R(x)))$
$\longrightarrow (\forall x.\ M(x) \longrightarrow \neg\ L(x))$›
$\langle proof \rangle$

¬¬28. AMENDED

lemma

$\langle(\forall x.\ P(x) \longrightarrow (\forall x.\ Q(x))) \land$
$(\neg\ \neg\ (\forall x.\ Q(x) \lor R(x)) \longrightarrow (\exists x.\ Q(x) \land S(x))) \land$
$(\neg\ \neg\ (\exists x.\ S(x)) \longrightarrow (\forall x.\ L(x) \longrightarrow M(x)))$
$\longrightarrow (\forall x.\ P(x) \land L(x) \longrightarrow M(x))\rangle$
$\langle proof \rangle$

29. Essentially the same as Principia Mathematica *11.71

lemma
$\langle(\exists x.\ P(x)) \land (\exists y.\ Q(y))$
$\longrightarrow ((\forall x.\ P(x) \longrightarrow R(x)) \land (\forall y.\ Q(y) \longrightarrow S(y)) \longleftrightarrow$
$(\forall x\ y.\ P(x) \land Q(y) \longrightarrow R(x) \land S(y)))\rangle$
$\langle proof \rangle$

¬¬30

lemma
$\langle(\forall x.\ (P(x) \lor Q(x)) \longrightarrow \neg\ R(x)) \land$
$(\forall x.\ (Q(x) \longrightarrow \neg\ S(x)) \longrightarrow P(x) \land R(x))$
$\longrightarrow (\forall x.\ \neg\ \neg\ S(x))\rangle$
$\langle proof \rangle$

31

lemma
$\langle\neg\ (\exists x.\ P(x) \land (Q(x) \lor R(x))) \land$
$(\exists x.\ L(x) \land P(x)) \land$
$(\forall x.\ \neg\ R(x) \longrightarrow M(x))$
$\longrightarrow (\exists x.\ L(x) \land M(x))\rangle$
$\langle proof \rangle$

32

lemma
$\langle(\forall x.\ P(x) \land (Q(x) \lor R(x)) \longrightarrow S(x)) \land$
$(\forall x.\ S(x) \land R(x) \longrightarrow L(x)) \land$
$(\forall x.\ M(x) \longrightarrow R(x))$
$\longrightarrow (\forall x.\ P(x) \land M(x) \longrightarrow L(x))\rangle$
$\langle proof \rangle$

¬¬33

lemma
$\langle(\forall x.\ \neg\ \neg\ (P(a) \land (P(x) \longrightarrow P(b)) \longrightarrow P(c))) \longleftrightarrow$
$(\forall x.\ \neg\ \neg\ ((\neg\ P(a) \lor P(x) \lor P(c)) \land (\neg\ P(a) \lor \neg\ P(b) \lor P(c))))\rangle$
$\langle proof \rangle$

36

lemma
$\langle(\forall x.\ \exists y.\ J(x,y)) \land$
$(\forall x.\ \exists y.\ G(x,y)) \land$
$(\forall x\ y.\ J(x,y) \lor G(x,y) \longrightarrow (\forall z.\ J(y,z) \lor G(y,z) \longrightarrow H(x,z)))$
$\longrightarrow (\forall x.\ \exists y.\ H(x,y))\rangle$

$\langle proof \rangle$

37

lemma
‹$(\forall z.\ \exists w.\ \forall x.\ \exists y.$
$\neg\ \neg\ (P(x,z) \longrightarrow P(y,w)) \wedge P(y,z) \wedge (P(y,w) \longrightarrow (\exists u.\ Q(u,w)))) \wedge$
$(\forall x\ z.\ \neg\ P(x,z) \longrightarrow (\exists y.\ Q(y,z))) \wedge$
$(\neg\ \neg\ (\exists x\ y.\ Q(x,y)) \longrightarrow (\forall x.\ R(x,x)))$
$\longrightarrow \neg\ \neg\ (\forall x.\ \exists y.\ R(x,y))$›
$\langle proof \rangle$

39

lemma ‹$\neg\ (\exists x.\ \forall y.\ F(y,x) \longleftrightarrow \neg\ F(y,y))$›
$\langle proof \rangle$

40. AMENDED

lemma
‹$(\exists y.\ \forall x.\ F(x,y) \longleftrightarrow F(x,x)) \longrightarrow$
$\neg\ (\forall x.\ \exists y.\ \forall z.\ F(z,y) \longleftrightarrow \neg\ F(z,x))$›
$\langle proof \rangle$

44

lemma
‹$(\forall x.\ f(x) \longrightarrow$
$(\exists y.\ g(y) \wedge h(x,y) \wedge (\exists y.\ g(y) \wedge \neg\ h(x,y)))) \wedge$
$(\exists x.\ j(x) \wedge (\forall y.\ g(y) \longrightarrow h(x,y)))$
$\longrightarrow (\exists x.\ j(x) \wedge \neg\ f(x))$›
$\langle proof \rangle$

48

lemma ‹$(a = b \vee c = d) \wedge (a = c \vee b = d) \longrightarrow a = d \vee b = c$›
$\langle proof \rangle$

51

lemma
‹$(\exists z\ w.\ \forall x\ y.\ P(x,y) \longleftrightarrow (x = z \wedge y = w)) \longrightarrow$
$(\exists z.\ \forall x.\ \exists w.\ (\forall y.\ P(x,y) \longleftrightarrow y = w) \longleftrightarrow x = z)$›
$\langle proof \rangle$

52

Almost the same as 51.

lemma
‹$(\exists z\ w.\ \forall x\ y.\ P(x,y) \longleftrightarrow (x = z \wedge y = w)) \longrightarrow$
$(\exists w.\ \forall y.\ \exists z.\ (\forall x.\ P(x,y) \longleftrightarrow x = z) \longleftrightarrow y = w)$›
$\langle proof \rangle$

56

lemma ‹$(\forall x.\ (\exists y.\ P(y) \land x = f(y)) \longrightarrow P(x)) \longleftrightarrow (\forall x.\ P(x) \longrightarrow P(f(x)))$›
⟨*proof*⟩

57

lemma
‹$P(f(a,b),\ f(b,c)) \land P(f(b,c),\ f(a,c)) \land$
$(\forall x\ y\ z.\ P(x,y) \land P(y,z) \longrightarrow P(x,z)) \longrightarrow P(f(a,b),\ f(a,c))$›
⟨*proof*⟩

60

lemma ‹$\forall x.\ P(x,f(x)) \longleftrightarrow (\exists y.\ (\forall z.\ P(z,y) \longrightarrow P(z,f(x))) \land P(x,y))$›
⟨*proof*⟩

end

9 First-Order Logic: propositional examples (intuitionistic version)

theory *Propositional-Int*
imports *IFOL*
begin

commutative laws of ∧ and ∨

lemma ‹$P \land Q \longrightarrow Q \land P$›
⟨*proof*⟩

lemma ‹$P \lor Q \longrightarrow Q \lor P$›
⟨*proof*⟩

associative laws of ∧ and ∨

lemma ‹$(P \land Q) \land R \longrightarrow P \land (Q \land R)$›
⟨*proof*⟩

lemma ‹$(P \lor Q) \lor R \longrightarrow P \lor (Q \lor R)$›
⟨*proof*⟩

distributive laws of ∧ and ∨

lemma ‹$(P \land Q) \lor R \longrightarrow (P \lor R) \land (Q \lor R)$›
⟨*proof*⟩

lemma ‹$(P \lor R) \land (Q \lor R) \longrightarrow (P \land Q) \lor R$›
⟨*proof*⟩

lemma ‹$(P \lor Q) \land R \longrightarrow (P \land R) \lor (Q \land R)$›
⟨*proof*⟩

lemma ‹$(P \land R) \lor (Q \land R) \longrightarrow (P \lor Q) \land R$›

$\langle proof \rangle$

Laws involving implication

lemma ‹$(P \longrightarrow R) \wedge (Q \longrightarrow R) \longleftrightarrow (P \vee Q \longrightarrow R)$›
$\langle proof \rangle$

lemma ‹$(P \wedge Q \longrightarrow R) \longleftrightarrow (P \longrightarrow (Q \longrightarrow R))$›
$\langle proof \rangle$

lemma ‹$((P \longrightarrow R) \longrightarrow R) \longrightarrow ((Q \longrightarrow R) \longrightarrow R) \longrightarrow (P \wedge Q \longrightarrow R) \longrightarrow R$›
$\langle proof \rangle$

lemma ‹$\neg (P \longrightarrow R) \longrightarrow \neg (Q \longrightarrow R) \longrightarrow \neg (P \wedge Q \longrightarrow R)$›
$\langle proof \rangle$

lemma ‹$(P \longrightarrow Q \wedge R) \longleftrightarrow (P \longrightarrow Q) \wedge (P \longrightarrow R)$›
$\langle proof \rangle$

