

Equivalents of the Axiom of Choice

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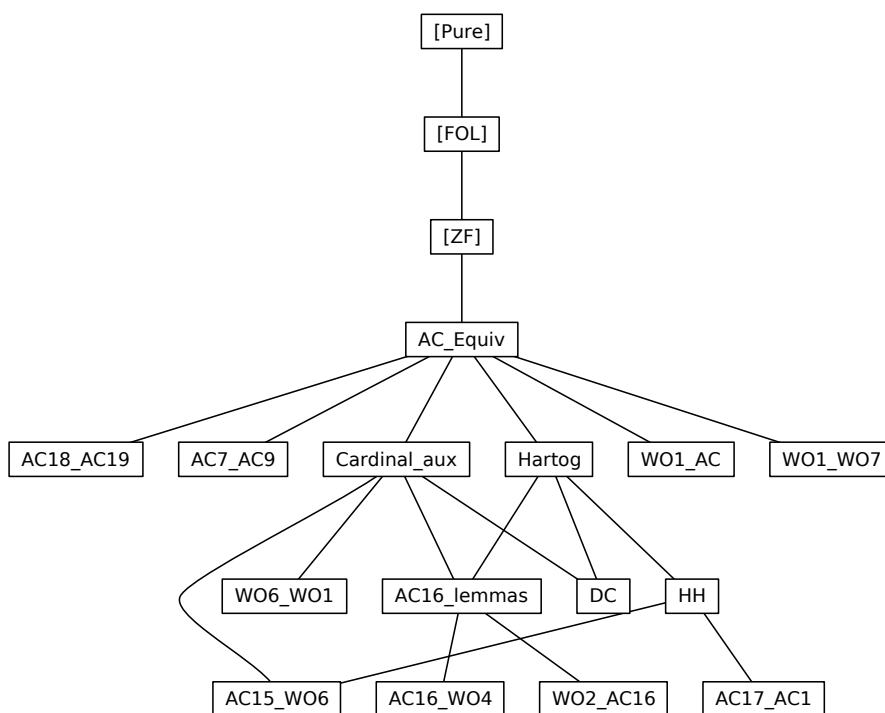
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Abstract

This development [1] proves the equivalence of seven formulations of the well-ordering theorem and twenty formulations of the axiom of choice. It formalizes the first two chapters of the monograph *Equivalents of the Axiom of Choice* by Rubin and Rubin [2]. Some of this material involves extremely complex techniques.

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theory AC_Equiv
imports ZF
begin

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definition

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"W01  $\equiv \forall A. \exists R. \text{well\_ord}(A,R)$ "

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definition

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"W02  $\equiv \forall A. \exists a. \text{Ord}(a) \wedge A \approx a$ "

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definition

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"W03  $\equiv \forall A. \exists a. \text{Ord}(a) \wedge (\exists b. b \subseteq a \wedge A \approx b)$ "

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definition

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"W04(m)  $\equiv \forall A. \exists a f. \text{Ord}(a) \wedge \text{domain}(f)=a \wedge$ 
 $(\bigcup b < a. f' b) = A \wedge (\forall b < a. f' b \lesssim m)$ "

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definition

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"W05  $\equiv \exists m \in \text{nat}. 1 \leq m \wedge \text{W04}(m)$ "

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definition

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"W06  $\equiv \forall A. \exists m \in \text{nat}. 1 \leq m \wedge (\exists a f. \text{Ord}(a) \wedge \text{domain}(f)=a$ 
 $\wedge (\bigcup b < a. f' b) = A \wedge (\forall b < a. f' b \lesssim m))$ "

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definition

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"W07  $\equiv \forall A. \text{Finite}(A) \longleftrightarrow (\forall R. \text{well\_ord}(A,R) \longrightarrow \text{well\_ord}(A,\text{converse}(R)))$ "

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definition

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"W08  $\equiv \forall A. (\exists f. f \in (\prod X \in A. X)) \longrightarrow (\exists R. \text{well\_ord}(A,R))$ "

```

definition

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pairwise_disjoint :: "i  $\Rightarrow$  o" where
"pairwise_disjoint(A)  $\equiv \forall A1 \in A. \forall A2 \in A. A1 \cap A2 \neq 0 \longrightarrow A1=A2$ "

```

definition

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sets_of_size_between :: "[i, i, i]  $\Rightarrow$  o" where
"sets_of_size_between(A,m,n)  $\equiv \forall B \in A. m \lesssim B \wedge B \lesssim n$ "

```

definition

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"AC0  $\equiv \forall A. \exists f. f \in (\prod X \in \text{Pow}(A)-\{0\}. X)$ "

```

definition

$$"AC1 \equiv \forall A. 0 \notin A \longrightarrow (\exists f. f \in (\prod X \in A. X))"$$

definition

$$"AC2 \equiv \forall A. 0 \notin A \wedge \text{pairwise_disjoint}(A) \longrightarrow (\exists C. \forall B \in A. \exists y. B \cap C = \{y\})"$$

definition

$$"AC3 \equiv \forall A B. \forall f \in A \rightarrow B. \exists g. g \in (\prod x \in \{a \in A. f'a \neq 0\}. f'x)"$$

definition

$$"AC4 \equiv \forall R A B. (R \subseteq A * B \longrightarrow (\exists f. f \in (\prod x \in \text{domain}(R). R'\{x\})))"$$

definition

$$"AC5 \equiv \forall A B. \forall f \in A \rightarrow B. \exists g \in \text{range}(f) \rightarrow A. \forall x \in \text{domain}(g). f'(g'x) = x"$$

definition

$$"AC6 \equiv \forall A. 0 \notin A \longrightarrow (\prod B \in A. B) \neq 0"$$

definition

$$"AC7 \equiv \forall A. 0 \notin A \wedge (\forall B1 \in A. \forall B2 \in A. B1 \approx B2) \longrightarrow (\prod B \in A. B) \neq 0"$$

definition

$$"AC8 \equiv \forall A. (\forall B \in A. \exists B1 B2. B = \langle B1, B2 \rangle \wedge B1 \approx B2) \longrightarrow (\exists f. \forall B \in A. f'B \in \text{bij}(\text{fst}(B), \text{snd}(B)))"$$

definition

$$"AC9 \equiv \forall A. (\forall B1 \in A. \forall B2 \in A. B1 \approx B2) \longrightarrow (\exists f. \forall B1 \in A. \forall B2 \in A. f'\langle B1, B2 \rangle \in \text{bij}(B1, B2))"$$

definition

$$"AC10(n) \equiv \forall A. (\forall B \in A. \neg \text{Finite}(B)) \longrightarrow (\exists f. \forall B \in A. (\text{pairwise_disjoint}(f'B) \wedge \text{sets_of_size_between}(f'B, 2, \text{succ}(n)) \wedge \bigcup (f'B) = B))"$$

definition

$$"AC11 \equiv \exists n \in \text{nat}. 1 \leq n \wedge AC10(n)"$$

definition

$$"AC12 \equiv \forall A. (\forall B \in A. \neg \text{Finite}(B)) \longrightarrow (\exists n \in \text{nat}. 1 \leq n \wedge (\exists f. \forall B \in A. (\text{pairwise_disjoint}(f'B) \wedge \text{sets_of_size_between}(f'B, 2, \text{succ}(n)) \wedge \bigcup (f'B) = B)))"$$

definition

$$"AC13(m) \equiv \forall A. 0 \notin A \longrightarrow (\exists f. \forall B \in A. f'B \neq 0 \wedge f'B \subseteq B \wedge f'B \lesssim m)"$$

definition

"AC14 $\equiv \exists m \in \text{nat}. 1 \leq m \wedge \text{AC13}(m)$ "

definition

"AC15 $\equiv \forall A. 0 \notin A \longrightarrow$
 $(\exists m \in \text{nat}. 1 \leq m \wedge (\exists f. \forall B \in A. f'B \neq 0 \wedge f'B \subseteq B \wedge$
 $f'B \lesssim m))$ "

definition

"AC16(n, k) \equiv
 $\forall A. \neg \text{Finite}(A) \longrightarrow$
 $(\exists T. T \subseteq \{X \in \text{Pow}(A). X \approx_{\text{succ}}(n)\} \wedge$
 $(\forall X \in \{X \in \text{Pow}(A). X \approx_{\text{succ}}(k)\}. \exists ! Y. Y \in T \wedge X \subseteq Y))$ "

definition

"AC17 $\equiv \forall A. \forall g \in (\text{Pow}(A) - \{0\} \rightarrow A) \rightarrow \text{Pow}(A) - \{0\}.$
 $\exists f \in \text{Pow}(A) - \{0\} \rightarrow A. f'(g'f) \in g'f$ "

locale AC18 =

assumes AC18: " $A \neq 0 \wedge (\forall a \in A. B(a) \neq 0) \longrightarrow$
 $((\bigcap a \in A. \bigcup b \in B(a). X(a, b)) =$
 $(\bigcup f \in \prod a \in A. B(a). \bigcap a \in A. X(a, f'a)))$ "
 — AC18 cannot be expressed within the object-logic

definition

"AC19 $\equiv \forall A. A \neq 0 \wedge 0 \notin A \longrightarrow ((\bigcap a \in A. \bigcup b \in a. b) =$
 $(\bigcup f \in (\prod B \in A. B). \bigcap a \in A. f'a))$ "

lemma rvimage_id: " $\text{rvimage}(A, \text{id}(A), r) = r \cap A * A$ "

unfolding rvimage_def

apply (rule equalityI, safe)

apply (drule_tac P = " $\lambda a. \langle \text{id}(A) \text{ 'xb, a} \rangle : r$ " in id_conv [THEN subst],
 assumption)

apply (drule_tac P = " $\lambda a. \langle a, \text{ya} \rangle : r$ " in id_conv [THEN subst], (assumption+))

apply (fast intro: id_conv [THEN ssubst])

done

lemma ordertype_Int:

" $\text{well_ord}(A, r) \implies \text{ordertype}(A, r \cap A * A) = \text{ordertype}(A, r)$ "

apply (rule_tac P = " $\lambda a. \text{ordertype}(A, a) = \text{ordertype}(A, r)$ " in rvimage_id
 [THEN subst])

```

apply (erule id_bij [THEN bij_ordertype_vimage])
done

lemma lam_sing_bij: " $(\lambda x \in A. \{x\}) \in \text{bij}(A, \{\{x\}. x \in A\})$ "
apply (rule_tac d = " $\lambda z. \text{THE } x. z = \{x\}$ " in lam_bijective)
apply (auto simp add: the_equality)
done

lemma inj_strengthen_type:
  " $\llbracket f \in \text{inj}(A, B); \bigwedge a. a \in A \implies f'a \in C \rrbracket \implies f \in \text{inj}(A, C)$ "
by (unfold inj_def, blast intro: Pi_type)

lemma ex1_two_eq: " $\llbracket \exists! x. P(x); P(x); P(y) \rrbracket \implies x=y$ "
by blast

lemma first_in_B:
  " $\llbracket \text{well\_ord}(\bigcup(A), r); 0 \notin A; B \in A \rrbracket \implies (\text{THE } b. \text{first}(b, B, r)) \in B$ "
by (blast dest!: well_ord_imp_ex1_first
    [THEN theI, THEN first_def [THEN def_imp_iff, THEN
    iffD1]])

lemma ex_choice_fun: " $\llbracket \text{well\_ord}(\bigcup(A), R); 0 \notin A \rrbracket \implies \exists f. f \in (\prod X \in A. X)$ "
by (fast elim!: first_in_B intro!: lam_type)

lemma ex_choice_fun_Pow: " $\text{well\_ord}(A, R) \implies \exists f. f \in (\prod X \in \text{Pow}(A) - \{0\}. X)$ "
by (fast elim!: well_ord_subset [THEN ex_choice_fun])

lemma lepoll_m_imp_domain_lepoll_m:
  " $\llbracket m \in \text{nat}; u \lesssim m \rrbracket \implies \text{domain}(u) \lesssim m$ "
  unfolding lepoll_def
  apply (erule exE)

```

```

apply (rule_tac x = " $\lambda x \in \text{domain}(u). \mu i. \exists y. \langle x, y \rangle \in u \wedge f' \langle x, y \rangle = i$ "
  in exI)
apply (rule_tac d = " $\lambda y. \text{fst} (\text{converse}(f) ' y)$ " in lam_injective)
apply (fast intro: LeastI2 nat_into_Ord [THEN Ord_in_Ord]
      inj_is_fun [THEN apply_type])
apply (erule domainE)
apply (frule inj_is_fun [THEN apply_type], assumption)
apply (rule LeastI2)
apply (auto elim!: nat_into_Ord [THEN Ord_in_Ord])
done

lemma rel_domain_ex1:
  " $\llbracket \text{succ}(m) \lesssim \text{domain}(r); r \lesssim \text{succ}(m); m \in \text{nat} \rrbracket \implies \text{function}(r)$ "
apply (unfold function_def, safe)
apply (rule ccontr)
apply (fast elim!: lepoll_trans [THEN succ_lepoll_natE]
      lepoll_m_imp_domain_lepoll_m [OF _ Diff_sing_lepoll]
      elim: domain_Diff_eq [OF _ not_sym, THEN subst])
done

lemma rel_is_fun:
  " $\llbracket \text{succ}(m) \lesssim \text{domain}(r); r \lesssim \text{succ}(m); m \in \text{nat};$   

 $r \subseteq A * B; A = \text{domain}(r) \rrbracket \implies r \in A \rightarrow B$ "
by (simp add: Pi_iff rel_domain_ex1)

end

theory Cardinal_aux imports AC_Equiv begin

lemma Diff_lepoll: " $\llbracket A \lesssim \text{succ}(m); B \subseteq A; B \neq 0 \rrbracket \implies A - B \lesssim m$ "
apply (rule not_emptyE, assumption)
apply (blast intro: lepoll_trans [OF subset_imp_lepoll Diff_sing_lepoll])
done

lemma lepoll_imp_ex_le_eqpoll:
  " $\llbracket A \lesssim i; \text{Ord}(i) \rrbracket \implies \exists j. j \leq i \wedge A \approx j$ "
by (blast intro!: lepoll_cardinal_le well_ord_Memrel
      well_ord_cardinal_eqpoll [THEN eqpoll_sym]
      dest: lepoll_well_ord)

```

```

lemma lesspoll_imp_ex_lt_eqpoll:
  "⟦A < i; Ord(i)⟧ ⟹ ∃ j. j < i ∧ A ≈ j"
by (unfold lesspoll_def, blast dest!: lepoll_imp_ex_le_eqpoll elim!: leE)

lemma Un_eqpoll_Inf_Ord:
  assumes A: "A ≈ i" and B: "B ≈ i" and NFI: "¬ Finite(i)" and i:
    "Ord(i)"
  shows "A ∪ B ≈ i"
proof (rule eqpollI)
  have AB: "A ≈ B" using A B by (blast intro: eqpoll_sym eqpoll_trans)

  have "2 ≲ nat"
    by (rule subset_imp_lepoll) (rule OrdmemD [OF nat_2I Ord_nat])
  also have "... ≲ i"
    by (simp add: nat_le_infinite_Ord le_imp_lepoll NFI i)+
  also have "... ≈ A" by (blast intro: eqpoll_sym A)
  finally have "2 ≲ A" .
  have ICI: "InfCard(|i|)"
    by (simp add: Inf_Card_is_InfCard Finite_cardinal_iff NFI i)
  have "A ∪ B ≲ A + B" by (rule Un_lepoll_sum)
  also have "... ≲ A × B"
    by (rule lepoll_imp_sum_lepoll_prod [OF AB [THEN eqpoll_imp_lepoll]
    <2 ≲ A>])
  also have "... ≈ i × i"
    by (blast intro: prod_eqpoll_cong eqpoll_imp_lepoll A B)
  also have "... ≈ i"
    by (blast intro: well_ord_InfCard_square_eq well_ord_Memrel ICI i)

  finally show "A ∪ B ≲ i" .
next
  have "i ≈ A" by (blast intro: A eqpoll_sym)
  also have "... ≲ A ∪ B" by (blast intro: subset_imp_lepoll)
  finally show "i ≲ A ∪ B" .
qed

schematic_goal paired_bij: "?f ∈ bij({y,z}. y ∈ x, x)"
apply (rule RepFun_bijective)
apply (simp add: doubleton_eq_iff, blast)
done

lemma paired_eqpoll: "{y,z}. y ∈ x ≈ x"
by (unfold eqpoll_def, fast intro!: paired_bij)

lemma ex_eqpoll_disjoint: "∃ B. B ≈ A ∧ B ∩ C = 0"
by (fast intro!: paired_eqpoll equalsOI elim: mem_asym)

```

```

lemma Un_lepoll_Inf_Ord:
  "[[A ≲ i; B ≲ i; ¬Finite(i); Ord(i)]] ⇒ A ∪ B ≲ i"
apply (rule_tac A1 = i and C1 = i in ex_eqpoll_disjoint [THEN exE])
apply (erule conjE)
apply (drule lepoll_trans)
apply (erule eqpoll_sym [THEN eqpoll_imp_lepoll])
apply (rule Un_lepoll_Un [THEN lepoll_trans], (assumption+))
apply (blast intro: eqpoll_refl Un_eqpoll_Inf_Ord eqpoll_imp_lepoll)
done

lemma Least_in_Ord: "[[P(i); i ∈ j; Ord(j)]] ⇒ (μ i. P(i)) ∈ j"
apply (erule Least_le [THEN leE])
apply (erule Ord_in_Ord, assumption)
apply (erule ltE)
apply (fast dest: OrdmemD)
apply (erule subst_elem, assumption)
done

lemma Diff_first_lepoll:
  "[[well_ord(x,r); y ⊆ x; y ≲ succ(n); n ∈ nat]]
   ⇒ y - {THE b. first(b,y,r)} ≲ n"
apply (case_tac "y=0", simp add: empty_lepollI)
apply (fast intro!: Diff_sing_lepoll the_first_in)
done

lemma UN_subset_split:
  "(⋃ x ∈ X. P(x)) ⊆ (⋃ x ∈ X. P(x)-Q(x)) ∪ (⋃ x ∈ X. Q(x))"
by blast

lemma UN_sing_lepoll: "Ord(a) ⇒ (⋃ x ∈ a. {P(x)}) ≲ a"
  unfolding lepoll_def
apply (rule_tac x = "λz. (⋃ x ∈ a. {P (x) }) . (μ i. P (i) =z) " in
exI)
apply (rule_tac d = "λz. P (z) " in lam_injective)
apply (fast intro!: Least_in_Ord)
apply (fast intro: LeastI elim!: Ord_in_Ord)
done

lemma UN_fun_lepoll_lemma [rule_format]:
  "[[well_ord(T, R); ¬Finite(a); Ord(a); n ∈ nat]]
   ⇒ ∀f. (∀b ∈ a. f' b ≲ n ∧ f' b ⊆ T) → (⋃ b ∈ a. f' b) ≲ a"
apply (induct_tac "n")
apply (rule allI)
apply (rule impI)
apply (rule_tac b = "⋃ b ∈ a. f' b" in subst)
apply (rule_tac [2] empty_lepollI)
apply (rule equalsOI [symmetric], clarify)
apply (fast dest: lepoll_0_is_0 [THEN subst])
apply (rule allI)

```



```

apply (rule impI)
apply (erule_tac x = "λx ∈ a. f'x - {THE b. first (b,f'x,R) }" in allE)
apply (erule impE, simp)
apply (fast intro!: Diff_first_lepoll, simp)
apply (rule UN_subset_split [THEN subset_imp_lepoll, THEN lepoll_trans])
apply (fast intro: Un_lepoll_Inf_Ord UN_sing_lepoll)
done

```

```

lemma UN_fun_lepoll:
  "⟦∀b ∈ a. f'b ≲ n ∧ f'b ⊆ T; well_ord(T, R);
    ¬Finite(a); Ord(a); n ∈ nat⟧ ⟹ (⋃b ∈ a. f'b) ≲ a"
by (blast intro: UN_fun_lepoll_lemma)

```

```

lemma UN_lepoll:
  "⟦∀b ∈ a. F(b) ≲ n ∧ F(b) ⊆ T; well_ord(T, R);
    ¬Finite(a); Ord(a); n ∈ nat⟧
  ⟹ (⋃b ∈ a. F(b)) ≲ a"
apply (rule rev_mp)
apply (rule_tac f="λb ∈ a. F (b)" in UN_fun_lepoll)
apply auto
done

```

```

lemma UN_eq_UN_Diffs:
  "Ord(a) ⟹ (⋃b ∈ a. F(b)) = (⋃b ∈ a. F(b) - (⋃c ∈ b. F(c)))"
apply (rule equalityI)
  prefer 2 apply fast
apply (rule subsetI)
apply (erule UN_E)
apply (rule UN_I)
  apply (rule_tac P = "λz. x ∈ F (z) " in Least_in_Ord, (assumption+))
apply (rule DiffI, best intro: Ord_in_Ord LeastI, clarify)
apply (erule_tac P = "λz. x ∈ F (z) " and i = c in less_LeastE)
apply (blast intro: Ord_Least ltI)
done

```

```

lemma lepoll_imp_eqpoll_subset:
  "a ≲ X ⟹ ∃Y. Y ⊆ X ∧ a ≈ Y"
apply (unfold lepoll_def eqpoll_def, clarify)
apply (blast intro: restrict_bij
  dest: inj_is_fun [THEN fun_is_rel, THEN image_subset])
done

```

```

lemma Diff_lesspoll_eqpoll_Card_lemma:
  "⟦A ≈ a; ¬Finite(a); Card(a); B < a; A-B < a⟧ ⟹ P"
apply (elim lesspoll_imp_ex_lt_eqpoll [THEN exE] Card_is_Ord conjE)

```

```

apply (frule_tac j=xa in Un_upper1_le [OF lt_Ord lt_Ord], assumption)
apply (frule_tac j=xa in Un_upper2_le [OF lt_Ord lt_Ord], assumption)
apply (drule Un_least_lt, assumption)
apply (drule eqpoll_imp_lepoll [THEN lepoll_trans],
       rule le_imp_lepoll, assumption)+
apply (case_tac "Finite(x  $\cup$  xa)")

finite case

  apply (drule Finite_Un [OF lepoll_Finite lepoll_Finite], assumption+)
  apply (drule subset_Un_Diff [THEN subset_imp_lepoll, THEN lepoll_Finite])
  apply (fast dest: eqpoll_sym [THEN eqpoll_imp_lepoll, THEN lepoll_Finite])

infinite case

  apply (drule Un_lepoll_Inf_Ord, (assumption+))
  apply (blast intro: le_Ord2)
  apply (drule lesspoll_trans1
         [OF subset_Un_Diff [THEN subset_imp_lepoll, THEN lepoll_trans]
          lt_Card_imp_lesspoll], assumption+)
  apply (simp add: lesspoll_def)
done

lemma Diff_lesspoll_eqpoll_Card:
  "[A  $\approx$  a;  $\neg$ Finite(a); Card(a); B  $\prec$  a]  $\implies$  A - B  $\approx$  a"
  apply (rule ccontr)
  apply (rule Diff_lesspoll_eqpoll_Card_lemma, (assumption+))
  apply (blast intro: lesspoll_def [THEN def_imp_iff, THEN iffD2]
         subset_imp_lepoll eqpoll_imp_lepoll lepoll_trans)
done

end

theory W06_W01
imports Cardinal_aux
begin

definition
  NN :: "i  $\Rightarrow$  i" where
    "NN(y)  $\equiv$  {m  $\in$  nat.  $\exists$  a.  $\exists$  f. Ord(a)  $\wedge$  domain(f)=a  $\wedge$ 
      ( $\bigcup$  b<a. f' b) = y  $\wedge$  ( $\forall$  b<a. f' b  $\lesssim$  m)}"

definition
  uu :: "[i, i, i, i]  $\Rightarrow$  i" where
    "uu(f, beta, gamma, delta)  $\equiv$  (f' beta * f' gamma)  $\cap$  f' delta"

definition

```

```

vv1 :: "[i, i, i] ⇒ i" where
  "vv1(f,m,b) ≡
    let g = μ g. (∃ d. Ord(d) ∧ (domain(uu(f,b,g,d)) ≠ 0 ∧
                                domain(uu(f,b,g,d)) ⋖ m));
    d = μ d. domain(uu(f,b,g,d)) ≠ 0 ∧
            domain(uu(f,b,g,d)) ⋖ m
    in if f' b ≠ 0 then domain(uu(f,b,g,d)) else 0"

