Equivalents of the Axiom of Choice

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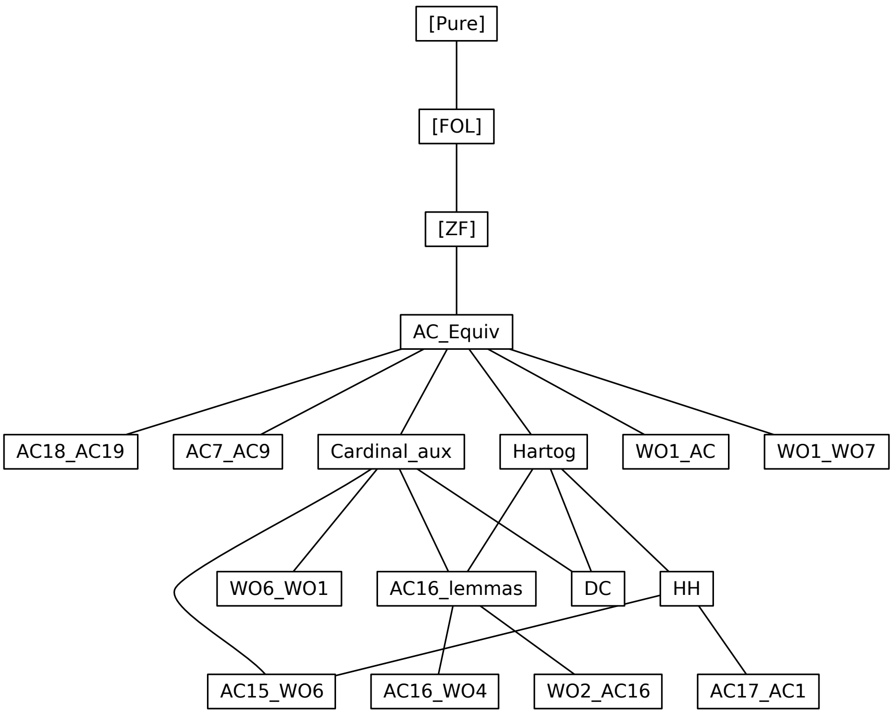
Abstract

This development [1] proves the equivalence of seven formulations of the well-ordering theorem and twenty formulations of the axiom of choice. It formalizes the first two chapters of the monograph *Equivalents of the Axiom of Choice* by Rubin and Rubin [2]. Some of this mmaterial involves extremely complex techniques.

Contents



theory *AC_Equiv*
imports *ZF*
begin

definition
```
"WO1 ≡ ∀A. ∃R. well_ord(A,R)"
```

definition
```
"WO2 ≡ ∀A. ∃a. Ord(a) ∧ A≈a"
```

definition
```
"WO3 ≡ ∀A. ∃a. Ord(a) ∧ (∃b. b ⊆ a ∧ A≈b)"
```

definition
```
"WO4(m) ≡ ∀A. ∃a f. Ord(a) ∧ domain(f)=a ∧
                   (⋃b<a. f`b) = A ∧ (∀b<a. f`b ≲ m)"
```

definition
```
"WO5 ≡ ∃m ∈ nat. 1≤m ∧ WO4(m)"
```

definition
```
"WO6 ≡ ∀A. ∃m ∈ nat. 1≤m ∧ (∃a f. Ord(a) ∧ domain(f)=a
                        ∧ (⋃b<a. f`b) = A ∧ (∀b<a. f`b ≲ m))"
```

definition
```
"WO7 ≡ ∀A. Finite(A) ⟷ (∀R. well_ord(A,R) ⟶ well_ord(A,converse(R)))"
```

definition
```
"WO8 ≡ ∀A. (∃f. f ∈ (∏X ∈ A. X)) ⟶ (∃R. well_ord(A,R))"
```

definition

pairwise_disjoint :: *"i ⇒ o"* **where**
```
"pairwise_disjoint(A) ≡ ∀A1 ∈ A. ∀A2 ∈ A. A1 ∩ A2 ≠ 0 ⟶ A1=A2"
```

definition
sets_of_size_between :: *"[i, i, i] ⇒ o"* **where**
```
"sets_of_size_between(A,m,n) ≡ ∀B ∈ A. m ≲ B ∧ B ≲ n"
```

definition
```
"AC0 ≡ ∀A. ∃f. f ∈ (∏X ∈ Pow(A)-{0}. X)"
```

definition
```
"AC1 ≡ ∀A. 0∉A ⟶ (∃f. f ∈ (∏X ∈ A. X))"
```

definition
```
"AC2 ≡ ∀A. 0∉A ∧ pairwise_disjoint(A)
                ⟶ (∃C. ∀B ∈ A. ∃y. B ∩ C = {y})"
```
definition
```
"AC3 ≡ ∀A B. ∀f ∈ A->B. ∃g. g ∈ (∏x ∈ {a ∈ A. f`a≠0}. f`x)"
```

definition
```
"AC4 ≡ ∀R A B. (R ⊆ A*B ⟶ (∃f. f ∈ (∏x ∈ domain(R). R``{x})))"
```

definition
```
"AC5 ≡ ∀A B. ∀f ∈ A->B. ∃g ∈ range(f)->A. ∀x ∈ domain(g). f`(g`x)
= x"
```

definition
```
"AC6 ≡ ∀A. 0∉A ⟶ (∏B ∈ A. B)≠0"
```

definition
```
"AC7 ≡ ∀A. 0∉A ∧ (∀B1 ∈ A. ∀B2 ∈ A. B1≈B2) ⟶ (∏B ∈ A. B) ≠
0"
```

definition
```
"AC8 ≡ ∀A. (∀B ∈ A. ∃B1 B2. B=⟨B1,B2⟩ ∧ B1≈B2)
                ⟶ (∃f. ∀B ∈ A. f`B ∈ bij(fst(B),snd(B)))"
```

definition
```
"AC9 ≡ ∀A. (∀B1 ∈ A. ∀B2 ∈ A. B1≈B2) ⟶
                (∃f. ∀B1 ∈ A. ∀B2 ∈ A. f`⟨B1,B2⟩ ∈ bij(B1,B2))"
```

definition
```
"AC10(n) ≡  ∀A. (∀B ∈ A. ¬Finite(B)) ⟶
                (∃f. ∀B ∈ A. (pairwise_disjoint(f`B) ∧
                sets_of_size_between(f`B, 2, succ(n)) ∧ ⋃(f`B)=B))"
```

definition
```
"AC11 ≡ ∃n ∈ nat. 1≤n ∧ AC10(n)"
```

definition
```
"AC12 ≡ ∀A. (∀B ∈ A. ¬Finite(B)) ⟶
              (∃n ∈ nat. 1≤n ∧ (∃f. ∀B ∈ A. (pairwise_disjoint(f`B)
∧
                 sets_of_size_between(f`B, 2, succ(n)) ∧ ⋃(f`B)=B)))"
```

definition
```
"AC13(m) ≡ ∀A. 0∉A ⟶ (∃f. ∀B ∈ A. f`B≠0 ∧ f`B ⊆ B ∧ f`B ≲
m)"
```

definition
"$AC14 \equiv \exists m \in nat.\ 1 \leq m \wedge AC13(m)$"

definition
"$AC15 \equiv \forall A.\ 0 \notin A \longrightarrow$
$(\exists m \in nat.\ 1 \leq m \wedge (\exists f.\ \forall B \in A.\ f`B \neq 0 \wedge f`B \subseteq B \wedge$
$f`B \lesssim m))$"

definition
"$AC16(n, k) \equiv$
$\forall A.\ \neg Finite(A) \longrightarrow$
$(\exists T.\ T \subseteq \{X \in Pow(A).\ X \approx succ(n)\} \wedge$
$(\forall X \in \{X \in Pow(A).\ X \approx succ(k)\}.\ \exists !\ Y.\ Y \in T \wedge X \subseteq Y))$"

definition
"$AC17 \equiv \forall A.\ \forall g \in (Pow(A)-\{0\} \text{ -> } A) \text{ -> } Pow(A)-\{0\}.$
$\exists f \in Pow(A)-\{0\} \text{ -> } A.\ f`(g`f) \in g`f$"

locale *AC18* =
assumes *AC18*: "$A \neq 0 \wedge (\forall a \in A.\ B(a) \neq 0) \longrightarrow$
$((\bigcap a \in A.\ \bigcup b \in B(a).\ X(a,b)) =$
$(\bigcup f \in \prod a \in A.\ B(a).\ \bigcap a \in A.\ X(a, f`a)))$"
— AC18 cannot be expressed within the object-logic

definition
"$AC19 \equiv \forall A.\ A \neq 0 \wedge 0 \notin A \longrightarrow ((\bigcap a \in A.\ \bigcup b \in a.\ b) =$
$(\bigcup f \in (\prod B \in A.\ B).\ \bigcap a \in A.\ f`a))$"

lemma *rvimage_id*: "$rvimage(A,id(A),r) = r \cap A*A$"
⟨*proof*⟩

lemma *ordertype_Int*:
"$well\_ord(A,r) \Longrightarrow ordertype(A,\ r \cap A*A) = ordertype(A,r)$"
⟨*proof*⟩

lemma *lam_sing_bij*: "$(\lambda x \in A.\ \{x\}) \in bij(A,\ \{\{x\}.\ x \in A\})$"
⟨*proof*⟩

lemma *inj_strengthen_type*:
"$[\![f \in inj(A, B);\ \bigwedge a.\ a \in A \Longrightarrow f`a \in C]\!] \Longrightarrow f \in inj(A,C)$"
⟨*proof*⟩

lemma *ex1_two_eq:* "⟦∃! x. P(x); P(x); P(y)⟧ ⟹ x=y"
⟨*proof*⟩

lemma *first_in_B:*
"⟦well_ord(⋃(A),r); 0 ∉ A; B ∈ A⟧ ⟹ (THE b. first(b,B,r)) ∈ B"
⟨*proof*⟩

lemma *ex_choice_fun:* "⟦well_ord(⋃(A), R); 0 ∉ A⟧ ⟹ ∃f. f ∈ (∏X ∈ A. X)"
⟨*proof*⟩

lemma *ex_choice_fun_Pow:* "well_ord(A, R) ⟹ ∃f. f ∈ (∏X ∈ Pow(A)-{0}. X)"
⟨*proof*⟩

lemma *lepoll_m_imp_domain_lepoll_m:*
"⟦m ∈ nat; u ≲ m⟧ ⟹ domain(u) ≲ m"
⟨*proof*⟩

lemma *rel_domain_ex1:*
"⟦succ(m) ≲ domain(r); r ≲ succ(m); m ∈ nat⟧ ⟹ function(r)"
⟨*proof*⟩

lemma *rel_is_fun:*
"⟦succ(m) ≲ domain(r); r ≲ succ(m); m ∈ nat;
r ⊆ A*B; A=domain(r)⟧ ⟹ r ∈ A->B"
⟨*proof*⟩

end

theory *Cardinal_aux* **imports** *AC_Equiv* **begin**

lemma *Diff_lepoll:* "$\llbracket$A $\lesssim$ succ(m); B $\subseteq$ A; B$\neq$0$\rrbracket$ $\Longrightarrow$ A-B $\lesssim$ m"
$\langle$*proof*$\rangle$

lemma *lepoll_imp_ex_le_eqpoll:*
"$\llbracket$A $\lesssim$ i; Ord(i)$\rrbracket$ $\Longrightarrow$ $\exists$j. j $\leq$ i $\wedge$ A $\approx$ j"
$\langle$*proof*$\rangle$

lemma *lesspoll_imp_ex_lt_eqpoll:*
"$\llbracket$A $\prec$ i; Ord(i)$\rrbracket$ $\Longrightarrow$ $\exists$j. j<i $\wedge$ A $\approx$ j"
$\langle$*proof*$\rangle$

lemma *Un_eqpoll_Inf_Ord:*
assumes *A:* "A $\approx$ i" **and** *B:* "B $\approx$ i" **and** *NFI:* "$\neg$ Finite(i)" **and** *i:* "Ord(i)"
shows "A $\cup$ B $\approx$ i"
$\langle$*proof*$\rangle$

schematic_goal *paired_bij:* "?f $\in$ bij({{y,z}. y $\in$ x}, x)"
$\langle$*proof*$\rangle$

lemma *paired_eqpoll:* "{{y,z}. y $\in$ x} $\approx$ x"
$\langle$*proof*$\rangle$

lemma *ex_eqpoll_disjoint:* "$\exists$B. B $\approx$ A $\wedge$ B $\cap$ C = 0"
$\langle$*proof*$\rangle$

lemma *Un_lepoll_Inf_Ord:*
"$\llbracket$A $\lesssim$ i; B $\lesssim$ i; $\neg$Finite(i); Ord(i)$\rrbracket$ $\Longrightarrow$ A $\cup$ B $\lesssim$ i"
$\langle$*proof*$\rangle$

lemma *Least_in_Ord:* "$\llbracket$P(i); i $\in$ j; Ord(j)$\rrbracket$ $\Longrightarrow$ (μ i. P(i)) $\in$ j"
$\langle$*proof*$\rangle$

lemma *Diff_first_lepoll:*
"$\llbracket$well_ord(x,r); y $\subseteq$ x; y $\lesssim$ succ(n); n $\in$ nat$\rrbracket$
$\Longrightarrow$ y - {THE b. first(b,y,r)} $\lesssim$ n"
$\langle$*proof*$\rangle$

lemma *UN_subset_split:*
"$(\bigcup x \in X.\ P(x)) \subseteq (\bigcup x \in X.\ P(x)\text{-}Q(x)) \cup (\bigcup x \in X.\ Q(x))$"
⟨*proof*⟩