Propositions-as-types

lemma ‹$P \longrightarrow (Q \longrightarrow P)$›
$\langle proof \rangle$
lemma ‹$(P \longrightarrow Q \longrightarrow R) \longrightarrow (P \longrightarrow Q) \longrightarrow (P \longrightarrow R)$›
$\langle proof \rangle$
lemma ‹$(P \longrightarrow Q) \vee (P \longrightarrow R) \longrightarrow (P \longrightarrow Q \vee R)$›
$\langle proof \rangle$

lemma ‹$(P \longrightarrow Q) \longrightarrow (\neg Q \longrightarrow \neg P)$›
$\langle proof \rangle$

Schwichtenberg's examples (via T. Nipkow)

lemma *stab-imp*: ‹$(((Q \longrightarrow R) \longrightarrow R) \longrightarrow Q) \longrightarrow (((P \longrightarrow Q) \longrightarrow R) \longrightarrow R)$
$\longrightarrow P \longrightarrow Q$›
$\langle proof \rangle$

lemma *stab-to-peirce*:
‹$(((P \longrightarrow R) \longrightarrow R) \longrightarrow P) \longrightarrow (((Q \longrightarrow R) \longrightarrow R) \longrightarrow Q)$
$\longrightarrow ((P \longrightarrow Q) \longrightarrow P) \longrightarrow P$›
$\langle proof \rangle$

lemma *peirce-imp1*:
‹$(((Q \longrightarrow R) \longrightarrow Q) \longrightarrow Q)$
$\longrightarrow (((P \longrightarrow Q) \longrightarrow R) \longrightarrow P \longrightarrow Q) \longrightarrow P \longrightarrow Q$›
$\langle proof \rangle$

lemma *peirce-imp2*: ‹$(((P \longrightarrow R) \longrightarrow P) \longrightarrow P) \longrightarrow ((P \longrightarrow Q \longrightarrow R) \longrightarrow P)$
$\longrightarrow P$›
$\langle proof \rangle$

lemma *mints*: ‹$((((P \longrightarrow Q) \longrightarrow P) \longrightarrow P) \longrightarrow Q) \longrightarrow Q$›

⟨*proof*⟩

lemma *mints-solovev*: ‹$(P \longrightarrow (Q \longrightarrow R) \longrightarrow Q) \longrightarrow ((P \longrightarrow Q) \longrightarrow R) \longrightarrow R$›
⟨*proof*⟩

lemma *tatsuta*:
‹$(((P7 \longrightarrow P1) \longrightarrow P10) \longrightarrow P4 \longrightarrow P5)$
$\longrightarrow (((P8 \longrightarrow P2) \longrightarrow P9) \longrightarrow P3 \longrightarrow P10)$
$\longrightarrow (P1 \longrightarrow P8) \longrightarrow P6 \longrightarrow P7$
$\longrightarrow (((P3 \longrightarrow P2) \longrightarrow P9) \longrightarrow P4)$
$\longrightarrow (P1 \longrightarrow P3) \longrightarrow (((P6 \longrightarrow P1) \longrightarrow P2) \longrightarrow P9) \longrightarrow P5$›
⟨*proof*⟩

lemma *tatsuta1*:
‹$(((P8 \longrightarrow P2) \longrightarrow P9) \longrightarrow P3 \longrightarrow P10)$
$\longrightarrow (((P3 \longrightarrow P2) \longrightarrow P9) \longrightarrow P4)$
$\longrightarrow (((P6 \longrightarrow P1) \longrightarrow P2) \longrightarrow P9)$
$\longrightarrow (((P7 \longrightarrow P1) \longrightarrow P10) \longrightarrow P4 \longrightarrow P5)$
$\longrightarrow (P1 \longrightarrow P3) \longrightarrow (P1 \longrightarrow P8) \longrightarrow P6 \longrightarrow P7 \longrightarrow P5$›
⟨*proof*⟩

end

10 First-Order Logic: quantifier examples (intuitionistic version)

theory *Quantifiers-Int*
imports *IFOL*
begin

lemma ‹$(\forall x\ y.\ P(x,y)) \longrightarrow (\forall y\ x.\ P(x,y))$›
⟨*proof*⟩

lemma ‹$(\exists x\ y.\ P(x,y)) \longrightarrow (\exists y\ x.\ P(x,y))$›
⟨*proof*⟩
lemma ‹$(\forall x.\ P(x)) \lor (\forall x.\ Q(x)) \longrightarrow (\forall x.\ P(x) \lor Q(x))$›
⟨*proof*⟩

lemma ‹$(\forall x.\ P \longrightarrow Q(x)) \longleftrightarrow (P \longrightarrow (\forall x.\ Q(x)))$›
⟨*proof*⟩

lemma ‹$(\forall x.\ P(x) \longrightarrow Q) \longleftrightarrow ((\exists x.\ P(x)) \longrightarrow Q)$›
⟨*proof*⟩

Some harder ones

lemma ‹$(\exists x.\ P(x) \lor Q(x)) \longleftrightarrow (\exists x.\ P(x)) \lor (\exists x.\ Q(x))$›
⟨*proof*⟩

lemma ‹$(\exists x.\ P(x) \wedge Q(x)) \longrightarrow (\exists x.\ P(x)) \wedge (\exists x.\ Q(x))$›
⟨*proof*⟩

Basic test of quantifier reasoning

lemma ‹$(\exists y.\ \forall x.\ Q(x,y)) \longrightarrow (\forall x.\ \exists y.\ Q(x,y))$›
⟨*proof*⟩

lemma ‹$(\forall x.\ Q(x)) \longrightarrow (\exists x.\ Q(x))$›
⟨*proof*⟩

The following should fail, as they are false!

lemma ‹$(\forall x.\ \exists y.\ Q(x,y)) \longrightarrow (\exists y.\ \forall x.\ Q(x,y))$›
⟨*proof*⟩

lemma ‹$(\exists x.\ Q(x)) \longrightarrow (\forall x.\ Q(x))$›
⟨*proof*⟩

schematic-goal ‹$P(?a) \longrightarrow (\forall x.\ P(x))$›
⟨*proof*⟩

schematic-goal ‹$(P(?a) \longrightarrow (\forall x.\ Q(x))) \longrightarrow (\forall x.\ P(x) \longrightarrow Q(x))$›
⟨*proof*⟩

Back to things that are provable . . .