```

definition

```

ww1 :: "[i, i, i] ⇒ i" where
  "ww1(f,m,b) ≡ f' b - vv1(f,m,b)"

```

definition

```

gg1 :: "[i, i, i] ⇒ i" where
  "gg1(f,a,m) ≡ λb ∈ a++a. if b < a then vv1(f,m,b) else ww1(f,m,b--a)"

```

definition

```

vv2 :: "[i, i, i, i] ⇒ i" where
  "vv2(f,b,g,s) ≡
    if f' g ≠ 0 then {uu(f, b, g, μ d. uu(f,b,g,d) ≠ 0)'s} else
0"

```

definition

```

ww2 :: "[i, i, i, i] ⇒ i" where
  "ww2(f,b,g,s) ≡ f' g - vv2(f,b,g,s)"

```

definition

```

gg2 :: "[i, i, i, i] ⇒ i" where
  "gg2(f,a,b,s) ≡
    λg ∈ a++a. if g < a then vv2(f,b,g,s) else ww2(f,b,g--a,s)"

```

```

lemma W02_W03: "W02 ⇒ W03"
by (unfold W02_def W03_def, fast)

```

```

lemma W03_W01: "W03 ⇒ W01"
  unfolding eqpoll_def W01_def W03_def
  apply (intro allI)
  apply (drule_tac x=A in spec)
  apply (blast intro: bij_is_inj well_ord_rvimage
                well_ord_Memrel [THEN well_ord_subset])
done

```

```

lemma W01_W02: "W01  $\implies$  W02"
  unfolding eqpoll_def W01_def W02_def
  apply (blast intro!: Ord_ordertype ordermap_bij)
done

```

```

lemma lam_sets: "f  $\in$  A $\rightarrow$ B  $\implies$  ( $\lambda x \in A. \{f'x\}$ ): A  $\rightarrow$   $\{\{b\}. b \in B\}$ "
by (fast intro!: lam_type apply_type)

```

```

lemma surj_imp_eq': "f  $\in$  surj(A,B)  $\implies$  ( $\bigcup a \in A. \{f'a\}$ ) = B"
  unfolding surj_def
  apply (fast elim!: apply_type)
done

```

```

lemma surj_imp_eq: "[f  $\in$  surj(A,B); Ord(A)]  $\implies$  ( $\bigcup a < A. \{f'a\}$ ) = B"
by (fast dest!: surj_imp_eq' intro!: ltI elim!: ltE)

```

```

lemma W01_W04: "W01  $\implies$  W04(1)"
  unfolding W01_def W04_def
  apply (rule allI)
  apply (erule_tac x = A in allE)
  apply (erule exE)
  apply (intro exI conjI)
  apply (erule Ord_ordertype)
  apply (erule ordermap_bij [THEN bij_converse_bij, THEN bij_is_fun, THEN
    lam_sets, THEN domain_of_fun])
  apply (simp_all add: singleton_eqpoll_1 eqpoll_imp_lepoll Ord_ordertype
    ordermap_bij [THEN bij_converse_bij, THEN bij_is_surj, THEN surj_imp_eq]
    ltD)
done

```

```

lemma W04_mono: "[m  $\leq$  n; W04(m)]  $\implies$  W04(n)"
  unfolding W04_def
  apply (blast dest!: spec intro: lepoll_trans [OF _ le_imp_lepoll])
done

```

```

lemma W04_W05: "[m  $\in$  nat; 1  $\leq$  m; W04(m)]  $\implies$  W05"
by (unfold W04_def W05_def, blast)

```

```

lemma W05_W06: "W05  $\implies$  W06"
by (unfold W04_def W05_def W06_def, blast)

```

```

lemma lt_oadd_odiff_disj:
  "⟦k < i++j; Ord(i); Ord(j)⟧
   ⇒ k < i | (¬ k < i ∧ k = i ++ (k--i) ∧ (k--i) < j)"
apply (rule_tac i = k and j = i in Ord_linear2)
prefer 4
  apply (drule odiff_lt_mono2, assumption)
  apply (simp add: oadd_odiff_inverse odiff_oadd_inverse)
apply (auto elim!: lt_Ord)
done

```

```

lemma domain_uu_subset: "domain(uu(f,b,g,d)) ⊆ f' b"
by (unfold uu_def, blast)

```

```

lemma quant_domain_uu_lepoll_m:
  "∀ b < a. f' b ≲ m ⇒ ∀ b < a. ∀ g < a. ∀ d < a. domain(uu(f,b,g,d)) ≲ m"
by (blast intro: domain_uu_subset [THEN subset_imp_lepoll] lepoll_trans)

```

```

lemma uu_subset1: "uu(f,b,g,d) ⊆ f' b * f' g"
by (unfold uu_def, blast)

```

```

lemma uu_subset2: "uu(f,b,g,d) ⊆ f' d"
by (unfold uu_def, blast)

```

```

lemma uu_lepoll_m: "⟦∀ b < a. f' b ≲ m; d < a⟧ ⇒ uu(f,b,g,d) ≲ m"
by (blast intro: uu_subset2 [THEN subset_imp_lepoll] lepoll_trans)

```

```

lemma cases:
  "∀ b < a. ∀ g < a. ∀ d < a. u(f,b,g,d) ≲ m
   ⇒ (∀ b < a. f' b ≠ 0 ⇒
      (∃ g < a. ∃ d < a. u(f,b,g,d) ≠ 0 ∧ u(f,b,g,d) < m))
    | (∃ b < a. f' b ≠ 0 ∧ (∀ g < a. ∀ d < a. u(f,b,g,d) ≠ 0 ⇒
      u(f,b,g,d) ≈ m))"

  unfolding lesspoll_def
  apply (blast del: equalityI)

```

done

```
lemma UN_oadd: "Ord(a)  $\implies$  ( $\bigcup b < a++a. C(b)$ ) = ( $\bigcup b < a. C(b) \cup C(a++b)$ )"
by (blast intro: ltI lt_oadd1 oadd_lt_mono2 dest!: lt_oadd_disj)
```

```
lemma vv1_subset: "vv1(f,m,b)  $\subseteq$  f' b"
by (simp add: vv1_def Let_def domain_uu_subset)
```

```
lemma UN_gg1_eq:
  "[[Ord(a); m  $\in$  nat]]  $\implies$  ( $\bigcup b < a++a. gg1(f,a,m) 'b$ ) = ( $\bigcup b < a. f 'b$ )"
by (simp add: gg1_def UN_oadd lt_oadd1 oadd_le_self [THEN le_imp_not_lt]

      lt_Ord odiff_oadd_inverse ltD vv1_subset [THEN Diff_partition]
      ww1_def)
```

```
lemma domain_gg1: "domain(gg1(f,a,m)) = a++a"
by (simp add: lam_funtype [THEN domain_of_fun] gg1_def)
```

```
lemma nested_LeastI:
  "[[P(a, b); Ord(a); Ord(b);
    Least_a = ( $\mu$  a.  $\exists x. Ord(x) \wedge P(a, x)$ )]]
 $\implies$  P(Least_a,  $\mu$  b. P(Least_a, b))"
apply (erule ssubst)
apply (rule_tac Q = " $\lambda z. P(z, \mu b. P(z, b))$ " in LeastI2)
apply (fast elim!: LeastI)+
done
```

```
lemmas nested_Least_instance =
  nested_LeastI [of " $\lambda g d. domain(uu(f,b,g,d)) \neq 0 \wedge$ 
    domain(uu(f,b,g,d))  $\lesssim$  m" for f b m]
```

```
lemma gg1_lepoll_m:
  "[[Ord(a); m  $\in$  nat;
     $\forall b < a. f 'b \neq 0 \implies$ 
    ( $\exists g < a. \exists d < a. domain(uu(f,b,g,d)) \neq 0 \wedge$ 

```

```

                                domain(uu(f,b,g,d))  $\lesssim$  m);
       $\forall b < a. f' b \lesssim \text{succ}(m); b < a++a]$ 
 $\implies \text{gg1}(f,a,m)'b \lesssim m$ 
apply (simp add: gg1_def empty_lepollI)
apply (safe dest!: lt_oadd_odiff_disj)

  apply (simp add: vv1_def Let_def empty_lepollI)
  apply (fast intro: nested_Least_instance [THEN conjunct2]
        elim!: lt_Ord)

apply (simp add: ww1_def empty_lepollI)
apply (case_tac "f' (b--a) = 0", simp add: empty_lepollI)
apply (rule Diff_lepoll, blast)
apply (rule vv1_subset)
apply (drule ospec [THEN mp], assumption+)
apply (elim oexE conjE)
apply (simp add: vv1_def Let_def lt_Ord nested_Least_instance [THEN conjunct1])
done

```

```

lemma ex_d_uu_not_empty:
  "[[b < a; g < a; f' b  $\neq$  0; f' g  $\neq$  0;
    y * y  $\subseteq$  y; ( $\bigcup b < a. f' b$ ) = y]]
 $\implies \exists d < a. \text{uu}(f,b,g,d) \neq 0$ "
by (unfold uu_def, blast)

lemma uu_not_empty:
  "[[b < a; g < a; f' b  $\neq$  0; f' g  $\neq$  0; y * y  $\subseteq$  y; ( $\bigcup b < a. f' b$ ) = y]]
 $\implies \text{uu}(f,b,g,\mu d. (\text{uu}(f,b,g,d) \neq 0)) \neq 0$ "
apply (drule ex_d_uu_not_empty, assumption+)
apply (fast elim!: LeastI lt_Ord)
done

```

```

lemma not_empty_rel_imp_domain: "[[r  $\subseteq$  A * B; r  $\neq$  0]]  $\implies \text{domain}(r) \neq 0$ "
by blast

```

```

lemma Least_uu_not_empty_lt_a:
  "[[b < a; g < a; f' b  $\neq$  0; f' g  $\neq$  0; y * y  $\subseteq$  y; ( $\bigcup b < a. f' b$ ) = y]]
 $\implies (\mu d. \text{uu}(f,b,g,d) \neq 0) < a$ "
apply (erule ex_d_uu_not_empty [THEN oexE], assumption+)
apply (blast intro: Least_le [THEN lt_trans1] lt_Ord)
done

```

```
lemma subset_Diff_sing: " $\llbracket B \subseteq A; a \notin B \rrbracket \implies B \subseteq A - \{a\}$ "
by blast
```

```
lemma supset_lepoll_imp_eq:
  " $\llbracket A \lesssim m; m \lesssim B; B \subseteq A; m \in \text{nat} \rrbracket \implies A=B$ "
apply (erule natE)
apply (fast dest!: lepoll_0_is_0 intro!: equalityI)
apply (safe intro!: equalityI)
apply (rule ccontr)
apply (rule succ_lepoll_natE)
  apply (erule lepoll_trans)
  apply (rule lepoll_trans)
  apply (erule subset_Diff_sing [THEN subset_imp_lepoll], assumption)
  apply (rule Diff_sing_lepoll, assumption+)
done
```

```
lemma uu_Least_is_fun:
  " $\llbracket \forall g < a. \forall d < a. \text{domain}(\text{uu}(f, b, g, d)) \neq 0 \longrightarrow$   

    $\text{domain}(\text{uu}(f, b, g, d)) \approx \text{succ}(m);$   

    $\forall b < a. f'b \lesssim \text{succ}(m); y*y \subseteq y;$   

    $(\bigcup b < a. f'b) = y; b < a; g < a; d < a;$   

    $f'b \neq 0; f'g \neq 0; m \in \text{nat}; s \in f'b \rrbracket$   

 $\implies \text{uu}(f, b, g, \mu d. \text{uu}(f, b, g, d) \neq 0) \in f'b \rightarrow f'g$ "
apply (drule_tac x2=g in ospec [THEN ospec, THEN mp])
  apply (rule_tac [3] not_empty_rel_imp_domain [OF uu_subset1 uu_not_empty])
  apply (rule_tac [2] Least_uu_not_empty_lt_a, assumption+)
apply (rule rel_is_fun)
  apply (erule eqpoll_sym [THEN eqpoll_imp_lepoll])
  apply (erule uu_lepoll_m)
  apply (rule Least_uu_not_empty_lt_a, assumption+)
apply (rule uu_subset1)
apply (rule supset_lepoll_imp_eq [OF _ eqpoll_sym [THEN eqpoll_imp_lepoll]])
apply (fast intro!: domain_uu_subset)+
done
```

```
lemma vv2_subset:
  " $\llbracket \forall g < a. \forall d < a. \text{domain}(\text{uu}(f, b, g, d)) \neq 0 \longrightarrow$   

    $\text{domain}(\text{uu}(f, b, g, d)) \approx \text{succ}(m);$   

    $\forall b < a. f'b \lesssim \text{succ}(m); y*y \subseteq y;$   

    $(\bigcup b < a. f'b) = y; b < a; g < a; m \in \text{nat}; s \in f'b \rrbracket$   

 $\implies \text{vv2}(f, b, g, s) \subseteq f'g$ "
apply (simp add: vv2_def)
apply (blast intro: uu_Least_is_fun [THEN apply_type])
done
```



```

lemma UN_gg2_eq:
  "⟦∀g<a. ∀d<a. domain(uu(f,b,g,d)) ≠ 0 ⟶
    domain(uu(f,b,g,d)) ≈ succ(m);
    ∀b<a. f' b ≲ succ(m); y*y ⊆ y;
    (⋃ b<a. f' b)=y; Ord(a); m ∈ nat; s ∈ f' b; b<a⟧
  ⟹ (⋃ g<a++a. gg2(f,a,b,s) ' g) = y"
  unfolding gg2_def
  apply (drule sym)
  apply (simp add: ltD UN_oadd oadd_le_self [THEN le_imp_not_lt]
    lt_Ord odiff_oadd_inverse ww2_def
    vv2_subset [THEN Diff_partition])
done

lemma domain_gg2: "domain(gg2(f,a,b,s)) = a++a"
by (simp add: lam_funtype [THEN domain_of_fun] gg2_def)

lemma vv2_lepoll: "⟦m ∈ nat; m≠0⟧ ⟹ vv2(f,b,g,s) ≲ m"
  unfolding vv2_def
  apply (simp add: empty_lepollI)
  apply (fast dest!: le_imp_subset [THEN subset_imp_lepoll, THEN lepoll_0_is_0]
    intro!: singleton_eqpoll_1 [THEN eqpoll_imp_lepoll, THEN lepoll_trans]
    not_lt_imp_le [THEN le_imp_subset, THEN subset_imp_lepoll]
    nat_into_Ord nat_1I)
done

lemma ww2_lepoll:
  "⟦∀b<a. f' b ≲ succ(m); g<a; m ∈ nat; vv2(f,b,g,d) ⊆ f' g⟧
  ⟹ ww2(f,b,g,d) ≲ m"
  unfolding ww2_def
  apply (case_tac "f' g = 0")
  apply (simp add: empty_lepollI)
  apply (drule ospec, assumption)
  apply (rule Diff_lepoll, assumption+)
  apply (simp add: vv2_def not_emptyI)
done

lemma gg2_lepoll_m:
  "⟦∀g<a. ∀d<a. domain(uu(f,b,g,d)) ≠ 0 ⟶
    domain(uu(f,b,g,d)) ≈ succ(m);
    ∀b<a. f' b ≲ succ(m); y*y ⊆ y;
    (⋃ b<a. f' b)=y; b<a; s ∈ f' b; m ∈ nat; m≠0; g<a++a⟧
  ⟹ gg2(f,a,b,s) ' g ≲ m"

```

```

apply (simp add: gg2_def empty_lepollI)
apply (safe elim!: lt_Ord2 dest!: lt_oadd_odiff_disj)
  apply (simp add: vv2_lepoll)
apply (simp add: ww2_lepoll vv2_subset)
done

```

```

lemma lemma_ii: "[[succ(m) ∈ NN(y); y*y ⊆ y; m ∈ nat; m≠0]] ⇒ m ∈
NN(y)"
  unfolding NN_def
  apply (elim CollectE exE conjE)
  apply (rule quant_domain_uu_lepoll_m [THEN cases, THEN disjE], assumption)

  apply (simp add: lesspoll_succ_iff)
  apply (rule_tac x = "a++a" in exI)
  apply (fast intro!: Ord_oadd domain_gg1 UN_gg1_eq gg1_lepoll_m)

  apply (elim oexE conjE)
  apply (rule_tac A = "f'B" for B in not_emptyE, assumption)
  apply (rule CollectI)
  apply (erule succ_natD)
  apply (rule_tac x = "a++a" in exI)
  apply (rule_tac x = "gg2 (f,a,b,x) " in exI)
  apply (simp add: Ord_oadd domain_gg2 UN_gg2_eq gg2_lepoll_m)
done

```

```

lemma z_n_subset_z_succ_n:
  "∀ n ∈ nat. rec(n, x, λk r. r ∪ r*r) ⊆ rec(succ(n), x, λk r. r ∪
r*r)"
by (fast intro: rec_succ [THEN ssubst])

```

```

lemma le_subsets:
  "[[∀ n ∈ nat. f(n) ≤ f(succ(n)); n ≤ m; n ∈ nat; m ∈ nat]]
⇒ f(n) ≤ f(m)"
  apply (erule_tac P = "n ≤ m" in rev_mp)
  apply (rule_tac P = "λz. n ≤ z → f (n) ⊆ f (z) " in nat_induct)
  apply (auto simp add: le_iff)
done

```

```

lemma le_imp_rec_subset:
  "[[n ≤ m; m ∈ nat]]
  ⇒ rec(n, x, λk r. r ∪ r*r) ⊆ rec(m, x, λk r. r ∪ r*r)"
apply (rule z_n_subset_z_succ_n [THEN le_subsets])
apply (blast intro: lt_nat_in_nat)+
done

lemma lemma_iv: "∃ y. x ∪ y*y ⊆ y"
apply (rule_tac x = "⋃ n ∈ nat. rec (n, x, λk r. r ∪ r*r) " in exI)
apply safe
apply (rule nat_0I [THEN UN_I], simp)
apply (rule_tac a = "succ (n ∪ na) " in UN_I)
apply (erule Un_nat_type [THEN nat_succI], assumption)
apply (auto intro: le_imp_rec_subset [THEN subsetD]
  intro!: Un_upper1_le Un_upper2_le Un_nat_type
  elim!: nat_into_Ord)
done

```

```

lemma W06_imp_NN_not_empty: "W06 ⇒ NN(y) ≠ 0"
by (unfold W06_def NN_def, clarify, blast)

```

```

lemma lemma1:
  "[[⋃ b < a. f' b = y; x ∈ y; ∀ b < a. f' b ≲ 1; Ord(a)]] ⇒ ∃ c < a. f' c = {x}"
by (fast elim!: lepoll_1_is_sing)

```

```

lemma lemma2:
  "[[⋃ b < a. f' b = y; x ∈ y; ∀ b < a. f' b ≲ 1; Ord(a)]]
  ⇒ f' (μ i. f' i = {x}) = {x}"

```

```

apply (drule lemma1, assumption+)
apply (fast elim!: lt_Ord intro: LeastI)
done

lemma NN_imp_ex_inj: "1 ∈ NN(y) ⇒ ∃ a f. Ord(a) ∧ f ∈ inj(y, a)"
  unfolding NN_def
  apply (elim CollectE exE conjE)
  apply (rule_tac x = a in exI)
  apply (rule_tac x = "λx ∈ y. μ i. f'i = {x}" in exI)
  apply (rule conjI, assumption)
  apply (rule_tac d = "λi. THE x. x ∈ f'i" in lam_injective)
  apply (drule lemma1, assumption+)
  apply (fast elim!: Least_le [THEN lt_trans1, THEN ltD] lt_Ord)
  apply (rule lemma2 [THEN ssubst], assumption+, blast)
done

lemma y_well_ord: "[y*y ⊆ y; 1 ∈ NN(y)] ⇒ ∃ r. well_ord(y, r)"
  apply (drule NN_imp_ex_inj)
  apply (fast elim!: well_ord_rvimage [OF _ well_ord_Memrel])
done

lemma rev_induct_lemma [rule_format]:
  "[n ∈ nat; ∧m. [m ∈ nat; m ≠ 0; P(succ(m))] ⇒ P(m)]
  ⇒ n ≠ 0 → P(n) → P(1)"
  by (erule nat_induct, blast+)

lemma rev_induct:
  "[n ∈ nat; P(n); n ≠ 0;
  ∧m. [m ∈ nat; m ≠ 0; P(succ(m))] ⇒ P(m)]
  ⇒ P(1)"
  by (rule rev_induct_lemma, blast+)

lemma NN_into_nat: "n ∈ NN(y) ⇒ n ∈ nat"
  by (simp add: NN_def)

lemma lemma3: "[n ∈ NN(y); y*y ⊆ y; n ≠ 0] ⇒ 1 ∈ NN(y)"
  apply (rule rev_induct [OF NN_into_nat], assumption+)
  apply (rule lemma_ii, assumption+)
  done

```

```

lemma NN_y_0: "0 ∈ NN(y) ⇒ y=0"
  unfolding NN_def
  apply (fast intro!: equalityI dest!: lepoll_0_is_0 elim: subst)
  done

lemma W06_imp_W01: "W06 ⇒ W01"
  unfolding W01_def
  apply (rule allI)
  apply (case_tac "A=0")
  apply (fast intro!: well_ord_Memrel nat_0I [THEN nat_into_Ord])
  apply (rule_tac x = A in lemma_iv [elim_format])
  apply (erule exE)
  apply (drule W06_imp_NN_not_empty)
  apply (erule Un_subset_iff [THEN iffD1, THEN conjE])
  apply (erule_tac A = "NN (y) " in not_emptyE)
  apply (frule y_well_ord)
  apply (fast intro!: lemma3 dest!: NN_y_0 elim!: not_emptyE)
  apply (fast elim: well_ord_subset)
  done

end

theory W01_W07
imports AC_Equiv
begin

definition
  "LEMMA ≡
    ∀ X. ¬Finite(X) → (∃ R. well_ord(X,R) ∧ ¬well_ord(X,converse(R)))"

lemma W07_iff_LEMMA: "W07 ↔ LEMMA"
  unfolding W07_def LEMMA_def
  apply (blast intro: Finite_well_ord_converse)
  done

lemma LEMMA_imp_W01: "LEMMA ⇒ W01"
  unfolding W01_def LEMMA_def Finite_def eqpoll_def
  apply (blast intro!: well_ord_rvimage [OF bij_is_inj nat_implies_well_ord])
  done

```

```

lemma converse_Memrel_not_wf_on:
  "[[Ord(a); ¬Finite(a)]] ⇒ ¬wf[a](converse(Memrel(a)))"
  unfolding wf_on_def wf_def
  apply (drule nat_le_infinite_Ord [THEN le_imp_subset], assumption)
  apply (rule notI)
  apply (erule_tac x = nat in allE, blast)
done

lemma converse_Memrel_not_well_ord:
  "[[Ord(a); ¬Finite(a)]] ⇒ ¬well_ord(a, converse(Memrel(a)))"
  unfolding well_ord_def
  apply (blast dest: converse_Memrel_not_wf_on)
done

lemma well_ord_rvimage_ordertype:
  "well_ord(A,r) ⇒
    rvimage (ordertype(A,r), converse(ordermap(A,r)),r) =
    Memrel(ordertype(A,r))"
  by (blast intro: ordertype_ord_iso [THEN ord_iso_sym] ord_iso_rvimage_eq
      Memrel_type [THEN subset_Int_iff [THEN iffD1]] trans)

lemma well_ord_converse_Memrel:
  "[[well_ord(A,r); well_ord(A,converse(r))]]
  ⇒ well_ord(ordertype(A,r), converse(Memrel(ordertype(A,r))))"

apply (subst well_ord_rvimage_ordertype [symmetric], assumption)
apply (rule rvimage_converse [THEN subst])
apply (blast intro: ordertype_ord_iso ord_iso_sym ord_iso_is_bij
    bij_is_inj well_ord_rvimage)
done

lemma W01_imp_LEMMA: "W01 ⇒ LEMMA"
apply (unfold W01_def LEMMA_def, clarify)
apply (blast dest: well_ord_converse_Memrel
    Ord_ordertype [THEN converse_Memrel_not_well_ord]
    intro: ordertype_ord_iso ord_iso_is_bij bij_is_inj lepoll_Finite