lemma *UN_sing_lepoll:* "$Ord(a) \Longrightarrow (\bigcup x \in a.\ \{P(x)\}) \lesssim a$"
⟨*proof*⟩

lemma *UN_fun_lepoll_lemma [rule_format]:*
"$[\![well\_ord(T, R);\ \neg Finite(a);\ Ord(a);\ n \in nat]\!]$
$\Longrightarrow \forall f.\ (\forall b \in a.\ f`b \lesssim n \wedge f`b \subseteq T) \longrightarrow (\bigcup b \in a.\ f`b) \lesssim a$"
⟨*proof*⟩

lemma *UN_fun_lepoll:*
"$[\![\forall b \in a.\ f`b \lesssim n \wedge f`b \subseteq T;\ well\_ord(T, R);$
$\neg Finite(a);\ Ord(a);\ n \in nat]\!] \Longrightarrow (\bigcup b \in a.\ f`b) \lesssim a$"
⟨*proof*⟩

lemma *UN_lepoll:*
"$[\![\forall b \in a.\ F(b) \lesssim n \wedge F(b) \subseteq T;\ well\_ord(T, R);$
$\neg Finite(a);\ Ord(a);\ n \in nat]\!]$
$\Longrightarrow (\bigcup b \in a.\ F(b)) \lesssim a$"
⟨*proof*⟩

lemma *UN_eq_UN_Diffs:*
"$Ord(a) \Longrightarrow (\bigcup b \in a.\ F(b)) = (\bigcup b \in a.\ F(b) - (\bigcup c \in b.\ F(c)))$"
⟨*proof*⟩

lemma *lepoll_imp_eqpoll_subset:*
"$a \lesssim X \Longrightarrow \exists Y.\ Y \subseteq X \wedge a \approx Y$"
⟨*proof*⟩

lemma *Diff_lesspoll_eqpoll_Card_lemma:*
"$[\![A{\approx}a;\ \neg Finite(a);\ Card(a);\ B \prec a;\ A\text{-}B \prec a]\!] \Longrightarrow P$"
⟨*proof*⟩

lemma *Diff_lesspoll_eqpoll_Card:*
"$[\![A \approx a;\ \neg Finite(a);\ Card(a);\ B \prec a]\!] \Longrightarrow A - B \approx a$"
⟨*proof*⟩

end

theory *WO6_WO1*
imports *Cardinal_aux*
begin

definition
```
  NN  :: "i ⇒ i"  where
    "NN(y) ≡ {m ∈ nat. ∃a. ∃f. Ord(a) ∧ domain(f)=a  ∧
                     (⋃b<a. f`b) = y ∧ (∀b<a. f`b ≲ m)}"
```

definition
```
  uu  :: "[i, i, i, i] ⇒ i"  where
    "uu(f, beta, gamma, delta) ≡ (f`beta * f`gamma) ∩ f`delta"
```

definition
```
  vv1 :: "[i, i, i] ⇒ i"  where
     "vv1(f,m,b) ≡
           let g = μ g. (∃d. Ord(d) ∧ (domain(uu(f,b,g,d)) ≠ 0 ∧
                                  domain(uu(f,b,g,d)) ≲ m));
               d = μ d. domain(uu(f,b,g,d)) ≠ 0 ∧
                             domain(uu(f,b,g,d)) ≲ m
           in  if f`b ≠ 0 then domain(uu(f,b,g,d)) else 0"
```

definition
```
  ww1 :: "[i, i, i] ⇒ i"  where
     "ww1(f,m,b) ≡ f`b - vv1(f,m,b)"
```

definition
```
  gg1 :: "[i, i, i] ⇒ i"  where
     "gg1(f,a,m) ≡ λb ∈ a++a. if b<a then vv1(f,m,b) else ww1(f,m,b--a)"
```

definition
```
  vv2 :: "[i, i, i, i] ⇒ i"  where
     "vv2(f,b,g,s) ≡
           if f`g ≠ 0 then {uu(f, b, g, μ d. uu(f,b,g,d) ≠ 0)`s} else
0"
```

definition
```
  ww2 :: "[i, i, i, i] ⇒ i"  where
     "ww2(f,b,g,s) ≡ f`g - vv2(f,b,g,s)"
```

definition
```
  gg2 :: "[i, i, i, i] ⇒ i"  where
     "gg2(f,a,b,s) ≡
              λg ∈ a++a. if g<a then vv2(f,b,g,s) else ww2(f,b,g--a,s)"
```

lemma `WO2_WO3: "WO2 ⟹ WO3"`

⟨proof⟩

lemma *WO3_WO1: "WO3 ⟹ WO1"*
⟨proof⟩

lemma *WO1_WO2: "WO1 ⟹ WO2"*
⟨proof⟩

lemma *lam_sets: "f ∈ A->B ⟹ (λx ∈ A. {f‘x}): A -> {{b}. b ∈ B}"*
⟨proof⟩

lemma *surj_imp_eq': "f ∈ surj(A,B) ⟹ (⋃a ∈ A. {f‘a}) = B"*
⟨proof⟩

lemma *surj_imp_eq: "⟦f ∈ surj(A,B); Ord(A)⟧ ⟹ (⋃a<A. {f‘a}) = B"*
⟨proof⟩

lemma *WO1_WO4: "WO1 ⟹ WO4(1)"*
⟨proof⟩

lemma *WO4_mono: "⟦m≤n; WO4(m)⟧ ⟹ WO4(n)"*
⟨proof⟩

lemma *WO4_WO5: "⟦m ∈ nat; 1≤m; WO4(m)⟧ ⟹ WO5"*
⟨proof⟩

lemma *WO5_WO6: "WO5 ⟹ WO6"*
⟨proof⟩

lemma *lt_oadd_odiff_disj:*
"⟦k < i++j; Ord(i); Ord(j)⟧
⟹ k < i | (¬ k<i ∧ k = i ++ (k--i) ∧ (k--i)<j)"
⟨proof⟩

lemma *domain_uu_subset:* "$domain(uu(f,b,g,d)) \subseteq f`b$"
⟨*proof*⟩

lemma *quant_domain_uu_lepoll_m:*
"$\forall b<a.\ f`b \lesssim m \Longrightarrow \forall b<a.\ \forall g<a.\ \forall d<a.\ domain(uu(f,b,g,d)) \lesssim m$"
⟨*proof*⟩

lemma *uu_subset1:* "$uu(f,b,g,d) \subseteq f`b * f`g$"
⟨*proof*⟩

lemma *uu_subset2:* "$uu(f,b,g,d) \subseteq f`d$"
⟨*proof*⟩

lemma *uu_lepoll_m:* "$[\![\forall b<a.\ f`b \lesssim m;\ \ d<a]\!] \Longrightarrow uu(f,b,g,d) \lesssim m$"
⟨*proof*⟩

lemma *cases:*
"$\forall b<a.\ \forall g<a.\ \forall d<a.\ u(f,b,g,d) \lesssim m$
$\Longrightarrow (\forall b<a.\ f`b \neq 0 \longrightarrow$
$(\exists g<a.\ \exists d<a.\ u(f,b,g,d) \neq 0 \wedge u(f,b,g,d) \prec m))$
$|\ (\exists b<a.\ f`b \neq 0 \wedge (\forall g<a.\ \forall d<a.\ u(f,b,g,d) \neq 0 \longrightarrow$
$u(f,b,g,d) \approx m))$"
⟨*proof*⟩

lemma *UN_oadd:* "$Ord(a) \Longrightarrow (\bigcup b<a++a.\ C(b)) = (\bigcup b<a.\ C(b) \cup C(a++b))$"
⟨*proof*⟩

lemma *vv1_subset:* "$vv1(f,m,b) \subseteq f`b$"
⟨*proof*⟩

lemma *UN_gg1_eq:*

```
"⟦Ord(a);  m ∈ nat⟧ ⟹ (⋃b<a++a. gg1(f,a,m)`b) = (⋃b<a. f`b)"
```

⟨*proof*⟩

lemma *domain_gg1:* `"domain(gg1(f,a,m)) = a++a"`

⟨*proof*⟩

lemma *nested_LeastI:*

```
"⟦P(a, b);  Ord(a);  Ord(b);
    Least_a = (μ a. ∃x. Ord(x) ∧ P(a, x))⟧
 ⟹ P(Least_a, μ b. P(Least_a, b))"
```

⟨*proof*⟩

lemmas *nested_Least_instance =*

```
nested_LeastI [of "λg d. domain(uu(f,b,g,d)) ≠ 0 ∧
                    domain(uu(f,b,g,d)) ≲ m"] for f b m
```

lemma *gg1_lepoll_m:*

```
"⟦Ord(a);  m ∈ nat;
    ∀b<a. f`b ≠0 ⟶
        (∃g<a. ∃d<a. domain(uu(f,b,g,d)) ≠ 0  ∧
                   domain(uu(f,b,g,d)) ≲ m);
    ∀b<a. f`b ≲ succ(m);  b<a++a⟧
 ⟹ gg1(f,a,m)`b ≲ m"
```

⟨*proof*⟩

lemma *ex_d_uu_not_empty:*

```
"⟦b<a;  g<a;  f`b≠0;  f`g≠0;
    y*y ⊆ y;  (⋃b<a. f`b)=y⟧
 ⟹ ∃d<a. uu(f,b,g,d) ≠ 0"
```

⟨*proof*⟩

lemma *uu_not_empty:*

```
"⟦b<a; g<a; f`b≠0; f`g≠0;  y*y ⊆ y;  (⋃b<a. f`b)=y⟧
```

⟹ uu(f,b,g,μ d. (uu(f,b,g,d) ≠ 0)) ≠ 0"
⟨*proof*⟩

lemma *not_empty_rel_imp_domain: "⟦r ⊆ A*B; r≠0⟧ ⟹ domain(r)≠0"*
⟨*proof*⟩

lemma *Least_uu_not_empty_lt_a:*
"⟦b<a; g<a; f‘b≠0; f‘g≠0; y*y ⊆ y; (⋃b<a. f‘b)=y⟧
⟹ (μ d. uu(f,b,g,d) ≠ 0) < a"
⟨*proof*⟩

lemma *subset_Diff_sing: "⟦B ⊆ A; a∉B⟧ ⟹ B ⊆ A-{a}"*
⟨*proof*⟩

lemma *supset_lepoll_imp_eq:*
"⟦A ≲ m; m ≲ B; B ⊆ A; m ∈ nat⟧ ⟹ A=B"
⟨*proof*⟩

lemma *uu_Least_is_fun:*
"⟦∀g<a. ∀d<a. domain(uu(f, b, g, d))≠0 ⟶
domain(uu(f, b, g, d)) ≈ succ(m);
∀b<a. f‘b ≲ succ(m); y*y ⊆ y;
(⋃b<a. f‘b)=y; b<a; g<a; d<a;
f‘b≠0; f‘g≠0; m ∈ nat; s ∈ f‘b⟧
⟹ uu(f, b, g, μ d. uu(f,b,g,d)≠0) ∈ f‘b -> f‘g"
⟨*proof*⟩

lemma *vv2_subset:*
"⟦∀g<a. ∀d<a. domain(uu(f, b, g, d))≠0 ⟶
domain(uu(f, b, g, d)) ≈ succ(m);
∀b<a. f‘b ≲ succ(m); y*y ⊆ y;
(⋃b<a. f‘b)=y; b<a; g<a; m ∈ nat; s ∈ f‘b⟧
⟹ vv2(f,b,g,s) ⊆ f‘g"
⟨*proof*⟩

lemma *UN_gg2_eq:*
"⟦∀g<a. ∀d<a. domain(uu(f,b,g,d)) ≠ 0 ⟶
domain(uu(f,b,g,d)) ≈ succ(m);
∀b<a. f‘b ≲ succ(m); y*y ⊆ y;
(⋃b<a. f‘b)=y; Ord(a); m ∈ nat; s ∈ f‘b; b<a⟧
⟹ (⋃g<a++a. gg2(f,a,b,s) ‘ g) = y"
⟨*proof*⟩

lemma *domain_gg2: "domain(gg2(f,a,b,s)) = a++a"*
⟨*proof*⟩

lemma *vv2_lepoll*: "$[\![m \in nat;\ m\neq 0]\!] \Longrightarrow vv2(f,b,g,s) \lesssim m$"
  ⟨*proof*⟩

lemma *ww2_lepoll*:
"$[\![\forall b<a.\ f`b \lesssim succ(m);\ \ g<a;\ \ m \in nat;\ \ vv2(f,b,g,d) \subseteq f`g]\!]$
$\Longrightarrow ww2(f,b,g,d) \lesssim m$"
  ⟨*proof*⟩

lemma *gg2_lepoll_m*:
"$[\![\forall g<a.\ \forall d<a.\ domain(uu(f,b,g,d)) \neq 0 \longrightarrow$
$domain(uu(f,b,g,d)) \approx succ(m);$
$\forall b<a.\ f`b \lesssim succ(m);\ \ y*y \subseteq y;$
$(\bigcup b<a.\ f`b)=y;\ \ b<a;\ \ s \in f`b;\ \ m \in nat;\ \ m\neq 0;\ \ g<a{+}{+}a]\!]$
$\Longrightarrow gg2(f,a,b,s)\ `\ g \lesssim m$"
⟨*proof*⟩

lemma *lemma_ii*: "$[\![succ(m) \in NN(y);\ y*y \subseteq y;\ m \in nat;\ m\neq 0]\!] \Longrightarrow m \in NN(y)$"
  ⟨*proof*⟩

lemma *z_n_subset_z_succ_n*:
"$\forall n \in nat.\ rec(n,\ x,\ \lambda k\ r.\ r \cup r*r) \subseteq rec(succ(n),\ x,\ \lambda k\ r.\ r \cup r*r)$"
⟨*proof*⟩

lemma *le_subsets*:
"$[\![\forall n \in nat.\ f(n)<=f(succ(n));\ n\leq m;\ n \in nat;\ m \in nat]\!]$
$\Longrightarrow f(n)<=f(m)$"
⟨*proof*⟩

lemma *le_imp_rec_subset*:

"⟦n≤m; m ∈ nat⟧
⟹ rec(n, x, λk r. r ∪ r*r) ⊆ rec(m, x, λk r. r ∪ r*r)"
⟨*proof*⟩

lemma *lemma_iv: "∃y. x ∪ y*y ⊆ y"*
⟨*proof*⟩

lemma *WO6_imp_NN_not_empty: "WO6 ⟹ NN(y) ≠ 0"*
⟨*proof*⟩

lemma *lemma1:*
"⟦(⋃b<a. f‘b)=y; x ∈ y; ∀b<a. f‘b ≲ 1; Ord(a)⟧ ⟹ ∃c<a. f‘c = {x}"
⟨*proof*⟩

lemma *lemma2:*
"⟦(⋃b<a. f‘b)=y; x ∈ y; ∀b<a. f‘b ≲ 1; Ord(a)⟧
⟹ f‘ (μ i. f‘i = {x}) = {x}"
⟨*proof*⟩

lemma *NN_imp_ex_inj: "1 ∈ NN(y) ⟹ ∃a f. Ord(a) ∧ f ∈ inj(y, a)"*
⟨*proof*⟩

lemma *y_well_ord: "⟦y*y ⊆ y; 1 ∈ NN(y)⟧ ⟹ ∃r. well_ord(y, r)"*
⟨*proof*⟩

lemma *rev_induct_lemma [rule_format]:*
"$[\![n \in nat; \bigwedge m.\ [\![m \in nat; m\neq 0; P(succ(m))]\!] \Longrightarrow P(m)]\!]$
$\Longrightarrow n\neq 0 \longrightarrow P(n) \longrightarrow P(1)$"
⟨*proof*⟩

lemma *rev_induct:*
"$[\![n \in nat;\ \ P(n);\ \ n\neq 0;$
$\bigwedge m.\ [\![m \in nat; m\neq 0; P(succ(m))]\!] \Longrightarrow P(m)]\!]$
$\Longrightarrow P(1)$"
⟨*proof*⟩

lemma *NN_into_nat:* "$n \in NN(y) \Longrightarrow n \in nat$"
⟨*proof*⟩

lemma *lemma3:* "$[\![n \in NN(y); y*y \subseteq y; n\neq 0]\!] \Longrightarrow 1 \in NN(y)$"
⟨*proof*⟩

lemma *NN_y_0:* "$0 \in NN(y) \Longrightarrow y=0$"
⟨*proof*⟩

lemma *WO6_imp_WO1:* "$WO6 \Longrightarrow WO1$"
⟨*proof*⟩

end

theory *WO1_WO7*
imports *AC_Equiv*
begin

definition
"$LEMMA \equiv$
$\forall X.\ \neg Finite(X) \longrightarrow (\exists R.\ well\_ord(X,R) \land \neg well\_ord(X,converse(R)))$"

lemma *WO7_iff_LEMMA:* "$WO7 \longleftrightarrow LEMMA$"
⟨*proof*⟩

lemma *LEMMA_imp_WO1: "LEMMA ⟹ WO1"*
⟨*proof*⟩

lemma *converse_Memrel_not_wf_on:*
"⟦Ord(a); ¬Finite(a)⟧ ⟹ ¬wf[a](converse(Memrel(a)))"
⟨*proof*⟩

lemma *converse_Memrel_not_well_ord:*
"⟦Ord(a); ¬Finite(a)⟧ ⟹ ¬well_ord(a,converse(Memrel(a)))"
⟨*proof*⟩

lemma *well_ord_rvimage_ordertype:*
"well_ord(A,r) ⟹
rvimage (ordertype(A,r), converse(ordermap(A,r)),r) =
Memrel(ordertype(A,r))"
⟨*proof*⟩

lemma *well_ord_converse_Memrel:*
"⟦well_ord(A,r); well_ord(A,converse(r))⟧
⟹ well_ord(ordertype(A,r), converse(Memrel(ordertype(A,r))))"

⟨*proof*⟩

lemma *WO1_imp_LEMMA: "WO1 ⟹ LEMMA"*
⟨*proof*⟩

lemma *WO1_iff_WO7: "WO1 ⟷ WO7"*
⟨*proof*⟩

lemma *WO1_WO8: "WO1 ⟹ WO8"*
⟨*proof*⟩

lemma *WO8_WO1: "WO8 ⟹ WO1"*
⟨*proof*⟩

end

theory *AC7_AC9*
imports *AC_Equiv*
begin

lemma *Sigma_fun_space_not0: "⟦0∉A; B ∈ A⟧ ⟹ (nat->⋃(A)) * B ≠ 0"*
⟨*proof*⟩

lemma *inj_lemma:*

```
"C ∈ A ⟹ (λg ∈ (nat->⋃(A))*C.
          (λn ∈ nat. if(n=0, snd(g), fst(g)`(n #- 1))))
         ∈ inj((nat->⋃(A))*C, (nat->⋃(A)) ) "
```

⟨*proof*⟩

lemma *Sigma_fun_space_eqpoll:*

```
"⟦C ∈ A; 0∉A⟧ ⟹ (nat->⋃(A)) * C ≈ (nat->⋃(A))"
```

⟨*proof*⟩

lemma *AC6_AC7: "AC6 ⟹ AC7"*
⟨*proof*⟩

lemma *lemma1_1: "y ∈ (∏B ∈ A. Y*B) ⟹ (λB ∈ A. snd(y`B)) ∈ (∏B ∈*

A. B)"
⟨*proof*⟩

lemma *lemma1_2:*
*"y ∈ (∏B ∈ {Y*C. C ∈ A}. B) ⟹ (λB ∈ A. y‘(Y*B)) ∈ (∏B ∈ A. Y*B)"*
⟨*proof*⟩

lemma *AC7_AC6_lemma1:*
*"(∏B ∈ {(nat->⋃(A))*C. C ∈ A}. B) ≠ 0 ⟹ (∏B ∈ A. B) ≠ 0"*
⟨*proof*⟩

lemma *AC7_AC6_lemma2: "0 ∉ A ⟹ 0 ∉ {(nat -> ⋃(A)) * C. C ∈ A}"*
⟨*proof*⟩

lemma *AC7_AC6: "AC7 ⟹ AC6"*
⟨*proof*⟩

lemma *AC1_AC8_lemma1:*
"∀B ∈ A. ∃B1 B2. B=⟨B1,B2⟩ ∧ B1 ≈ B2
⟹ 0 ∉ { bij(fst(B),snd(B)). B ∈ A }"
⟨*proof*⟩

lemma *AC1_AC8_lemma2:*
"⟦f ∈ (∏X ∈ RepFun(A,p). X); D ∈ A⟧ ⟹ (λx ∈ A. f‘p(x))‘D ∈ p(D)"
⟨*proof*⟩

lemma *AC1_AC8: "AC1 ⟹ AC8"*
⟨*proof*⟩

lemma *AC8_AC9_lemma:*
"∀B1 ∈ A. ∀B2 ∈ A. B1 ≈ B2
*⟹ ∀B ∈ A*A. ∃B1 B2. B=⟨B1,B2⟩ ∧ B1 ≈ B2"*
⟨*proof*⟩

lemma *AC8_AC9: "AC8 ⟹ AC9"*

⟨proof⟩

lemma *snd_lepoll_SigmaI: "b ∈ B ⟹ X ≲ B × X"*
⟨proof⟩

lemma *nat_lepoll_lemma:*
"⟦0 ∉ A; B ∈ A⟧ ⟹ nat ≲ ((nat → ⋃(A)) × B) × nat"
⟨proof⟩

lemma *AC9_AC1_lemma1:*
"⟦0∉A; A≠0;
*C = {((nat->⋃(A))*B)*nat. B ∈ A} ∪*
*{cons(0,((nat->⋃(A))*B)*nat). B ∈ A};*
B1 ∈ C; B2 ∈ C⟧
⟹ B1 ≈ B2"
⟨proof⟩

lemma *AC9_AC1_lemma2:*
*"∀B1 ∈ {(F*B)*N. B ∈ A} ∪ {cons(0,(F*B)*N). B ∈ A}.*
*∀B2 ∈ {(F*B)*N. B ∈ A} ∪ {cons(0,(F*B)*N). B ∈ A}.*
f‘⟨B1,B2⟩ ∈ bij(B1, B2)
*⟹ (λB ∈ A. snd(fst((f‘<cons(0,(F*B)*N),(F*B)*N>)‘0))) ∈ (∏X ∈ A. X)"*
⟨proof⟩

lemma *AC9_AC1: "AC9 ⟹ AC1"*
⟨proof⟩

end

theory *WO1_AC*
imports *AC_Equiv*
begin

theorem *WO1_AC1: "WO1 ⟹ AC1"*

⟨proof⟩

lemma *lemma1:* "⟦WO1; ∀B ∈ A. ∃C ∈ D(B). P(C,B)⟧ ⟹ ∃f. ∀B ∈ A. P(f`B,B)"
⟨proof⟩

lemma *lemma2_1:* "⟦¬Finite(B); WO1⟧ ⟹ |B| + |B| ≈ B"
⟨proof⟩

lemma *lemma2_2:*
"f ∈ bij(D+D, B) ⟹ {{f`Inl(i), f`Inr(i)}. i ∈ D} ∈ Pow(Pow(B))"
⟨proof⟩

lemma *lemma2_3:*
"f ∈ bij(D+D, B) ⟹ pairwise_disjoint({{f`Inl(i), f`Inr(i)}. i ∈ D})"
⟨proof⟩

lemma *lemma2_4:*
"⟦f ∈ bij(D+D, B); 1≤n⟧
⟹ sets_of_size_between({{f`Inl(i), f`Inr(i)}. i ∈ D}, 2, succ(n))"
⟨proof⟩

lemma *lemma2_5:*
"f ∈ bij(D+D, B) ⟹ ⋃({{f`Inl(i), f`Inr(i)}. i ∈ D})=B"
⟨proof⟩

lemma *lemma2:*
"⟦WO1; ¬Finite(B); 1≤n⟧
⟹ ∃C ∈ Pow(Pow(B)). pairwise_disjoint(C) ∧
sets_of_size_between(C, 2, succ(n)) ∧
⋃(C)=B"
⟨proof⟩

theorem *WO1_AC10:* "⟦WO1; 1≤n⟧ ⟹ AC10(n)"
⟨proof⟩

end

theory *Hartog*
imports *AC_Equiv*
begin

definition

```
  Hartog :: "i ⇒ i"  where
   "Hartog(X) ≡ μ i. ¬ i ≲ X"
```

lemma *Ords_in_set:* "∀a. Ord(a) ⟶ a ∈ X ⟹ P"
⟨*proof*⟩

lemma *Ord_lepoll_imp_ex_well_ord:*
```
     "⟦Ord(a); a ≲ X⟧
      ⟹ ∃Y. Y ⊆ X ∧ (∃R. well_ord(Y,R) ∧ ordertype(Y,R)=a)"
```
⟨*proof*⟩

lemma *Ord_lepoll_imp_eq_ordertype:*
```
     "⟦Ord(a); a ≲ X⟧ ⟹ ∃Y. Y ⊆ X ∧ (∃R. R ⊆ X*X ∧ ordertype(Y,R)=a)"
```
⟨*proof*⟩

lemma *Ords_lepoll_set_lemma:*
```
     "(∀a. Ord(a) ⟶ a ≲ X) ⟹
       ∀a. Ord(a) ⟶
        a ∈ {b. Z ∈ Pow(X)*Pow(X*X), ∃Y R. Z=⟨Y,R⟩ ∧ ordertype(Y,R)=b}"
```
⟨*proof*⟩

lemma *Ords_lepoll_set:* "∀a. Ord(a) ⟶ a ≲ X ⟹ P"
⟨*proof*⟩

lemma *ex_Ord_not_lepoll:* "∃a. Ord(a) ∧ ¬a ≲ X"
⟨*proof*⟩

lemma *not_Hartog_lepoll_self:* "¬ Hartog(A) ≲ A"
⟨*proof*⟩

lemmas *Hartog_lepoll_selfE = not_Hartog_lepoll_self [THEN notE]*

lemma *Ord_Hartog:* "Ord(Hartog(A))"
⟨*proof*⟩

lemma *less_HartogE1:* "⟦i < Hartog(A); ¬ i ≲ A⟧ ⟹ P"
⟨*proof*⟩

lemma *less_HartogE:* "⟦i < Hartog(A); i ≈ Hartog(A)⟧ ⟹ P"
⟨*proof*⟩

lemma *Card_Hartog:* "Card(Hartog(A))"
⟨*proof*⟩

end

theory *HH*
imports *AC_Equiv Hartog*

begin

definition
```
  HH :: "[i, i, i] ⇒ i"  where
    "HH(f,x,a) ≡ transrec(a, λb r. let z = x - (⋃c ∈ b. r`c)
                                  in  if f`z ∈ Pow(z)-{0} then f`z else
{x})"
```