lemma ‹$(\forall x.\ P(x) \longrightarrow Q(x)) \wedge (\exists x.\ P(x)) \longrightarrow (\exists x.\ Q(x))$›
⟨*proof*⟩
lemma ‹$(P \longrightarrow (\exists x.\ Q(x))) \wedge P \longrightarrow (\exists x.\ Q(x))$›
⟨*proof*⟩

schematic-goal ‹$(\forall x.\ P(x) \longrightarrow Q(f(x))) \wedge (\forall x.\ Q(x) \longrightarrow R(g(x))) \wedge P(d) \longrightarrow R(?a)$›
⟨*proof*⟩

lemma ‹$(\forall x.\ Q(x)) \longrightarrow (\exists x.\ Q(x))$›
⟨*proof*⟩

Some slow ones

lemma ‹$(\forall x\ y.\ P(x) \longrightarrow Q(y)) \longleftrightarrow ((\exists x.\ P(x)) \longrightarrow (\forall y.\ Q(y)))$›
⟨*proof*⟩

lemma ‹$(\exists x\ y.\ P(x) \wedge Q(x,y)) \longleftrightarrow (\exists x.\ P(x) \wedge (\exists y.\ Q(x,y)))$›
⟨*proof*⟩

lemma ‹$(\exists y.\ \forall x.\ P(x) \longrightarrow Q(x,y)) \longrightarrow (\forall x.\ P(x) \longrightarrow (\exists y.\ Q(x,y)))$›
⟨*proof*⟩

end

11 Classical Predicate Calculus Problems

theory *Classical*
imports *FOL*
begin

lemma ‹$(P \longrightarrow Q \vee R) \longrightarrow (P \longrightarrow Q) \vee (P \longrightarrow R)$›
⟨*proof*⟩

11.0.1 If and only if

lemma ‹$(P \longleftrightarrow Q) \longleftrightarrow (Q \longleftrightarrow P)$›
⟨*proof*⟩

lemma ‹$\neg (P \longleftrightarrow \neg P)$›
⟨*proof*⟩

11.1 Pelletier's examples

Sample problems from

- F. J. Pelletier, Seventy-Five Problems for Testing Automatic Theorem Provers, J. Automated Reasoning 2 (1986), 191-216. Errata, JAR 4 (1988), 236-236.

The hardest problems – judging by experience with several theorem provers, including matrix ones – are 34 and 43.

1

lemma ‹$(P \longrightarrow Q) \longleftrightarrow (\neg Q \longrightarrow \neg P)$›
⟨*proof*⟩

2

lemma ‹$\neg \neg P \longleftrightarrow P$›
⟨*proof*⟩

3

lemma ‹$\neg (P \longrightarrow Q) \longrightarrow (Q \longrightarrow P)$›
⟨*proof*⟩

4

lemma ‹$(\neg P \longrightarrow Q) \longleftrightarrow (\neg Q \longrightarrow P)$›
⟨*proof*⟩

5

lemma ‹$((P \vee Q) \longrightarrow (P \vee R)) \longrightarrow (P \vee (Q \longrightarrow R))$›
⟨*proof*⟩

6

lemma ‹$P \vee \neg P$›
  ⟨*proof*⟩

7

lemma ‹$P \vee \neg \neg \neg P$›
  ⟨*proof*⟩

8. Peirce's law

lemma ‹$((P \longrightarrow Q) \longrightarrow P) \longrightarrow P$›
  ⟨*proof*⟩

9

lemma ‹$((P \vee Q) \wedge (\neg P \vee Q) \wedge (P \vee \neg Q)) \longrightarrow \neg (\neg P \vee \neg Q)$›
  ⟨*proof*⟩

10

lemma ‹$(Q \longrightarrow R) \wedge (R \longrightarrow P \wedge Q) \wedge (P \longrightarrow Q \vee R) \longrightarrow (P \longleftrightarrow Q)$›
  ⟨*proof*⟩

11. Proved in each direction (incorrectly, says Pelletier!!)

lemma ‹$P \longleftrightarrow P$›
  ⟨*proof*⟩

12. "Dijkstra's law"

lemma ‹$((P \longleftrightarrow Q) \longleftrightarrow R) \longleftrightarrow (P \longleftrightarrow (Q \longleftrightarrow R))$›
  ⟨*proof*⟩

13. Distributive law

lemma ‹$P \vee (Q \wedge R) \longleftrightarrow (P \vee Q) \wedge (P \vee R)$›
  ⟨*proof*⟩

14

lemma ‹$(P \longleftrightarrow Q) \longleftrightarrow ((Q \vee \neg P) \wedge (\neg Q \vee P))$›
  ⟨*proof*⟩

15

lemma ‹$(P \longrightarrow Q) \longleftrightarrow (\neg P \vee Q)$›
  ⟨*proof*⟩

16

lemma ‹$(P \longrightarrow Q) \vee (Q \longrightarrow P)$›
  ⟨*proof*⟩

17

lemma ‹$((P \wedge (Q \longrightarrow R)) \longrightarrow S) \longleftrightarrow ((\neg P \vee Q \vee S) \wedge (\neg P \vee \neg R \vee S))$›
  ⟨*proof*⟩

11.2 Classical Logic: examples with quantifiers

lemma ‹$(\forall x.\ P(x) \wedge Q(x)) \longleftrightarrow (\forall x.\ P(x)) \wedge (\forall x.\ Q(x))$›
  ⟨*proof*⟩

lemma ‹$(\exists x.\ P \longrightarrow Q(x)) \longleftrightarrow (P \longrightarrow (\exists x.\ Q(x)))$›
  ⟨*proof*⟩

lemma ‹$(\exists x.\ P(x) \longrightarrow Q) \longleftrightarrow (\forall x.\ P(x)) \longrightarrow Q$›
  ⟨*proof*⟩

lemma ‹$(\forall x.\ P(x)) \vee Q \longleftrightarrow (\forall x.\ P(x) \vee Q)$›
  ⟨*proof*⟩

Discussed in Avron, Gentzen-Type Systems, Resolution and Tableaux, JAR 10 (265-281), 1993. Proof is trivial!

lemma ‹$\neg ((\exists x.\ \neg P(x)) \wedge ((\exists x.\ P(x)) \vee (\exists x.\ P(x) \wedge Q(x))) \wedge \neg (\exists x.\ P(x)))$›
  ⟨*proof*⟩

11.3 Problems requiring quantifier duplication

Theorem B of Peter Andrews, Theorem Proving via General Matings, JACM 28 (1981).

lemma ‹$(\exists x.\ \forall y.\ P(x) \longleftrightarrow P(y)) \longrightarrow ((\exists x.\ P(x)) \longleftrightarrow (\forall y.\ P(y)))$›
  ⟨*proof*⟩

Needs multiple instantiation of ALL.

lemma ‹$(\forall x.\ P(x) \longrightarrow P(f(x))) \wedge P(d) \longrightarrow P(f(f(f(d))))$›
  ⟨*proof*⟩

Needs double instantiation of the quantifier

lemma ‹$\exists x.\ P(x) \longrightarrow P(a) \wedge P(b)$›
  ⟨*proof*⟩

lemma ‹$\exists z.\ P(z) \longrightarrow (\forall x.\ P(x))$›
  ⟨*proof*⟩

lemma ‹$\exists x.\ (\exists y.\ P(y)) \longrightarrow P(x)$›
  ⟨*proof*⟩

V. Lifschitz, What Is the Inverse Method?, JAR 5 (1989), 1–23. NOT PROVED.

lemma
  ‹$\exists x\ x'.\ \forall y.\ \exists z\ z'.$
    $(\neg P(y,y) \vee P(x,x) \vee \neg S(z,x)) \wedge$
    $(S(x,y) \vee \neg S(y,z) \vee Q(z',z')) \wedge$
    $(Q(x',y) \vee \neg Q(y,z') \vee S(x',x'))$›
  ⟨*proof*⟩