```

```

                                lepoll_def [THEN def_imp_iff, THEN iffD2] )
done

lemma W01_iff_W07: "W01  $\longleftrightarrow$  W07"
apply (simp add: W07_iff_LEMMA)
apply (blast intro: LEMMA_imp_W01 W01_imp_LEMMA)
done

lemma W01_W08: "W01  $\implies$  W08"
by (unfold W01_def W08_def, fast)

lemma W08_W01: "W08  $\implies$  W01"
  unfolding W01_def W08_def
apply (rule allI)
apply (erule_tac x = "{x}. x  $\in$  A" in allE)
apply (erule impE)
  apply (rule_tac x = " $\lambda a \in \{x\}. x \in A$ . THE x. a={x}" in exI)
  apply (force intro!: lam_type simp add: singleton_eq_iff the_equality)
apply (blast intro: lam_sing_bij bij_is_inj well_ord_rvimage)
done

end

theory AC7_AC9
imports AC_Equiv
begin

lemma Sigma_fun_space_not0: "[0  $\notin$  A; B  $\in$  A]  $\implies$  (nat  $\rightarrow$   $\bigcup$  (A)) * B  $\neq$  0"
by (blast dest!: Sigma_empty_iff [THEN iffD1] Union_empty_iff [THEN iffD1])

lemma inj_lemma:
  "C  $\in$  A  $\implies$  ( $\lambda g \in$  (nat  $\rightarrow$   $\bigcup$  (A)) * C.
    ( $\lambda n \in$  nat. if (n=0, snd(g), fst(g)^(n #- 1))))
     $\in$  inj((nat  $\rightarrow$   $\bigcup$  (A)) * C, (nat  $\rightarrow$   $\bigcup$  (A))) "
```

```

    unfolding inj_def
  apply (rule CollectI)
  apply (fast intro!: lam_type if_type apply_type fst_type snd_type, auto)

  apply (rule fun_extension, assumption+)
  apply (drule lam_eqE [OF _ nat_succI], assumption, simp)
  apply (drule lam_eqE [OF _ nat_0I], simp)
  done

lemma Sigma_fun_space_eqpoll:
  " $\llbracket C \in A; 0 \notin A \rrbracket \implies (\text{nat} \rightarrow \bigcup (A)) * C \approx (\text{nat} \rightarrow \bigcup (A))$ "
  apply (rule eqpollI)
  apply (simp add: lepoll_def)
  apply (fast intro!: inj_lemma)
  apply (fast elim!: prod_lepoll_self not_sym [THEN not_emptyE] subst_elem

    elim: swap)
  done

```

```

lemma AC6_AC7: "AC6  $\implies$  AC7"
by (unfold AC6_def AC7_def, blast)

```

```

lemma lemma1_1: " $y \in (\prod B \in A. Y*B) \implies (\lambda B \in A. \text{snd}(y'B)) \in (\prod B \in A. B)$ "
by (fast intro!: lam_type snd_type apply_type)

```

```

lemma lemma1_2:
  " $y \in (\prod B \in \{Y*C. C \in A\}. B) \implies (\lambda B \in A. y'(Y*B)) \in (\prod B \in A. Y*B)$ "
  apply (fast intro!: lam_type apply_type)
  done

```

```

lemma AC7_AC6_lemma1:
  " $(\prod B \in \{(\text{nat} \rightarrow \bigcup (A)) * C. C \in A\}. B) \neq 0 \implies (\prod B \in A. B) \neq 0$ "
  by (fast intro!: equals0I lemma1_1 lemma1_2)

```

```

lemma AC7_AC6_lemma2: " $0 \notin A \implies 0 \notin \{(\text{nat} \rightarrow \bigcup (A)) * C. C \in A\}$ "
by (blast dest: Sigma_fun_space_not0)

```



```

lemma AC7_AC6: "AC7  $\implies$  AC6"
  unfolding AC6_def AC7_def
  apply (rule allI)
  apply (rule impI)
  apply (case_tac "A=0", simp)
  apply (rule AC7_AC6_lemma1)
  apply (erule allE)
  apply (blast del: notI
    intro!: AC7_AC6_lemma2 intro: eqpoll_sym eqpoll_trans
      Sigma_fun_space_eqpoll)
done

```

```

lemma AC1_AC8_lemma1:
  " $\forall B \in A. \exists B1\ B2. B=\langle B1,B2 \rangle \wedge B1 \approx B2$ 
 $\implies 0 \notin \{ \text{bij}(\text{fst}(B), \text{snd}(B)).\ B \in A \}$ "
  apply (unfold eqpoll_def, auto)
done

```

```

lemma AC1_AC8_lemma2:
  " $\llbracket f \in (\prod X \in \text{RepFun}(A,p).\ X); D \in A \rrbracket \implies (\lambda x \in A. f'p(x))'D \in$ 
 $p(D)$ "
  apply (simp, fast elim!: apply_type)
done

```

```

lemma AC1_AC8: "AC1  $\implies$  AC8"
  unfolding AC1_def AC8_def
  apply (fast dest: AC1_AC8_lemma1 AC1_AC8_lemma2)
done

```

```

lemma AC8_AC9_lemma:
  " $\forall B1 \in A. \forall B2 \in A. B1 \approx B2$ 
 $\implies \forall B \in A * A. \exists B1\ B2. B=\langle B1,B2 \rangle \wedge B1 \approx B2$ "
  by fast

```

```

lemma AC8_AC9: "AC8  $\implies$  AC9"
  unfolding AC8_def AC9_def

```

```

apply (intro allI impI)
apply (erule allE)
apply (erule impE, erule AC8_AC9_lemma, force)
done

```

```

lemma snd_lepoll_SigmaI: "b ∈ B ⇒ X ≲ B × X"
by (blast intro: lepoll_trans prod_lepoll_self eqpoll_imp_lepoll
    prod_commute_eqpoll)

```

```

lemma nat_lepoll_lemma:
  "0 ∉ A; B ∈ A ⇒ nat ≲ ((nat → ⋃ (A)) × B) × nat"
by (blast dest: Sigma_fun_space_not0 intro: snd_lepoll_SigmaI)

```

```

lemma AC9_AC1_lemma1:
  "0 ∉ A; A ≠ 0;
   C = {(nat → ⋃ (A)) * B} * nat. B ∈ A} ∪
   {cons(0, (nat → ⋃ (A)) * B) * nat. B ∈ A};
   B1 ∈ C; B2 ∈ C ⇒
   ⇒ B1 ≈ B2"
by (blast intro!: nat_lepoll_lemma Sigma_fun_space_eqpoll
    nat_cons_eqpoll [THEN eqpoll_trans]
    prod_eqpoll_cong [OF _ eqpoll_refl]
    intro: eqpoll_trans eqpoll_sym )

```

```

lemma AC9_AC1_lemma2:
  "∀ B1 ∈ {(F*B)*N. B ∈ A} ∪ {cons(0, (F*B)*N). B ∈ A}.
   ∀ B2 ∈ {(F*B)*N. B ∈ A} ∪ {cons(0, (F*B)*N). B ∈ A}.
   f'⟨B1, B2⟩ ∈ bij(B1, B2)
   ⇒ (λ B ∈ A. snd(fst((f'⟨cons(0, (F*B)*N), (F*B)*N⟩) '0))) ∈ (∏ X
   ∈ A. X)"
apply (intro lam_type snd_type fst_type)
apply (rule apply_type [OF _ consI1])
apply (fast intro!: fun_weaken_type bij_is_fun)
done

```

```

lemma AC9_AC1: "AC9 ⇒ AC1"
  unfolding AC1_def AC9_def
apply (intro allI impI)
apply (erule allE)
apply (case_tac "A ≠ 0")

```

```

apply (blast dest: AC9_AC1_lemma1 AC9_AC1_lemma2, force)
done

end

theory W01_AC
imports AC_Equiv
begin

theorem W01_AC1: "W01  $\implies$  AC1"
by (unfold AC1_def W01_def, fast elim!: ex_choice_fun)

lemma lemma1: " $\llbracket W01; \forall B \in A. \exists C \in D(B). P(C,B) \rrbracket \implies \exists f. \forall B \in A. P(f'B,B)$ "
  unfolding W01_def
  apply (erule_tac x = " $\bigcup (\{ \{ C \in D(B) . P(C,B) \} . B \in A \})$ " in allE)
  apply (erule exE, drule ex_choice_fun, fast)
  apply (erule exE)
  apply (rule_tac x = " $\lambda x \in A. f'\{ C \in D(x) . P(C,x) \}$ " in exI)
  apply (simp, blast dest!: apply_type [OF _ RepFunI])
  done

lemma lemma2_1: " $\llbracket \neg \text{Finite}(B); W01 \rrbracket \implies |B| + |B| \approx B$ "
  unfolding W01_def
  apply (rule eqpoll_trans)
  prefer 2 apply (fast elim!: well_ord_cardinal_eqpoll)
  apply (rule eqpoll_sym [THEN eqpoll_trans])
  apply (fast elim!: well_ord_cardinal_eqpoll)
  apply (drule spec [of _ B])
  apply (clarify dest!: eqpoll_imp_Finite_iff [OF well_ord_cardinal_eqpoll])

  apply (simp add: cadd_def [symmetric]
    eqpoll_refl InfCard_cdouble_eq Card_cardinal Inf_Card_is_InfCard)

done

lemma lemma2_2:
  " $f \in \text{bij}(D+D, B) \implies \{ \{ f' \text{Inl}(i), f' \text{Inr}(i) \} . i \in D \} \in \text{Pow}(\text{Pow}(B))$ "
by (fast elim!: bij_is_fun [THEN apply_type])

```

```

lemma lemma2_3:
  "f ∈ bij(D+D, B) ⇒ pairwise_disjoint({{f'Inl(i), f'Inr(i)}.
i ∈ D})"
  unfolding pairwise_disjoint_def
apply (blast dest: bij_is_inj [THEN inj_apply_equality])
done

lemma lemma2_4:
  "[f ∈ bij(D+D, B); 1 ≤ n]
  ⇒ sets_of_size_between({{f'Inl(i), f'Inr(i)}. i ∈ D}, 2, succ(n))"
apply (simp (no_asm_simp) add: sets_of_size_between_def succ_def)
apply (blast intro!: cons_lepoll_cong
  intro: singleton_eqpoll_1 [THEN eqpoll_imp_lepoll]
  le_imp_subset [THEN subset_imp_lepoll] lepoll_trans
  dest: bij_is_inj [THEN inj_apply_equality] elim!: mem_irrefl)
done

lemma lemma2_5:
  "f ∈ bij(D+D, B) ⇒ ⋃ ({f'Inl(i), f'Inr(i)}. i ∈ D)=B"
  unfolding bij_def surj_def
apply (fast elim!: inj_is_fun [THEN apply_type])
done

lemma lemma2:
  "[W01; ¬Finite(B); 1 ≤ n]
  ⇒ ∃ C ∈ Pow(Pow(B)). pairwise_disjoint(C) ∧
  sets_of_size_between(C, 2, succ(n)) ∧
  ⋃ (C)=B"
apply (drule lemma2_1 [THEN eqpoll_def [THEN def_imp_iff, THEN iffD1]],
  assumption)
apply (blast intro!: lemma2_2 lemma2_3 lemma2_4 lemma2_5)
done

theorem W01_AC10: "[W01; 1 ≤ n] ⇒ AC10(n)"
  unfolding AC10_def
apply (fast intro!: lemma1 elim!: lemma2)
done

end

theory Hartog
imports AC_Equiv
begin

definition
  Hartog :: "i ⇒ i" where

```

```

    "Hartog(X)  $\equiv \mu i. \neg i \lesssim X$ "

lemma Ords_in_set: " $\forall a. \text{Ord}(a) \longrightarrow a \in X \implies P$ "
apply (rule_tac X = "{y  $\in$  X. Ord (y) }" in ON_class [elim_format])
apply fast
done

lemma Ord_lepoll_imp_ex_well_ord:
  "[Ord(a); a  $\lesssim$  X]
 $\implies \exists Y. Y \subseteq X \wedge (\exists R. \text{well\_ord}(Y,R) \wedge \text{ordertype}(Y,R)=a)$ "
  unfolding lepoll_def
apply (erule exE)
apply (intro exI conjI)
  apply (erule inj_is_fun [THEN fun_is_rel, THEN image_subset])
  apply (rule well_ord_rvimage [OF bij_is_inj well_ord_Memrel])
  apply (erule restrict_bij [THEN bij_converse_bij])
apply (rule subset_refl, assumption)
apply (rule trans)
apply (rule bij_ordertype_vimage)
apply (erule restrict_bij [THEN bij_converse_bij])
apply (rule subset_refl)
apply (erule well_ord_Memrel)
apply (erule ordertype_Memrel)
done

lemma Ord_lepoll_imp_eq_ordertype:
  "[Ord(a); a  $\lesssim$  X]  $\implies \exists Y. Y \subseteq X \wedge (\exists R. R \subseteq X \times X \wedge \text{ordertype}(Y,R)=a)$ "
apply (drule Ord_lepoll_imp_ex_well_ord, assumption, clarify)
apply (intro exI conjI)
apply (erule_tac [3] ordertype_Int, auto)
done

lemma Ords_lepoll_set_lemma:
  " $(\forall a. \text{Ord}(a) \longrightarrow a \lesssim X) \implies$ 
 $\forall a. \text{Ord}(a) \longrightarrow$ 
 $a \in \{b. Z \in \text{Pow}(X) * \text{Pow}(X \times X), \exists Y R. Z = \langle Y, R \rangle \wedge \text{ordertype}(Y,R)=b\}$ "
apply (intro allI impI)
apply (elim allE impE, assumption)
apply (blast dest!: Ord_lepoll_imp_eq_ordertype intro: sym)
done

lemma Ords_lepoll_set: " $\forall a. \text{Ord}(a) \longrightarrow a \lesssim X \implies P$ "
by (erule Ords_lepoll_set_lemma [THEN Ords_in_set])

lemma ex_Ord_not_lepoll: " $\exists a. \text{Ord}(a) \wedge \neg a \lesssim X$ "
apply (rule ccontr)
apply (best intro: Ords_lepoll_set)
done

```

```

lemma not_Hartog_lepoll_self: " $\neg \text{Hartog}(A) \lesssim A$ "
  unfolding Hartog_def
  apply (rule ex_Ord_not_lepoll [THEN exE])
  apply (rule LeastI, auto)
  done

lemmas Hartog_lepoll_selfE = not_Hartog_lepoll_self [THEN notE]

lemma Ord_Hartog: " $\text{Ord}(\text{Hartog}(A))$ "
  by (unfold Hartog_def, rule Ord_Least)

lemma less_HartogE1: " $\llbracket i < \text{Hartog}(A); \neg i \lesssim A \rrbracket \implies P$ "
  by (unfold Hartog_def, fast elim: less_LeastE)

lemma less_HartogE: " $\llbracket i < \text{Hartog}(A); i \approx \text{Hartog}(A) \rrbracket \implies P$ "
  by (blast intro: less_HartogE1 eqpoll_sym eqpoll_imp_lepoll
    lepoll_trans [THEN Hartog_lepoll_selfE])

lemma Card_Hartog: " $\text{Card}(\text{Hartog}(A))$ "
  by (fast intro!: CardI Ord_Hartog elim: less_HartogE)

end

theory HH
imports AC_Equiv Hartog
begin

definition
  HH :: "[i, i, i]  $\Rightarrow$  i" where
    "HH(f,x,a)  $\equiv$  transrec(a,  $\lambda b$  r. let z = x - ( $\bigcup c \in b.$  r'c)
      in if f'z  $\in$  Pow(z)-{0} then f'z else
{x})"

0.1 Lemmas useful in each of the three proofs

lemma HH_def_satisfies_eq:
  "HH(f,x,a) = (let z = x - ( $\bigcup b \in a.$  HH(f,x,b))
    in if f'z  $\in$  Pow(z)-{0} then f'z else {x})"
  by (rule HH_def [THEN def_transrec, THEN trans], simp)

lemma HH_values: " $\text{HH}(f,x,a) \in \text{Pow}(x) - \{0\} \mid \text{HH}(f,x,a) = \{x\}$ "
  apply (rule HH_def_satisfies_eq [THEN ssubst])
  apply (simp add: Let_def Diff_subset [THEN PowI], fast)
  done

lemma subset_imp_Diff_eq:
  " $B \subseteq A \implies X - (\bigcup a \in A. P(a)) = X - (\bigcup a \in A - B. P(a)) - (\bigcup b \in B. P(b))$ "
  by fast

```

```

lemma Ord_DiffE: "[c ∈ a-b; b<a] ⇒ c=b / b<c ∧ c<a"
apply (erule ltE)
apply (drule Ord_linear [of _ c])
apply (fast elim: Ord_in_Ord)
apply (fast intro!: ltI intro: Ord_in_Ord)
done

lemma Diff_UN_eq_self: "(⋀y. y∈A ⇒ P(y) = {x}) ⇒ x - (⋃y ∈ A.
P(y)) = x"
by (simp, fast elim!: mem_irrefl)

lemma HH_eq: "x - (⋃b ∈ a. HH(f,x,b)) = x - (⋃b ∈ a1. HH(f,x,b))
⇒ HH(f,x,a) = HH(f,x,a1)"
apply (subst HH_def_satisfies_eq [of _ a1])
apply (rule HH_def_satisfies_eq [THEN trans], simp)
done

lemma HH_is_x_gt_too: "[HH(f,x,b)={x}; b<a] ⇒ HH(f,x,a)={x}"
apply (rule_tac P = "b<a" in impE)
prefer 2 apply assumption+
apply (erule lt_Ord2 [THEN trans_induct])
apply (rule impI)
apply (rule HH_eq [THEN trans])
prefer 2 apply assumption+
apply (rule leI [THEN le_imp_subset, THEN subset_imp_Diff_eq, THEN ssubst],
      assumption)
apply (rule_tac t = "λz. z-X" for X in subst_context)
apply (rule Diff_UN_eq_self)
apply (drule Ord_DiffE, assumption)
apply (fast elim: ltE, auto)
done

lemma HH_subset_x_lt_too:
  "[HH(f,x,a) ∈ Pow(x)-{0}; b<a] ⇒ HH(f,x,b) ∈ Pow(x)-{0}"
apply (rule HH_values [THEN disjE], assumption)
apply (drule HH_is_x_gt_too, assumption)
apply (drule subst, assumption)
apply (fast elim!: mem_irrefl)
done

lemma HH_subset_x_imp_subset_Diff_UN:
  "HH(f,x,a) ∈ Pow(x)-{0} ⇒ HH(f,x,a) ∈ Pow(x - (⋃b ∈ a. HH(f,x,b)))-{0}"
apply (drule HH_def_satisfies_eq [THEN subst])
apply (rule HH_def_satisfies_eq [THEN ssubst])
apply (simp add: Let_def Diff_subset [THEN PowI])
apply (drule split_if [THEN iffD1])
apply (fast elim!: mem_irrefl)

```

done

lemma HH_eq_arg_lt:

" $\llbracket HH(f, x, v) = HH(f, x, w); HH(f, x, v) \in Pow(x) - \{0\}; v \in w \rrbracket \implies P$ "
 apply (frule_tac P = " $\lambda y. y \in Pow(x) - \{0\}$ " in subst, assumption)
 apply (drule_tac a = w in HH_subset_x_imp_subset_Diff_UN)
 apply (drule subst_elem, assumption)
 apply (fast intro!: singleton_iff [THEN iffD2] equals0I)
 done

lemma HH_eq_imp_arg_eq:

" $\llbracket HH(f, x, v) = HH(f, x, w); HH(f, x, w) \in Pow(x) - \{0\}; Ord(v); Ord(w) \rrbracket \implies v = w$ "
 apply (rule_tac j = w in Ord_linear_lt)
 apply (simp_all (no_asm_simp))
 apply (drule subst_elem, assumption)
 apply (blast dest: ltD HH_eq_arg_lt)
 apply (blast dest: HH_eq_arg_lt [OF sym] ltD)
 done

lemma HH_subset_x_imp_lepoll:

" $\llbracket HH(f, x, i) \in Pow(x) - \{0\}; Ord(i) \rrbracket \implies i \lesssim Pow(x) - \{0\}$ "
 unfolding lepoll_def inj_def
 apply (rule_tac x = " $\lambda j \in i. HH(f, x, j)$ " in exI)
 apply (simp (no_asm_simp))
 apply (fast del: DiffE
 elim!: HH_eq_imp_arg_eq Ord_in_Ord HH_subset_x_lt_too
 intro!: lam_type ballI ltI intro: bexI)

done

lemma HH_Hartog_is_x: " $HH(f, x, Hartog(Pow(x) - \{0\})) = \{x\}$ "

apply (rule HH_values [THEN disjE])
 prefer 2 apply assumption
 apply (fast del: DiffE
 intro!: Ord_Hartog
 dest!: HH_subset_x_imp_lepoll
 elim!: Hartog_lepoll_selfE)

done

lemma HH_Least_eq_x: " $HH(f, x, \mu i. HH(f, x, i) = \{x\}) = \{x\}$ "

by (fast intro!: Ord_Hartog HH_Hartog_is_x LeastI)

lemma less_Least_subset_x:

" $a \in (\mu i. HH(f, x, i) = \{x\}) \implies HH(f, x, a) \in Pow(x) - \{0\}$ "
 apply (rule HH_values [THEN disjE], assumption)
 apply (rule less_LeastE)
 apply (erule_tac [2] ltI [OF _ Ord_Least], assumption)
 done

0.2 Lemmas used in the proofs of $AC1 \implies W02$ and $AC17 \implies AC1$

```

lemma lam_Least_HH_inj_Pow:
  "(\lambda a \in (\mu i. HH(f,x,i)={x}). HH(f,x,a))
   \in inj(\mu i. HH(f,x,i)={x}, Pow(x)-{0})"
apply (unfold inj_def, simp)
apply (fast intro!: lam_type dest: less_Least_subset_x
      elim!: HH_eq_imp_arg_eq Ord_Least [THEN Ord_in_Ord])
done

lemma lam_Least_HH_inj:
  "\forall a \in (\mu i. HH(f,x,i)={x}). \exists z \in x. HH(f,x,a) = {z}
   \implies (\lambda a \in (\mu i. HH(f,x,i)={x}). HH(f,x,a))
    \in inj(\mu i. HH(f,x,i)={x}, {\{y\}. y \in x})"
by (rule lam_Least_HH_inj_Pow [THEN inj_strengthen_type], simp)

lemma lam_surj_sing:
  "[x - (\bigcup a \in A. F(a)) = 0; \forall a \in A. \exists z \in x. F(a) = {z}]
   \implies (\lambda a \in A. F(a)) \in surj(A, {\{y\}. y \in x})"
apply (simp add: surj_def lam_type Diff_eq_0_iff)
apply (blast elim: equalityE)
done

lemma not_emptyI2: "y \in Pow(x)-{0} \implies x \neq 0"
by auto

lemma f_subset_imp_HH_subset:
  "f'(x - (\bigcup j \in i. HH(f,x,j))) \in Pow(x - (\bigcup j \in i. HH(f,x,j)))-{0}
   \implies HH(f, x, i) \in Pow(x) - {0}"
apply (rule HH_def_satisfies_eq [THEN ssubst])
apply (simp add: Let_def Diff_subset [THEN PowI] not_emptyI2 [THEN if_P],
      fast)
done

lemma f_subsets_imp_UN_HH_eq_x:
  "\forall z \in Pow(x)-{0}. f'z \in Pow(z)-{0}
   \implies x - (\bigcup j \in (\mu i. HH(f,x,i)={x}). HH(f,x,j)) = 0"
apply (case_tac "P \in {0}" for P, fast)
apply (drule Diff_subset [THEN PowI, THEN DiffI])
apply (drule bspec, assumption)
apply (drule f_subset_imp_HH_subset)
apply (blast dest!: subst_elem [OF _ HH_Least_eq_x [symmetric]]
      elim!: mem_irrefl)
done

lemma HH_values2: "HH(f,x,i) = f'(x - (\bigcup j \in i. HH(f,x,j))) \mid HH(f,x,i)={x}"
apply (rule HH_def_satisfies_eq [THEN ssubst])
apply (simp add: Let_def Diff_subset [THEN PowI])
done