0.1 Lemmas useful in each of the three proofs

lemma *HH_def_satisfies_eq:*
```
     "HH(f,x,a) = (let z = x - (⋃b ∈ a. HH(f,x,b))
                   in  if f`z ∈ Pow(z)-{0} then f`z else {x})"
```
⟨*proof*⟩

lemma *HH_values:* `"HH(f,x,a) ∈ Pow(x)-{0} | HH(f,x,a)={x}"`
⟨*proof*⟩

lemma *subset_imp_Diff_eq:*
```
     "B ⊆ A ⟹ X-(⋃a ∈ A. P(a)) = X-(⋃a ∈ A-B. P(a))-(⋃b ∈ B. P(b))"
```
⟨*proof*⟩

lemma *Ord_DiffE:* `"⟦c ∈ a-b; b<a⟧ ⟹ c=b | b<c ∧ c<a"`
⟨*proof*⟩

lemma *Diff_UN_eq_self:* `"(⋀y. y∈A ⟹ P(y) = {x}) ⟹ x - (⋃y ∈ A.`
`P(y)) = x"`
⟨*proof*⟩

lemma *HH_eq:* `"x - (⋃b ∈ a. HH(f,x,b)) = x - (⋃b ∈ a1. HH(f,x,b))`
```
             ⟹ HH(f,x,a) = HH(f,x,a1)"
```
⟨*proof*⟩

lemma *HH_is_x_gt_too:* `"⟦HH(f,x,b)={x}; b<a⟧ ⟹ HH(f,x,a)={x}"`
⟨*proof*⟩

lemma *HH_subset_x_lt_too:*
```
     "⟦HH(f,x,a) ∈ Pow(x)-{0}; b<a⟧ ⟹ HH(f,x,b) ∈ Pow(x)-{0}"
```
⟨*proof*⟩

lemma *HH_subset_x_imp_subset_Diff_UN:*
```
    "HH(f,x,a) ∈ Pow(x)-{0} ⟹ HH(f,x,a) ∈ Pow(x - (⋃b ∈ a. HH(f,x,b)))-{0}"
```
⟨*proof*⟩

lemma *HH_eq_arg_lt:*
```
     "⟦HH(f,x,v)=HH(f,x,w); HH(f,x,v) ∈ Pow(x)-{0}; v ∈ w⟧ ⟹ P"
```
⟨*proof*⟩

lemma *HH_eq_imp_arg_eq:*

"⟦HH(f,x,v)=HH(f,x,w); HH(f,x,w) ∈ Pow(x)-{0}; Ord(v); Ord(w)⟧ ⟹ v=w"
⟨proof⟩

lemma *HH_subset_x_imp_lepoll:*
"⟦HH(f, x, i) ∈ Pow(x)-{0}; Ord(i)⟧ ⟹ i ≲ Pow(x)-{0}"
⟨proof⟩

lemma *HH_Hartog_is_x:* "HH(f, x, Hartog(Pow(x)-{0})) = {x}"
⟨proof⟩

lemma *HH_Least_eq_x:* "HH(f, x, μ i. HH(f, x, i) = {x}) = {x}"
⟨proof⟩

lemma *less_Least_subset_x:*
"a ∈ (μ i. HH(f,x,i)={x}) ⟹ HH(f,x,a) ∈ Pow(x)-{0}"
⟨proof⟩

0.2 Lemmas used in the proofs of AC1 ⟹ WO2 and AC17 ⟹ AC1

lemma *lam_Least_HH_inj_Pow:*
"(λa ∈ (μ i. HH(f,x,i)={x}). HH(f,x,a))
∈ inj(μ i. HH(f,x,i)={x}, Pow(x)-{0})"
⟨proof⟩

lemma *lam_Least_HH_inj:*
"∀ a ∈ (μ i. HH(f,x,i)={x}). ∃ z ∈ x. HH(f,x,a) = {z}
⟹ (λa ∈ (μ i. HH(f,x,i)={x}). HH(f,x,a))
∈ inj(μ i. HH(f,x,i)={x}, {{y}. y ∈ x})"
⟨proof⟩

lemma *lam_surj_sing:*
"⟦x - (⋃a ∈ A. F(a)) = 0; ∀ a ∈ A. ∃ z ∈ x. F(a) = {z}⟧
⟹ (λa ∈ A. F(a)) ∈ surj(A, {{y}. y ∈ x})"
⟨proof⟩

lemma *not_emptyI2:* "y ∈ Pow(x)-{0} ⟹ x ≠ 0"
⟨proof⟩

lemma *f_subset_imp_HH_subset:*
"f‘(x - (⋃j ∈ i. HH(f,x,j))) ∈ Pow(x - (⋃j ∈ i. HH(f,x,j)))-{0}

⟹ HH(f, x, i) ∈ Pow(x) - {0}"
⟨proof⟩

lemma *f_subsets_imp_UN_HH_eq_x:*
"∀ z ∈ Pow(x)-{0}. f‘z ∈ Pow(z)-{0}
⟹ x - (⋃j ∈ (μ i. HH(f,x,i)={x}). HH(f,x,j)) = 0"
⟨proof⟩

lemma *HH_values2:* "$HH(f,x,i) = f`(x - (\bigcup j \in i.\ HH(f,x,j)))$ | $HH(f,x,i)=\{x\}$"
⟨*proof*⟩

lemma *HH_subset_imp_eq:*
"$HH(f,x,i): Pow(x)-\{0\} \Longrightarrow HH(f,x,i)=f`(x - (\bigcup j \in i.\ HH(f,x,j)))$"
⟨*proof*⟩

lemma *f_sing_imp_HH_sing:*
"$[\![f \in (Pow(x)-\{0\}) \to \{\{z\}.\ z \in x\};$
$a \in (\mu\ i.\ HH(f,x,i)=\{x\})]\!] \Longrightarrow \exists z \in x.\ HH(f,x,a) = \{z\}$"
⟨*proof*⟩

lemma *f_sing_lam_bij:*
"$[\![x - (\bigcup j \in (\mu\ i.\ HH(f,x,i)=\{x\}).\ HH(f,x,j)) = 0;$
$f \in (Pow(x)-\{0\}) \to \{\{z\}.\ z \in x\}]\!]$
$\Longrightarrow (\lambda a \in (\mu\ i.\ HH(f,x,i)=\{x\}).\ HH(f,x,a))$
$\in bij(\mu\ i.\ HH(f,x,i)=\{x\}, \{\{y\}.\ y \in x\})$"
⟨*proof*⟩

lemma *lam_singI:*
"$f \in (\prod X \in Pow(x)-\{0\}.\ F(X))$
$\Longrightarrow (\lambda X \in Pow(x)-\{0\}.\ \{f`X\}) \in (\prod X \in Pow(x)-\{0\}.\ \{\{z\}.\ z \in F(X)\})$"
⟨*proof*⟩

lemmas *bij_Least_HH_x =*
comp_bij [OF f_sing_lam_bij [OF _ lam_singI]
lam_sing_bij [THEN bij_converse_bij]]

0.3 The proof of *AC1* $\Longrightarrow$ *WO2*

lemma *bijection:*
"$f \in (\prod X \in Pow(x) - \{0\}.\ X)$
$\Longrightarrow \exists g.\ g \in bij(x, \mu\ i.\ HH(\lambda X \in Pow(x)-\{0\}.\ \{f`X\}, x, i) = \{x\})$"
⟨*proof*⟩

lemma *AC1_WO2:* "*AC1* $\Longrightarrow$ *WO2*"
⟨*proof*⟩

end

theory *AC15_WO6*
imports *HH Cardinal_aux*
begin

lemma *lepoll_Sigma:* "$A \neq 0 \Longrightarrow B \lesssim A*B$"
$\langle proof \rangle$

lemma *cons_times_nat_not_Finite:*
"$0 \notin A \Longrightarrow \forall B \in \{cons(0,x*nat).\ x \in A\}.\ \neg Finite(B)$"
$\langle proof \rangle$

lemma *lemma1:* "$[\![\bigcup(C)=A;\ a \in A]\!] \Longrightarrow \exists B \in C.\ a \in B \wedge B \subseteq A$"
$\langle proof \rangle$

lemma *lemma2:*
"$[\![pairwise\_disjoint(A);\ B \in A;\ C \in A;\ a \in B;\ a \in C]\!] \Longrightarrow B=C$"
$\langle proof \rangle$

lemma *lemma3:*
"$\forall B \in \{cons(0,\ x*nat).\ x \in A\}.\ pairwise\_disjoint(f`B) \wedge$
$sets\_of\_size\_between(f`B,\ 2,\ n) \wedge \bigcup(f`B)=B$
$\Longrightarrow \forall B \in A.\ \exists!\ u.\ u \in f`cons(0,\ B*nat) \wedge u \subseteq cons(0,\ B*nat) \wedge$

$0 \in u \wedge 2 \lesssim u \wedge u \lesssim n$"
$\langle proof \rangle$

lemma *lemma4:* "$[\![A \lesssim i;\ Ord(i)]\!] \Longrightarrow \{P(a).\ a \in A\} \lesssim i$"
$\langle proof \rangle$

lemma *lemma5_1:*
"$[\![B \in A;\ 2 \lesssim u(B)]\!] \Longrightarrow (\lambda x \in A.\ \{fst(x).\ x \in u(x)-\{0\}\})`B \neq 0$"
$\langle proof \rangle$

lemma *lemma5_2:*
"$[\![B \in A;\ u(B) \subseteq cons(0,\ B*nat)]\!]$
$\Longrightarrow (\lambda x \in A.\ \{fst(x).\ x \in u(x)-\{0\}\})`B \subseteq B$"
$\langle proof \rangle$

lemma *lemma5_3:*
"$[\![n \in nat;\ B \in A;\ 0 \in u(B);\ u(B) \lesssim succ(n)]\!]$
$\Longrightarrow (\lambda x \in A.\ \{fst(x).\ x \in u(x)-\{0\}\})`B \lesssim n$"
$\langle proof \rangle$

lemma *ex_fun_AC13_AC15:*
"$[\![\forall B \in \{cons(0,\ x*nat).\ x \in A\}.$
$pairwise\_disjoint(f`B) \wedge$
$sets\_of\_size\_between(f`B,\ 2,\ succ(n)) \wedge \bigcup(f`B)=B;$
$n \in nat]\!]$

$\Longrightarrow$ $\exists$ *f. $\forall$B $\in$ A. f`B $\neq$ 0 $\wedge$ f`B $\subseteq$ B $\wedge$ f`B $\lesssim$ n"*
⟨*proof*⟩

theorem *AC10_AC11: "⟦n $\in$ nat; 1$\leq$n; AC10(n)⟧ $\Longrightarrow$ AC11"*
⟨*proof*⟩

theorem *AC11_AC12: "AC11 $\Longrightarrow$ AC12"*
⟨*proof*⟩

theorem *AC12_AC15: "AC12 $\Longrightarrow$ AC15"*
⟨*proof*⟩

lemma *OUN_eq_UN: "Ord(x) $\Longrightarrow$ ($\bigcup$a<x. F(a)) = ($\bigcup$a $\in$ x. F(a))"*
⟨*proof*⟩

lemma *AC15_WO6_aux1:*
"$\forall$x $\in$ Pow(A)-{0}. f`x$\neq$0 $\wedge$ f`x $\subseteq$ x $\wedge$ f`x $\lesssim$ m
$\Longrightarrow$ ($\bigcup$i<μ x. HH(f,A,x)={A}. HH(f,A,i)) = A"
⟨*proof*⟩

lemma *AC15_WO6_aux2:*
"$\forall$x $\in$ Pow(A)-{0}. f`x$\neq$0 $\wedge$ f`x $\subseteq$ x $\wedge$ f`x $\lesssim$ m
$\Longrightarrow$ $\forall$x < (μ x. HH(f,A,x)={A}). HH(f,A,x) $\lesssim$ m"
⟨*proof*⟩

theorem *AC15_WO6: "AC15 $\Longrightarrow$ WO6"*
⟨*proof*⟩

theorem *AC10_AC13:* "$[\![n \in nat;\ 1 \le n;\ AC10(n)]\!] \Longrightarrow AC13(n)$"
$\langle proof \rangle$

lemma *AC1_AC13:* "$AC1 \Longrightarrow AC13(1)$"
$\langle proof \rangle$

lemma *AC13_mono:* "$[\![m \le n;\ AC13(m)]\!] \Longrightarrow AC13(n)$"
$\langle proof \rangle$

theorem *AC13_AC14:* "$[\![n \in nat;\ 1 \le n;\ AC13(n)]\!] \Longrightarrow AC14$"
$\langle proof \rangle$

theorem *AC14_AC15: "AC14 $\Longrightarrow$ AC15"*
⟨*proof*⟩

lemma *lemma_aux: "⟦A≠0; A $\lesssim$ 1⟧ $\Longrightarrow$ ∃a. A={a}"*
⟨*proof*⟩

lemma *AC13_AC1_lemma:*
"∀B ∈ A. f(B)≠0 ∧ f(B)<=B ∧ f(B) $\lesssim$ 1
$\Longrightarrow$ (λx ∈ A. THE y. f(x)={y}) ∈ (∏X ∈ A. X)"
⟨*proof*⟩

theorem *AC13_AC1: "AC13(1) $\Longrightarrow$ AC1"*
⟨*proof*⟩

theorem *AC11_AC14: "AC11 $\Longrightarrow$ AC14"*
⟨*proof*⟩

end

theory *AC16_lemmas*
imports *AC_Equiv Hartog Cardinal_aux*
begin

lemma *cons_Diff_eq: "a∉A $\Longrightarrow$ cons(a,A)-{a}=A"*
⟨*proof*⟩

lemma *nat_1_lepoll_iff: "1$\lesssim$X ⟷ (∃x. x ∈ X)"*
⟨*proof*⟩

lemma *eqpoll_1_iff_singleton: "X≈1 ⟷ (∃x. X={x})"*
⟨*proof*⟩

lemma *cons_eqpoll_succ: "⟦x≈n; y∉x⟧ $\Longrightarrow$ cons(y,x)≈succ(n)"*
⟨*proof*⟩