11.4 Hard examples with quantifiers

18

lemma ‹$\exists y.\ \forall x.\ P(y) \longrightarrow P(x)$›
  ⟨*proof*⟩

19

lemma ‹$\exists x.\ \forall y\ z.\ (P(y) \longrightarrow Q(z)) \longrightarrow (P(x) \longrightarrow Q(x))$›
  ⟨*proof*⟩

20

lemma ‹$(\forall x\ y.\ \exists z.\ \forall w.\ (P(x) \land Q(y) \longrightarrow R(z) \land S(w)))$
  $\longrightarrow (\exists x\ y.\ P(x) \land Q(y)) \longrightarrow (\exists z.\ R(z))$›
  ⟨*proof*⟩

21

lemma ‹$(\exists x.\ P \longrightarrow Q(x)) \land (\exists x.\ Q(x) \longrightarrow P) \longrightarrow (\exists x.\ P \longleftrightarrow Q(x))$›
  ⟨*proof*⟩

22

lemma ‹$(\forall x.\ P \longleftrightarrow Q(x)) \longrightarrow (P \longleftrightarrow (\forall x.\ Q(x)))$›
  ⟨*proof*⟩

23

lemma ‹$(\forall x.\ P \lor Q(x)) \longleftrightarrow (P \lor (\forall x.\ Q(x)))$›
  ⟨*proof*⟩

24

lemma
  ‹$\neg\ (\exists x.\ S(x) \land Q(x)) \land (\forall x.\ P(x) \longrightarrow Q(x) \lor R(x)) \land$
    $(\neg\ (\exists x.\ P(x)) \longrightarrow (\exists x.\ Q(x))) \land (\forall x.\ Q(x) \lor R(x) \longrightarrow S(x))$
    $\longrightarrow (\exists x.\ P(x) \land R(x))$›
  ⟨*proof*⟩

25

lemma
  ‹$(\exists x.\ P(x)) \land$
    $(\forall x.\ L(x) \longrightarrow \neg\ (M(x) \land R(x))) \land$
    $(\forall x.\ P(x) \longrightarrow (M(x) \land L(x))) \land$
    $((\forall x.\ P(x) \longrightarrow Q(x)) \lor (\exists x.\ P(x) \land R(x)))$
    $\longrightarrow (\exists x.\ Q(x) \land P(x))$›
  ⟨*proof*⟩

26

lemma
  ‹$((\exists x.\ p(x)) \longleftrightarrow (\exists x.\ q(x))) \land$
    $(\forall x.\ \forall y.\ p(x) \land q(y) \longrightarrow (r(x) \longleftrightarrow s(y)))$

$\longrightarrow ((\forall x.\ p(x) \longrightarrow r(x)) \longleftrightarrow (\forall x.\ q(x) \longrightarrow s(x)))$›
$\langle proof \rangle$

27

lemma
‹$(\exists x.\ P(x) \wedge \neg\ Q(x)) \wedge$
$(\forall x.\ P(x) \longrightarrow R(x)) \wedge$
$(\forall x.\ M(x) \wedge L(x) \longrightarrow P(x)) \wedge$
$((\exists x.\ R(x) \wedge \neg\ Q(x)) \longrightarrow (\forall x.\ L(x) \longrightarrow \neg\ R(x)))$
$\longrightarrow (\forall x.\ M(x) \longrightarrow \neg\ L(x))$›
$\langle proof \rangle$

28. AMENDED

lemma
‹$(\forall x.\ P(x) \longrightarrow (\forall x.\ Q(x))) \wedge$
$((\forall x.\ Q(x) \vee R(x)) \longrightarrow (\exists x.\ Q(x) \wedge S(x))) \wedge$
$((\exists x.\ S(x)) \longrightarrow (\forall x.\ L(x) \longrightarrow M(x)))$
$\longrightarrow (\forall x.\ P(x) \wedge L(x) \longrightarrow M(x))$›
$\langle proof \rangle$

29. Essentially the same as Principia Mathematica *11.71

lemma
‹$(\exists x.\ P(x)) \wedge (\exists y.\ Q(y))$
$\longrightarrow ((\forall x.\ P(x) \longrightarrow R(x)) \wedge (\forall y.\ Q(y) \longrightarrow S(y)) \longleftrightarrow$
$(\forall x\ y.\ P(x) \wedge Q(y) \longrightarrow R(x) \wedge S(y)))$›
$\langle proof \rangle$

30

lemma
‹$(\forall x.\ P(x) \vee Q(x) \longrightarrow \neg\ R(x)) \wedge$
$(\forall x.\ (Q(x) \longrightarrow \neg\ S(x)) \longrightarrow P(x) \wedge R(x))$
$\longrightarrow (\forall x.\ S(x))$›
$\langle proof \rangle$

31

lemma
‹$\neg\ (\exists x.\ P(x) \wedge (Q(x) \vee R(x))) \wedge$
$(\exists x.\ L(x) \wedge P(x)) \wedge$
$(\forall x.\ \neg\ R(x) \longrightarrow M(x))$
$\longrightarrow (\exists x.\ L(x) \wedge M(x))$›
$\langle proof \rangle$

32

lemma
‹$(\forall x.\ P(x) \wedge (Q(x) \vee R(x)) \longrightarrow S(x)) \wedge$
$(\forall x.\ S(x) \wedge R(x) \longrightarrow L(x)) \wedge$
$(\forall x.\ M(x) \longrightarrow R(x))$
$\longrightarrow (\forall x.\ P(x) \wedge M(x) \longrightarrow L(x))$›

⟨*proof*⟩

33

lemma
‹$(\forall x.\ P(a) \wedge (P(x) \longrightarrow P(b)) \longrightarrow P(c)) \longleftrightarrow$
$(\forall x.\ (\neg\ P(a) \vee P(x) \vee P(c)) \wedge (\neg\ P(a) \vee \neg\ P(b) \vee P(c)))$›
⟨*proof*⟩

34. AMENDED (TWICE!!). Andrews's challenge.

lemma
‹$((\exists x.\ \forall y.\ p(x) \longleftrightarrow p(y)) \longleftrightarrow ((\exists x.\ q(x)) \longleftrightarrow (\forall y.\ p(y)))) \longleftrightarrow$
$((\exists x.\ \forall y.\ q(x) \longleftrightarrow q(y)) \longleftrightarrow ((\exists x.\ p(x)) \longleftrightarrow (\forall y.\ q(y))))$›
⟨*proof*⟩

35

lemma ‹$\exists x\ y.\ P(x,y) \longrightarrow (\forall u\ v.\ P(u,v))$›
⟨*proof*⟩

36

lemma
‹$(\forall x.\ \exists y.\ J(x,y)) \wedge$
$(\forall x.\ \exists y.\ G(x,y)) \wedge$
$(\forall x\ y.\ J(x,y) \vee G(x,y) \longrightarrow (\forall z.\ J(y,z) \vee G(y,z) \longrightarrow H(x,z)))$
$\longrightarrow (\forall x.\ \exists y.\ H(x,y))$›
⟨*proof*⟩

37

lemma
‹$(\forall z.\ \exists w.\ \forall x.\ \exists y.$
$(P(x,z) \longrightarrow P(y,w)) \wedge P(y,z) \wedge (P(y,w) \longrightarrow (\exists u.\ Q(u,w)))) \wedge$
$(\forall x\ z.\ \neg\ P(x,z) \longrightarrow (\exists y.\ Q(y,z))) \wedge$
$((\exists x\ y.\ Q(x,y)) \longrightarrow (\forall x.\ R(x,x)))$
$\longrightarrow (\forall x.\ \exists y.\ R(x,y))$›
⟨*proof*⟩