```

```

lemma HH_subset_imp_eq:
  "HH(f,x,i): Pow(x)-{0}  $\implies$  HH(f,x,i)=f'(x - ( $\bigcup j \in i$ . HH(f,x,j)))"
apply (rule HH_values2 [THEN disjE], assumption)
apply (fast elim!: equalityE mem_irrefl dest!: singleton_subsetD)
done

lemma f_sing_imp_HH_sing:
  "[f  $\in$  (Pow(x)-{0})  $\rightarrow$  {z}. z  $\in$  x];
   a  $\in$  ( $\mu$  i. HH(f,x,i)={x})]  $\implies \exists z \in x$ . HH(f,x,a) = {z}"
apply (drule least_subset_x)
apply (frule HH_subset_imp_eq)
apply (drule apply_type)
apply (rule Diff_subset [THEN PowI, THEN DiffI])
apply (fast dest!: HH_subset_x_imp_subset_Diff_UN [THEN not_emptyI2],
force)
done

lemma f_sing_lam_bij:
  "[x - ( $\bigcup j \in$  ( $\mu$  i. HH(f,x,i)={x}). HH(f,x,j)) = 0;
   f  $\in$  (Pow(x)-{0})  $\rightarrow$  {z}. z  $\in$  x]]
 $\implies$  ( $\lambda a \in$  ( $\mu$  i. HH(f,x,i)={x}). HH(f,x,a))
 $\in$  bij( $\mu$  i. HH(f,x,i)={x}, {y}. y  $\in$  x)"
  unfolding bij_def
apply (fast intro!: lam_Least_HH_inj lam_surj_sing f_sing_imp_HH_sing)
done

lemma lam_singI:
  "f  $\in$  ( $\prod X \in$  Pow(x)-{0}. F(X))
 $\implies$  ( $\lambda X \in$  Pow(x)-{0}. {f'X})  $\in$  ( $\prod X \in$  Pow(x)-{0}. {z}. z  $\in$  F(X))"
by (fast del: DiffI DiffE
intro!: lam_type singleton_eq_iff [THEN iffD2] dest: apply_type)

lemmas bij_Least_HH_x =
  comp_bij [OF f_sing_lam_bij [OF lam_singI]
lam_sing_bij [THEN bij_converse_bij]]



### 0.3 The proof of AC1 $\implies$ W02



lemma bijection:
  "f  $\in$  ( $\prod X \in$  Pow(x) - {0}. X)
 $\implies \exists g$ . g  $\in$  bij(x,  $\mu$  i. HH( $\lambda X \in$  Pow(x)-{0}. {f'X}, x, i) = {x})"
apply (rule exI)
apply (rule bij_Least_HH_x [THEN bij_converse_bij])
apply (rule f_subsets_imp_UN_HH_eq_x)
apply (intro ballI apply_type)
apply (fast intro: lam_type apply_type del: DiffE, assumption)
apply (fast intro: Pi_weaken_type)

```

done

```
lemma AC1_W02: "AC1  $\implies$  W02"
  unfolding AC1_def W02_def eqpoll_def
  apply (intro allI)
  apply (drule_tac x = "Pow(A) - {0}" in spec)
  apply (blast dest: bijection)
  done
```

end

```
theory AC15_W06
imports HH Cardinal_aux
begin
```

```
lemma lepoll_Sigma: "A  $\neq 0 \implies B \lesssim A*B"$ 
  unfolding lepoll_def
  apply (erule not_emptyE)
  apply (rule_tac x = " $\lambda z \in B. \langle x, z \rangle$ " in exI)
  apply (fast intro!: snd_conv lam_injective)
  done
```

```
lemma cons_times_nat_not_Finite:
  " $0 \notin A \implies \forall B \in \{\text{cons}(0, x*\text{nat}). x \in A\}. \neg \text{Finite}(B)$ "
  apply clarify
  apply (rule nat_not_Finite [THEN notE] )
  apply (subgoal_tac " $x \neq 0$ ")
  apply (blast intro: lepoll_Sigma [THEN lepoll_Finite])
  done
```

```
lemma lemma1: " $\bigcup (C)=A; a \in A \implies \exists B \in C. a \in B \wedge B \subseteq A$ "
  by fast
```

```
lemma lemma2:
  " $\llbracket \text{pairwise\_disjoint}(A); B \in A; C \in A; a \in B; a \in C \rrbracket \implies B=C$ "
  by (unfold pairwise_disjoint_def, blast)
```

```
lemma lemma3:
  " $\forall B \in \{\text{cons}(0, x*\text{nat}). x \in A\}. \text{pairwise\_disjoint}(f'B) \wedge$   

 $\text{sets\_of\_size\_between}(f'B, 2, n) \wedge \bigcup (f'B)=B$ "
```

```

     $\implies \forall B \in A. \exists ! u. u \in f'cons(0, B*nat) \wedge u \subseteq cons(0, B*nat) \wedge$ 
     $0 \in u \wedge 2 \lesssim u \wedge u \lesssim n$ 
    unfolding sets_of_size_between_def
  apply (rule ballI)
  apply (erule_tac x="cons(0, B*nat)" in ballE)
  apply (blast dest: lemma1 intro!: lemma2, blast)
done

lemma lemma4: " $\llbracket A \lesssim i; Ord(i) \rrbracket \implies \{P(a). a \in A\} \lesssim i$ "
  unfolding lepoll_def
  apply (erule exE)
  apply (rule_tac x = " $\lambda x \in RepFun(A,P). \mu j. \exists a \in A. x=P(a) \wedge f'a=j$ "
    in exI)
  apply (rule_tac d = " $\lambda y. P (converse (f) 'y)$ " in lam_injective)
  apply (erule RepFunE)
  apply (frule inj_is_fun [THEN apply_type], assumption)
  apply (fast intro: LeastI2 elim!: Ord_in_Ord inj_is_fun [THEN apply_type])
  apply (erule RepFunE)
  apply (rule LeastI2)
  apply fast
  apply (fast elim!: Ord_in_Ord inj_is_fun [THEN apply_type])
  apply (fast elim: sym left_inverse [THEN ssubst])
done

lemma lemma5_1:
  " $\llbracket B \in A; 2 \lesssim u(B) \rrbracket \implies (\lambda x \in A. \{fst(x). x \in u(x)-\{0\}\})'B \neq 0$ "
  apply simp
  apply (fast dest: lepoll_Diff_sing
    elim: lepoll_trans [THEN succ_lepoll_natE] ssubst
    intro!: lepoll_refl)
done

lemma lemma5_2:
  " $\llbracket B \in A; u(B) \subseteq cons(0, B*nat) \rrbracket$ 
 $\implies (\lambda x \in A. \{fst(x). x \in u(x)-\{0\}\})'B \subseteq B$ "
  apply auto
done

lemma lemma5_3:
  " $\llbracket n \in nat; B \in A; 0 \in u(B); u(B) \lesssim succ(n) \rrbracket$ 
 $\implies (\lambda x \in A. \{fst(x). x \in u(x)-\{0\}\})'B \lesssim n$ "
  apply simp
  apply (fast elim!: Diff_lepoll [THEN lemma4 [OF _ nat_into_Ord]])
done

lemma ex_fun_AC13_AC15:
  " $\llbracket \forall B \in \{cons(0, x*nat). x \in A\}. pairwise\_disjoint(f'B) \wedge$ "

```

```

      sets_of_size_between(f'B, 2, succ(n))  $\wedge$   $\bigcup$  (f'B)=B;
    n  $\in$  nat]]
 $\implies \exists f. \forall B \in A. f'B \neq 0 \wedge f'B \subseteq B \wedge f'B \lesssim n$ 
by (fast del: subsetI notI
    dest!: lemma3 theI intro!: lemma5_1 lemma5_2 lemma5_3)

```

```

theorem AC10_AC11: "[n  $\in$  nat; 1 $\leq$ n; AC10(n)]  $\implies$  AC11"
by (unfold AC10_def AC11_def, blast)

```

```

theorem AC11_AC12: "AC11  $\implies$  AC12"
by (unfold AC10_def AC11_def AC11_def AC12_def, blast)

```

```

theorem AC12_AC15: "AC12  $\implies$  AC15"
  unfolding AC12_def AC15_def
  apply (blast del: ballI
    intro!: cons_times_nat_not_Finite ex_fun_AC13_AC15)
done

```

```

lemma OUN_eq_UN: "Ord(x)  $\implies$  ( $\bigcup$  a<x. F(a)) = ( $\bigcup$  a  $\in$  x. F(a))"
by (fast intro!: ltI dest!: ltD)

```

```

lemma AC15_W06_aux1:
  " $\forall x \in \text{Pow}(A) - \{0\}. f'x \neq 0 \wedge f'x \subseteq x \wedge f'x \lesssim m$ 
 $\implies$  ( $\bigcup_{i < \mu} x. \text{HH}(f, A, x) = \{A\}. \text{HH}(f, A, i)$ ) = A"
  apply (simp add: Ord_Least [THEN OUN_eq_UN])
  apply (rule equalityI)
  apply (fast dest!: less_Least_subset_x)
  apply (blast del: subsetI)

```

```

      intro!: f_subsets_imp_UN_HH_eq_x [THEN Diff_eq_0_iff [THEN
iffD1]]))
done

lemma AC15_W06_aux2:
  "∀ x ∈ Pow(A) - {0}. f'x ≠ 0 ∧ f'x ⊆ x ∧ f'x ≲ m
  ⇒ ∀ x < (μ x. HH(f,A,x) = {A}). HH(f,A,x) ≲ m"
apply (rule oallI)
apply (drule ltD [THEN less_Least_subset_x])
apply (frule HH_subset_imp_eq)
apply (erule ssubst)
apply (blast dest!: HH_subset_x_imp_subset_Diff_UN [THEN not_emptyI2])

done

theorem AC15_W06: "AC15 ⇒ W06"
  unfolding AC15_def W06_def
apply (rule allI)
apply (erule_tac x = "Pow (A) - {0}" in allE)
apply (erule impE, fast)
apply (elim bexE conjE exE)
apply (rule bexI)
  apply (rule conjI, assumption)
  apply (rule_tac x = "μ i. HH (f,A,i) = {A}" in exI)
  apply (rule_tac x = "λ j ∈ (μ i. HH (f,A,i) = {A}) . HH (f,A,j) " in exI)
  apply (simp_all add: ltD)
apply (fast intro!: Ord_Least lam_type [THEN domain_of_fun]
  elim!: less_Least_subset_x AC15_W06_aux1 AC15_W06_aux2)
done

```

```

theorem AC10_AC13: "[n ∈ nat; 1 ≤ n; AC10(n)] ⇒ AC13(n)"
apply (unfold AC10_def AC13_def, safe)
apply (erule allE)
apply (erule impE [OF _ cons_times_nat_not_Finite], assumption)
apply (fast elim!: impE [OF _ cons_times_nat_not_Finite]
  dest!: ex_fun_AC13_AC15)

```

done

```
lemma AC1_AC13: "AC1  $\implies$  AC13(1)"
  unfolding AC1_def AC13_def
  apply (rule allI)
  apply (erule allE)
  apply (rule impI)
  apply (drule mp, assumption)
  apply (elim exE)
  apply (rule_tac x = " $\lambda x \in A. \{f'x\}$ " in exI)
  apply (simp add: singleton_eqpoll_1 [THEN eqpoll_imp_lepoll])
done
```

```
lemma AC13_mono: " $\llbracket m \leq n; AC13(m) \rrbracket \implies AC13(n)$ "
  unfolding AC13_def
  apply (drule le_imp_lepoll)
  apply (fast elim!: lepoll_trans)
done
```

```
theorem AC13_AC14: " $\llbracket n \in \text{nat}; 1 \leq n; AC13(n) \rrbracket \implies AC14$ "
by (unfold AC13_def AC14_def, auto)
```

```
theorem AC14_AC15: "AC14  $\implies$  AC15"
by (unfold AC13_def AC14_def AC15_def, fast)
```

```

lemma lemma_aux: " $\llbracket A \neq 0; A \lesssim 1 \rrbracket \implies \exists a. A = \{a\}$ "
by (fast elim!: not_emptyE lepoll_1_is_sing)

lemma AC13_AC1_lemma:
  " $\forall B \in A. f(B) \neq 0 \wedge f(B) \leq B \wedge f(B) \lesssim 1$ 
 $\implies (\lambda x \in A. \text{THE } y. f(x) = \{y\}) \in (\prod X \in A. X)$ "
apply (rule lam_type)
apply (drule bspec, assumption)
apply (elim conjE)
apply (erule lemma_aux [THEN exE], assumption)
apply (simp add: the_equality)
done

theorem AC13_AC1: "AC13(1)  $\implies$  AC1"
  unfolding AC13_def AC1_def
apply (fast elim!: AC13_AC1_lemma)
done

theorem AC11_AC14: "AC11  $\implies$  AC14"
  unfolding AC11_def AC14_def
apply (fast intro!: AC10_AC13)
done

end

theory AC16_lemmas
imports AC_Equiv Hartog Cardinal_aux
begin

lemma cons_Diff_eq: " $a \notin A \implies \text{cons}(a, A) - \{a\} = A$ "
by fast

lemma nat_1_lepoll_iff: " $1 \lesssim X \iff (\exists x. x \in X)$ "
  unfolding lepoll_def
apply (rule iffI)
apply (fast intro: inj_is_fun [THEN apply_type])

```



```

apply (erule exE)
apply (rule_tac x = "λa ∈ 1. x" in exI)
apply (fast intro!: lam_injective)
done

lemma eqpoll_1_iff_singleton: " $X \approx 1 \iff (\exists x. X = \{x\})$ "
apply (rule iffI)
apply (erule eqpollE)
apply (drule nat_1_lepoll_iff [THEN iffD1])
apply (fast intro!: lepoll_1_is_sing)
apply (fast intro!: singleton_eqpoll_1)
done

lemma cons_eqpoll_succ: " $\llbracket x \approx n; y \notin x \rrbracket \implies \text{cons}(y, x) \approx \text{succ}(n)$ "
  unfolding succ_def
apply (fast elim!: cons_eqpoll_cong mem_irrefl)
done

lemma subsets_eqpoll_1_eq: " $\{Y \in \text{Pow}(X). Y \approx 1\} = \{\{x\}. x \in X\}$ "
apply (rule equalityI)
apply (rule subsetI)
apply (erule CollectE)
apply (drule eqpoll_1_iff_singleton [THEN iffD1])
apply (fast intro!: RepFunI)
apply (rule subsetI)
apply (erule RepFunE)
apply (rule CollectI, fast)
apply (fast intro!: singleton_eqpoll_1)
done

lemma eqpoll_RepFun_sing: " $X \approx \{\{x\}. x \in X\}$ "
  unfolding eqpoll_def bij_def
apply (rule_tac x = "λx ∈ X. {x}" in exI)
apply (rule IntI)
apply (unfold inj_def surj_def, simp)
apply (fast intro!: lam_type RepFunI intro: singleton_eq_iff [THEN iffD1],
  simp)
apply (fast intro!: lam_type)
done

lemma subsets_eqpoll_1_eqpoll: " $\{Y \in \text{Pow}(X). Y \approx 1\} \approx X$ "
apply (rule subsets_eqpoll_1_eq [THEN ssubst])
apply (rule eqpoll_RepFun_sing [THEN eqpoll_sym])
done

lemma InfCard_Least_in:
  " $\llbracket \text{InfCard}(x); y \subseteq x; y \approx \text{succ}(z) \rrbracket \implies (\mu i. i \in y) \in y$ "
apply (erule eqpoll_sym [THEN eqpoll_imp_lepoll,
  THEN succ_lepoll_imp_not_empty, THEN not_emptyE])

```

```

apply (fast intro: LeastI
      dest!: InfCard_is_Card [THEN Card_is_Ord]
      elim: Ord_in_Ord)
done

lemma subsets_lepoll_lemma1:
  "[[InfCard(x); n ∈ nat]]
  ⇒ {y ∈ Pow(x). y≈succ(succ(n))} ≲ x*{y ∈ Pow(x). y≈succ(n)}"
  unfolding lepoll_def
apply (rule_tac x = "λy ∈ {y ∈ Pow(x) . y≈succ (succ (n))}.
  <μ i. i ∈ y, y-{μ i. i ∈ y}>" in exI)
apply (rule_tac d = "λz. cons (fst(z), snd(z))" in lam_injective)
  apply (blast intro!: Diff_sing_eqpoll intro: InfCard_Least_in)
  apply (simp, blast intro: InfCard_Least_in)
done

lemma set_of_Ord_succ_Union: "(∀y ∈ z. Ord(y)) ⇒ z ⊆ succ(⋃ (z))"
apply (rule subsetI)
apply (case_tac "∀y ∈ z. y ⊆ x", blast )
apply (simp, erule bexE)
apply (rule_tac i=y and j=x in Ord_linear_le)
apply (blast dest: le_imp_subset elim: leE ltE)+
done

lemma subset_not_mem: "j ⊆ i ⇒ i ∉ j"
by (fast elim!: mem_irrefl)

lemma succ_Union_not_mem:
  "(⋀y. y ∈ z ⇒ Ord(y)) ⇒ succ(⋃ (z)) ∉ z"
apply (rule set_of_Ord_succ_Union [THEN subset_not_mem], blast)
done

lemma Union_cons_eq_succ_Union:
  "⋃ (cons(succ(⋃ (z)),z)) = succ(⋃ (z))"
by fast

lemma Un_Ord_disj: "[[Ord(i); Ord(j)] ⇒ i ∪ j = i / i ∪ j = j"
by (fast dest!: le_imp_subset elim: Ord_linear_le)

lemma Union_eq_Un: "x ∈ X ⇒ ⋃ (X) = x ∪ ⋃ (X-{x})"
by fast

lemma Union_in_lemma [rule_format]:
  "n ∈ nat ⇒ ∀z. (∀y ∈ z. Ord(y)) ∧ z≈n ∧ z≠0 → ⋃ (z) ∈ z"
apply (induct_tac "n")
apply (fast dest!: eqpoll_imp_lepoll [THEN lepoll_0_is_0])
apply (intro allI impI)
apply (erule natE)
apply (fast dest!: eqpoll_1_iff_singleton [THEN iffD1])

```

```

      intro!: Union_singleton, clarify)
apply (elim not_emptyE)
apply (erule_tac x = "z-{xb}" in allE)
apply (erule impE)
apply (fast elim!: Diff_sing_eqpoll
      Diff_sing_eqpoll [THEN eqpoll_succ_imp_not_empty])
apply (subgoal_tac "xb  $\cup$   $\bigcup$  (z - {xb})  $\in$  z")
apply (simp add: Union_eq_Un [symmetric])
apply (frule bspec, assumption)
apply (drule bspec)
apply (erule Diff_subset [THEN subsetD])
apply (drule Un_Ord_disj, assumption, auto)
done

lemma Union_in: " $\llbracket \forall x \in z. \text{Ord}(x); z \approx n; z \neq 0; n \in \text{nat} \rrbracket \implies \bigcup (z) \in z$ "
by (blast intro: Union_in_lemma)

lemma succ_Union_in_x:
  " $\llbracket \text{InfCard}(x); z \in \text{Pow}(x); z \approx n; n \in \text{nat} \rrbracket \implies \text{succ}(\bigcup (z)) \in x$ "
apply (rule Limit_has_succ [THEN ltE])
prefer 3 apply assumption
apply (erule InfCard_is_Limit)
apply (case_tac "z=0")
apply (simp, fast intro!: InfCard_is_Limit [THEN Limit_has_0])
apply (rule ltI [OF PowD [THEN subsetD] InfCard_is_Card [THEN Card_is_Ord]],
  assumption)
apply (blast intro: Union_in
      InfCard_is_Card [THEN Card_is_Ord, THEN Ord_in_Ord])
done

lemma succ_lepoll_succ_succ:
  " $\llbracket \text{InfCard}(x); n \in \text{nat} \rrbracket$ 
 $\implies \{y \in \text{Pow}(x). y \approx \text{succ}(n)\} \lesssim \{y \in \text{Pow}(x). y \approx \text{succ}(\text{succ}(n))\}$ "
  unfolding lepoll_def
apply (rule_tac x = " $\lambda z \in \{y \in \text{Pow}(x). y \approx \text{succ}(n)\}. \text{cons}(\text{succ}(\bigcup (z)), z)$ "
  in exI)
apply (rule_tac d = " $\lambda z. z - \{\bigcup (z)\}$ " in lam_injective)
apply (blast intro!: succ_Union_in_x succ_Union_not_mem
  intro: cons_eqpoll_succ Ord_in_Ord
  dest!: InfCard_is_Card [THEN Card_is_Ord])
apply (simp only: Union_cons_eq_succ_Union)
apply (rule cons_Diff_eq)
apply (fast dest!: InfCard_is_Card [THEN Card_is_Ord]
  elim: Ord_in_Ord
  intro!: succ_Union_not_mem)
done

lemma subsets_eqpoll_X:

```

```

    "[InfCard(X); n ∈ nat] ⇒ {Y ∈ Pow(X). Y≈succ(n)} ≈ X"
  apply (induct_tac "n")
  apply (rule subsets_eqpoll_1_eqpoll)
  apply (rule eqpollI)
  apply (rule subsets_lepoll_lemma1 [THEN lepoll_trans], assumption+)
  apply (rule eqpoll_trans [THEN eqpoll_imp_lepoll])
    apply (erule eqpoll_refl [THEN prod_eqpoll_cong])
  apply (erule InfCard_square_eqpoll)
  apply (fast elim: eqpoll_sym [THEN eqpoll_imp_lepoll, THEN lepoll_trans]

        intro!: succ_lepoll_succ_succ)
done

lemma image_vimage_eq:
  "[f ∈ surj(A,B); y ⊆ B] ⇒ f“(converse(f)“y) = y"
  unfolding surj_def
  apply (fast dest: apply_equality2 elim: apply_iff [THEN iffD2])
done

lemma vimage_image_eq: "[f ∈ inj(A,B); y ⊆ A] ⇒ converse(f)“(f“y)
= y"
by (fast elim!: inj_is_fun [THEN apply_Pair] dest: inj_equality)

lemma subsets_eqpoll:
  "A≈B ⇒ {Y ∈ Pow(A). Y≈n}≈{Y ∈ Pow(B). Y≈n}"
  unfolding eqpoll_def
  apply (erule exE)
  apply (rule_tac x = "λX ∈ {Y ∈ Pow (A) . ∃f. f ∈ bij (Y, n) }. f“X"
in exI)
  apply (rule_tac d = "λZ. converse (f) “Z" in lam_bijective)
  apply (fast intro!: bij_is_inj [THEN restrict_bij, THEN bij_converse_bij,

        THEN comp_bij]
        elim!: bij_is_fun [THEN fun_is_rel, THEN image_subset])
  apply (blast intro!: bij_is_inj [THEN restrict_bij]
        comp_bij bij_converse_bij
        bij_is_fun [THEN fun_is_rel, THEN image_subset])
  apply (fast elim!: bij_is_inj [THEN vimage_image_eq])
  apply (fast elim!: bij_is_surj [THEN image_vimage_eq])
done

lemma W02_imp_ex_Card: "W02 ⇒ ∃a. Card(a) ∧ X≈a"
  unfolding W02_def
  apply (drule spec [of _ X])
  apply (blast intro: Card_cardinal eqpoll_trans
        well_ord_Memrel [THEN well_ord_cardinal_eqpoll, THEN eqpoll_sym])
done

lemma lepoll_infinite: "[X⋖Y; ¬Finite(X)] ⇒ ¬Finite(Y)"