lemma *subsets_eqpoll_1_eq: "{Y ∈ Pow(X). Y≈1} = {{x}. x ∈ X}"*

⟨*proof*⟩

lemma *eqpoll_RepFun_sing: "X≈{{x}. x ∈ X}"*
⟨*proof*⟩

lemma *subsets_eqpoll_1_eqpoll: "{Y ∈ Pow(X). Y≈1}≈X"*
⟨*proof*⟩

lemma *InfCard_Least_in:*
"⟦InfCard(x); y ⊆ x; y ≈ succ(z)⟧ ⟹ (μ i. i ∈ y) ∈ y"
⟨*proof*⟩

lemma *subsets_lepoll_lemma1:*
"⟦InfCard(x); n ∈ nat⟧
⟹ {y ∈ Pow(x). y≈succ(succ(n))} ≲ x{y ∈ Pow(x). y≈succ(n)}"*
⟨*proof*⟩

lemma *set_of_Ord_succ_Union: "(∀y ∈ z. Ord(y)) ⟹ z ⊆ succ(⋃(z))"*
⟨*proof*⟩

lemma *subset_not_mem: "j ⊆ i ⟹ i ∉ j"*
⟨*proof*⟩

lemma *succ_Union_not_mem:*
"(⋀y. y ∈ z ⟹ Ord(y)) ⟹ succ(⋃(z)) ∉ z"
⟨*proof*⟩

lemma *Union_cons_eq_succ_Union:*
"⋃(cons(succ(⋃(z)),z)) = succ(⋃(z))"
⟨*proof*⟩

lemma *Un_Ord_disj: "⟦Ord(i); Ord(j)⟧ ⟹ i ∪ j = i | i ∪ j = j"*
⟨*proof*⟩

lemma *Union_eq_Un: "x ∈ X ⟹ ⋃(X) = x ∪ ⋃(X-{x})"*
⟨*proof*⟩

lemma *Union_in_lemma [rule_format]:*
"n ∈ nat ⟹ ∀z. (∀y ∈ z. Ord(y)) ∧ z≈n ∧ z≠0 ⟶ ⋃(z) ∈ z"
⟨*proof*⟩

lemma *Union_in: "⟦∀x ∈ z. Ord(x); z≈n; z≠0; n ∈ nat⟧ ⟹ ⋃(z) ∈ z"*
⟨*proof*⟩

lemma *succ_Union_in_x:*
"⟦InfCard(x); z ∈ Pow(x); z≈n; n ∈ nat⟧ ⟹ succ(⋃(z)) ∈ x"
⟨*proof*⟩

lemma *succ_lepoll_succ_succ:*

"⟦*InfCard(x); n ∈ nat*⟧
⟹ *{y ∈ Pow(x). y≈succ(n)} ≲ {y ∈ Pow(x). y≈succ(succ(n))}*"
⟨*proof*⟩

lemma *subsets_eqpoll_X:*
"⟦*InfCard(X); n ∈ nat*⟧ ⟹ *{Y ∈ Pow(X). Y≈succ(n)} ≈ X*"
⟨*proof*⟩

lemma *image_vimage_eq:*
"⟦*f ∈ surj(A,B); y ⊆ B*⟧ ⟹ *f``(converse(f)``y) = y*"
⟨*proof*⟩

lemma *vimage_image_eq:* "⟦*f ∈ inj(A,B); y ⊆ A*⟧ ⟹ *converse(f)``(f``y) = y*"
⟨*proof*⟩

lemma *subsets_eqpoll:*
"*A≈B* ⟹ *{Y ∈ Pow(A). Y≈n}≈{Y ∈ Pow(B). Y≈n}*"
⟨*proof*⟩

lemma *WO2_imp_ex_Card:* "*WO2* ⟹ ∃*a. Card(a)* ∧ *X≈a*"
⟨*proof*⟩

lemma *lepoll_infinite:* "⟦*X≲Y;* ¬*Finite(X)*⟧ ⟹ ¬*Finite(Y)*"
⟨*proof*⟩

lemma *infinite_Card_is_InfCard:* "⟦¬*Finite(X); Card(X)*⟧ ⟹ *InfCard(X)*"
⟨*proof*⟩

lemma *WO2_infinite_subsets_eqpoll_X:* "⟦*WO2; n ∈ nat;* ¬*Finite(X)*⟧
⟹ *{Y ∈ Pow(X). Y≈succ(n)}≈X*"
⟨*proof*⟩

lemma *well_ord_imp_ex_Card:* "*well_ord(X,R)* ⟹ ∃*a. Card(a)* ∧ *X≈a*"
⟨*proof*⟩

lemma *well_ord_infinite_subsets_eqpoll_X:*
"⟦*well_ord(X,R); n ∈ nat;* ¬*Finite(X)*⟧ ⟹ *{Y ∈ Pow(X). Y≈succ(n)}≈X*"
⟨*proof*⟩

end

theory *WO2_AC16* **imports** *AC_Equiv AC16_lemmas Cardinal_aux* **begin**

definition
recfunAC16 :: "*[i,i,i,i]* ⇒ *i*" **where**

"*recfunAC16(f,h,i,a)* ≡
transrec2(i, 0,
λg r. if (∃y ∈ r. h'g ⊆ y) then r
else r ∪ {f'(μ i. h'g ⊆ f'i ∧
(∀b<a. (h'b ⊆ f'i ⟶ (∀t ∈ r. ¬ h'b ⊆ t))))})"

lemma *recfunAC16_0: "recfunAC16(f,h,0,a) = 0"*
⟨*proof*⟩

lemma *recfunAC16_succ:*
"recfunAC16(f,h,succ(i),a) =
(if (∃y ∈ recfunAC16(f,h,i,a). h ' i ⊆ y) then recfunAC16(f,h,i,a)

else recfunAC16(f,h,i,a) ∪
{f ' (μ j. h ' i ⊆ f ' j ∧
(∀b<a. (h'b ⊆ f'j
⟶ (∀t ∈ recfunAC16(f,h,i,a). ¬ h'b ⊆ t))))})"
⟨*proof*⟩

lemma *recfunAC16_Limit: "Limit(i)*
⟹ recfunAC16(f,h,i,a) = (⋃j<i. recfunAC16(f,h,j,a))"
⟨*proof*⟩

lemma *transrec2_mono_lemma [rule_format]:*
"⟦⋀g r. r ⊆ B(g,r); Ord(i)⟧
⟹ j<i ⟶ transrec2(j, 0, B) ⊆ transrec2(i, 0, B)"
⟨*proof*⟩

lemma *transrec2_mono:*
"⟦⋀g r. r ⊆ B(g,r); j≤i⟧
⟹ transrec2(j, 0, B) ⊆ transrec2(i, 0, B)"
⟨*proof*⟩

lemma *recfunAC16_mono:*
"i≤j ⟹ recfunAC16(f, g, i, a) ⊆ recfunAC16(f, g, j, a)"
⟨*proof*⟩

lemma *lemma3_1:*

```
"⟦∀y<x. ∀z<a. z<y | (∃Y ∈ F(y). f(z)<=Y) ⟶ (∃! Y. Y ∈ F(y) ∧
f(z)<=Y);
       ∀i j. i≤j ⟶ F(i) ⊆ F(j); j≤i; i<x; z<a;
       V ∈ F(i); f(z)<=V; W ∈ F(j); f(z)<=W⟧
    ⟹ V = W"
```

⟨*proof*⟩

lemma *lemma3:*

```
"⟦∀y<x. ∀z<a. z<y | (∃Y ∈ F(y). f(z)<=Y) ⟶ (∃! Y. Y ∈ F(y) ∧
f(z)<=Y);
       ∀i j. i≤j ⟶ F(i) ⊆ F(j); i<x; j<x; z<a;
       V ∈ F(i); f(z)<=V; W ∈ F(j); f(z)<=W⟧
    ⟹ V = W"
```

⟨*proof*⟩

lemma *lemma4:*

```
"⟦∀y<x. F(y) ⊆ X ∧
          (∀x<a. x < y | (∃Y ∈ F(y). h(x) ⊆ Y) ⟶
           (∃! Y. Y ∈ F(y) ∧ h(x) ⊆ Y));
    x < a⟧
 ⟹ ∀y<x. ∀z<a. z < y | (∃Y ∈ F(y). h(z) ⊆ Y) ⟶
               (∃! Y. Y ∈ F(y) ∧ h(z) ⊆ Y)"
```

⟨*proof*⟩

lemma *lemma5:*

```
"⟦∀y<x. F(y) ⊆ X ∧
          (∀x<a. x < y | (∃Y ∈ F(y). h(x) ⊆ Y) ⟶
           (∃! Y. Y ∈ F(y) ∧ h(x) ⊆ Y));
    x < a; Limit(x); ∀i j. i≤j ⟶ F(i) ⊆ F(j)⟧
 ⟹ (⋃x<x. F(x)) ⊆ X ∧
    (∀xa<a. xa < x | (∃x ∈ ⋃x<x. F(x). h(xa) ⊆ x)
         ⟶ (∃! Y. Y ∈ (⋃x<x. F(x)) ∧ h(xa) ⊆ Y))"
```

⟨*proof*⟩

lemma *dbl_Diff_eqpoll_Card:*
"⟦A≈a; Card(a); ¬Finite(a); B≺a; C≺a⟧ ⟹ A - B - C≈a"
⟨*proof*⟩

lemma *Finite_lesspoll_infinite_Ord:*
"⟦Finite(X); ¬Finite(a); Ord(a)⟧ ⟹ X≺a"
⟨*proof*⟩

lemma *Union_lesspoll:*
"⟦∀x ∈ X. x ≲ n ∧ x ⊆ T; well_ord(T, R); X ≲ b;
b<a; ¬Finite(a); Card(a); n ∈ nat⟧
⟹ ⋃(X)≺a"
⟨*proof*⟩

lemma *Un_sing_eq_cons:* "A ∪ {a} = cons(a, A)"
⟨*proof*⟩

lemma *Un_lepoll_succ:* "A ≲ B ⟹ A ∪ {a} ≲ succ(B)"
⟨*proof*⟩

lemma *Diff_UN_succ_empty:* "Ord(a) ⟹ F(a) - (⋃b<succ(a). F(b)) = 0"
⟨*proof*⟩

lemma *Diff_UN_succ_subset:* "Ord(a) ⟹ F(a) ∪ X - (⋃b<succ(a). F(b)) ⊆ X"
⟨*proof*⟩

lemma *recfunAC16_Diff_lepoll_1:*
"Ord(x)
⟹ recfunAC16(f, g, x, a) - (⋃i<x. recfunAC16(f, g, i, a)) ≲ 1"
⟨*proof*⟩

lemma *in_Least_Diff:*
"⟦z ∈ F(x); Ord(x)⟧
⟹ z ∈ F(μ i. z ∈ F(i)) - (⋃j<(μ i. z ∈ F(i)). F(j))"
⟨*proof*⟩

lemma *Least_eq_imp_ex:*
```
"⟦(μ i. w ∈ F(i)) = (μ i. z ∈ F(i));
    w ∈ (⋃i<a. F(i)); z ∈ (⋃i<a. F(i))⟧
 ⟹ ∃b<a. w ∈ (F(b) - (⋃c<b. F(c))) ∧ z ∈ (F(b) - (⋃c<b. F(c)))"
```
⟨*proof*⟩

lemma *two_in_lepoll_1:* `"⟦A ≲ 1; a ∈ A; b ∈ A⟧ ⟹ a=b"`
⟨*proof*⟩

lemma *UN_lepoll_index:*
```
"⟦∀i<a. F(i)-(⋃j<i. F(j)) ≲ 1; Limit(a)⟧
 ⟹ (⋃x<a. F(x)) ≲ a"
```
⟨*proof*⟩

lemma *recfunAC16_lepoll_index:* `"Ord(y) ⟹ recfunAC16(f, h, y, a) ≲ y"`
⟨*proof*⟩

lemma *Union_recfunAC16_lesspoll:*
```
"⟦recfunAC16(f,g,y,a) ⊆ {X ∈ Pow(A). X≈n};
    A≈a;  y<a;  ¬Finite(a);  Card(a);  n ∈ nat⟧
 ⟹ ⋃(recfunAC16(f,g,y,a))≺a"
```
⟨*proof*⟩

lemma *dbl_Diff_eqpoll:*
```
"⟦recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
    Card(a); ¬ Finite(a); A≈a;
    k ∈ nat;  y<a;
    h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)})⟧
 ⟹ A - ⋃(recfunAC16(f, h, y, a)) - h`y≈a"
```
⟨*proof*⟩

lemmas *disj_Un_eqpoll_nat_sum =*
```
  eqpoll_trans [THEN eqpoll_trans,
                OF disj_Un_eqpoll_sum sum_eqpoll_cong nat_sum_eqpoll_sum]
```

lemma *Un_in_Collect:* `"⟦x ∈ Pow(A - B - h`i); x≈m;`
```
    h ∈ bij(a, {x ∈ Pow(A) . x≈k}); i<a; k ∈ nat; m ∈ nat⟧
    ⟹ h ` i ∪ x ∈ {x ∈ Pow(A) . x≈k #+ m}"
```
⟨*proof*⟩

lemma *lemma6:*
"⟦∀y<succ(j). F(y)<=X ∧ (∀x<a. x<y | P(x,y) ⟶ Q(x,y)); succ(j)<a⟧
⟹ F(j)<=X ∧ (∀x<a. x<j | P(x,j) ⟶ Q(x,j))"
⟨*proof*⟩

lemma *lemma7:*
"⟦∀x<a. x<j | P(x,j) ⟶ Q(x,j); succ(j)<a⟧
⟹ P(j,j) ⟶ (∀x<a. x≤j | P(x,j) ⟶ Q(x,j))"
⟨*proof*⟩