38

lemma
‹$(\forall x.\ p(a) \wedge (p(x) \longrightarrow (\exists y.\ p(y) \wedge r(x,y))) \longrightarrow$
$(\exists z.\ \exists w.\ p(z) \wedge r(x,w) \wedge r(w,z)))\ \longleftrightarrow$
$(\forall x.\ (\neg\ p(a) \vee p(x) \vee (\exists z.\ \exists w.\ p(z) \wedge r(x,w) \wedge r(w,z))) \wedge$
$(\neg\ p(a) \vee \neg\ (\exists y.\ p(y) \wedge r(x,y)) \vee$
$(\exists z.\ \exists w.\ p(z) \wedge r(x,w) \wedge r(w,z))))$›
⟨*proof*⟩

39

lemma ‹$\neg\ (\exists x.\ \forall y.\ F(y,x) \longleftrightarrow \neg\ F(y,y))$›
⟨*proof*⟩

40. AMENDED

lemma
$\langle(\exists y.\ \forall x.\ F(x,y) \longleftrightarrow F(x,x)) \longrightarrow$
$\neg\ (\forall x.\ \exists y.\ \forall z.\ F(z,y) \longleftrightarrow \neg\ F(z,x))\rangle$
$\langle proof\rangle$

41

lemma
$\langle(\forall z.\ \exists y.\ \forall x.\ f(x,y) \longleftrightarrow f(x,z) \wedge \neg\ f(x,x))$
$\longrightarrow \neg\ (\exists z.\ \forall x.\ f(x,z))\rangle$
$\langle proof\rangle$

42

lemma $\langle\neg\ (\exists y.\ \forall x.\ p(x,y) \longleftrightarrow \neg\ (\exists z.\ p(x,z) \wedge p(z,x)))\rangle$
$\langle proof\rangle$

43

lemma
$\langle(\forall x.\ \forall y.\ q(x,y) \longleftrightarrow (\forall z.\ p(z,x) \longleftrightarrow p(z,y)))$
$\longrightarrow (\forall x.\ \forall y.\ q(x,y) \longleftrightarrow q(y,x))\rangle$
$\langle proof\rangle$

Other proofs: Can use *auto*, which cheats by using rewriting! *Deepen-tac* alone requires 253 secs. Or *by* (*mini-tac 1 THEN Deepen-tac 5 1*).

44

lemma
$\langle(\forall x.\ f(x) \longrightarrow (\exists y.\ g(y) \wedge h(x,y) \wedge (\exists y.\ g(y) \wedge \neg\ h(x,y)))) \wedge$
$(\exists x.\ j(x) \wedge (\forall y.\ g(y) \longrightarrow h(x,y)))$
$\longrightarrow (\exists x.\ j(x) \wedge \neg\ f(x))\rangle$
$\langle proof\rangle$

45

lemma
$\langle(\forall x.\ f(x) \wedge (\forall y.\ g(y) \wedge h(x,y) \longrightarrow j(x,y))$
$\longrightarrow (\forall y.\ g(y) \wedge h(x,y) \longrightarrow k(y))) \wedge$
$\neg\ (\exists y.\ l(y) \wedge k(y)) \wedge$
$(\exists x.\ f(x) \wedge (\forall y.\ h(x,y) \longrightarrow l(y)) \wedge (\forall y.\ g(y) \wedge h(x,y) \longrightarrow j(x,y)))$
$\longrightarrow (\exists x.\ f(x) \wedge \neg\ (\exists y.\ g(y) \wedge h(x,y)))\rangle$
$\langle proof\rangle$

46

lemma
$\langle(\forall x.\ f(x) \wedge (\forall y.\ f(y) \wedge h(y,x) \longrightarrow g(y)) \longrightarrow g(x)) \wedge$
$((\exists x.\ f(x) \wedge \neg\ g(x)) \longrightarrow$
$(\exists x.\ f(x) \wedge \neg\ g(x) \wedge (\forall y.\ f(y) \wedge \neg\ g(y) \longrightarrow j(x,y)))) \wedge$
$(\forall x\ y.\ f(x) \wedge f(y) \wedge h(x,y) \longrightarrow \neg\ j(y,x))$
$\longrightarrow (\forall x.\ f(x) \longrightarrow g(x))\rangle$
$\langle proof\rangle$

11.5 Problems (mainly) involving equality or functions

48

lemma ‹$(a = b \lor c = d) \land (a = c \lor b = d) \longrightarrow a = d \lor b = c$›
⟨*proof*⟩

49. NOT PROVED AUTOMATICALLY. Hard because it involves substitution for Vars; the type constraint ensures that x,y,z have the same type as a,b,u.

lemma
‹$(\exists x\ y{::}'a.\ \forall z.\ z = x \lor z = y) \land P(a) \land P(b) \land a \neq b \longrightarrow (\forall u{::}'a.\ P(u))$›
⟨*proof*⟩

50. (What has this to do with equality?)

lemma ‹$(\forall x.\ P(a,x) \lor (\forall y.\ P(x,y))) \longrightarrow (\exists x.\ \forall y.\ P(x,y))$›
⟨*proof*⟩

51

lemma
‹$(\exists z\ w.\ \forall x\ y.\ P(x,y) \longleftrightarrow (x = z \land y = w)) \longrightarrow$
$(\exists z.\ \forall x.\ \exists w.\ (\forall y.\ P(x,y) \longleftrightarrow y{=}w) \longleftrightarrow x = z)$›
⟨*proof*⟩

52

Almost the same as 51.

lemma
‹$(\exists z\ w.\ \forall x\ y.\ P(x,y) \longleftrightarrow (x = z \land y = w)) \longrightarrow$
$(\exists w.\ \forall y.\ \exists z.\ (\forall x.\ P(x,y) \longleftrightarrow x = z) \longleftrightarrow y = w)$›
⟨*proof*⟩

55

Non-equational version, from Manthey and Bry, CADE-9 (Springer, 1988). fast DISCOVERS who killed Agatha.

schematic-goal
‹$lives(agatha) \land lives(butler) \land lives(charles) \land$
$(killed(agatha,agatha) \lor killed(butler,agatha) \lor killed(charles,agatha)) \land$
$(\forall x\ y.\ killed(x,y) \longrightarrow hates(x,y) \land \neg\ richer(x,y)) \land$
$(\forall x.\ hates(agatha,x) \longrightarrow \neg\ hates(charles,x)) \land$
$(hates(agatha,agatha) \land hates(agatha,charles)) \land$
$(\forall x.\ lives(x) \land \neg\ richer(x,agatha) \longrightarrow hates(butler,x)) \land$
$(\forall x.\ hates(agatha,x) \longrightarrow hates(butler,x)) \land$
$(\forall x.\ \neg\ hates(x,agatha) \lor \neg\ hates(x,butler) \lor \neg\ hates(x,charles)) \longrightarrow$
$killed(?who,agatha)$›
⟨*proof*⟩

56

lemma ‹$(\forall x.\ (\exists y.\ P(y) \wedge x = f(y)) \longrightarrow P(x)) \longleftrightarrow (\forall x.\ P(x) \longrightarrow P(f(x)))$›
  ⟨*proof*⟩

57

lemma
  ‹$P(f(a,b),\ f(b,c)) \wedge P(f(b,c),\ f(a,c)) \wedge$
    $(\forall x\ y\ z.\ P(x,y) \wedge P(y,z) \longrightarrow P(x,z)) \longrightarrow P(f(a,b),\ f(a,c))$›
  ⟨*proof*⟩

58 NOT PROVED AUTOMATICALLY

lemma ‹$(\forall x\ y.\ f(x) = g(y)) \longrightarrow (\forall x\ y.\ f(f(x)) = f(g(y)))$›
  ⟨*proof*⟩

59

lemma ‹$(\forall x.\ P(x) \longleftrightarrow \neg\ P(f(x))) \longrightarrow (\exists x.\ P(x) \wedge \neg\ P(f(x)))$›
  ⟨*proof*⟩