```

```

by (blast intro: lepoll_Finite)

lemma infinite_Card_is_InfCard: "[¬Finite(X); Card(X)] ⇒ InfCard(X)"
  unfolding InfCard_def
  apply (fast elim!: Card_is_Ord [THEN nat_le_infinite_Ord])
  done

lemma W02_infinite_subsets_eqpoll_X: "[W02; n ∈ nat; ¬Finite(X)]
  ⇒ {Y ∈ Pow(X). Y ≈ succ(n)} ≈ X"
  apply (drule W02_imp_ex_Card)
  apply (elim allE exE conjE)
  apply (frule eqpoll_imp_lepoll [THEN lepoll_infinite], assumption)
  apply (drule infinite_Card_is_InfCard, assumption)
  apply (blast intro: subsets_eqpoll subsets_eqpoll_X eqpoll_sym eqpoll_trans)

done

lemma well_ord_imp_ex_Card: "well_ord(X,R) ⇒ ∃ a. Card(a) ∧ X ≈ a"
  by (fast elim!: well_ord_cardinal_eqpoll [THEN eqpoll_sym]
      intro!: Card_cardinal)

lemma well_ord_infinite_subsets_eqpoll_X:
  "[well_ord(X,R); n ∈ nat; ¬Finite(X)] ⇒ {Y ∈ Pow(X). Y ≈ succ(n)} ≈ X"
  apply (drule well_ord_imp_ex_Card)
  apply (elim allE exE conjE)
  apply (frule eqpoll_imp_lepoll [THEN lepoll_infinite], assumption)
  apply (drule infinite_Card_is_InfCard, assumption)
  apply (blast intro: subsets_eqpoll subsets_eqpoll_X eqpoll_sym eqpoll_trans)

done

end

theory W02_AC16 imports AC_Equiv AC16_lemmas Cardinal_aux begin

definition
  recfunAC16 :: "[i,i,i,i] ⇒ i" where
    "recfunAC16(f,h,i,a) ≡
      transrec2(i, 0,
        λg r. if (∃ y ∈ r. h'g ⊆ y) then r
          else r ∪ {f'(μ i. h'g ⊆ f'i ∧
            (∀ b < a. (h'b ⊆ f'i → (∀ t ∈ r. ¬ h'b ⊆ t))))})"

```

```

lemma recfunAC16_0: "recfunAC16(f,h,0,a) = 0"
by (simp add: recfunAC16_def)

lemma recfunAC16_succ:
  "recfunAC16(f,h,succ(i),a) =
    (if ( $\exists y \in \text{recfunAC16}(f,h,i,a). h \text{ ' } i \subseteq y$ ) then  $\text{recfunAC16}(f,h,i,a)$ 

      else  $\text{recfunAC16}(f,h,i,a) \cup$ 
        { $f \text{ ' } (\mu j. h \text{ ' } i \subseteq f \text{ ' } j \wedge$ 
          ( $\forall b < a. (h \text{ ' } b \subseteq f \text{ ' } j$ 
             $\longrightarrow (\forall t \in \text{recfunAC16}(f,h,i,a). \neg h \text{ ' } b \subseteq t))$ ))})}"
apply (simp add: recfunAC16_def)
done

lemma recfunAC16_Limit: "Limit(i)
 $\implies \text{recfunAC16}(f,h,i,a) = (\bigcup_{j < i} \text{recfunAC16}(f,h,j,a))"$ 
by (simp add: recfunAC16_def transrec2_Limit)

lemma transrec2_mono_lemma [rule_format]:
  " $\llbracket \bigwedge g r. r \subseteq B(g,r); \text{Ord}(i) \rrbracket$ 
 $\implies j < i \longrightarrow \text{transrec2}(j, 0, B) \subseteq \text{transrec2}(i, 0, B)"$ 
apply (erule trans_induct)
apply (rule Ord_cases, assumption+, fast)
apply (simp (no_asm_simp))
apply (blast elim!: leE)
apply (simp add: transrec2_Limit)
apply (blast intro: OUN_I ltI Ord_in_Ord [THEN le_refl]
  elim!: Limit_has_succ [THEN ltE])
done

lemma transrec2_mono:
  " $\llbracket \bigwedge g r. r \subseteq B(g,r); j \leq i \rrbracket$ 
 $\implies \text{transrec2}(j, 0, B) \subseteq \text{transrec2}(i, 0, B)"$ 
apply (erule leE)
apply (rule transrec2_mono_lemma)
apply (auto intro: lt_Ord2 )
done

lemma recfunAC16_mono:
  " $i \leq j \implies \text{recfunAC16}(f, g, i, a) \subseteq \text{recfunAC16}(f, g, j, a)"$ 

```

```

    unfolding recfunAC16_def
  apply (rule transrec2_mono, auto)
done

```

```

lemma lemma3_1:
  "[ $\forall y < x. \forall z < a. z < y \mid (\exists Y \in F(y). f(z) \leq Y) \longrightarrow (\exists ! Y. Y \in F(y) \wedge f(z) \leq Y)$ ];
    [ $\forall i j. i \leq j \longrightarrow F(i) \subseteq F(j); j \leq i; i < x; z < a;$ 
     $V \in F(i); f(z) \leq V; W \in F(j); f(z) \leq W$ ]]
   $\implies V = W$ "
  apply (erule asm_rl allE impE)+
  apply (drule subsetD, assumption, blast)
done

```

```

lemma lemma3:
  "[ $\forall y < x. \forall z < a. z < y \mid (\exists Y \in F(y). f(z) \leq Y) \longrightarrow (\exists ! Y. Y \in F(y) \wedge f(z) \leq Y)$ ];
    [ $\forall i j. i \leq j \longrightarrow F(i) \subseteq F(j); i < x; j < x; z < a;$ 
     $V \in F(i); f(z) \leq V; W \in F(j); f(z) \leq W$ ]]
   $\implies V = W$ "
  apply (rule_tac j=j in Ord_linear_le [OF lt_Ord lt_Ord], assumption+)
  apply (erule lemma3_1 [symmetric], assumption+)
  apply (erule lemma3_1, assumption+)
done

```

```

lemma lemma4:
  "[ $\forall y < x. F(y) \subseteq X \wedge$ 
    ( $\forall x < a. x < y \mid (\exists Y \in F(y). h(x) \subseteq Y) \longrightarrow$ 
    ( $\exists ! Y. Y \in F(y) \wedge h(x) \subseteq Y$ ));
     $x < a$ ]]
   $\implies \forall y < x. \forall z < a. z < y \mid (\exists Y \in F(y). h(z) \subseteq Y) \longrightarrow$ 
    ( $\exists ! Y. Y \in F(y) \wedge h(z) \subseteq Y$ )"
  apply (intro oallI impI)
  apply (drule ospec, assumption, clarify)
  apply (blast elim!: oallE )
done

```

```

lemma lemma5:
  "[ $\forall y < x. F(y) \subseteq X \wedge$ 
    ( $\forall x < a. x < y \mid (\exists Y \in F(y). h(x) \subseteq Y) \longrightarrow$ 
    ( $\exists ! Y. Y \in F(y) \wedge h(x) \subseteq Y$ ));
     $x < a; \text{Limit}(x); \forall i j. i \leq j \longrightarrow F(i) \subseteq F(j)$ ]]
   $\implies (\bigcup_{x < x. F(x)} \subseteq X \wedge$ 

```

```

      
$$(\forall xa < a. xa < x \mid (\exists x \in \bigcup_{x < x}. F(x). h(xa) \subseteq x) \\ \longrightarrow (\exists! Y. Y \in (\bigcup_{x < x}. F(x)) \wedge h(xa) \subseteq Y))"$$

    apply (rule conjI)
    apply (rule subsetI)
    apply (erule OUN_E)
    apply (drule ospec, assumption, fast)
    apply (drule lemma4, assumption)
    apply (rule oallI)
    apply (rule impI)
    apply (erule disjE)
    apply (frule ospec, erule Limit_has_succ, assumption)
    apply (drule_tac A = a and x = xa in ospec, assumption)
    apply (erule impE, rule le_refl [THEN disjI1], erule lt_Ord)
    apply (blast intro: lemma3 Limit_has_succ)
    apply (blast intro: lemma3)
  done

```

```

lemma dbl_Diff_eqpoll_Card:
  "[A ≈ a; Card(a); ¬Finite(a); B < a; C < a] ⇒ A - B - C ≈ a"
by (blast intro: Diff_lesspoll_eqpoll_Card)

```

```

lemma Finite_lesspoll_infinite_Ord:
  "[Finite(X); ¬Finite(a); Ord(a)] ⇒ X < a"
  unfolding lesspoll_def
  apply (rule conjI)
  apply (drule nat_le_infinite_Ord [THEN le_imp_lepoll], assumption)
  unfolding Finite_def
  apply (blast intro: leI [THEN le_imp_subset, THEN subset_imp_lepoll]
    ltI eqpoll_imp_lepoll lepoll_trans)
  apply (blast intro: eqpoll_sym [THEN eqpoll_trans])
  done

```

```

lemma Union_lesspoll:

```



```

"[[ $\forall x \in X. x \lesssim n \wedge x \subseteq T; \text{well\_ord}(T, R); X \lesssim b;$ 
 $b < a; \neg \text{Finite}(a); \text{Card}(a); n \in \text{nat}]$ 
 $\implies \bigcup (X) < a$ "
apply (case_tac "Finite (X)")
apply (blast intro: Card_is_Ord Finite_lesspoll_infinite_Ord
               lepoll_nat_imp_Finite Finite_Union)
apply (drule lepoll_imp_ex_le_eqpoll)
apply (erule lt_Ord)
apply (elim exE conjE)
apply (frule eqpoll_imp_lepoll [THEN lepoll_infinite], assumption)
apply (erule eqpoll_sym [THEN eqpoll_def [THEN def_imp_iff, THEN iffD1],
               THEN exE])
apply (frule bij_is_surj [THEN surj_image_eq])
apply (drule image_fun [OF bij_is_fun subset_refl])
apply (drule sym [THEN trans], assumption)
apply (blast intro: lt_Ord UN_lepoll lt_Card_imp_lesspoll
               lt_trans1 lespoll_trans1)
done

```

```

lemma Un_sing_eq_cons: "A  $\cup$  {a} = cons(a, A)"
by fast

```

```

lemma Un_lepoll_succ: "A  $\lesssim$  B  $\implies$  A  $\cup$  {a}  $\lesssim$  succ(B)"
apply (simp add: Un_sing_eq_cons succ_def)
apply (blast elim!: mem_irrefl intro: cons_lepoll_cong)
done

```

```

lemma Diff_UN_succ_empty: "Ord(a)  $\implies$  F(a) - ( $\bigcup b < \text{succ}(a). F(b)$ ) = 0"
by (fast intro!: le_refl)

```

```

lemma Diff_UN_succ_subset: "Ord(a)  $\implies$  F(a)  $\cup$  X - ( $\bigcup b < \text{succ}(a). F(b)$ )
 $\subseteq$  X"
by blast

```

```

lemma recfunAC16_Diff_lepoll_1:
  "Ord(x)
 $\implies \text{recfunAC16}(f, g, x, a) - (\bigcup i < x. \text{recfunAC16}(f, g, i, a)) \lesssim 1$ "
apply (erule Ord_cases)
  apply (simp add: recfunAC16_0 empty_subsetI [THEN subset_imp_lepoll])

prefer 2 apply (simp add: recfunAC16_Limit Diff_cancel
                        empty_subsetI [THEN subset_imp_lepoll])

apply (simp add: recfunAC16_succ
               Diff_UN_succ_empty [of _ " $\lambda j. \text{recfunAC16}(f, g, j, a)$ "])

```

```

      empty_subsetI [THEN subset_imp_lepoll])
apply (best intro: Diff_UN_succ_subset [THEN subset_imp_lepoll]
      singleton_eqpoll_1 [THEN eqpoll_imp_lepoll] lepoll_trans)
done

```

```

lemma in_Least_Diff:
  "[z ∈ F(x); Ord(x)]
  ⇒ z ∈ F(μ i. z ∈ F(i)) - (⋃ j<(μ i. z ∈ F(i)). F(j))"
by (fast elim: less_LeastE elim!: LeastI)

```

```

lemma Least_eq_imp_ex:
  "[(μ i. w ∈ F(i)) = (μ i. z ∈ F(i));
   w ∈ (⋃ i<a. F(i)); z ∈ (⋃ i<a. F(i))]"
  ⇒ ∃ b<a. w ∈ (F(b) - (⋃ c<b. F(c))) ∧ z ∈ (F(b) - (⋃ c<b. F(c)))"
apply (elim OUN_E)
apply (drule in_Least_Diff, erule lt_Ord)
apply (frule in_Least_Diff, erule lt_Ord)
apply (rule oexI, force)
apply (blast intro: lt_Ord Least_le [THEN lt_trans1])
done

```

```

lemma two_in_lepoll_1: "[A ≲ 1; a ∈ A; b ∈ A] ⇒ a=b"
by (fast dest!: lepoll_1_is_sing)

```

```

lemma UN_lepoll_index:
  "[∀ i<a. F(i)-(⋃ j<i. F(j)) ≲ 1; Limit(a)]
  ⇒ (⋃ x<a. F(x)) ≲ a"
apply (rule lepoll_def [THEN def_imp_iff [THEN iffD2]])
apply (rule_tac x = "λz ∈ (⋃ x<a. F(x)). μ i. z ∈ F(i)" in exI)
  unfolding inj_def
apply (rule CollectI)
apply (rule lam_type)
apply (erule OUN_E)
apply (erule Least_in_Ord)
apply (erule ltD)
apply (erule lt_Ord2)
apply (intro ballI)
apply (simp (no_asm_simp))
apply (rule impI)
apply (drule Least_eq_imp_ex, assumption+)
apply (fast elim!: two_in_lepoll_1)
done

```

```

lemma recfunAC16_lepoll_index: "Ord(y) ⇒ recfunAC16(f, h, y, a) ≲
y"
apply (erule trans_induct3)

```

```

apply (simp (no_asm_simp) add: recfunAC16_0 lepoll_refl)

apply (simp (no_asm_simp) add: recfunAC16_succ)
apply (blast dest!: succI1 [THEN rev_bspec]
        intro: subset_succI [THEN subset_imp_lepoll] Un_lepoll_succ

        lepoll_trans)
apply (simp (no_asm_simp) add: recfunAC16_Limit)
apply (blast intro: lt_Ord [THEN recfunAC16_Diff_lepoll_1] UN_lepoll_index)
done

```

```

lemma Union_recfunAC16_lesspoll:
  "[recfunAC16(f,g,y,a)  $\subseteq$  {X  $\in$  Pow(A). X $\approx$ n};
   A $\approx$ a; y<a;  $\neg$ Finite(a); Card(a); n  $\in$  nat]
 $\implies \bigcup$  (recfunAC16(f,g,y,a))<a"
apply (erule eqpoll_def [THEN def_imp_iff, THEN iffD1, THEN exE])
apply (rule_tac T=A in Union_lesspoll, simp_all)
apply (blast intro!: eqpoll_imp_lepoll)
apply (blast intro: bij_is_inj Card_is_Ord [THEN well_ord_Memrel]
        well_ord_rvimage)
apply (erule lt_Ord [THEN recfunAC16_lepoll_index])
done

```

```

lemma dbl_Diff_eqpoll:
  "[recfunAC16(f, h, y, a)  $\subseteq$  {X  $\in$  Pow(A) . X $\approx$ succ(k #+ m)};
   Card(a);  $\neg$  Finite(a); A $\approx$ a;
   k  $\in$  nat; y<a;
   h  $\in$  bij(a, {Y  $\in$  Pow(A). Y $\approx$ succ(k)})]
 $\implies A - \bigcup$  (recfunAC16(f, h, y, a)) - h'y $\approx$ a"
apply (rule dbl_Diff_eqpoll_Card, simp_all)
apply (simp add: Union_recfunAC16_lesspoll)
apply (rule Finite_lesspoll_infinite_Ord)
apply (rule Finite_def [THEN def_imp_iff, THEN iffD2])
apply (blast dest: ltD bij_is_fun [THEN apply_type], assumption)
apply (blast intro: Card_is_Ord)
done

```

```

lemmas disj_Un_eqpoll_nat_sum =
  eqpoll_trans [THEN eqpoll_trans,
    OF disj_Un_eqpoll_sum sum_eqpoll_cong nat_sum_eqpoll_sum]

```

```

lemma Un_in_Collect: "[x  $\in$  Pow(A - B - h'i); x $\approx$ m;
  h  $\in$  bij(a, {x  $\in$  Pow(A) . x $\approx$ k}); i<a; k  $\in$  nat; m  $\in$  nat]
 $\implies h \text{ ' } i \cup x \in \{x \in \text{Pow}(A) . x \approx k \text{ \#+ } m\}"$ 
```

```

by (blast intro: disj_Un_eqpoll_nat_sum
    dest: ltD bij_is_fun [THEN apply_type])

```

```

lemma lemma6:
  "[[ $\forall y < \text{succ}(j). F(y) \leq X \wedge (\forall x < a. x < y \mid P(x, y) \longrightarrow Q(x, y))$ ;  $\text{succ}(j) < a$ ]]
   $\implies F(j) \leq X \wedge (\forall x < a. x < j \mid P(x, j) \longrightarrow Q(x, j))$ "
by (blast intro!: lt_Ord succI1 [THEN ltI, THEN lt_Ord, THEN le_refl])

```

```

lemma lemma7:
  "[[ $\forall x < a. x < j \mid P(x, j) \longrightarrow Q(x, j)$ ;  $\text{succ}(j) < a$ ]]
   $\implies P(j, j) \longrightarrow (\forall x < a. x \leq j \mid P(x, j) \longrightarrow Q(x, j))$ "
by (fast elim!: leE)

```

```

lemma ex_subset_eqpoll:
  "[[ $A \approx a$ ;  $\neg \text{Finite}(a)$ ;  $\text{Ord}(a)$ ;  $m \in \text{nat}$ ]]  $\implies \exists X \in \text{Pow}(A). X \approx_m$ "
apply (rule lepoll_imp_eqpoll_subset [of m A, THEN exE])
  apply (rule lepoll_trans, rule leI [THEN le_imp_lepoll])
    apply (blast intro: lt_trans2 [OF ltI nat_le_infinite_Ord] Ord_nat)
  apply (erule eqpoll_sym [THEN eqpoll_imp_lepoll])
  apply (fast elim!: eqpoll_sym)
done

```

```

lemma subset_Un_disjoint: "[[ $A \subseteq B \cup C$ ;  $A \cap C = 0$ ]]  $\implies A \subseteq B$ "
by blast

```

```

lemma Int_empty:
  "[[ $X \in \text{Pow}(A - \bigcup (B) - C)$ ;  $T \in B$ ;  $F \subseteq T$ ]]  $\implies F \cap X = 0$ "
by blast

```

```

lemma subset_imp_eq_lemma:

```

```

      "m ∈ nat ⇒ ∀ A B. A ⊆ B ∧ m ≲ A ∧ B ≲ m → A=B"
    apply (induct_tac "m")
    apply (fast dest!: lepoll_0_is_0)
    apply (intro allI impI)
    apply (elim conjE)
    apply (rule succ_lepoll_imp_not_empty [THEN not_emptyE], assumption)
    apply (frule subsetD [THEN Diff_sing_lepoll], assumption+)
    apply (frule lepoll_Diff_sing)
    apply (erule allE impE)+
    apply (rule conjI)
    prefer 2 apply fast
    apply fast
    apply (blast elim: equalityE)
  done

lemma subset_imp_eq: "[A ⊆ B; m ≲ A; B ≲ m; m ∈ nat] ⇒ A=B"
by (blast dest!: subset_imp_eq_lemma)

lemma bij_imp_arg_eq:
  "[f ∈ bij(a, {Y ∈ X. Y ≈ succ(k)}); k ∈ nat; f' b ⊆ f' y; b < a; y < a]
    ⇒ b=y"
  apply (drule subset_imp_eq)
  apply (erule_tac [3] nat_succI)
  unfolding bij_def inj_def
  apply (blast intro: eqpoll_sym eqpoll_imp_lepoll
    dest: ltD apply_type)+
  done

lemma ex_next_set:
  "[recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X ≈ succ(k #+ m)};
    Card(a); ¬ Finite(a); A ≈ a;
    k ∈ nat; m ∈ nat; y < a;
    h ∈ bij(a, {Y ∈ Pow(A) . Y ≈ succ(k)});
    ¬ (∃ Y ∈ recfunAC16(f, h, y, a). h' y ⊆ Y)]
    ⇒ ∃ X ∈ {Y ∈ Pow(A) . Y ≈ succ(k #+ m)}. h' y ⊆ X ∧
      (∀ b < a. h' b ⊆ X →
        (∀ T ∈ recfunAC16(f, h, y, a). ¬ h' b ⊆ T))"
  apply (erule_tac m1=m in dbl_Diff_eqpoll [THEN ex_subset_eqpoll, THEN
    bexE],
    assumption+)
  apply (erule Card_is_Ord, assumption)
  apply (frule Un_in_Collect, (erule asm_rl nat_succI)+)
  apply (erule CollectE)
  apply (rule rev_bexI, simp)
  apply (rule conjI, blast)

```

```

apply (intro ballI impI oallI notI)
apply (drule subset_Un_disjoint, rule Int_empty, assumption+)
apply (blast dest: bij_imp_arg_eq)
done

```

```

lemma ex_next_Ord:
  "[[recfunAC16(f, h, y, a)  $\subseteq$  {X  $\in$  Pow(A) . X $\approx$ succ(k #+ m)};
    Card(a);  $\neg$  Finite(a); A $\approx$ a;
    k  $\in$  nat; m  $\in$  nat; y<a;
    h  $\in$  bij(a, {Y  $\in$  Pow(A). Y $\approx$ succ(k)});
    f  $\in$  bij(a, {Y  $\in$  Pow(A). Y $\approx$ succ(k #+ m)});
     $\neg$  ( $\exists$  Y  $\in$  recfunAC16(f, h, y, a). h'y  $\subseteq$  Y)]]
   $\implies \exists$  c<a. h'y  $\subseteq$  f'c  $\wedge$ 
    ( $\forall$  b<a. h'b  $\subseteq$  f'c  $\longrightarrow$ 
      ( $\forall$  T  $\in$  recfunAC16(f, h, y, a).  $\neg$  h'b  $\subseteq$  T))"
apply (drule ex_next_set, assumption+)
apply (erule bexE)
apply (rule_tac x="converse(f)'X" in oexI)
apply (simp add: right_inverse_bij)
apply (blast intro: bij_converse_bij bij_is_fun [THEN apply_type] ltI
      Card_is_Ord)
done

```

```

lemma lemma8:
  "[[ $\forall$  x<a. x<j / ( $\exists$  xa  $\in$  F(j). P(x, xa))
     $\longrightarrow$  ( $\exists$  ! Y. Y  $\in$  F(j)  $\wedge$  P(x, Y)); F(j)  $\subseteq$  X;
    L  $\in$  X; P(j, L)  $\wedge$  ( $\forall$  x<a. P(x, L)  $\longrightarrow$  ( $\forall$  xa  $\in$  F(j).  $\neg$  P(x, xa)))]
   $\implies$  F(j)  $\cup$  {L}  $\subseteq$  X  $\wedge$ 
    ( $\forall$  x<a. x $\leq$ j / ( $\exists$  xa  $\in$  (F(j)  $\cup$  {L}). P(x, xa))  $\longrightarrow$ 
      ( $\exists$  ! Y. Y  $\in$  (F(j)  $\cup$  {L})  $\wedge$  P(x, Y)))"
```

```

apply (rule conjI)
apply (fast intro!: singleton_subsetI)
apply (rule oallI)
apply (blast elim!: leE oallE)
done

```

```

lemma main_induct:
  "[[b < a; f ∈ bij(a, {Y ∈ Pow(A) . Y ≈ succ(k #+ m)}});
   h ∈ bij(a, {Y ∈ Pow(A) . Y ≈ succ(k)}});
   ¬Finite(a); Card(a); A ≈ a; k ∈ nat; m ∈ nat]]
  ⇒ recfunAC16(f, h, b, a) ⊆ {X ∈ Pow(A) . X ≈ succ(k #+ m)} ∧
    (∀x<a. x < b | (∃Y ∈ recfunAC16(f, h, b, a). h ' x ⊆ Y) →

      (∃! Y. Y ∈ recfunAC16(f, h, b, a) ∧ h ' x ⊆ Y))"
apply (erule lt_induct)
apply (frule lt_Ord)
apply (erule Ord_cases)

apply (simp add: recfunAC16_0)

prefer 2 apply (simp add: recfunAC16_Limit)
          apply (rule lemma5, assumption+)
          apply (blast dest!: recfunAC16_mono)

apply clarify
apply (erule lemma6 [THEN conjE], assumption)
apply (simp (no_asm_simp) split del: split_if add: recfunAC16_succ)
apply (rule conjI [THEN split_if [THEN iffD2]])
  apply (simp, erule lemma7, assumption)
apply (rule impI)
apply (rule ex_next_Ord [THEN oexE],
      assumption+, rule le_refl [THEN lt_trans], assumption+)
apply (erule lemma8, assumption)
  apply (rule bij_is_fun [THEN apply_type], assumption)
  apply (erule Least_le [THEN lt_trans2, THEN ltD])
    apply (erule lt_Ord)
    apply (erule succ_leI)
  apply (erule LeastI)
  apply (erule lt_Ord)
done