lemma *ex_subset_eqpoll:*
"⟦A≈a; ¬ Finite(a); Ord(a); m ∈ nat⟧ ⟹ ∃X ∈ Pow(A). X≈m"
⟨*proof*⟩

lemma *subset_Un_disjoint:* "⟦A ⊆ B ∪ C; A ∩ C = 0⟧ ⟹ A ⊆ B"
⟨*proof*⟩

lemma *Int_empty:*
"⟦X ∈ Pow(A - ⋃(B) -C); T ∈ B; F ⊆ T⟧ ⟹ F ∩ X = 0"
⟨*proof*⟩

lemma *subset_imp_eq_lemma:*
"m ∈ nat ⟹ ∀A B. A ⊆ B ∧ m ≲ A ∧ B ≲ m ⟶ A=B"
⟨*proof*⟩

lemma *subset_imp_eq:* "⟦A ⊆ B; m ≲ A; B ≲ m; m ∈ nat⟧ ⟹ A=B"
⟨*proof*⟩

lemma *bij_imp_arg_eq:*
"⟦f ∈ bij(a, {Y ∈ X. Y≈succ(k)}); k ∈ nat; f`b ⊆ f`y; b<a; y<a⟧

```
        ⟹ b=y"
⟨proof⟩
```

lemma *ex_next_set:*

```
     "⟦recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
         Card(a); ¬ Finite(a); A≈a;
         k ∈ nat; m ∈ nat; y<a;
         h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)});
         ¬ (∃Y ∈ recfunAC16(f, h, y, a). h`y ⊆ Y)⟧
      ⟹ ∃X ∈ {Y ∈ Pow(A). Y≈succ(k #+ m)}. h`y ⊆ X ∧
                (∀b<a. h`b ⊆ X ⟶
                (∀T ∈ recfunAC16(f, h, y, a). ¬ h`b ⊆ T))"
⟨proof⟩
```

lemma *ex_next_Ord:*

```
     "⟦recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
         Card(a); ¬ Finite(a); A≈a;
         k ∈ nat; m ∈ nat; y<a;
         h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)});
         f ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k #+ m)});
         ¬ (∃Y ∈ recfunAC16(f, h, y, a). h`y ⊆ Y)⟧
      ⟹ ∃c<a. h`y ⊆ f`c ∧
                (∀b<a. h`b ⊆ f`c ⟶
                (∀T ∈ recfunAC16(f, h, y, a). ¬ h`b ⊆ T))"
⟨proof⟩
```

lemma *lemma8:*

```
     "⟦∀x<a. x<j | (∃xa ∈ F(j). P(x, xa))
         ⟶ (∃! Y. Y ∈ F(j) ∧ P(x, Y)); F(j) ⊆ X;
         L ∈ X; P(j, L) ∧ (∀x<a. P(x, L) ⟶ (∀xa ∈ F(j). ¬P(x, xa)))⟧

      ⟹ F(j) ∪ {L} ⊆ X ∧
          (∀x<a. x≤j | (∃xa ∈ (F(j) ∪ {L}). P(x, xa)) ⟶
                 (∃! Y. Y ∈ (F(j) ∪ {L}) ∧ P(x, Y)))"
⟨proof⟩
```

lemma *main_induct:*
"⟦b < a; f ∈ bij(a, {Y ∈ Pow(A) . Y≈succ(k #+ m)});
h ∈ bij(a, {Y ∈ Pow(A) . Y≈succ(k)});
¬Finite(a); Card(a); A≈a; k ∈ nat; m ∈ nat⟧
⟹ recfunAC16(f, h, b, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)} ∧
(∀x<a. x < b | (∃Y ∈ recfunAC16(f, h, b, a). h ‘ x ⊆ Y) ⟶

(∃! Y. Y ∈ recfunAC16(f, h, b, a) ∧ h ‘ x ⊆ Y))"
⟨*proof*⟩

lemma *lemma_simp_induct:*
"⟦∀b. b<a ⟶ F(b) ⊆ S ∧ (∀x<a. (x<b | (∃Y ∈ F(b). f‘x ⊆ Y))
⟶ (∃! Y. Y ∈ F(b) ∧ f‘x ⊆ Y));
f ∈ a->f‘‘(a); Limit(a);
∀i j. i≤j ⟶ F(i) ⊆ F(j)⟧
⟹ (⋃j<a. F(j)) ⊆ S ∧
(∀x ∈ f‘‘a. ∃! Y. Y ∈ (⋃j<a. F(j)) ∧ x ⊆ Y)"
⟨*proof*⟩

theorem *WO2_AC16:* "⟦WO2; 0<m; k ∈ nat; m ∈ nat⟧ ⟹ AC16(k #+ m,k)"
⟨*proof*⟩

end

theory *AC16_WO4*
imports *AC16_lemmas*
begin

lemma *lemma1:*
"⟦Finite(A); 0<m; m ∈ nat⟧
⟹ ∃a f. Ord(a) ∧ domain(f) = a ∧

$(\bigcup b<a.\ f`b) = A \wedge (\forall b<a.\ f`b \lesssim m)$"
⟨proof⟩

lemmas *well_ord_paired = paired_bij [THEN bij_is_inj, THEN well_ord_rvimage]*

lemma *lepoll_trans1:* "$[\![A \lesssim B; \neg A \lesssim C]\!] \Longrightarrow \neg B \lesssim C$"
⟨proof⟩

lemmas *lepoll_paired = paired_eqpoll [THEN eqpoll_sym, THEN eqpoll_imp_lepoll]*

lemma *lemma2:* "$\exists y\ R.\ well\_ord(y,R) \wedge x \cap y = 0 \wedge \neg y \lesssim z \wedge \neg Finite(y)$"
⟨proof⟩

lemma *infinite_Un:* "$\neg Finite(B) \Longrightarrow \neg Finite(A \cup B)$"
⟨proof⟩

lemma *succ_not_lepoll_lemma:*
"$[\![\neg(\exists x \in A.\ f`x=y); f \in inj(A, B); y \in B]\!]$
$\Longrightarrow (\lambda a \in succ(A).\ if(a=A, y, f`a)) \in inj(succ(A), B)$"
⟨proof⟩

lemma *succ_not_lepoll_imp_eqpoll:* "$[\![\neg A \approx B; A \lesssim B]\!] \Longrightarrow succ(A) \lesssim B$"
⟨proof⟩

lemmas *ordertype_eqpoll =*
ordermap_bij [THEN exI [THEN eqpoll_def [THEN def_imp_iff, THEN iffD2]]]

lemma *cons_cons_subset:*
"⟦a ⊆ y; b ∈ y-a; u ∈ x⟧ ⟹ cons(b, cons(u, a)) ∈ Pow(x ∪ y)"
⟨*proof*⟩

lemma *cons_cons_eqpoll:*
"⟦a ≈ k; a ⊆ y; b ∈ y-a; u ∈ x; x ∩ y = 0⟧
⟹ cons(b, cons(u, a)) ≈ succ(succ(k))"
⟨*proof*⟩

lemma *set_eq_cons:*
"⟦succ(k) ≈ A; k ≈ B; B ⊆ A; a ∈ A-B; k ∈ nat⟧ ⟹ A = cons(a, B)"
⟨*proof*⟩

lemma *cons_eqE:* "⟦cons(x,a) = cons(y,a); x ∉ a⟧ ⟹ x = y "
⟨*proof*⟩

lemma *eq_imp_Int_eq:* "A = B ⟹ A ∩ C = B ∩ C"
⟨*proof*⟩

lemma *eqpoll_sum_imp_Diff_lepoll_lemma [rule_format]:*
"⟦k ∈ nat; m ∈ nat⟧
⟹ ∀A B. A ≈ k #+ m ∧ k ≲ B ∧ B ⊆ A ⟶ A-B ≲ m"
⟨*proof*⟩

lemma *eqpoll_sum_imp_Diff_lepoll:*
"⟦A ≈ succ(k #+ m); B ⊆ A; succ(k) ≲ B; k ∈ nat; m ∈ nat⟧
⟹ A-B ≲ m"
⟨*proof*⟩

lemma *eqpoll_sum_imp_Diff_eqpoll_lemma [rule_format]:*
"⟦k ∈ nat; m ∈ nat⟧
⟹ ∀A B. A ≈ k #+ m ∧ k ≈ B ∧ B ⊆ A ⟶ A-B ≈ m"
⟨*proof*⟩

lemma *eqpoll_sum_imp_Diff_eqpoll:*
"⟦A ≈ succ(k #+ m); B ⊆ A; succ(k) ≈ B; k ∈ nat; m ∈ nat⟧
⟹ A-B ≈ m"
⟨*proof*⟩

lemma *subsets_lepoll_0_eq_unit:* "{x ∈ Pow(X). x ≲ 0} = {0}"
⟨*proof*⟩

lemma *subsets_lepoll_succ:*
"n ∈ nat ⟹ {z ∈ Pow(y). z ≲ succ(n)} =
{z ∈ Pow(y). z ≲ n} ∪ {z ∈ Pow(y). z ≈ succ(n)}"
⟨*proof*⟩

lemma *Int_empty:*
"n ∈ nat ⟹ {z ∈ Pow(y). z ≲ n} ∩ {z ∈ Pow(y). z ≈ succ(n)} = 0"
⟨*proof*⟩

```
locale AC16 =
  fixes x and y and k and l and m and t_n and R and MM and LL and
GG and s
  defines k_def:     "k   ≡ succ(l)"
      and MM_def:    "MM  ≡ {v ∈ t_n. succ(k) ≲ v ∩ y}"
      and LL_def:    "LL  ≡ {v ∩ y. v ∈ MM}"
      and GG_def:    "GG  ≡ λv ∈ LL. (THE w. w ∈ MM ∧ v ⊆ w) - v"
      and s_def:     "s(u) ≡ {v ∈ t_n. u ∈ v ∧ k ≲ v ∩ y}"
  assumes all_ex:    "∀z ∈ {z ∈ Pow(x ∪ y) . z ≈ succ(k)}.
                      ∃! w. w ∈ t_n ∧ z ⊆ w "
    and disjoint[iff]:  "x ∩ y = 0"
    and "includes":  "t_n ⊆ {v ∈ Pow(x ∪ y). v ≈ succ(k #+ m)}"
    and WO_R[iff]:      "well_ord(y,R)"
    and lnat[iff]:      "l ∈ nat"
    and mnat[iff]:      "m ∈ nat"
    and mpos[iff]:      "0<m"
    and Infinite[iff]:  "¬ Finite(y)"
    and noLepoll:  "¬ y ≲ {v ∈ Pow(x). v ≈ m}"
begin
```

lemma *knat [iff]:* "k ∈ nat"
⟨*proof*⟩

lemma *Diff_Finite_eqpoll:* "$[\![$1 $\approx$ a; a $\subseteq$ y$]\!]$ $\Longrightarrow$ y - a $\approx$ y"
$\langle proof \rangle$

lemma *s_subset:* "s(u) $\subseteq$ t_n"
$\langle proof \rangle$

lemma *sI:*
"$[\![$w $\in$ t_n; cons(b,cons(u,a)) $\subseteq$ w; a $\subseteq$ y; b $\in$ y-a; 1 $\approx$ a$]\!]$
$\Longrightarrow$ w $\in$ s(u)"
$\langle proof \rangle$

lemma *in_s_imp_u_in:* "v $\in$ s(u) $\Longrightarrow$ u $\in$ v"
$\langle proof \rangle$

lemma *ex1_superset_a:*
"$[\![$1 $\approx$ a; a $\subseteq$ y; b $\in$ y - a; u $\in$ x$]\!]$
$\Longrightarrow$ $\exists$! c. c $\in$ s(u) $\wedge$ a $\subseteq$ c $\wedge$ b $\in$ c"
$\langle proof \rangle$

lemma *the_eq_cons:*
"$[\![\forall$ v $\in$ s(u). succ(1) $\approx$ v $\cap$ y;
1 $\approx$ a; a $\subseteq$ y; b $\in$ y - a; u $\in$ x$]\!]$
$\Longrightarrow$ (THE c. c $\in$ s(u) $\wedge$ a $\subseteq$ c $\wedge$ b $\in$ c) $\cap$ y = cons(b, a)"
$\langle proof \rangle$

lemma *y_lepoll_subset_s:*
"$[\![\forall$ v $\in$ s(u). succ(1) $\approx$ v $\cap$ y;
1 $\approx$ a; a $\subseteq$ y; u $\in$ x$]\!]$
$\Longrightarrow$ y $\lesssim$ {v $\in$ s(u). a $\subseteq$ v}"
$\langle proof \rangle$

lemma *x_imp_not_y [dest]:* "a $\in$ x $\Longrightarrow$ a $\notin$ y"
$\langle proof \rangle$

lemma *w_Int_eq_w_Diff:*
"w $\subseteq$ x $\cup$ y $\Longrightarrow$ w $\cap$ (x - {u}) = w - cons(u, w $\cap$ y)"
$\langle proof \rangle$

lemma *w_Int_eqpoll_m:*
```
"⟦w ∈ {v ∈ s(u). a ⊆ v};
    l ≈ a;  u ∈ x;
    ∀v ∈ s(u). succ(l) ≈ v ∩ y⟧
 ⟹ w ∩ (x - {u}) ≈ m"
```
⟨*proof*⟩

lemma *eqpoll_m_not_empty:* `"a ≈ m ⟹ a ≠ 0"`
⟨*proof*⟩

lemma *cons_cons_in:*
```
"⟦z ∈ xa ∩ (x - {u}); l ≈ a; a ⊆ y; u ∈ x⟧
 ⟹ ∃! w. w ∈ t_n ∧ cons(z, cons(u, a)) ⊆ w"
```
⟨*proof*⟩