60

lemma ‹$\forall x.\ P(x,f(x)) \longleftrightarrow (\exists y.\ (\forall z.\ P(z,y) \longrightarrow P(z,f(x))) \wedge P(x,y))$›
  ⟨*proof*⟩

62 as corrected in JAR 18 (1997), page 135

lemma
  ‹$(\forall x.\ p(a) \wedge (p(x) \longrightarrow p(f(x))) \longrightarrow p(f(f(x)))) \longleftrightarrow$
    $(\forall x.\ (\neg\ p(a) \vee p(x) \vee p(f(f(x)))) \wedge$
      $(\neg\ p(a) \vee \neg\ p(f(x)) \vee p(f(f(x)))))$›
  ⟨*proof*⟩

From Davis, Obvious Logical Inferences, IJCAI-81, 530-531 fast indeed copes!

lemma
  ‹$(\forall x.\ F(x) \wedge \neg\ G(x) \longrightarrow (\exists y.\ H(x,y) \wedge J(y))) \wedge$
    $(\exists x.\ K(x) \wedge F(x) \wedge (\forall y.\ H(x,y) \longrightarrow K(y))) \wedge$
    $(\forall x.\ K(x) \longrightarrow \neg\ G(x)) \longrightarrow (\exists x.\ K(x) \wedge J(x))$›
  ⟨*proof*⟩

From Rudnicki, Obvious Inferences, JAR 3 (1987), 383-393. It does seem obvious!

lemma
  ‹$(\forall x.\ F(x) \wedge \neg\ G(x) \longrightarrow (\exists y.\ H(x,y) \wedge J(y))) \wedge$
    $(\exists x.\ K(x) \wedge F(x) \wedge (\forall y.\ H(x,y) \longrightarrow K(y))) \wedge$
    $(\forall x.\ K(x) \longrightarrow \neg\ G(x)) \longrightarrow (\exists x.\ K(x) \longrightarrow \neg\ G(x))$›
  ⟨*proof*⟩

Halting problem: Formulation of Li Dafa (AAR Newsletter 27, Oct 1994.) author U. Egly.

lemma
  ‹$((\exists x.\ A(x) \wedge (\forall y.\ C(y) \longrightarrow (\forall z.\ D(x,y,z)))) \longrightarrow$

$(\exists\, w.\ C(w) \wedge (\forall\, y.\ C(y) \longrightarrow (\forall\, z.\ D(w,y,z)))))$
$\wedge$
$(\forall\, w.\ C(w) \wedge (\forall\, u.\ C(u) \longrightarrow (\forall\, v.\ D(w,u,v))) \longrightarrow$
$(\forall\, y\ z.$
$(C(y) \wedge P(y,z) \longrightarrow Q(w,y,z) \wedge OO(w,g)) \wedge$
$(C(y) \wedge \neg\, P(y,z) \longrightarrow Q(w,y,z) \wedge OO(w,b))))$
$\wedge$
$(\forall\, w.\ C(w) \wedge$
$(\forall\, y\ z.$
$(C(y) \wedge P(y,z) \longrightarrow Q(w,y,z) \wedge OO(w,g)) \wedge$
$(C(y) \wedge \neg\, P(y,z) \longrightarrow Q(w,y,z) \wedge OO(w,b))) \longrightarrow$
$(\exists\, v.\ C(v) \wedge$
$(\forall\, y.\ ((C(y) \wedge Q(w,y,y)) \wedge OO(w,g) \longrightarrow \neg\, P(v,y)) \wedge$
$((C(y) \wedge Q(w,y,y)) \wedge OO(w,b) \longrightarrow P(v,y) \wedge OO(v,b)))))$
$\longrightarrow \neg\, (\exists\, x.\ A(x) \wedge (\forall\, y.\ C(y) \longrightarrow (\forall\, z.\ D(x,y,z))))$›
⟨*proof*⟩

Halting problem II: credited to M. Bruschi by Li Dafa in JAR 18(1), p. 105.

lemma
‹$((\exists\, x.\ A(x) \wedge (\forall\, y.\ C(y) \longrightarrow (\forall\, z.\ D(x,y,z)))) \longrightarrow$
$(\exists\, w.\ C(w) \wedge (\forall\, y.\ C(y) \longrightarrow (\forall\, z.\ D(w,y,z)))))$
$\wedge$
$(\forall\, w.\ C(w) \wedge (\forall\, u.\ C(u) \longrightarrow (\forall\, v.\ D(w,u,v))) \longrightarrow$
$(\forall\, y\ z.$
$(C(y) \wedge P(y,z) \longrightarrow Q(w,y,z) \wedge OO(w,g)) \wedge$
$(C(y) \wedge \neg\, P(y,z) \longrightarrow Q(w,y,z) \wedge OO(w,b))))$
$\wedge$
$((\exists\, w.\ C(w) \wedge (\forall\, y.\ (C(y) \wedge P(y,y) \longrightarrow Q(w,y,y) \wedge OO(w,g)) \wedge$
$(C(y) \wedge \neg\, P(y,y) \longrightarrow Q(w,y,y) \wedge OO(w,b))))$
$\longrightarrow$
$(\exists\, v.\ C(v) \wedge (\forall\, y.\ (C(y) \wedge P(y,y) \longrightarrow P(v,y) \wedge OO(v,g)) \wedge$
$(C(y) \wedge \neg\, P(y,y) \longrightarrow P(v,y) \wedge OO(v,b)))))$
$\longrightarrow$
$((\exists\, v.\ C(v) \wedge (\forall\, y.\ (C(y) \wedge P(y,y) \longrightarrow P(v,y) \wedge OO(v,g)) \wedge$
$(C(y) \wedge \neg\, P(y,y) \longrightarrow P(v,y) \wedge OO(v,b))))$
$\longrightarrow$
$(\exists\, u.\ C(u) \wedge (\forall\, y.\ (C(y) \wedge P(y,y) \longrightarrow \neg\, P(u,y)) \wedge$
$(C(y) \wedge \neg\, P(y,y) \longrightarrow P(u,y) \wedge OO(u,b)))))$
$\longrightarrow \neg\, (\exists\, x.\ A(x) \wedge (\forall\, y.\ C(y) \longrightarrow (\forall\, z.\ D(x,y,z))))$›
⟨*proof*⟩

Challenge found on info-hol.

lemma ‹$\forall\, x.\ \exists\, v\ w.\ \forall\, y\ z.\ P(x) \wedge Q(y) \longrightarrow (P(v) \vee R(w)) \wedge (R(z) \longrightarrow Q(v))$›
⟨*proof*⟩

Attributed to Lewis Carroll by S. G. Pulman. The first or last assumption can be deleted.

lemma
‹$(\forall\, x.\ \mathit{honest}(x) \wedge \mathit{industrious}(x) \longrightarrow \mathit{healthy}(x)) \wedge$

$\neg (\exists x.\ grocer(x) \wedge healthy(x)) \wedge$
$(\forall x.\ industrious(x) \wedge grocer(x) \longrightarrow honest(x)) \wedge$
$(\forall x.\ cyclist(x) \longrightarrow industrious(x)) \wedge$
$(\forall x.\ \neg\ healthy(x) \wedge cyclist(x) \longrightarrow \neg\ honest(x))$
$\longrightarrow (\forall x.\ grocer(x) \longrightarrow \neg\ cyclist(x))$›
⟨*proof*⟩

end

12 First-Order Logic: propositional examples (classical version)

theory *Propositional-Cla*
imports *FOL*
begin

commutative laws of ∧ and ∨

lemma ‹$P \wedge Q \longrightarrow Q \wedge P$›
⟨*proof*⟩

lemma ‹$P \vee Q \longrightarrow Q \vee P$›
⟨*proof*⟩

associative laws of ∧ and ∨

lemma ‹$(P \wedge Q) \wedge R \longrightarrow P \wedge (Q \wedge R)$›
⟨*proof*⟩

lemma ‹$(P \vee Q) \vee R \longrightarrow P \vee (Q \vee R)$›
⟨*proof*⟩

distributive laws of ∧ and ∨

lemma ‹$(P \wedge Q) \vee R \longrightarrow (P \vee R) \wedge (Q \vee R)$›
⟨*proof*⟩

lemma ‹$(P \vee R) \wedge (Q \vee R) \longrightarrow (P \wedge Q) \vee R$›
⟨*proof*⟩

lemma ‹$(P \vee Q) \wedge R \longrightarrow (P \wedge R) \vee (Q \wedge R)$›
⟨*proof*⟩

lemma ‹$(P \wedge R) \vee (Q \wedge R) \longrightarrow (P \vee Q) \wedge R$›
⟨*proof*⟩