```

```

lemma lemma_simp_induct:
  "[[∀b. b<a → F(b) ⊆ S ∧ (∀x<a. (x<b | (∃Y ∈ F(b). f'x ⊆ Y))
    → (∃! Y. Y ∈ F(b) ∧ f'x ⊆ Y));
   f ∈ a->f''(a); Limit(a);
   ∀i j. i ≤ j → F(i) ⊆ F(j)]
  ⇒ (⋃j<a. F(j)) ⊆ S ∧

```

```

      ( $\forall x \in f^{-1}a. \exists! Y. Y \in (\bigcup_{j < a} F(j)) \wedge x \subseteq Y$ )
apply (rule conjI)
apply (rule subsetI)
apply (erule OUN_E, blast)
apply (rule ballI)
apply (erule imageE)
apply (drule ltI, erule Limit_is_Ord)
apply (drule Limit_has_succ, assumption)
apply (frule_tac x1="succ(xa)" in spec [THEN mp], assumption)
apply (erule conjE)
apply (drule ospec)

apply (erule leI [THEN succ_leE])
apply (erule impE)
apply (fast elim!: leI [THEN succ_leE, THEN lt_Ord, THEN le_refl])
apply (drule apply_equality, assumption)
apply (elim conjE ex1E)

apply (rule ex1I, blast)
apply (elim conjE OUN_E)
apply (erule_tac i="succ(xa)" and j=aa
      in Ord_linear_le [OF lt_Ord lt_Ord], assumption)
prefer 2
apply (drule spec [THEN spec, THEN mp, THEN subsetD], assumption+, blast)

apply (drule_tac x1=aa in spec [THEN mp], assumption)
apply (frule succ_leE)
apply (drule spec [THEN spec, THEN mp, THEN subsetD], assumption+, blast)

done

```

```

theorem W02_AC16: "[W02; 0 < m; k ∈ nat; m ∈ nat] ⇒ AC16(k #+ m, k)"
  unfolding AC16_def
  apply (rule allI)
  apply (rule impI)
  apply (frule W02_infinite_subsets_eqpoll_X, assumption+)
  apply (frule_tac n="k #+ m" in W02_infinite_subsets_eqpoll_X, simp, simp)

  apply (frule W02_imp_ex_Card)
  apply (elim exE conjE)
  apply (drule eqpoll_trans [THEN eqpoll_sym,
    THEN eqpoll_def [THEN def_imp_iff, THEN iffD1]],
    assumption)
  apply (drule eqpoll_trans [THEN eqpoll_sym,

```



```

      THEN eqpoll_def [THEN def_imp_iff, THEN iffD1]],

      assumption+)
apply (elim exE)
apply (rename_tac h)
apply (rule_tac x = "⋃ j<a. recfunAC16 (h,f,j,a) " in exI)
apply (rule_tac P="λz. Y ∧ (∀x ∈ z. Z(x))" for Y Z
      in bij_is_surj [THEN surj_image_eq, THEN subst], assumption)
apply (rule lemma_simp_induct)
apply (blast del: conjI notI
      intro!: main_induct eqpoll_imp_lepoll [THEN lepoll_infinite]
    )
apply (blast intro: bij_is_fun [THEN surj_image, THEN surj_is_fun])
apply (erule eqpoll_imp_lepoll [THEN lepoll_infinite,
      THEN infinite_Card_is_InfCard,
      THEN InfCard_is_Limit],
      assumption+)
apply (blast dest!: recfunAC16_mono)
done

end

theory AC16_W04
imports AC16_lemmas
begin

lemma lemma1:
  "[Finite(A); 0<m; m ∈ nat]
  ⇒ ∃ a f. Ord(a) ∧ domain(f) = a ∧
      (⋃ b<a. f' b) = A ∧ (∀ b<a. f' b ≲ m)"
  unfolding Finite_def
apply (erule bexE)
apply (drule eqpoll_sym [THEN eqpoll_def [THEN def_imp_iff, THEN iffD1]])
apply (erule exE)
apply (rule_tac x = n in exI)
apply (rule_tac x = "λi ∈ n. {f' i}" in exI)
apply (simp add: ltD bij_def surj_def)
apply (fast intro!: ltI nat_into_Ord lam_funtype [THEN domain_of_fun]
      singleton_eqpoll_1 [THEN eqpoll_imp_lepoll, THEN lepoll_trans]
      nat_1_lepoll_iff [THEN iffD2]
    elim!: apply_type ltE)
done

```

```
lemmas well_ord_paired = paired_bij [THEN bij_is_inj, THEN well_ord_rvimage]
```

```
lemma lepoll_trans1: "[A  $\lesssim$  B;  $\neg A \lesssim C$ ]  $\implies \neg B \lesssim C$ "
by (blast intro: lepoll_trans)
```

```
lemmas lepoll_paired = paired_eqpoll [THEN eqpoll_sym, THEN eqpoll_imp_lepoll]
```

```
lemma lemma2: " $\exists y R. \text{well\_ord}(y, R) \wedge x \cap y = 0 \wedge \neg y \lesssim z \wedge \neg \text{Finite}(y)$ "
apply (rule_tac x = "{a, x}. a  $\in$  nat  $\cup$  Hartog (z) }" in exI)
apply (rule well_ord_Un [OF Ord_nat [THEN well_ord_Memrel]
      Ord_Hartog [THEN well_ord_Memrel], THEN exE])
apply (blast intro!: Ord_Hartog well_ord_Memrel well_ord_paired
      lepoll_trans1 [OF _ not_Hartog_lepoll_self]
      lepoll_trans [OF subset_imp_lepoll lepoll_paired]
      elim!: nat_not_Finite [THEN notE]
      elim: mem_asym
      dest!: Un_upper1 [THEN subset_imp_lepoll, THEN lepoll_Finite]
      lepoll_paired [THEN lepoll_Finite])
done
```

```
lemma infinite_Un: " $\neg \text{Finite}(B) \implies \neg \text{Finite}(A \cup B)$ "
by (blast intro: subset_Finite)
```

```
lemma succ_not_lepoll_lemma:
  "[ $\neg (\exists x \in A. f'x=y); f \in \text{inj}(A, B); y \in B$ ]
 $\implies (\lambda a \in \text{succ}(A). \text{if}(a=A, y, f'a)) \in \text{inj}(\text{succ}(A), B)$ "
```

```

apply (rule_tac d = "λz. if (z=y, A, converse (f) 'z) " in lam_injective)
apply (force simp add: inj_is_fun [THEN apply_type])

apply (simp (no_asm_simp))
apply force
done

lemma succ_not_lepoll_imp_eqpoll: "[¬A ≈ B; A ≲ B] ⇒ succ(A) ≲ B"
  unfolding lepoll_def eqpoll_def bij_def surj_def
apply (fast elim!: succ_not_lepoll_lemma inj_is_fun)
done

lemmas ordertype_eqpoll =
  ordermap_bij [THEN exI [THEN eqpoll_def [THEN def_imp_iff, THEN
iffD2]]]

lemma cons_cons_subset:
  "[a ⊆ y; b ∈ y-a; u ∈ x] ⇒ cons(b, cons(u, a)) ∈ Pow(x ∪ y)"
by fast

lemma cons_cons_eqpoll:
  "[a ≈ k; a ⊆ y; b ∈ y-a; u ∈ x; x ∩ y = 0]
  ⇒ cons(b, cons(u, a)) ≈ succ(succ(k))"
by (fast intro!: cons_eqpoll_succ)

lemma set_eq_cons:
  "[succ(k) ≈ A; k ≈ B; B ⊆ A; a ∈ A-B; k ∈ nat] ⇒ A = cons(a,
B)"
apply (rule equalityI)
prefer 2 apply fast
apply (rule Diff_eq_0_iff [THEN iffD1])
apply (rule equalsOI)
apply (drule eqpoll_sym [THEN eqpoll_imp_lepoll])
apply (drule eqpoll_sym [THEN cons_eqpoll_succ], fast)
apply (drule cons_eqpoll_succ, fast)
apply (fast elim!: lepoll_trans [THEN lepoll_trans, THEN succ_lepoll_natE,
OF eqpoll_sym [THEN eqpoll_imp_lepoll] subset_imp_lepoll])
done

lemma cons_eqE: "[cons(x,a) = cons(y,a); x ∉ a] ⇒ x = y"
by (fast elim!: equalityE)

lemma eq_imp_Int_eq: "A = B ⇒ A ∩ C = B ∩ C"
by blast

```

```

lemma eqpoll_sum_imp_Diff_lepoll_lemma [rule_format]:
  "[[k ∈ nat; m ∈ nat]]
   ⇒ ∀ A B. A ≈ k #+ m ∧ k ≲ B ∧ B ⊆ A → A-B ≲ m"
apply (induct_tac "k")
apply (simp add: add_0)
apply (blast intro: eqpoll_imp_lepoll lepoll_trans
              Diff_subset [THEN subset_imp_lepoll])
apply (intro allI impI)
apply (rule succ_lepoll_imp_not_empty [THEN not_emptyE], fast)
apply (erule_tac x = "A - {xa}" in allE)
apply (erule_tac x = "B - {xa}" in allE)
apply (erule impE)
apply (simp add: add_succ)
apply (fast intro!: Diff_sing_eqpoll lepoll_Diff_sing)
apply (subgoal_tac "A - {xa} - (B - {xa}) = A - B", simp)
apply blast
done

lemma eqpoll_sum_imp_Diff_lepoll:
  "[[A ≈ succ(k #+ m); B ⊆ A; succ(k) ≲ B; k ∈ nat; m ∈ nat]]
   ⇒ A-B ≲ m"
apply (simp only: add_succ [symmetric])
apply (blast intro: eqpoll_sum_imp_Diff_lepoll_lemma)
done

lemma eqpoll_sum_imp_Diff_eqpoll_lemma [rule_format]:
  "[[k ∈ nat; m ∈ nat]]
   ⇒ ∀ A B. A ≈ k #+ m ∧ k ≈ B ∧ B ⊆ A → A-B ≈ m"
apply (induct_tac "k")
apply (force dest!: eqpoll_sym [THEN eqpoll_imp_lepoll, THEN lepoll_0_is_0])
apply (intro allI impI)
apply (rule succ_lepoll_imp_not_empty [THEN not_emptyE])
apply (fast elim!: eqpoll_imp_lepoll)
apply (erule_tac x = "A - {xa}" in allE)
apply (erule_tac x = "B - {xa}" in allE)
apply (erule impE)
apply (force intro: eqpoll_sym intro!: Diff_sing_eqpoll)
apply (subgoal_tac "A - {xa} - (B - {xa}) = A - B", simp)
apply blast
done

```

```

lemma eqpoll_sum_imp_Diff_eqpoll:
  "⌊A ≈ succ(k #+ m); B ⊆ A; succ(k) ≈ B; k ∈ nat; m ∈ nat⌋
  ⇒ A-B ≈ m"
apply (simp only: add_succ [symmetric])
apply (blast intro: eqpoll_sum_imp_Diff_eqpoll_lemma)
done

lemma subsets_lepoll_0_eq_unit: "{x ∈ Pow(X). x ≲ 0} = {0}"
by (fast dest!: lepoll_0_is_0 intro!: lepoll_refl)

lemma subsets_lepoll_succ:
  "n ∈ nat ⇒ {z ∈ Pow(y). z ≲ succ(n)} =
    {z ∈ Pow(y). z ≲ n} ∪ {z ∈ Pow(y). z ≈ succ(n)}"
by (blast intro: leI le_imp_lepoll nat_into_Ord
    lepoll_trans eqpoll_imp_lepoll
    dest!: lepoll_succ_disj)

lemma Int_empty:
  "n ∈ nat ⇒ {z ∈ Pow(y). z ≲ n} ∩ {z ∈ Pow(y). z ≈ succ(n)} =
  0"
by (blast intro: eqpoll_sym [THEN eqpoll_imp_lepoll, THEN lepoll_trans]
    succ_lepoll_natE)

locale AC16 =
  fixes x and y and k and l and m and t_n and R and MM and LL and
  GG and s
  defines k_def:      "k ≡ succ(l)"
    and MM_def:      "MM ≡ {v ∈ t_n. succ(k) ≲ v ∩ y}"
    and LL_def:      "LL ≡ {v ∩ y. v ∈ MM}"
    and GG_def:      "GG ≡ λv ∈ LL. (THE w. w ∈ MM ∧ v ⊆ w) - v"
    and s_def:       "s(u) ≡ {v ∈ t_n. u ∈ v ∧ k ≲ v ∩ y}"
  assumes all_ex:    "∀ z ∈ {z ∈ Pow(x ∪ y) . z ≈ succ(k)}.
    ∃ ! w. w ∈ t_n ∧ z ⊆ w"
    and disjoint[iff]: "x ∩ y = 0"
    and "includes":  "t_n ⊆ {v ∈ Pow(x ∪ y). v ≈ succ(k #+ m)}"
    and WO_R[iff]:   "well_ord(y,R)"
    and lnat[iff]:   "l ∈ nat"
    and mnat[iff]:   "m ∈ nat"
    and mpos[iff]:   "0 < m"
    and Infinite[iff]: "¬ Finite(y)"
    and noLepoll:    "¬ y ≲ {v ∈ Pow(x). v ≈ m}"
begin

```

```
lemma knat [iff]: "k ∈ nat"
by (simp add: k_def)
```

```
lemma Diff_Finite_eqpoll: "[l ≈ a; a ⊆ y] ⇒ y - a ≈ y"
apply (insert WO_R Infinite lnat)
apply (rule eqpoll_trans)
apply (rule Diff_lesspoll_eqpoll_Card)
apply (erule well_ord_cardinal_eqpoll [THEN eqpoll_sym])
apply (blast intro: lesspoll_trans1
      intro!: Card_cardinal
            Card_cardinal [THEN Card_is_Ord, THEN nat_le_infinite_Ord,
                          THEN le_imp_lepoll]
      dest: well_ord_cardinal_eqpoll
            eqpoll_sym eqpoll_imp_lepoll
            n_lesspoll_nat [THEN lesspoll_trans2]
            well_ord_cardinal_eqpoll [THEN eqpoll_sym,
                                      THEN eqpoll_imp_lepoll, THEN lepoll_infinite])
done
```

```
lemma s_subset: "s(u) ⊆ t_n"
by (unfold s_def, blast)
```

```
lemma sI:
  "[w ∈ t_n; cons(b, cons(u, a)) ⊆ w; a ⊆ y; b ∈ y - a; l ≈ a]
  ⇒ w ∈ s(u)"
  unfolding s_def succ_def k_def
apply (blast intro!: eqpoll_imp_lepoll [THEN cons_lepoll_cong]
      intro: subset_imp_lepoll lepoll_trans)
done
```

```
lemma in_s_imp_u_in: "v ∈ s(u) ⇒ u ∈ v"
by (unfold s_def, blast)
```

```
lemma ex1_superset_a:
  "[l ≈ a; a ⊆ y; b ∈ y - a; u ∈ x]
  ⇒ ∃! c. c ∈ s(u) ∧ a ⊆ c ∧ b ∈ c"
apply (rule all_ex [simplified k_def, THEN ballE])
apply (erule ex1E)
apply (rule_tac a = w in ex1I, blast intro: sI)
apply (blast dest: s_subset [THEN subsetD] in_s_imp_u_in)
```

```

apply (blast del: PowI
      intro!: cons_cons_subset eqpoll_sym [THEN cons_cons_eqpoll])
done

lemma the_eq_cons:
  "⟦∀v ∈ s(u). succ(l) ≈ v ∩ y;
    l ≈ a; a ⊆ y; b ∈ y - a; u ∈ x⟧
   ⇒ (THE c. c ∈ s(u) ∧ a ⊆ c ∧ b ∈ c) ∩ y = cons(b, a)"
apply (frule ex1_superset_a [THEN theI], assumption+)
apply (rule set_eq_cons)
apply (fast+)
done

lemma y_lepoll_subset_s:
  "⟦∀v ∈ s(u). succ(l) ≈ v ∩ y;
    l ≈ a; a ⊆ y; u ∈ x⟧
   ⇒ y ≲ {v ∈ s(u). a ⊆ v}"
apply (rule Diff_Finite_eqpoll [THEN eqpoll_sym, THEN eqpoll_imp_lepoll,
                                THEN lepoll_trans], fast+)
apply (rule_tac f3 = "λb ∈ y-a. THE c. c ∈ s (u) ∧ a ⊆ c ∧ b ∈ c"
      in exI [THEN lepoll_def [THEN def_imp_iff, THEN iffD2]])
apply (simp add: inj_def)
apply (rule conjI)
apply (rule lam_type)
apply (frule ex1_superset_a [THEN theI], fast+, clarify)
apply (rule cons_eqE [of _ a])
apply (drule_tac A = "THE c. P (c)" and C = y for P in eq_imp_Int_eq)
apply (simp_all add: the_eq_cons)
done

```

```

lemma x_imp_not_y [dest]: "a ∈ x ⇒ a ∉ y"
by (blast dest: disjoint [THEN equalityD1, THEN subsetD, OF IntI])

```

```

lemma w_Int_eq_w_Diff:
  "w ⊆ x ∪ y ⇒ w ∩ (x - {u}) = w - cons(u, w ∩ y)"
by blast

```

```

lemma w_Int_eqpoll_m:
  "⟦w ∈ {v ∈ s(u). a ⊆ v};
    l ≈ a; u ∈ x;

```

```

       $\forall v \in s(u). \text{succ}(l) \approx v \cap y$ 
 $\implies w \cap (x - \{u\}) \approx m$ 
    apply (erule CollectE)
    apply (subst w_Int_eq_w_Diff)
    apply (fast dest!: s_subset [THEN subsetD]
      "includes" [simplified k_def, THEN subsetD])
    apply (blast dest: s_subset [THEN subsetD]
      "includes" [simplified k_def, THEN subsetD]
      dest: eqpoll_sym [THEN cons_eqpoll_succ, THEN eqpoll_sym]

      in_s_imp_u_in
    intro!: eqpoll_sum_imp_Diff_eqpoll)
  done

lemma eqpoll_m_not_empty: "a  $\approx$  m  $\implies$  a  $\neq$  0"
  apply (insert mpos)
  apply (fast elim!: zero_lt_natE dest!: eqpoll_succ_imp_not_empty)
  done

lemma cons_cons_in:
  " $\llbracket z \in xa \cap (x - \{u\}); l \approx a; a \subseteq y; u \in x \rrbracket$ 
 $\implies \exists ! w. w \in t_n \wedge \text{cons}(z, \text{cons}(u, a)) \subseteq w$ "
  apply (rule all_ex [THEN bspec])
  unfolding k_def
  apply (fast intro!: cons_eqpoll_succ elim: eqpoll_sym)
  done

lemma subset_s_lepoll_w:
  " $\llbracket \forall v \in s(u). \text{succ}(l) \approx v \cap y; a \subseteq y; l \approx a; u \in x \rrbracket$ 
 $\implies \{v \in s(u). a \subseteq v\} \lesssim \{v \in \text{Pow}(x). v \approx m\}$ "
  apply (rule_tac f3 = " $\lambda w \in \{v \in s(u). a \subseteq v\}. w \cap (x - \{u\})$ "
    in exI [THEN lepoll_def [THEN def_imp_iff, THEN iffD2]])
  apply (simp add: inj_def)
  apply (intro conjI lam_type CollectI)
  apply fast
  apply (blast intro: w_Int_eqpoll_m)
  apply (intro ballI impI)

  apply (rule w_Int_eqpoll_m [THEN eqpoll_m_not_empty, THEN not_emptyE])
  apply (blast, assumption+)
  apply (drule equalityD1 [THEN subsetD], assumption)
  apply (frule cons_cons_in, assumption+)
  apply (blast dest: ex1_two_eq intro: s_subset [THEN subsetD] in_s_imp_u_in)+

```


done

```
lemma well_ord_subsets_eqpoll_n:
  "n ∈ nat ⇒ ∃ S. well_ord({z ∈ Pow(y) . z ≈ succ(n)}, S)"
apply (rule WO_R [THEN well_ord_infinite_subsets_eqpoll_X,
  THEN eqpoll_def [THEN def_imp_iff, THEN iffD1], THEN
  exE])
apply (fast intro: bij_is_inj [THEN well_ord_rvimage])
done
```

```
lemma well_ord_subsets_lepoll_n:
  "n ∈ nat ⇒ ∃ R. well_ord({z ∈ Pow(y). z ≲ n}, R)"
apply (induct_tac "n")
apply (force intro!: well_ord_unit simp add: subsets_lepoll_0_eq_unit)
apply (erule exE)
apply (rule well_ord_subsets_eqpoll_n [THEN exE], assumption)
apply (simp add: subsets_lepoll_succ)
apply (drule well_ord_radd, assumption)
apply (erule Int_empty [THEN disj_Un_eqpoll_sum,
  THEN eqpoll_def [THEN def_imp_iff, THEN iffD1], THEN
  exE])
apply (fast elim!: bij_is_inj [THEN well_ord_rvimage])
done
```

```
lemma LL_subset: "LL ⊆ {z ∈ Pow(y). z ≲ succ(k #+ m)}"
  unfolding LL_def MM_def
  apply (insert "includes")
  apply (blast intro: subset_imp_lepoll eqpoll_imp_lepoll lepoll_trans)
done
```

```
lemma well_ord_LL: "∃ S. well_ord(LL, S)"
  apply (rule well_ord_subsets_lepoll_n [THEN exE, of "succ(k #+ m)"])
  apply simp
  apply (blast intro: well_ord_subset [OF _ LL_subset])
done
```

```
lemma unique_superset_in_MM:
  "v ∈ LL ⇒ ∃! w. w ∈ MM ∧ v ⊆ w"
  apply (unfold MM_def LL_def, safe, fast)
```

```

apply (rule lepoll_imp_eqpoll_subset [THEN exE], assumption)
apply (rule_tac x = x in all_ex [THEN ballE])
apply (blast intro: eqpoll_sym)+
done

```

```

lemma Int_in_LL: "w ∈ MM ⇒ w ∩ y ∈ LL"
by (unfold LL_def, fast)

```

```

lemma in_LL_eq_Int:
  "v ∈ LL ⇒ v = (THE x. x ∈ MM ∧ v ⊆ x) ∩ y"
apply (unfold LL_def, clarify)
apply (subst unique_superset_in_MM [THEN the_equality2])
apply (auto simp add: Int_in_LL)
done

```

```

lemma unique_superset1: "a ∈ LL ⇒ (THE x. x ∈ MM ∧ a ⊆ x) ∈ MM"
by (erule unique_superset_in_MM [THEN theI, THEN conjunct1])

```

```

lemma the_in_MM_subset:
  "v ∈ LL ⇒ (THE x. x ∈ MM ∧ v ⊆ x) ⊆ x ∪ y"
apply (drule unique_superset1)
  unfolding MM_def
apply (fast dest!: unique_superset1 "includes" [THEN subsetD])
done

```

```

lemma GG_subset: "v ∈ LL ⇒ GG ' v ⊆ x"
  unfolding GG_def
apply (frule the_in_MM_subset)
apply (frule in_LL_eq_Int)
apply (force elim: equalityE)
done

```

```

lemma nat_lepoll_ordertype: "nat ≲ ordertype(y, R)"
apply (rule nat_le_infinite_Ord [THEN le_imp_lepoll])
  apply (rule Ord_ordertype [OF WO_R])
apply (rule ordertype_eqpoll [THEN eqpoll_imp_lepoll, THEN lepoll_infinite])

  apply (rule WO_R)
apply (rule Infinite)
done