lemma *subset_s_lepoll_w:*
```
"⟦∀v ∈ s(u). succ(l) ≈ v ∩ y; a ⊆ y; l ≈ a; u ∈ x⟧
 ⟹ {v ∈ s(u). a ⊆ v} ≲ {v ∈ Pow(x). v ≈ m}"
```
⟨*proof*⟩

lemma *well_ord_subsets_eqpoll_n:*
```
"n ∈ nat ⟹ ∃S. well_ord({z ∈ Pow(y) . z ≈ succ(n)}, S)"
```
⟨*proof*⟩

lemma *well_ord_subsets_lepoll_n:*
```
"n ∈ nat ⟹ ∃R. well_ord({z ∈ Pow(y). z ≲ n}, R)"
```
⟨*proof*⟩

lemma *LL_subset:* `"LL ⊆ {z ∈ Pow(y). z ≲ succ(k #+ m)}"`
⟨*proof*⟩

lemma *well_ord_LL:* `"∃S. well_ord(LL,S)"`
⟨*proof*⟩

lemma *unique_superset_in_MM:*
"$v \in LL \Longrightarrow \exists! \ w.\ w \in MM \wedge v \subseteq w$"
⟨*proof*⟩

lemma *Int_in_LL:* "$w \in MM \Longrightarrow w \cap y \in LL$"
⟨*proof*⟩

lemma *in_LL_eq_Int:*
"$v \in LL \Longrightarrow v = (\text{THE } x.\ x \in MM \wedge v \subseteq x) \cap y$"
⟨*proof*⟩

lemma *unique_superset1:* "$a \in LL \Longrightarrow (\text{THE } x.\ x \in MM \wedge a \subseteq x) \in MM$"
⟨*proof*⟩

lemma *the_in_MM_subset:*
"$v \in LL \Longrightarrow (\text{THE } x.\ x \in MM \wedge v \subseteq x) \subseteq x \cup y$"
⟨*proof*⟩

lemma *GG_subset:* "$v \in LL \Longrightarrow GG\ `\ v \subseteq x$"
⟨*proof*⟩

lemma *nat_lepoll_ordertype:* "$nat \lesssim ordertype(y, R)$"
⟨*proof*⟩

lemma *ex_subset_eqpoll_n:* "$n \in nat \Longrightarrow \exists z.\ z \subseteq y \wedge n \approx z$"
⟨*proof*⟩

lemma *exists_proper_in_s:* "$u \in x \Longrightarrow \exists v \in s(u).\ succ(k) \lesssim v \cap y$"
⟨*proof*⟩

lemma *exists_in_MM:* "$u \in x \Longrightarrow \exists w \in MM.\ u \in w$"
⟨*proof*⟩

lemma *exists_in_LL:* "$u \in x \Longrightarrow \exists w \in LL.\ u \in GG`w$"
⟨*proof*⟩

lemma *OUN_eq_x:* "$well\_ord(LL,S) \Longrightarrow$

($\bigcup$b<ordertype(LL,S). GG ‘ (converse(ordermap(LL,S)) ‘ b)) = x"
⟨proof⟩

lemma *in_MM_eqpoll_n:* "w ∈ MM ⟹ w ≈ succ(k #+ m)"
⟨proof⟩

lemma *in_LL_eqpoll_n:* "w ∈ LL ⟹ succ(k) ≲ w"
⟨proof⟩

lemma *in_LL:* "w ∈ LL ⟹ w ⊆ (THE x. x ∈ MM ∧ w ⊆ x)"
⟨proof⟩

lemma *all_in_lepoll_m:*
"well_ord(LL,S) ⟹
∀b<ordertype(LL,S). GG ‘ (converse(ordermap(LL,S)) ‘ b) ≲ m"
⟨proof⟩

lemma *"conclusion":*
"∃a f. Ord(a) ∧ domain(f) = a ∧ ($\bigcup$b<a. f ‘ b) = x ∧ (∀b<a. f ‘ b ≲ m)"
⟨proof⟩

end

theorem *AC16_WO4:*
"⟦AC_Equiv.AC16(k #+ m, k); 0 < k; 0 < m; k ∈ nat; m ∈ nat⟧ ⟹ WO4(m)"
⟨proof⟩

end

theory *AC17_AC1*
imports *HH*
begin

lemma *AC0_AC1_lemma:* "⟦f:(∏X ∈ A. X); D ⊆ A⟧ ⟹ ∃g. g:(∏X ∈ D.

X)"
⟨*proof*⟩

lemma *AC0_AC1: "AC0 $\Longrightarrow$ AC1"*
⟨*proof*⟩

lemma *AC1_AC0: "AC1 $\Longrightarrow$ AC0"*
⟨*proof*⟩

lemma *AC1_AC17_lemma: "f $\in$ ($\prod$X $\in$ Pow(A) - {0}. X) $\Longrightarrow$ f $\in$ (Pow(A) - {0} -> A)"*
⟨*proof*⟩

lemma *AC1_AC17: "AC1 $\Longrightarrow$ AC17"*
⟨*proof*⟩

lemma *UN_eq_imp_well_ord:*
"⟦x - ($\bigcup$j $\in$ μ i. HH(λX $\in$ Pow(x)-{0}. {f`X}, x, i) = {x}.
HH(λX $\in$ Pow(x)-{0}. {f`X}, x, j)) = 0;
f $\in$ Pow(x)-{0} -> x⟧
$\Longrightarrow$ $\exists$r. well_ord(x,r)"
⟨*proof*⟩

lemma *not_AC1_imp_ex:*
"$\neg$AC1 $\Longrightarrow$ $\exists$A. $\forall$f $\in$ Pow(A)-{0} -> A. $\exists$u $\in$ Pow(A)-{0}. f`u $\notin$ u"
⟨*proof*⟩

lemma *AC17_AC1_aux1:*
"⟦$\forall$f $\in$ Pow(x) - {0} -> x. $\exists$u $\in$ Pow(x) - {0}. f`u$\notin$u;
$\exists$f $\in$ Pow(x)-{0}->x.
x - ($\bigcup$a $\in$ (μ i. HH(λX $\in$ Pow(x)-{0}. {f`X},x,i)={x}).
HH(λX $\in$ Pow(x)-{0}. {f`X},x,a)) = 0⟧
$\Longrightarrow$ P"
⟨*proof*⟩

lemma *AC17_AC1_aux2:*
```
"¬ (∃f ∈ Pow(x)-{0}->x. x - F(f) = 0)
  ⟹ (λf ∈ Pow(x)-{0}->x . x - F(f))
      ∈ (Pow(x) -{0} -> x) -> Pow(x) - {0}"
```
⟨*proof*⟩

lemma *AC17_AC1_aux3:*
```
"⟦f`Z ∈ Z; Z ∈ Pow(x)-{0}⟧
 ⟹ (λX ∈ Pow(x)-{0}. {f`X})`Z ∈ Pow(Z)-{0}"
```
⟨*proof*⟩

lemma *AC17_AC1_aux4:*
```
"∃f ∈ F. f`((λf ∈ F. Q(f))`f) ∈ (λf ∈ F. Q(f))`f
 ⟹ ∃f ∈ F. f`Q(f) ∈ Q(f)"
```
⟨*proof*⟩

lemma *AC17_AC1: "AC17 ⟹ AC1"*
⟨*proof*⟩

lemma *AC1_AC2_aux1:*
```
"⟦f:(∏X ∈ A. X);  B ∈ A;  0∉A⟧ ⟹ {f`B} ⊆ B ∩ {f`C. C ∈ A}"
```
⟨*proof*⟩

lemma *AC1_AC2_aux2:*
```
"⟦pairwise_disjoint(A); B ∈ A; C ∈ A; D ∈ B; D ∈ C⟧ ⟹ f`B
= f`C"
```
⟨*proof*⟩

lemma *AC1_AC2: "AC1 ⟹ AC2"*
⟨*proof*⟩

lemma *AC2_AC1_aux1: "0∉A ⟹ 0 ∉ {B*{B}. B ∈ A}"*
⟨*proof*⟩

lemma *AC2_AC1_aux2:* *"⟦X*{X} ∩ C = {y}; X ∈ A⟧*
```
          ⟹ (THE y. X*{X} ∩ C = {y}): X*A"
```
⟨*proof*⟩

lemma *AC2_AC1_aux3:*
"$\forall D \in$ {E*{E}. E $\in$ A}. $\exists y$. D $\cap$ C = {y}
$\Longrightarrow$ (λx $\in$ A. fst(THE z. (x*{x} $\cap$ C = {z}))) $\in$ ($\prod$X $\in$ A. X)"
⟨*proof*⟩

lemma *AC2_AC1:* "AC2 $\Longrightarrow$ AC1"
⟨*proof*⟩

lemma *empty_notin_images:* "0 $\notin$ {R``{x}. x $\in$ domain(R)}"
⟨*proof*⟩

lemma *AC1_AC4:* "AC1 $\Longrightarrow$ AC4"
⟨*proof*⟩

lemma *AC4_AC3_aux1:* "f $\in$ A->B $\Longrightarrow$ ($\bigcup$z $\in$ A. {z}*f`z) $\subseteq$ A*$\bigcup$(B)"
⟨*proof*⟩

lemma *AC4_AC3_aux2:* "domain($\bigcup$z $\in$ A. {z}*f(z)) = {a $\in$ A. f(a)$\neq$0}"
⟨*proof*⟩

lemma *AC4_AC3_aux3:* "x $\in$ A $\Longrightarrow$ ($\bigcup$z $\in$ A. {z}*f(z))``{x} = f(x)"
⟨*proof*⟩

lemma *AC4_AC3:* "AC4 $\Longrightarrow$ AC3"
⟨*proof*⟩

lemma *AC3_AC1_lemma:*
"b$\notin$A $\Longrightarrow$ ($\prod$x $\in$ {a $\in$ A. id(A)`a$\neq$b}. id(A)`x) = ($\prod$x $\in$ A. x)"
⟨*proof*⟩

lemma *AC3_AC1:* "AC3 $\Longrightarrow$ AC1"
⟨*proof*⟩

lemma *AC4_AC5: "AC4 ⟹ AC5"*
⟨*proof*⟩

lemma *AC5_AC4_aux1: "R ⊆ A*B ⟹ (λx ∈ R. fst(x)) ∈ R -> A"*
⟨*proof*⟩

lemma *AC5_AC4_aux2: "R ⊆ A*B ⟹ range(λx ∈ R. fst(x)) = domain(R)"*
⟨*proof*⟩

lemma *AC5_AC4_aux3: "⟦∃f ∈ A->C. P(f,domain(f)); A=B⟧ ⟹ ∃f ∈ B->C. P(f,B)"*
⟨*proof*⟩

lemma *AC5_AC4_aux4: "⟦R ⊆ A*B; g ∈ C->R; ∀x ∈ C. (λz ∈ R. fst(z))‘ (g‘x) = x⟧
⟹ (λx ∈ C. snd(g‘x)): (∏x ∈ C. R‘‘{x})"*
⟨*proof*⟩

lemma *AC5_AC4: "AC5 ⟹ AC4"*
⟨*proof*⟩

lemma *AC1_iff_AC6: "AC1 ⟷ AC6"*
⟨*proof*⟩

end

theory *AC18_AC19*
imports *AC_Equiv*
begin

definition
uu :: "i ⇒ i" **where**
"uu(a) ≡ {c ∪ {0}. c ∈ a}"

lemma *PROD_subsets:*
$"[\![f \in (\prod b \in \{P(a).\ a \in A\}.\ b);\ \ \forall a \in A.\ P(a) \text{<=} Q(a)]\!]$
$\Longrightarrow (\lambda a \in A.\ f`P(a)) \in (\prod a \in A.\ Q(a))"$
⟨*proof*⟩

lemma *lemma_AC18:*
$"[\![\forall A.\ 0 \notin A \longrightarrow (\exists f.\ f \in (\prod X \in A.\ X));\ A \neq 0]\!]$
$\Longrightarrow (\bigcap a \in A.\ \bigcup b \in B(a).\ X(a,\ b)) \subseteq$
$(\bigcup f \in \prod a \in A.\ B(a).\ \bigcap a \in A.\ X(a,\ f`a))"$
⟨*proof*⟩

lemma *AC1_AC18:* $"AC1 \Longrightarrow PROP\ AC18"$
⟨*proof*⟩

theorem **(in** *AC18)* *AC19*
⟨*proof*⟩

lemma *RepRep_conj:*
$"[\![A \neq 0;\ 0 \notin A]\!] \Longrightarrow \{uu(a).\ a \in A\} \neq 0 \wedge 0 \notin \{uu(a).\ a \in A\}"$
⟨*proof*⟩

lemma *lemma1_1:* $"[\![c \in a;\ x = c \cup \{0\};\ x \notin a]\!] \Longrightarrow x - \{0\} \in a"$
⟨*proof*⟩

lemma *lemma1_2:*
$"[\![f`(uu(a)) \notin a;\ f \in (\prod B \in \{uu(a).\ a \in A\}.\ B);\ a \in A]\!]$
$\Longrightarrow f`(uu(a))-\{0\} \in a"$
⟨*proof*⟩

lemma *lemma1:* $"\exists f.\ f \in (\prod B \in \{uu(a).\ a \in A\}.\ B) \Longrightarrow \exists f.\ f \in (\prod B$
$\in A.\ B)"$
⟨*proof*⟩

lemma *lemma2_1:* $"a \neq 0 \Longrightarrow 0 \in (\bigcup b \in uu(a).\ b)"$
⟨*proof*⟩

lemma *lemma2:* $"[\![A \neq 0;\ 0 \notin A]\!] \Longrightarrow (\bigcap x \in \{uu(a).\ a \in A\}.\ \bigcup b \in x.\ b) \neq$