Laws involving implication

lemma ‹$(P \longrightarrow R) \wedge (Q \longrightarrow R) \longleftrightarrow (P \vee Q \longrightarrow R)$›

$\langle proof \rangle$

lemma ‹$(P \wedge Q \longrightarrow R) \longleftrightarrow (P \longrightarrow (Q \longrightarrow R))$›
$\langle proof \rangle$

lemma ‹$((P \longrightarrow R) \longrightarrow R) \longrightarrow ((Q \longrightarrow R) \longrightarrow R) \longrightarrow (P \wedge Q \longrightarrow R) \longrightarrow R$›
$\langle proof \rangle$

lemma ‹$\neg (P \longrightarrow R) \longrightarrow \neg (Q \longrightarrow R) \longrightarrow \neg (P \wedge Q \longrightarrow R)$›
$\langle proof \rangle$

lemma ‹$(P \longrightarrow Q \wedge R) \longleftrightarrow (P \longrightarrow Q) \wedge (P \longrightarrow R)$›
$\langle proof \rangle$

Propositions-as-types

lemma ‹$P \longrightarrow (Q \longrightarrow P)$›
$\langle proof \rangle$
lemma ‹$(P \longrightarrow Q \longrightarrow R) \longrightarrow (P \longrightarrow Q) \longrightarrow (P \longrightarrow R)$›
$\langle proof \rangle$
lemma ‹$(P \longrightarrow Q) \vee (P \longrightarrow R) \longrightarrow (P \longrightarrow Q \vee R)$›
$\langle proof \rangle$

lemma ‹$(P \longrightarrow Q) \longrightarrow (\neg Q \longrightarrow \neg P)$›
$\langle proof \rangle$

Schwichtenberg's examples (via T. Nipkow)

lemma *stab-imp*: ‹$(((Q \longrightarrow R) \longrightarrow R) \longrightarrow Q) \longrightarrow (((P \longrightarrow Q) \longrightarrow R) \longrightarrow R)$
$\longrightarrow P \longrightarrow Q$›
$\langle proof \rangle$

lemma *stab-to-peirce*:
‹$(((P \longrightarrow R) \longrightarrow R) \longrightarrow P) \longrightarrow (((Q \longrightarrow R) \longrightarrow R) \longrightarrow Q)$
$\longrightarrow ((P \longrightarrow Q) \longrightarrow P) \longrightarrow P$›
$\langle proof \rangle$

lemma *peirce-imp1*:
‹$(((Q \longrightarrow R) \longrightarrow Q) \longrightarrow Q)$
$\longrightarrow (((P \longrightarrow Q) \longrightarrow R) \longrightarrow P \longrightarrow Q) \longrightarrow P \longrightarrow Q$›
$\langle proof \rangle$

lemma *peirce-imp2*: ‹$(((P \longrightarrow R) \longrightarrow P) \longrightarrow P) \longrightarrow ((P \longrightarrow Q \longrightarrow R) \longrightarrow P)$
$\longrightarrow P$›
$\langle proof \rangle$

lemma *mints*: ‹$((((P \longrightarrow Q) \longrightarrow P) \longrightarrow P) \longrightarrow Q) \longrightarrow Q$›
$\langle proof \rangle$

lemma *mints-solovev*: ‹$(P \longrightarrow (Q \longrightarrow R) \longrightarrow Q) \longrightarrow ((P \longrightarrow Q) \longrightarrow R) \longrightarrow R$›
$\langle proof \rangle$

lemma *tatsuta*:
$‹(((P7 \longrightarrow P1) \longrightarrow P10) \longrightarrow P4 \longrightarrow P5)$
$\longrightarrow (((P8 \longrightarrow P2) \longrightarrow P9) \longrightarrow P3 \longrightarrow P10)$
$\longrightarrow (P1 \longrightarrow P8) \longrightarrow P6 \longrightarrow P7$
$\longrightarrow (((P3 \longrightarrow P2) \longrightarrow P9) \longrightarrow P4)$
$\longrightarrow (P1 \longrightarrow P3) \longrightarrow (((P6 \longrightarrow P1) \longrightarrow P2) \longrightarrow P9) \longrightarrow P5›$
⟨*proof*⟩

lemma *tatsuta1*:
$‹(((P8 \longrightarrow P2) \longrightarrow P9) \longrightarrow P3 \longrightarrow P10)$
$\longrightarrow (((P3 \longrightarrow P2) \longrightarrow P9) \longrightarrow P4)$
$\longrightarrow (((P6 \longrightarrow P1) \longrightarrow P2) \longrightarrow P9)$
$\longrightarrow (((P7 \longrightarrow P1) \longrightarrow P10) \longrightarrow P4 \longrightarrow P5)$
$\longrightarrow (P1 \longrightarrow P3) \longrightarrow (P1 \longrightarrow P8) \longrightarrow P6 \longrightarrow P7 \longrightarrow P5›$
⟨*proof*⟩

end

13 First-Order Logic: quantifier examples (classical version)

theory *Quantifiers-Cla*
imports *FOL*
begin

lemma $‹(\forall x\ y.\ P(x,y)) \longrightarrow (\forall y\ x.\ P(x,y))›$
⟨*proof*⟩

lemma $‹(\exists x\ y.\ P(x,y)) \longrightarrow (\exists y\ x.\ P(x,y))›$
⟨*proof*⟩

Converse is false.

lemma $‹(\forall x.\ P(x)) \vee (\forall x.\ Q(x)) \longrightarrow (\forall x.\ P(x) \vee Q(x))›$
⟨*proof*⟩

lemma $‹(\forall x.\ P \longrightarrow Q(x)) \longleftrightarrow (P \longrightarrow (\forall x.\ Q(x)))›$
⟨*proof*⟩

lemma $‹(\forall x.\ P(x) \longrightarrow Q) \longleftrightarrow ((\exists x.\ P(x)) \longrightarrow Q)›$
⟨*proof*⟩

Some harder ones.

lemma $‹(\exists x.\ P(x) \vee Q(x)) \longleftrightarrow (\exists x.\ P(x)) \vee (\exists x.\ Q(x))›$
⟨*proof*⟩
lemma $‹(\exists x.\ P(x) \wedge Q(x)) \longrightarrow (\exists x.\ P(x)) \wedge (\exists x.\ Q(x))›$
⟨*proof*⟩

Basic test of quantifier reasoning.

lemma ‹$(\exists y.\ \forall x.\ Q(x,y)) \longrightarrow (\forall x.\ \exists y.\ Q(x,y))$›
  ⟨*proof*⟩

lemma ‹$(\forall x.\ Q(x)) \longrightarrow (\exists x.\ Q(x))$›
  ⟨*proof*⟩

The following should fail, as they are false!