```

```

lemma ex_subset_eqpoll_n: "n ∈ nat ⇒ ∃ z. z ⊆ y ∧ n ≈ z"
apply (erule nat_lepoll_imp_ex_eqpoll_n)
apply (rule lepoll_trans [OF nat_lepoll_ordertype])
apply (rule ordertype_eqpoll [THEN eqpoll_sym, THEN eqpoll_imp_lepoll])

apply (rule WO_R)
done

lemma exists_proper_in_s: "u ∈ x ⇒ ∃ v ∈ s(u). succ(k) ≲ v ∩ y"
apply (rule ccontr)
apply (subgoal_tac "∀ v ∈ s (u) . k ≈ v ∩ y")
prefer 2 apply (simp add: s_def, blast intro: succ_not_lepoll_imp_eqpoll)
  unfolding k_def
apply (insert all_ex "includes" lnat)
apply (rule ex_subset_eqpoll_n [THEN exE], assumption)
apply (rule noLepoll [THEN notE])
apply (blast intro: lepoll_trans [OF y_lepoll_subset_s subset_s_lepoll_w])
done

lemma exists_in_MM: "u ∈ x ⇒ ∃ w ∈ MM. u ∈ w"
apply (erule exists_proper_in_s [THEN bexE])
apply (unfold MM_def s_def, fast)
done

lemma exists_in_LL: "u ∈ x ⇒ ∃ w ∈ LL. u ∈ GG'w"
apply (rule exists_in_MM [THEN bexE], assumption)
apply (rule bexI)
apply (erule_tac [2] Int_in_LL)
  unfolding GG_def
apply (simp add: Int_in_LL)
apply (subst unique_superset_in_MM [THEN the_equality2])
apply (fast elim!: Int_in_LL)+
done

lemma OUN_eq_x: "well_ord(LL,S) ⇒
  (⋃ b<ordertype(LL,S). GG ' (converse(ordermap(LL,S)) ' b)) = x"
apply (rule equalityI)
apply (rule subsetI)
apply (erule OUN_E)
apply (rule GG_subset [THEN subsetD])
prefer 2 apply assumption
apply (blast intro: ltD ordermap_bij [THEN bij_converse_bij, THEN bij_is_fun,
  THEN apply_type])

apply (rule subsetI)
apply (erule exists_in_LL [THEN bexE])
apply (force intro: ltI [OF _ Ord_ordertype]
  ordermap_type [THEN apply_type])

```

```

      simp add: ordermap_bij [THEN bij_is_inj, THEN left_inverse])
done

lemma in_MM_eqpoll_n: "w ∈ MM ⇒ w ≈ succ(k #+ m)"
  unfolding MM_def
  apply (fast dest: "includes" [THEN subsetD])
done

lemma in_LL_eqpoll_n: "w ∈ LL ⇒ succ(k) ≲ w"
  by (unfold LL_def MM_def, fast)

lemma in_LL: "w ∈ LL ⇒ w ⊆ (THE x. x ∈ MM ∧ w ⊆ x)"
  by (erule subset_trans [OF in_LL_eq_Int [THEN equalityD1] Int_lower1])

lemma all_in_lepoll_m:
  "well_ord(LL,S) ⇒
    ∀ b < ordertype(LL,S). GG ' (converse(ordermap(LL,S)) ' b) ≲ m"
  unfolding GG_def
  apply (rule oallI)
  apply (simp add: ltD ordermap_bij [THEN bij_converse_bij, THEN bij_is_fun,
    THEN apply_type])
  apply (insert "includes")
  apply (rule eqpoll_sum_imp_Diff_lepoll)
  apply (blast del: subsetI
    dest!: ltD
    intro!: eqpoll_sum_imp_Diff_lepoll in_LL_eqpoll_n
    intro: in_LL unique_superset1 [THEN in_MM_eqpoll_n]
    ordermap_bij [THEN bij_converse_bij, THEN bij_is_fun,
      THEN apply_type])+)
done

lemma "conclusion":
  "∃ a f. Ord(a) ∧ domain(f) = a ∧ (⋃ b < a. f ' b) = x ∧ (∀ b < a. f '
    b ≲ m)"
  apply (rule well_ord_LL [THEN exE])
  apply (rename_tac S)
  apply (rule_tac x = "ordertype (LL,S)" in exI)
  apply (rule_tac x = "λb ∈ ordertype(LL,S).
    GG ' (converse (ordermap (LL,S)) ' b)" in exI)
  apply (simp add: ltD)
  apply (blast intro: lam_funtype [THEN domain_of_fun]
    Ord_ordertype OUN_eq_x all_in_lepoll_m [THEN ospec])
done

```

end

```

theorem AC16_W04:
  "⟦AC_Equiv.AC16(k #+ m, k); 0 < k; 0 < m; k ∈ nat; m ∈ nat⟧ ⇒
  W04(m) "
  unfolding AC_Equiv.AC16_def W04_def
  apply (rule allI)
  apply (case_tac "Finite (A)")
  apply (rule lemma1, assumption+)
  apply (cut_tac lemma2)
  apply (elim exE conjE)
  apply (erule_tac x = "A ∪ y" in allE)
  apply (frule infinite_Un, drule mp, assumption)
  apply (erule zero_lt_natE, assumption, clarify)
  apply (blast intro: AC16.conclusion [OF AC16.intro])
  done

end

```

```

theory AC17_AC1
imports HH
begin

```

```

lemma AC0_AC1_lemma: "⟦f: (∏ X ∈ A. X); D ⊆ A⟧ ⇒ ∃ g. g: (∏ X ∈ D.
X) "
by (fast intro!: lam_type apply_type)

```

```

lemma AC0_AC1: "AC0 ⇒ AC1"
  unfolding AC0_def AC1_def
  apply (blast intro: AC0_AC1_lemma)
  done

```

```

lemma AC1_AC0: "AC1 ⇒ AC0"
by (unfold AC0_def AC1_def, blast)

```

```

lemma AC1_AC17_lemma: "f ∈ (∏ X ∈ Pow(A) - {0}. X) ⇒ f ∈ (Pow(A)
- {0} → A) "

```

```

apply (rule Pi_type, assumption)
apply (drule apply_type, assumption, fast)
done

lemma AC1_AC17: "AC1  $\implies$  AC17"
  unfolding AC1_def AC17_def
  apply (rule allI)
  apply (rule ballI)
  apply (erule_tac x = "Pow (A) -{0}" in allE)
  apply (erule impE, fast)
  apply (erule exE)
  apply (rule bexI)
  apply (erule_tac [2] AC1_AC17_lemma)
  apply (rule apply_type, assumption)
  apply (fast dest!: AC1_AC17_lemma elim!: apply_type)
done

```

```

lemma UN_eq_imp_well_ord:
  "[x - ( $\bigcup j \in \mu i. HH(\lambda X \in Pow(x) - \{0\}. \{f'X\}, x, i) = \{x\}.$ 
     $HH(\lambda X \in Pow(x) - \{0\}. \{f'X\}, x, j)) = 0;$ 
     $f \in Pow(x) - \{0\} \rightarrow x]$ 
     $\implies \exists r. well\_ord(x, r)"]$ 
  apply (rule exI)
  apply (erule well_ord_rvimage
    [OF bij_Least_HH_x [THEN bij_converse_bij, THEN bij_is_inj]
      Ord_Least [THEN well_ord_Memrel]], assumption)
done

```

```

lemma not_AC1_imp_ex:
  " $\neg AC1 \implies \exists A. \forall f \in Pow(A) - \{0\} \rightarrow A. \exists u \in Pow(A) - \{0\}. f'u \notin u$ "
  unfolding AC1_def
  apply (erule swap)
  apply (rule allI)
  apply (erule swap)
  apply (rule_tac x = " $\bigcup (A)$ " in exI)
  apply (blast intro: lam_type)
done

```

```

lemma AC17_AC1_aux1:
  "[[ $\forall f \in \text{Pow}(x) - \{0\} \rightarrow x. \exists u \in \text{Pow}(x) - \{0\}. f'u \notin u;$ 
     $\exists f \in \text{Pow}(x) - \{0\} \rightarrow x.$ 
     $x - (\bigcup a \in (\mu i. \text{HH}(\lambda X \in \text{Pow}(x) - \{0\}. \{f'X\}, x, i) = \{x\}).$ 
     $\text{HH}(\lambda X \in \text{Pow}(x) - \{0\}. \{f'X\}, x, a)) = 0]$ 
     $\Rightarrow P$ "
  apply (erule bexE)
  apply (erule UN_eq_imp_well_ord [THEN exE], assumption)
  apply (erule ex_choice_fun_Pow [THEN exE])
  apply (erule ballE)
  apply (fast intro: apply_type del: DiffE)
  apply (erule notE)
  apply (rule Pi_type, assumption)
  apply (blast dest: apply_type)
done

lemma AC17_AC1_aux2:
  " $\neg (\exists f \in \text{Pow}(x) - \{0\} \rightarrow x. x - F(f) = 0)$ 
     $\Rightarrow (\lambda f \in \text{Pow}(x) - \{0\} \rightarrow x. x - F(f))$ 
     $\in (\text{Pow}(x) - \{0\} \rightarrow x) \rightarrow \text{Pow}(x) - \{0\}$ "
  by (fast intro!: lam_type dest!: Diff_eq_0_iff [THEN iffD1])

lemma AC17_AC1_aux3:
  "[[ $f'Z \in Z; Z \in \text{Pow}(x) - \{0\}]$ 
     $\Rightarrow (\lambda X \in \text{Pow}(x) - \{0\}. \{f'X\})'Z \in \text{Pow}(Z) - \{0\}$ "
  by auto

lemma AC17_AC1_aux4:
  " $\exists f \in F. f'((\lambda f \in F. Q(f))'f) \in (\lambda f \in F. Q(f))'f$ 
     $\Rightarrow \exists f \in F. f'Q(f) \in Q(f)$ "
  by simp

lemma AC17_AC1: "AC17  $\Rightarrow$  AC1"
  unfolding AC17_def
  apply (rule classical)
  apply (erule not_AC1_imp_ex [THEN exE])
  apply (case_tac
    " $\exists f \in \text{Pow}(x) - \{0\} \rightarrow x.$ 
     $x - (\bigcup a \in (\mu i. \text{HH}(\lambda X \in \text{Pow}(x) - \{0\}. \{f'X\}, x, i) = \{x\}).$ 
     $\text{HH}(\lambda X \in \text{Pow}(x) - \{0\}. \{f'X\}, x, a)) = 0$ ")
  apply (erule AC17_AC1_aux1, assumption)
  apply (drule AC17_AC1_aux2)
  apply (erule allE)
  apply (drule bspec, assumption)
  apply (drule AC17_AC1_aux4)
  apply (erule bexE)
  apply (drule apply_type, assumption)
  apply (simp add: HH_Least_eq_x del: Diff_iff)
  apply (drule AC17_AC1_aux3, assumption)

```

```

apply (fast dest!: subst_elem [OF _ HH_Least_eq_x [symmetric]]
      f_subset_imp_HH_subset elim!: mem_irrefl)
done

```

```

lemma AC1_AC2_aux1:
  "⟦f: (⟦X ∈ A. X⟧); B ∈ A; 0 ∉ A⟧ ⟹ {f'B} ⊆ B ∩ {f'C. C ∈ A}"
by (fast elim!: apply_type)

```

```

lemma AC1_AC2_aux2:
  "⟦pairwise_disjoint(A); B ∈ A; C ∈ A; D ∈ B; D ∈ C⟧ ⟹ f'B
= f'C"
by (unfold pairwise_disjoint_def, fast)

```

```

lemma AC1_AC2: "AC1 ⟹ AC2"
  unfolding AC1_def AC2_def
  apply (rule allI)
  apply (rule impI)
  apply (elim asm_rl conjE allE exE impE, assumption)
  apply (intro exI ballI equalityI)
  prefer 2 apply (rule AC1_AC2_aux1, assumption+)
  apply (fast elim!: AC1_AC2_aux2 elim: apply_type)
done

```

```

lemma AC2_AC1_aux1: "0 ∉ A ⟹ 0 ∉ {B*{B}. B ∈ A}"
by (fast dest!: sym [THEN Sigma_empty_iff [THEN iffD1]])

```

```

lemma AC2_AC1_aux2: "⟦X*{X} ∩ C = {y}; X ∈ A⟧
  ⟹ (THE y. X*{X} ∩ C = {y}): X*A"
  apply (rule subst_elem [of y])
  apply (blast elim!: equalityE)
  apply (auto simp add: singleton_eq_iff)
done

```

```

lemma AC2_AC1_aux3:
  "∀D ∈ {E*{E}. E ∈ A}. ∃y. D ∩ C = {y}
  ⟹ (λx ∈ A. fst(THE z. (x*{x} ∩ C = {z}))) ∈ (⟦X ∈ A. X⟧)"
  apply (rule lam_type)

```



```

apply (drule bspec, blast)
apply (blast intro: AC2_AC1_aux2 fst_type)
done

```

```

lemma AC2_AC1: "AC2  $\implies$  AC1"
  unfolding AC1_def AC2_def pairwise_disjoint_def
  apply (intro allI impI)
  apply (elim allE impE)
  prefer 2 apply (fast elim!: AC2_AC1_aux3)
  apply (blast intro!: AC2_AC1_aux1)
done

```

```

lemma empty_notin_images: "0  $\notin$  {R' '{x}. x  $\in$  domain(R)}"
by blast

```

```

lemma AC1_AC4: "AC1  $\implies$  AC4"
  unfolding AC1_def AC4_def
  apply (intro allI impI)
  apply (drule spec, drule mp [OF _ empty_notin_images])
  apply (best intro!: lam_type elim!: apply_type)
done

```

```

lemma AC4_AC3_aux1: "f  $\in$  A $\rightarrow$ B  $\implies$  ( $\bigcup$  z  $\in$  A. {z}*f'z)  $\subseteq$  A* $\bigcup$  (B)"
by (fast dest!: apply_type)

```

```

lemma AC4_AC3_aux2: "domain( $\bigcup$  z  $\in$  A. {z}*f(z)) = {a  $\in$  A. f(a) $\neq$ 0}"
by blast

```

```

lemma AC4_AC3_aux3: "x  $\in$  A  $\implies$  ( $\bigcup$  z  $\in$  A. {z}*f(z))' '{x} = f(x)"
by fast

```

```

lemma AC4_AC3: "AC4  $\implies$  AC3"
  unfolding AC3_def AC4_def
  apply (intro allI ballI)
  apply (elim allE impE)
  apply (erule AC4_AC3_aux1)
  apply (simp add: AC4_AC3_aux2 AC4_AC3_aux3 cong add: Pi_cong)
done

```

```

lemma AC3_AC1_lemma:
  "b ∉ A ⇒ (∏ x ∈ {a ∈ A. id(A) 'a ≠ b}. id(A) 'x) = (∏ x ∈ A. x)"
apply (simp add: id_def cong add: Pi_cong)
apply (rule_tac b = A in subst_context, fast)
done

lemma AC3_AC1: "AC3 ⇒ AC1"
  unfolding AC1_def AC3_def
apply (fast intro!: id_type elim: AC3_AC1_lemma [THEN subst])
done

lemma AC4_AC5: "AC4 ⇒ AC5"
  unfolding range_def AC4_def AC5_def
apply (intro allI ballI)
apply (elim allE impE)
apply (erule fun_is_rel [THEN converse_type])
apply (erule exE)
apply (rename_tac g)
apply (rule_tac x=g in bexI)
apply (blast dest: apply_equality range_type)
apply (blast intro: Pi_type dest: apply_type fun_is_rel)
done

lemma AC5_AC4_aux1: "R ⊆ A*B ⇒ (λx ∈ R. fst(x)) ∈ R -> A"
by (fast intro!: lam_type fst_type)

lemma AC5_AC4_aux2: "R ⊆ A*B ⇒ range(λx ∈ R. fst(x)) = domain(R)"
by (unfold lam_def, force)

lemma AC5_AC4_aux3: "⟦∃ f ∈ A->C. P(f, domain(f)); A=B⟧ ⇒ ∃ f ∈ B->C.
P(f, B)"
apply (erule bexE)
apply (frule domain_of_fun, fast)
done

lemma AC5_AC4_aux4: "⟦R ⊆ A*B; g ∈ C->R; ∀ x ∈ C. (λz ∈ R. fst(z)) '

```

```

(g'x) = x]]
       $\implies (\lambda x \in C. \text{snd}(g'x)) : (\prod x \in C. R'\{x\})"$ 
apply (rule lam_type)
apply (force dest: apply_type)
done

```

```

lemma AC5_AC4: "AC5  $\implies$  AC4"
apply (unfold AC4_def AC5_def, clarify)
apply (elim allE ballE)
apply (drule AC5_AC4_aux3 [OF _ AC5_AC4_aux2], assumption)
apply (fast elim!: AC5_AC4_aux4)
apply (blast intro: AC5_AC4_aux1)
done

```

```

lemma AC1_iff_AC6: "AC1  $\longleftrightarrow$  AC6"
by (unfold AC1_def AC6_def, blast)

```

end

```

theory AC18_AC19
imports AC_Equiv
begin

```

```

definition
  uu      :: "i  $\Rightarrow$  i" where
    "uu(a)  $\equiv$  {c  $\cup$  {0}. c  $\in$  a}"

```

```

lemma PROD_subsets:
  "[f  $\in$  ( $\prod b \in \{P(a). a \in A\}. b$ );  $\forall a \in A. P(a) \leq Q(a)$ ]"
   $\implies (\lambda a \in A. f'P(a)) \in (\prod a \in A. Q(a))"$ 
by (rule lam_type, drule apply_type, auto)

```

```

lemma lemma_AC18:
  "[ $\forall A. 0 \notin A \longrightarrow (\exists f. f \in (\prod X \in A. X))$ ;  $A \neq 0$ ]"
   $\implies (\bigcap a \in A. \bigcup b \in B(a). X(a, b)) \subseteq$ 
     $(\bigcup f \in \prod a \in A. B(a). \bigcap a \in A. X(a, f'a))"$ 
apply (rule subsetI)
apply (erule_tac x = "{b  $\in$  B (a) . x  $\in$  X (a,b) }. a  $\in$  A}" in allE)

```

```

apply (erule impE, fast)
apply (erule exE)
apply (rule UN_I)
  apply (fast elim!: PROD_subsets)
apply (simp, fast elim!: not_emptyE dest: apply_type [OF _ RepFunI])
done

lemma AC1_AC18: "AC1  $\implies$  PROP AC18"
  unfolding AC1_def
  apply (rule AC18.intro)
  apply (fast elim!: lemma_AC18 apply_type intro!: equalityI INT_I UN_I)
done

theorem (in AC18) AC19
  unfolding AC19_def
  apply (intro allI impI)
  apply (rule AC18 [of _ " $\lambda x. x$ ", THEN mp], blast)
done

lemma RepRep_conj:
  " $\llbracket A \neq 0; 0 \notin A \rrbracket \implies \{uu(a). a \in A\} \neq 0 \wedge 0 \notin \{uu(a). a \in A\}$ "
  apply (unfold uu_def, auto)
  apply (blast dest!: sym [THEN RepFun_eq_0_iff [THEN iffD1]])
done

lemma lemma1_1: " $\llbracket c \in a; x = c \cup \{0\}; x \notin a \rrbracket \implies x - \{0\} \in a$ "
  apply clarify
  apply (rule subst_elem, assumption)
  apply (fast elim: notE subst_elem)
done

lemma lemma1_2:
  " $\llbracket f'(uu(a)) \notin a; f \in (\prod B \in \{uu(a). a \in A\}. B); a \in A \rrbracket$   

 $\implies f'(uu(a)) - \{0\} \in a$ "
  apply (unfold uu_def, fast elim!: lemma1_1 dest!: apply_type)
done

lemma lemma1: " $\exists f. f \in (\prod B \in \{uu(a). a \in A\}. B) \implies \exists f. f \in (\prod B$   

 $\in A. B)$ "
  apply (erule exE)
  apply (rule_tac x = " $\lambda a \in A. \text{if } (f'(uu(a)) \in a, f'(uu(a)), f'(uu(a)) - \{0\})$ "

```

```

      in exI)
    apply (rule lam_type)
    apply (simp add: lemma1_2)
  done

lemma lemma2_1: "a ≠ 0 ⇒ 0 ∈ (⋃ b ∈ uu(a). b)"
by (unfold uu_def, auto)

lemma lemma2: "[[A ≠ 0; 0 ∉ A]] ⇒ (⋂ x ∈ {uu(a). a ∈ A}. ⋃ b ∈ x. b) ≠ 0"
  apply (erule not_emptyE)
  apply (rule_tac a = 0 in not_emptyI)
  apply (fast intro!: lemma2_1)
done

lemma AC19_AC1: "AC19 ⇒ AC1"
  apply (unfold AC19_def AC1_def, clarify)
  apply (case_tac "A=0", force)
  apply (erule_tac x = "{uu (a) . a ∈ A}" in allE)
  apply (erule impE)
  apply (erule RepRep_conj, assumption)
  apply (rule lemma1)
  apply (drule lemma2, assumption, auto)
done

end

theory DC
imports AC_Equiv Hartog Cardinal_aux
begin

lemma RepFun_lepoll: "Ord(a) ⇒ {P(b). b ∈ a} ≲ a"
  unfolding lepoll_def
  apply (rule_tac x = "λz ∈ RepFun (a,P) . μ i. z=P (i)" in exI)
  apply (rule_tac d="λz. P (z)" in lam_injective)
  apply (fast intro!: Least_in_Ord)
  apply (rule sym)
  apply (fast intro: LeastI Ord_in_Ord)
done

Trivial in the presence of AC, but here we need a wellordering of X

lemma image_Ord_lepoll: "[[f ∈ X→Y; Ord(X)]] ⇒ f'X ≲ X"
  unfolding lepoll_def
  apply (rule_tac x = "λx ∈ f'X. μ y. f'y = x" in exI)
  apply (rule_tac d = "λz. f'z" in lam_injective)
  apply (fast intro!: Least_in_Ord apply_equality, clarify)
  apply (rule LeastI)

```

```

    apply (erule apply_equality, assumption+)
  apply (blast intro: Ord_in_Ord)
done

lemma range_subset_domain:
  "[[R ⊆ X*X; ⋀g. g ∈ X ⇒ ∃u. ⟨g,u⟩ ∈ R]]
  ⇒ range(R) ⊆ domain(R)"
by blast

lemma cons_fun_type: "g ∈ n->X ⇒ cons(⟨n,x⟩, g) ∈ succ(n) -> cons(x,
X)"
  unfolding succ_def
  apply (fast intro!: fun_extend elim!: mem_irrefl)
done

lemma cons_fun_type2:
  "[[g ∈ n->X; x ∈ X]] ⇒ cons(⟨n,x⟩, g) ∈ succ(n) -> X"
by (erule cons_absorb [THEN subst], erule cons_fun_type)

lemma cons_image_n: "n ∈ nat ⇒ cons(⟨n,x⟩, g) 'n = g 'n"
by (fast elim!: mem_irrefl)

lemma cons_val_n: "g ∈ n->X ⇒ cons(⟨n,x⟩, g) 'n = x"
by (fast intro!: apply_equality elim!: cons_fun_type)

lemma cons_image_k: "k ∈ n ⇒ cons(⟨n,x⟩, g) 'k = g 'k"
by (fast elim: mem_asym)

lemma cons_val_k: "[[k ∈ n; g ∈ n->X]] ⇒ cons(⟨n,x⟩, g) 'k = g 'k"
by (fast intro!: apply_equality consI2 elim!: cons_fun_type apply_Pair)

lemma domain_cons_eq_succ: "domain(f)=x ⇒ domain(cons(⟨x,y⟩, f)) =
succ(x)"
by (simp add: domain_cons succ_def)

lemma restrict_cons_eq: "g ∈ n->X ⇒ restrict(cons(⟨n,x⟩, g), n) =
g"
  apply (simp add: restrict_def Pi_iff)
  apply (blast intro: elim: mem_irrefl)
done

lemma succ_in_succ: "[[Ord(k); i ∈ k]] ⇒ succ(i) ∈ succ(k)"
  apply (rule Ord_linear [of "succ(i)" "succ(k)", THEN disjE])
  apply (fast elim: Ord_in_Ord mem_irrefl mem_asym)+
done

lemma restrict_eq_imp_val_eq:
  "[[restrict(f, domain(g)) = g; x ∈ domain(g)]
  ⇒ f 'x = g 'x"