0"
⟨*proof*⟩

lemma *AC19_AC1: "AC19 ⟹ AC1"*
⟨*proof*⟩

end

theory *DC*
imports *AC_Equiv Hartog Cardinal_aux*
begin

lemma *RepFun_lepoll: "Ord(a) ⟹ {P(b). b ∈ a} ≲ a"*
⟨*proof*⟩

Trivial in the presence of AC, but here we need a wellordering of X

lemma *image_Ord_lepoll: "⟦f ∈ X->Y; Ord(X)⟧ ⟹ f``X ≲ X"*
⟨*proof*⟩

lemma *range_subset_domain:*
*"⟦R ⊆ X*X; ⋀g. g ∈ X ⟹ ∃u. ⟨g,u⟩ ∈ R⟧*
⟹ range(R) ⊆ domain(R)"
⟨*proof*⟩

lemma *cons_fun_type: "g ∈ n->X ⟹ cons(⟨n,x⟩, g) ∈ succ(n) -> cons(x, X)"*
⟨*proof*⟩

lemma *cons_fun_type2:*
"⟦g ∈ n->X; x ∈ X⟧ ⟹ cons(⟨n,x⟩, g) ∈ succ(n) -> X"
⟨*proof*⟩

lemma *cons_image_n: "n ∈ nat ⟹ cons(⟨n,x⟩, g)``n = g``n"*
⟨*proof*⟩

lemma *cons_val_n: "g ∈ n->X ⟹ cons(⟨n,x⟩, g)`n = x"*
⟨*proof*⟩

lemma *cons_image_k: "k ∈ n ⟹ cons(⟨n,x⟩, g)``k = g``k"*
⟨*proof*⟩

lemma *cons_val_k: "⟦k ∈ n; g ∈ n->X⟧ ⟹ cons(⟨n,x⟩, g)`k = g`k"*
⟨*proof*⟩

lemma *domain_cons_eq_succ: "domain(f)=x ⟹ domain(cons(⟨x,y⟩, f)) = succ(x)"*
⟨*proof*⟩

lemma *restrict_cons_eq: "g ∈ n->X ⟹ restrict(cons(⟨n,x⟩, g), n) = g"*
⟨*proof*⟩

lemma *succ_in_succ: "⟦Ord(k); i ∈ k⟧ ⟹ succ(i) ∈ succ(k)"*
⟨*proof*⟩

lemma *restrict_eq_imp_val_eq:*
"⟦restrict(f, domain(g)) = g; x ∈ domain(g)⟧
⟹ f‘x = g‘x"
⟨*proof*⟩

lemma *domain_eq_imp_fun_type: "⟦domain(f)=A; f ∈ B->C⟧ ⟹ f ∈ A->C"*
⟨*proof*⟩

lemma *ex_in_domain: "⟦R ⊆ A * B; R ≠ 0⟧ ⟹ ∃x. x ∈ domain(R)"*
⟨*proof*⟩

definition
DC :: "i ⇒ o" **where**
*"DC(a) ≡ ∀X R. R ⊆ Pow(X)*X ∧*
(∀Y ∈ Pow(X). Y ≺ a ⟶ (∃x ∈ X. ⟨Y,x⟩ ∈ R))
⟶ (∃f ∈ a->X. ∀b<a. <f‘‘b,f‘b> ∈ R)"

definition
DC0 :: o **where**
*"DC0 ≡ ∀A B R. R ⊆ A*B ∧ R≠0 ∧ range(R) ⊆ domain(R)*
⟶ (∃f ∈ nat->domain(R). ∀n ∈ nat. <f‘n,f‘succ(n)>:R)"

definition
ff :: "[i, i, i, i] ⇒ i" **where**
"ff(b, X, Q, R) ≡
transrec(b, λc r. THE x. first(x, {x ∈ X. <r‘‘c, x> ∈ R}, Q))"

locale *DC0_imp =*
fixes *XX* **and** *RR* **and** *X* **and** *R*

assumes *all_ex: "∀Y ∈ Pow(X). Y ≺ nat ⟶ (∃x ∈ X. ⟨Y, x⟩ ∈ R)"*

defines *XX_def: "XX ≡ (⋃n ∈ nat. {f ∈ n->X. ∀k ∈ n. <f‘‘k, f‘k> ∈ R})"*
and *RR_def: "RR ≡ {⟨z1,z2⟩:XX*XX. domain(z2)=succ(domain(z1))*
∧ restrict(z2, domain(z1)) = z1}"
begin

lemma *lemma1_1: "RR ⊆ XX*XX"*
⟨*proof*⟩

lemma *lemma1_2: "RR ≠ 0"*
⟨*proof*⟩

lemma *lemma1_3: "range(RR) ⊆ domain(RR)"*
⟨*proof*⟩

lemma *lemma2:*
"⟦∀n ∈ nat. <f‘n, f‘succ(n)> ∈ RR; f ∈ nat -> XX; n ∈ nat⟧
⟹ ∃k ∈ nat. f‘succ(n) ∈ k -> X ∧ n ∈ k
∧ <f‘succ(n)‘‘n, f‘succ(n)‘n> ∈ R"
⟨*proof*⟩

lemma *lemma3_1:*
"⟦∀n ∈ nat. <f‘n, f‘succ(n)> ∈ RR; f ∈ nat -> XX; m ∈ nat⟧
⟹ {f‘succ(x)‘x. x ∈ m} = {f‘succ(m)‘x. x ∈ m}"
⟨*proof*⟩

lemma *lemma3:*
"⟦∀n ∈ nat. <f‘n, f‘succ(n)> ∈ RR; f ∈ nat -> XX; m ∈ nat⟧
⟹ (λx ∈ nat. f‘succ(x)‘x) ‘‘ m = f‘succ(m)‘‘m"
⟨*proof*⟩

end

theorem *DC0_imp_DC_nat: "DC0 ⟹ DC(nat)"*
⟨*proof*⟩

lemma *singleton_in_funs:*
```
 "x ∈ X ⟹ {⟨0,x⟩} ∈
          (⋃n ∈ nat. {f ∈ succ(n)->X. ∀k ∈ n. <f`k, f`succ(k)> ∈
R})"
```
⟨*proof*⟩

locale *imp_DC0 =*
fixes *XX* **and** *RR* **and** *x* **and** *R* **and** *f* **and** *allRR*
defines *XX_def:*
```
 "XX ≡ (⋃n ∈ nat.
                  {f ∈ succ(n)->domain(R). ∀k ∈ n. <f`k, f`succ(k)>
∈ R})"
```
and *RR_def:*
```
   "RR ≡ {⟨z1,z2⟩:Fin(XX)*XX.
             (domain(z2)=succ(⋃f ∈ z1. domain(f))
               ∧ (∀f ∈ z1. restrict(z2, domain(f)) = f))
             | (¬ (∃g ∈ XX. domain(g)=succ(⋃f ∈ z1. domain(f))

                ∧ (∀f ∈ z1. restrict(g, domain(f)) = f)) ∧ z2={⟨0,x⟩})}"
```
and *allRR_def:*
```
  "allRR ≡ ∀b<nat.
              <f``b, f`b> ∈
               {⟨z1,z2⟩∈Fin(XX)*XX. (domain(z2)=succ(⋃f ∈ z1. domain(f))
                             ∧ (⋃f ∈ z1. domain(f)) = b
                             ∧ (∀f ∈ z1. restrict(z2,domain(f))
= f))}"
```
begin

lemma *lemma4:*
```
   "⟦range(R) ⊆ domain(R);  x ∈ domain(R)⟧
    ⟹ RR ⊆ Pow(XX)*XX ∧
         (∀Y ∈ Pow(XX). Y ≺ nat ⟶ (∃x ∈ XX. ⟨Y,x⟩:RR))"
```
⟨*proof*⟩

lemma *UN_image_succ_eq:*
```
   "⟦f ∈ nat->X; n ∈ nat⟧
    ⟹ (⋃x ∈ f``succ(n). P(x)) =  P(f`n) ∪ (⋃x ∈ f``n. P(x))"
```
⟨*proof*⟩

lemma *UN_image_succ_eq_succ:*

```
"⟦(⋃x ∈ f``n. P(x)) = y; P(f`n) = succ(y);
    f ∈ nat -> X; n ∈ nat⟧ ⟹ (⋃x ∈ f``succ(n). P(x)) = succ(y)"
⟨proof⟩
```

lemma *apply_domain_type:*
```
"⟦h ∈ succ(n) -> D;  n ∈ nat; domain(h)=succ(y)⟧ ⟹ h`y ∈ D"
⟨proof⟩
```

lemma *image_fun_succ:*
```
"⟦h ∈ nat -> X; n ∈ nat⟧ ⟹ h``succ(n) = cons(h`n, h``n)"
⟨proof⟩
```

lemma *f_n_type:*
```
"⟦domain(f`n) = succ(k); f ∈ nat -> XX;  n ∈ nat⟧
 ⟹ f`n ∈ succ(k) -> domain(R)"
⟨proof⟩
```

lemma *f_n_pairs_in_R [rule_format]:*
```
"⟦h ∈ nat -> XX;  domain(h`n) = succ(k);  n ∈ nat⟧
 ⟹ ∀i ∈ k. <h`n`i, h`n`succ(i)> ∈ R"
⟨proof⟩
```

lemma *restrict_cons_eq_restrict:*
```
"⟦restrict(h, domain(u))=u;  h ∈ n->X;  domain(u) ⊆ n⟧
 ⟹ restrict(cons(⟨n, y⟩, h), domain(u)) = u"
⟨proof⟩
```

lemma *all_in_image_restrict_eq:*
```
"⟦∀x ∈ f``n. restrict(f`n, domain(x))=x;
    f ∈ nat -> XX;
    n ∈ nat;  domain(f`n) = succ(n);
    (⋃x ∈ f``n. domain(x)) ⊆ n⟧
 ⟹ ∀x ∈ f``succ(n). restrict(cons(<succ(n),y>, f`n), domain(x))
= x"
⟨proof⟩
```

lemma *simplify_recursion:*
```
"⟦∀b<nat. <f``b, f`b> ∈ RR;
    f ∈ nat -> XX; range(R) ⊆ domain(R); x ∈ domain(R)⟧
 ⟹ allRR"
⟨proof⟩
```

lemma *lemma2:*
```
"⟦allRR; f ∈ nat->XX; range(R) ⊆ domain(R); x ∈ domain(R); n ∈
nat⟧
 ⟹ f`n ∈ succ(n) -> domain(R) ∧ (∀i ∈ n. <f`n`i, f`n`succ(i)>:R)"
⟨proof⟩
```

lemma *lemma3*:
"⟦allRR; f ∈ nat->XX; n∈nat; range(R) ⊆ domain(R); x ∈ domain(R)⟧
⟹ f‘n‘n = f‘succ(n)‘n"
⟨*proof*⟩

end

theorem *DC_nat_imp_DC0*: "DC(nat) ⟹ DC0"
⟨*proof*⟩

lemma *fun_Ord_inj*:
"⟦f ∈ a->X; Ord(a);
⋀b c. ⟦b<c; c ∈ a⟧ ⟹ f‘b≠f‘c⟧
⟹ f ∈ inj(a, X)"
⟨*proof*⟩

lemma *value_in_image*: "⟦f ∈ X->Y; A ⊆ X; a ∈ A⟧ ⟹ f‘a ∈ f‘‘A"
⟨*proof*⟩

lemma *lesspoll_lemma*: "⟦¬ A ≺ B; C ≺ B⟧ ⟹ A - C ≠ 0"
⟨*proof*⟩

theorem *DC_WO3*: "(∀K. Card(K) ⟶ DC(K)) ⟹ WO3"
⟨*proof*⟩

lemma *images_eq*:
"⟦∀x ∈ A. f‘x=g‘x; f ∈ Df->Cf; g ∈ Dg->Cg; A ⊆ Df; A ⊆ Dg⟧
⟹ f‘‘A = g‘‘A"
⟨*proof*⟩

lemma *lam_images_eq*:
"⟦Ord(a); b ∈ a⟧ ⟹ (λx ∈ a. h(x))‘‘b = (λx ∈ b. h(x))‘‘b"
⟨*proof*⟩

lemma *lam_type_RepFun*: "(λb ∈ a. h(b)) ∈ a -> {h(b). b ∈ a}"
⟨*proof*⟩

lemma *lemmaX*:
"⟦∀Y ∈ Pow(X). Y ≺ K ⟶ (∃x ∈ X. ⟨Y, x⟩ ∈ R);
b ∈ K; Z ∈ Pow(X); Z ≺ K⟧

```
    ⟹ {x ∈ X. ⟨Z,x⟩ ∈ R} ≠ 0"
⟨proof⟩
```

lemma `WO1_DC_lemma:`

```
  "⟦Card(K); well_ord(X,Q);
    ∀Y ∈ Pow(X). Y ≺ K ⟶ (∃x ∈ X. ⟨Y, x⟩ ∈ R); b ∈ K⟧
    ⟹ ff(b, X, Q, R) ∈ {x ∈ X. <(λc ∈ b. ff(c, X, Q, R))``b, x> ∈
R}"
⟨proof⟩
```

theorem `WO1_DC_Card: "WO1 ⟹ ∀K. Card(K) ⟶ DC(K)"`
⟨*proof*⟩

end

References

[1] Lawrence C. Paulson and Krzysztof Grąbczewski. Mechanizing set theory: Cardinal arithmetic and the axiom of choice. *Journal of Automated Reasoning*, 17(3):291–323, December 1996.

[2] Herman Rubin and Jean E. Rubin. *Equivalents of the Axiom of Choice, II.* North-Holland, 1985.