lemma ‹$(\forall x.\ \exists y.\ Q(x,y)) \longrightarrow (\exists y.\ \forall x.\ Q(x,y))$›
  ⟨*proof*⟩

lemma ‹$(\exists x.\ Q(x)) \longrightarrow (\forall x.\ Q(x))$›
  ⟨*proof*⟩

schematic-goal ‹$P(?a) \longrightarrow (\forall x.\ P(x))$›
  ⟨*proof*⟩

schematic-goal ‹$(P(?a) \longrightarrow (\forall x.\ Q(x))) \longrightarrow (\forall x.\ P(x) \longrightarrow Q(x))$›
  ⟨*proof*⟩

Back to things that are provable . . .

lemma ‹$(\forall x.\ P(x) \longrightarrow Q(x)) \wedge (\exists x.\ P(x)) \longrightarrow (\exists x.\ Q(x))$›
  ⟨*proof*⟩

An example of why *exI* should be delayed as long as possible.

lemma ‹$(P \longrightarrow (\exists x.\ Q(x))) \wedge P \longrightarrow (\exists x.\ Q(x))$›
  ⟨*proof*⟩

schematic-goal ‹$(\forall x.\ P(x) \longrightarrow Q(f(x))) \wedge (\forall x.\ Q(x) \longrightarrow R(g(x))) \wedge P(d) \longrightarrow R(?a)$›
  ⟨*proof*⟩

lemma ‹$(\forall x.\ Q(x)) \longrightarrow (\exists x.\ Q(x))$›
  ⟨*proof*⟩

Some slow ones

Principia Mathematica *11.53

lemma ‹$(\forall x\ y.\ P(x) \longrightarrow Q(y)) \longleftrightarrow ((\exists x.\ P(x)) \longrightarrow (\forall y.\ Q(y)))$›
  ⟨*proof*⟩

lemma ‹$(\exists x\ y.\ P(x) \wedge Q(x,y)) \longleftrightarrow (\exists x.\ P(x) \wedge (\exists y.\ Q(x,y)))$›
  ⟨*proof*⟩

lemma ‹$(\exists y.\ \forall x.\ P(x) \longrightarrow Q(x,y)) \longrightarrow (\forall x.\ P(x) \longrightarrow (\exists y.\ Q(x,y)))$›
  ⟨*proof*⟩

end

theory *Miniscope*
imports *FOL*
begin

lemmas *ccontr* = *FalseE* [*THEN classical*]

13.1 Negation Normal Form

13.1.1 de Morgan laws

lemma *demorgans1*:
$‹\neg (P \wedge Q) \longleftrightarrow \neg P \vee \neg Q›$
$‹\neg (P \vee Q) \longleftrightarrow \neg P \wedge \neg Q›$
$‹\neg \neg P \longleftrightarrow P›$
⟨*proof*⟩

lemma *demorgans2*:
$‹\bigwedge P. \neg (\forall x. P(x)) \longleftrightarrow (\exists x. \neg P(x))›$
$‹\bigwedge P. \neg (\exists x. P(x)) \longleftrightarrow (\forall x. \neg P(x))›$
⟨*proof*⟩

lemmas *demorgans* = *demorgans1 demorgans2*

lemma *nnf-simps*:
$‹(P \longrightarrow Q) \longleftrightarrow (\neg P \vee Q)›$
$‹\neg (P \longrightarrow Q) \longleftrightarrow (P \wedge \neg Q)›$
$‹(P \longleftrightarrow Q) \longleftrightarrow (\neg P \vee Q) \wedge (\neg Q \vee P)›$
$‹\neg (P \longleftrightarrow Q) \longleftrightarrow (P \vee Q) \wedge (\neg P \vee \neg Q)›$
⟨*proof*⟩

13.1.2 Pushing in the existential quantifiers

lemma *ex-simps*:
$‹(\exists x. P) \longleftrightarrow P›$
$‹\bigwedge P\ Q. (\exists x. P(x) \wedge Q) \longleftrightarrow (\exists x. P(x)) \wedge Q›$
$‹\bigwedge P\ Q. (\exists x. P \wedge Q(x)) \longleftrightarrow P \wedge (\exists x. Q(x))›$
$‹\bigwedge P\ Q. (\exists x. P(x) \vee Q(x)) \longleftrightarrow (\exists x. P(x)) \vee (\exists x. Q(x))›$
$‹\bigwedge P\ Q. (\exists x. P(x) \vee Q) \longleftrightarrow (\exists x. P(x)) \vee Q›$
$‹\bigwedge P\ Q. (\exists x. P \vee Q(x)) \longleftrightarrow P \vee (\exists x. Q(x))›$
⟨*proof*⟩

13.1.3 Pushing in the universal quantifiers

lemma *all-simps*:

‹$(\forall x.\ P) \longleftrightarrow P$›
‹$\bigwedge P\ Q.\ (\forall x.\ P(x) \wedge Q(x)) \longleftrightarrow (\forall x.\ P(x)) \wedge (\forall x.\ Q(x))$›
‹$\bigwedge P\ Q.\ (\forall x.\ P(x) \wedge Q) \longleftrightarrow (\forall x.\ P(x)) \wedge Q$›
‹$\bigwedge P\ Q.\ (\forall x.\ P \wedge Q(x)) \longleftrightarrow P \wedge (\forall x.\ Q(x))$›
‹$\bigwedge P\ Q.\ (\forall x.\ P(x) \vee Q) \longleftrightarrow (\forall x.\ P(x)) \vee Q$›
‹$\bigwedge P\ Q.\ (\forall x.\ P \vee Q(x)) \longleftrightarrow P \vee (\forall x.\ Q(x))$›
⟨*proof*⟩

lemmas *mini-simps* = *demorgans nnf-simps ex-simps all-simps*

⟨*ML*⟩

end

14 First-Order Logic: the 'if' example

theory *If*
imports *FOL*
begin

definition *if* :: ‹[*o*,*o*,*o*]=>*o*›
where ‹$if(P,Q,R) \equiv P \wedge Q \vee \neg P \wedge R$›

lemma *ifI*: ‹$\llbracket P \Longrightarrow Q;\ \neg P \Longrightarrow R \rrbracket \Longrightarrow if(P,Q,R)$›
⟨*proof*⟩

lemma *ifE*: ‹$\llbracket if(P,Q,R);\ \llbracket P;\ Q \rrbracket \Longrightarrow S;\ \llbracket \neg P;\ R \rrbracket \Longrightarrow S \rrbracket \Longrightarrow S$›
⟨*proof*⟩

lemma *if-commute*: ‹$if(P,\ if(Q,A,B),\ if(Q,C,D)) \longleftrightarrow if(Q,\ if(P,A,C),\ if(P,B,D))$›
⟨*proof*⟩

Trying again from the beginning in order to use *blast*

declare *ifI* [*intro*!]
declare *ifE* [*elim*!]

lemma *if-commute*: ‹$if(P,\ if(Q,A,B),\ if(Q,C,D)) \longleftrightarrow if(Q,\ if(P,A,C),\ if(P,B,D))$›
⟨*proof*⟩

lemma ‹$if(if(P,Q,R),\ A,\ B) \longleftrightarrow if(P,\ if(Q,A,B),\ if(R,A,B))$›
⟨*proof*⟩

Trying again from the beginning in order to prove from the definitions

lemma ‹$if(if(P,Q,R),\ A,\ B) \longleftrightarrow if(P,\ if(Q,A,B),\ if(R,A,B))$›
⟨*proof*⟩

An invalid formula. High-level rules permit a simpler diagnosis.

lemma ‹$if(if(P,Q,R),\ A,\ B) \longleftrightarrow if(P,\ if(Q,A,B),\ if(R,B,A))$›
  ⟨*proof*⟩

Trying again from the beginning in order to prove from the definitions.

lemma ‹$if(if(P,Q,R),\ A,\ B) \longleftrightarrow if(P,\ if(Q,A,B),\ if(R,B,A))$›
  ⟨*proof*⟩

end