```

by (erule subst, simp add: restrict)

lemma domain_eq_imp_fun_type: " $\llbracket \text{domain}(f)=A; f \in B \rightarrow C \rrbracket \implies f \in A \rightarrow C$ "
by (frule domain_of_fun, fast)

lemma ex_in_domain: " $\llbracket R \subseteq A * B; R \neq 0 \rrbracket \implies \exists x. x \in \text{domain}(R)$ "
by (fast elim!: not_emptyE)

definition

DC :: "i \Rightarrow o" where

$$\text{"DC}(a) \equiv \forall X R. R \subseteq \text{Pow}(X) * X \wedge$$

$$(\forall Y \in \text{Pow}(X). Y \prec a \longrightarrow (\exists x \in X. \langle Y, x \rangle \in R))$$

$$\longrightarrow (\exists f \in a \rightarrow X. \forall b \prec a. \langle f' b, f' b \rangle \in R) \text{"}$$

definition

DCO :: o where

$$\text{"DCO} \equiv \forall A B R. R \subseteq A * B \wedge R \neq 0 \wedge \text{range}(R) \subseteq \text{domain}(R)$$

$$\longrightarrow (\exists f \in \text{nat} \rightarrow \text{domain}(R). \forall n \in \text{nat}. \langle f' n, f' \text{succ}(n) \rangle \in R) \text{"}$$

definition

ff :: "[i, i, i, i] \Rightarrow i" where

$$\text{"ff}(b, X, Q, R) \equiv$$

$$\text{transrec}(b, \lambda c r. \text{THE } x. \text{first}(x, \{x \in X. \langle r' c, x \rangle \in R\},$$

$$Q)) \text{"}$$

locale DCO_imp =

fixes XX and RR and X and R

assumes all_ex: " $\forall Y \in \text{Pow}(X). Y \prec \text{nat} \longrightarrow (\exists x \in X. \langle Y, x \rangle \in R)$ "

defines XX_def: " $XX \equiv (\bigcup n \in \text{nat}. \{f \in n \rightarrow X. \forall k \in n. \langle f' k, f' k \rangle \in R\})$ "

and RR_def: " $RR \equiv \{\langle z1, z2 \rangle : XX * XX. \text{domain}(z2) = \text{succ}(\text{domain}(z1))$
 $\wedge \text{restrict}(z2, \text{domain}(z1)) = z1\}$ "

begin

```
lemma lemma1_1: "RR  $\subseteq$  XX*XX"
by (unfold RR_def, fast)
```

```
lemma lemma1_2: "RR  $\neq$  0"
  unfolding RR_def XX_def
  apply (rule all_ex [THEN ballE])
  apply (erule_tac [2] notE [OF _ empty_subsetI [THEN PowI]])
  apply (erule_tac impE [OF _ nat_OI [THEN n_lesspoll_nat]])
  apply (erule bexE)
  apply (rule_tac a = "<0, {<0, x>}>" in not_emptyI)
  apply (rule CollectI)
  apply (rule SigmaI)
  apply (rule nat_OI [THEN UN_I])
  apply (simp (no_asm_simp) add: nat_OI [THEN UN_I])
  apply (rule nat_1I [THEN UN_I])
  apply (force intro!: singleton_fun [THEN Pi_type]
    simp add: singleton_0 [symmetric])
  apply (simp add: singleton_0)
done
```

```
lemma lemma1_3: "range(RR)  $\subseteq$  domain(RR)"
  unfolding RR_def XX_def
  apply (rule range_subset_domain, blast, clarify)
  apply (frule fun_is_rel [THEN image_subset, THEN PowI,
    THEN all_ex [THEN bspec]])
  apply (erule impE[OF _ lesspoll_trans1[OF image_Ord_lepoll
    [OF _ nat_into_Ord] n_lesspoll_nat]],
    assumption+)
  apply (erule bexE)
  apply (rule_tac x = "cons (<n,x>, g) " in exI)
  apply (rule CollectI)
  apply (force elim!: cons_fun_type2
    simp add: cons_image_n cons_val_n cons_image_k cons_val_k)
  apply (simp add: domain_of_fun succ_def restrict_cons_eq)
```


done

lemma lemma2:

" $\forall n \in \text{nat}. \langle f'n, f'\text{succ}(n) \rangle \in \text{RR}; f \in \text{nat} \rightarrow \text{XX}; n \in \text{nat}$ "
 $\implies \exists k \in \text{nat}. f'\text{succ}(n) \in k \rightarrow X \wedge n \in k$
 $\wedge \langle f'\text{succ}(n)'n, f'\text{succ}(n)'n \rangle \in R$ "

apply (induct_tac "n")
 apply (drule apply_type [OF _ nat_1I])
 apply (drule bspec [OF _ nat_0I])
 apply (simp add: XX_def, safe)
 apply (rule rev_bexI, assumption)
 apply (subgoal_tac "0 ∈ y", force)
 apply (force simp add: RR_def
 intro: ltD elim!: nat_0_le [THEN leE])

 apply (drule bspec [OF _ nat_succI], assumption)
 apply (subgoal_tac "f ' succ (succ (x)) ∈ succ (k) → X")
 apply (drule apply_type [OF _ nat_succI [THEN nat_succI]], assumption)
 apply (simp (no_asm_use) add: XX_def RR_def)
 apply safe
 apply (frule_tac a="succ(k)" in domain_of_fun [symmetric, THEN trans],
 assumption)
 apply (frule_tac a=y in domain_of_fun [symmetric, THEN trans],
 assumption)
 apply (fast elim!: nat_into_Ord [THEN succ_in_succ]
 dest!: bspec [OF _ nat_into_Ord [THEN succ_in_succ]])
 apply (drule domain_of_fun)
 apply (simp add: XX_def RR_def, clarify)
 apply (blast dest: domain_of_fun [symmetric, THEN trans])
 done

lemma lemma3_1:

" $\forall n \in \text{nat}. \langle f'n, f'\text{succ}(n) \rangle \in \text{RR}; f \in \text{nat} \rightarrow \text{XX}; m \in \text{nat}$ "
 $\implies \{f'\text{succ}(x)'x. x \in m\} = \{f'\text{succ}(m)'x. x \in m\}$ "

apply (subgoal_tac " $\forall x \in m. f'\text{succ}(m)'x = f'\text{succ}(x)'x$ ")
 apply simp
 apply (induct_tac "m", blast)
 apply (rule ballI)
 apply (erule succE)
 apply (rule restrict_eq_imp_val_eq)
 apply (drule bspec [OF _ nat_succI], assumption)
 apply (simp add: RR_def)
 apply (drule lemma2, assumption+)
 apply (fast dest!: domain_of_fun)
 apply (drule_tac x = xa in bspec, assumption)
 apply (erule sym [THEN trans, symmetric])
 apply (rule restrict_eq_imp_val_eq [symmetric])
 apply (drule bspec [OF _ nat_succI], assumption)

```

    apply (simp add: RR_def)
  apply (drule lemma2, assumption+)
  apply (blast dest!: domain_of_fun
    intro: nat_into_Ord OrdmemD [THEN subsetD])
done

lemma lemma3:
  "⟦∀ n ∈ nat. <f'n, f'succ(n)> ∈ RR; f ∈ nat -> XX; m ∈ nat⟧
  ⇒ (λx ∈ nat. f'succ(x)'x) 'm = f'succ(m)'m"
  apply (erule natE, simp)
  apply (subst image_lam)
  apply (fast elim!: OrdmemD [OF nat_succI Ord_nat])
  apply (subst lemma3_1, assumption+)
  apply fast
  apply (fast dest!: lemma2
    elim!: image_fun [symmetric, OF _ OrdmemD [OF _ nat_into_Ord]])
done

end

theorem DCO_imp_DC_nat: "DC0 ⇒ DC(nat)"
  apply (unfold DC_def DCO_def, clarify)
  apply (elim allE)
  apply (erule impE)

  apply (blast intro!: DCO_imp.lemma1_1 [OF DCO_imp.intro] DCO_imp.lemma1_2
    [OF DCO_imp.intro] DCO_imp.lemma1_3 [OF DCO_imp.intro])
  apply (erule bexE)
  apply (rule_tac x = "λn ∈ nat. f'succ (n) 'n" in rev_bexI)
  apply (rule lam_type, blast dest!: DCO_imp.lemma2 [OF DCO_imp.intro]
    intro: fun_weaken_type)
  apply (rule oallI)
  apply (frule DCO_imp.lemma2 [OF DCO_imp.intro], assumption)
  apply (blast intro: fun_weaken_type)
  apply (erule ltD)

  apply (subst DCO_imp.lemma3 [OF DCO_imp.intro], assumption+)
  apply (fast elim!: fun_weaken_type)
  apply (erule ltD)
  apply (force simp add: lt_def)
done

lemma singleton_in_funs:
  "x ∈ X ⇒ {⟨0,x⟩} ∈
    (⋃ n ∈ nat. {f ∈ succ(n)->X. ∀ k ∈ n. <f'k, f'succ(k)> ∈
  R})"

```

```

apply (rule nat_OI [THEN UN_I])
apply (force simp add: singleton_0 [symmetric]
        intro!: singleton_fun [THEN Pi_type])
done

locale imp_DCO =
  fixes XX and RR and x and R and f and allRR
  defines XX_def: "XX  $\equiv$  ( $\bigcup n \in \text{nat}.$ 
    {f  $\in$  succ(n)  $\rightarrow$  domain(R).  $\forall k \in n.$  <f'k, f'succ(k)>
 $\in$  R})"
  and RR_def:
    "RR  $\equiv$  {<z1,z2>:Fin(XX)*XX.
      (domain(z2)=succ( $\bigcup f \in z1.$  domain(f))
       $\wedge$  ( $\forall f \in z1.$  restrict(z2, domain(f)) = f))
      | ( $\neg$  ( $\exists g \in XX.$  domain(g)=succ( $\bigcup f \in z1.$  domain(f))
       $\wedge$  ( $\forall f \in z1.$  restrict(g, domain(f)) = f))  $\wedge$  z2={<0,x>})}"
  and allRR_def:
    "allRR  $\equiv$   $\forall b < \text{nat}.$ 
      <f' 'b, f' b>  $\in$ 
      {<z1,z2>:Fin(XX)*XX. (domain(z2)=succ( $\bigcup f \in z1.$  domain(f))
       $\wedge$  ( $\bigcup f \in z1.$  domain(f)) = b
       $\wedge$  ( $\forall f \in z1.$  restrict(z2, domain(f))
      = f))}"
begin

lemma lemma4:
  "[range(R)  $\subseteq$  domain(R); x  $\in$  domain(R)]
 $\implies$  RR  $\subseteq$  Pow(XX)*XX  $\wedge$ 
  ( $\forall Y \in \text{Pow}(XX).$  Y  $\prec$  nat  $\longrightarrow$  ( $\exists x \in XX.$  <Y,x>:RR))"
apply (rule conjI)
apply (force dest!: FinD [THEN PowI] simp add: RR_def)
apply (rule impI [THEN ballI])
apply (drule Finite_Fin [OF lesspoll_nat_is_Finite PowD], assumption)
apply (case_tac
  "  $\exists g \in XX.$  domain (g) =
    succ( $\bigcup f \in Y.$  domain(f))  $\wedge$  ( $\forall f \in Y.$  restrict(g, domain(f))
  = f)")
apply (simp add: RR_def, blast)
apply (safe del: domainE)
  unfolding XX_def RR_def
apply (rule rev_bexI, erule singleton_in_funs)
apply (simp add: nat_OI [THEN rev_bexI] cons_fun_type2)
done

lemma UN_image_succ_eq:
  "[f  $\in$  nat  $\rightarrow$  X; n  $\in$  nat]
 $\implies$  ( $\bigcup x \in f' \text{'succ(n)}. P(x)$ ) = P(f'n)  $\cup$  ( $\bigcup x \in f' \text{'n}. P(x)$ )"

```

```

by (simp add: image_fun OrdmemD)

lemma UN_image_succ_eq_succ:
  "[( $\bigcup x \in f' n. P(x) = y; P(f' n) = \text{succ}(y);$ 
     $f \in \text{nat} \rightarrow X; n \in \text{nat}$ )]  $\implies$  [ $\bigcup x \in f' \text{succ}(n). P(x) = \text{succ}(y)$ ]"
by (simp add: UN_image_succ_eq, blast)

lemma apply_domain_type:
  "[( $h \in \text{succ}(n) \rightarrow D; n \in \text{nat}; \text{domain}(h) = \text{succ}(y)$ )]  $\implies h' y \in D$ "
by (fast elim: apply_type dest!: trans [OF sym domain_of_fun])

lemma image_fun_succ:
  "[( $h \in \text{nat} \rightarrow X; n \in \text{nat}$ )]  $\implies h' \text{succ}(n) = \text{cons}(h' n, h' n)$ "
by (simp add: image_fun OrdmemD)

lemma f_n_type:
  "[( $\text{domain}(f' n) = \text{succ}(k); f \in \text{nat} \rightarrow XX; n \in \text{nat}$ )]
 $\implies f' n \in \text{succ}(k) \rightarrow \text{domain}(R)$ "
  unfolding XX_def
  apply (drule apply_type, assumption)
  apply (fast elim: domain_eq_imp_fun_type)
  done

lemma f_n_pairs_in_R [rule_format]:
  "[( $h \in \text{nat} \rightarrow XX; \text{domain}(h' n) = \text{succ}(k); n \in \text{nat}$ )]
 $\implies \forall i \in k. \langle h' n' i, h' n' \text{succ}(i) \rangle \in R$ "
  unfolding XX_def
  apply (drule apply_type, assumption)
  apply (elim UN_E CollectE)
  apply (drule domain_of_fun [symmetric, THEN trans], assumption, simp)
  done

lemma restrict_cons_eq_restrict:
  "[( $\text{restrict}(h, \text{domain}(u)) = u; h \in n \rightarrow X; \text{domain}(u) \subseteq n$ )]
 $\implies \text{restrict}(\text{cons}(\langle n, y \rangle, h), \text{domain}(u)) = u$ "
  unfolding restrict_def
  apply (simp add: restrict_def Pi_iff)
  apply (erule sym [THEN trans, symmetric])
  apply (blast elim: mem_irrefl)
  done

lemma all_in_image_restrict_eq:
  "[( $\forall x \in f' n. \text{restrict}(f' n, \text{domain}(x)) = x;$ 
     $f \in \text{nat} \rightarrow XX;$ 
     $n \in \text{nat}; \text{domain}(f' n) = \text{succ}(n);$ 
     $(\bigcup x \in f' n. \text{domain}(x)) \subseteq n$ )]
 $\implies \forall x \in f' \text{succ}(n). \text{restrict}(\text{cons}(\langle \text{succ}(n), y \rangle, f' n), \text{domain}(x))$ 
 $= x$ "
  apply (rule ballI)

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apply (simp add: image_fun_succ)
apply (drule f_n_type, assumption+)
apply (erule disjE)
  apply (simp add: domain_of_fun restrict_cons_eq)
apply (blast intro!: restrict_cons_eq_restrict)
done

lemma simplify_recursion:
  "⟦∀ b<nat. <f'`b, f'b> ∈ RR;
    f ∈ nat -> XX; range(R) ⊆ domain(R); x ∈ domain(R)⟧
  ⇒ allRR"
  unfolding RR_def allRR_def
apply (rule oallI, drule ltD)
apply (erule nat_induct)
apply (drule_tac x=0 in ospec, blast intro: Limit_has_0)
apply (force simp add: singleton_fun [THEN domain_of_fun] singleton_in_funs)

apply (simp only: separation split)
apply (drule_tac x="succ(xa)" in ospec, blast intro: ltI)
apply (elim conjE disjE)
apply (force elim!: trans subst_context
  intro!: UN_image_succ_eq_succ)
apply (erule notE)
apply (simp add: XX_def UN_image_succ_eq_succ)
apply (elim conjE bexE)
apply (drule apply_domain_type, assumption+)
apply (erule domainE)+
apply (frule f_n_type)
apply (simp add: XX_def, assumption+)
apply (rule rev_bexI, erule nat_succI)
apply (rename_tac m i j y z)
apply (rule_tac x = "cons(<succ(m), z>, f'm)" in bexI)
prefer 2 apply (blast intro: cons_fun_type2)
apply (rule conjI)
prefer 2 apply (fast del: ballI subsetI
  elim: trans [OF _ subst_context, THEN domain_cons_eq_succ]
  subst_context
  all_in_image_restrict_eq [simplified XX_def]
  trans equalityD1)

apply (rule ballI)
apply (erule succE)

  apply (simp add: cons_val_n cons_val_k)

apply (drule f_n_pairs_in_R [simplified XX_def, OF _ domain_of_fun],
  assumption, assumption, assumption)

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apply (simp add: nat_into_Ord [THEN succ_in_succ] succI2 cons_val_k)
done

lemma lemma2:
  "[[allRR; f ∈ nat->XX; range(R) ⊆ domain(R); x ∈ domain(R); n ∈
nat]]
  ⇒ f'n ∈ succ(n) -> domain(R) ∧ (∀ i ∈ n. <f'n'i, f'n'succ(i)>:R)"
  unfolding allRR_def
  apply (drule ospec)
  apply (erule ltI [OF _ Ord_nat])
  apply (erule CollectE, simp)
  apply (rule conjI)
  prefer 2 apply (fast elim!: f_n_pairs_in_R trans subst_context)
  unfolding XX_def
  apply (fast elim!: trans [THEN domain_eq_imp_fun_type] subst_context)
  done

lemma lemma3:
  "[[allRR; f ∈ nat->XX; n ∈ nat; range(R) ⊆ domain(R); x ∈ domain(R)]
  ⇒ f'n'n = f'succ(n)'n"
  apply (frule lemma2 [THEN conjunct1, THEN domain_of_fun], assumption+)
  unfolding allRR_def
  apply (drule ospec)
  apply (drule ltI [OF nat_succI Ord_nat], assumption, simp)
  apply (elim conjE ballE)
  apply (erule restrict_eq_imp_val_eq [symmetric], force)
  apply (simp add: image_fun OrdmemD)
  done

end

theorem DC_nat_imp_DC0: "DC(nat) ⇒ DC0"
  unfolding DC_def DC0_def
  apply (intro allI impI)
  apply (erule asm_rl conjE ex_in_domain [THEN exE] allE)+
  apply (erule impE [OF _ imp_DC0.lemma4], assumption+)
  apply (erule bexE)
  apply (drule imp_DC0.simplify_recursion, assumption+)
  apply (rule_tac x = "λn ∈ nat. f'n'n" in bexI)
  apply (rule_tac [2] lam_type)
  apply (erule_tac [2] apply_type [OF imp_DC0.lemma2 [THEN conjunct1] succI1])
  apply (rule ballI)
  apply (frule_tac n="succ(n)" in imp_DC0.lemma2,
  (assumption/erule nat_succI)+)
  apply (drule imp_DC0.lemma3, auto)
  done

```

```

lemma fun_Ord_inj:
  "[[f ∈ a->X; Ord(a);
    ∧ b c. [[b < c; c ∈ a]] ⇒ f' b ≠ f' c]]
  ⇒ f ∈ inj(a, X)"
apply (unfold inj_def, simp)
apply (intro ballI impI)
apply (rule_tac j=x in Ord_in_Ord [THEN Ord_linear_lt], assumption+)
apply (blast intro: Ord_in_Ord, auto)
apply (atomize, blast dest: not_sym)
done

lemma value_in_image: "[[f ∈ X->Y; A ⊆ X; a ∈ A]] ⇒ f'a ∈ f''A"
by (fast elim!: image_fun [THEN ssubst])

lemma lesspoll_lemma: "[[¬ A < B; C < B]] ⇒ A - C ≠ 0"
  unfolding lesspoll_def
apply (fast dest!: Diff_eq_0_iff [THEN iffD1, THEN subset_imp_lepoll]
  intro!: eqpollI elim: notE
  elim!: eqpollE lepoll_trans)
done

theorem DC_W03: "(∀ K. Card(K) → DC(K)) ⇒ W03"
  unfolding DC_def W03_def
apply (rule allI)
apply (case_tac "A < Hartog (A)")
apply (fast dest!: lesspoll_imp_ex_lt_eqpoll
  intro!: Ord_Hartog leI [THEN le_imp_subset])
apply (erule allE impE)+
apply (rule Card_Hartog)
apply (erule_tac x = A in allE)
apply (erule_tac x = "{⟨z1,z2⟩ ∈ Pow (A) *A . z1 < Hartog (A) ∧ z2 ∉
z1}"
  in allE)
apply simp
apply (erule impE, fast elim: lesspoll_lemma [THEN not_emptyE])
apply (erule bexE)
apply (rule Hartog_lepoll_selfE)
apply (rule lepoll_def [THEN def_imp_iff, THEN iffD2])
apply (rule exI, rule fun_Ord_inj, assumption, rule Ord_Hartog)
apply (drule value_in_image)
apply (drule OrdmemD, rule Ord_Hartog, assumption+, erule ltD)
apply (drule ospec)
apply (blast intro: ltI Ord_Hartog, force)
done

```

```

lemma images_eq:
  "⟦∀x ∈ A. f'x=g'x; f ∈ Df->Cf; g ∈ Dg->Cg; A ⊆ Df; A ⊆ Dg⟧
  ⇒ f' 'A = g' 'A"
apply (simp (no_asm_simp) add: image_fun)
done

lemma lam_images_eq:
  "⟦Ord(a); b ∈ a⟧ ⇒ (λx ∈ a. h(x))' 'b = (λx ∈ b. h(x))' 'b"
apply (rule images_eq)
  apply (rule ballI)
  apply (drule OrdmemD [THEN subsetD], assumption+)
  apply simp
  apply (fast elim!: RepFunI OrdmemD intro!: lam_type)+
done

lemma lam_type_RepFun: "(λb ∈ a. h(b)) ∈ a -> {h(b). b ∈ a}"
by (fast intro!: lam_type RepFunI)

lemma lemmaX:
  "⟦∀Y ∈ Pow(X). Y < K ⟶ (∃x ∈ X. ⟨Y, x⟩ ∈ R);
  b ∈ K; Z ∈ Pow(X); Z < K⟧
  ⇒ {x ∈ X. ⟨Z, x⟩ ∈ R} ≠ 0"
by blast

lemma W01_DC_lemma:
  "⟦Card(K); well_ord(X, Q);
  ∀Y ∈ Pow(X). Y < K ⟶ (∃x ∈ X. ⟨Y, x⟩ ∈ R); b ∈ K⟧
  ⇒ ff(b, X, Q, R) ∈ {x ∈ X. <(λc ∈ b. ff(c, X, Q, R))' 'b, x> ∈
  R}"
apply (rule_tac P = "b ∈ K" in impE, (erule_tac [2] asm_rl)+)
apply (rule_tac i=b in Card_is_Ord [THEN Ord_in_Ord, THEN trans_induct],
  assumption+)
apply (rule impI)
apply (rule ff_def [THEN def_transrec, THEN ssubst])
apply (erule the_first_in, fast)
apply (simp add: image_fun [OF lam_type_RepFun subset_refl])
apply (erule lemmaX, assumption)
  apply (blast intro: Card_is_Ord OrdmemD [THEN subsetD])
apply (blast intro: lesspoll_trans1 in_Card_imp_lesspoll RepFun_1epoll)
done

theorem W01_DC_Card: "W01 ⟹ ∀K. Card(K) ⟶ DC(K)"
  unfolding DC_def W01_def

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apply (rule allI impI)+
apply (erule allE exE conjE)+
apply (rule_tac x = " $\lambda b \in K. \text{ff } (b, X, Ra, R)$ " in bexI)
  apply (simp add: lam_images_eq [OF Card_is_Ord ltD])
  apply (fast elim!: ltE W01_DC_lemma [THEN CollectD2])
apply (rule_tac lam_type)
apply (rule W01_DC_lemma [THEN CollectD1], assumption+)
done

end

```

References

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- [2] Herman Rubin and Jean E. Rubin. *Equivalents of the Axiom of Choice, II*. North-Holland, 1985.