

IMP — A WHILE-language and two semantics

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Abstract

The formalization of the denotational and operational semantics of a simple while-language together with an equivalence proof between the two semantics. The whole development essentially formalizes/transcribes chapters 2 and 5 of [1]. A much extended version of this development is found in HOL/IMP of the Isabelle distribution.

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1 Arithmetic expressions, boolean expressions, commands

theory Com imports ZF begin

1.1 Arithmetic expressions

consts

loc :: i

aexp :: i

datatype \subseteq "univ(loc \cup (nat \rightarrow nat) \cup ((nat \times nat) \rightarrow nat))"

aexp = N ("n \in nat")

```

| X ("x ∈ loc")
| Op1 ("f ∈ nat -> nat", "a ∈ aexp")
| Op2 ("f ∈ (nat × nat) -> nat", "a0 ∈ aexp", "a1 ∈ aexp")

consts evala :: i

abbreviation
  evala_syntax :: "[i, i] ⇒ o"      (infixl <-a-> 50)
  where "p -a-> n ≡ ⟨p,n⟩ ∈ evala"

inductive
  domains "evala" ⊆ "(aexp × (loc -> nat)) × nat"
  intros
    N: "[n ∈ nat; sigma ∈ loc->nat] ⇒ ⟨N(n),sigma⟩ -a-> n"
    X: "[x ∈ loc; sigma ∈ loc->nat] ⇒ ⟨X(x),sigma⟩ -a-> sigma'x"
    Op1: "[⟨e,sigma⟩ -a-> n; f ∈ nat -> nat] ⇒ ⟨Op1(f,e),sigma⟩ -a-> f'n"
    Op2: "[⟨e0,sigma⟩ -a-> n0; ⟨e1,sigma⟩ -a-> n1; f ∈ (nat×nat) -> nat]
           ⇒ ⟨Op2(f,e0,e1),sigma⟩ -a-> f'⟨n0,n1⟩"
  type_intros aexp.intros apply_funtype

```

1.2 Boolean expressions

```

consts bexp :: i

datatype ⊆ "univ(aexp ∪ ((nat × nat)->bool))"
  bexp = true
        | false
        | ROp ("f ∈ (nat × nat)->bool", "a0 ∈ aexp", "a1 ∈ aexp")
        | noti ("b ∈ bexp")
        | andi ("b0 ∈ bexp", "b1 ∈ bexp")      (infixl <andi> 60)
        | ori ("b0 ∈ bexp", "b1 ∈ bexp")      (infixl <ori> 60)

consts evalb :: i

abbreviation
  evalb_syntax :: "[i,i] ⇒ o"      (infixl <-b-> 50)
  where "p -b-> b ≡ ⟨p,b⟩ ∈ evalb"

inductive
  domains "evalb" ⊆ "(bexp × (loc -> nat)) × bool"
  intros
    true: "[sigma ∈ loc -> nat] ⇒ ⟨true,sigma⟩ -b-> 1"
    false: "[sigma ∈ loc -> nat] ⇒ ⟨false,sigma⟩ -b-> 0"
    ROp: "[⟨a0,sigma⟩ -a-> n0; ⟨a1,sigma⟩ -a-> n1; f ∈ (nat*nat)->bool]
           ⇒ ⟨ROp(f,a0,a1),sigma⟩ -b-> f'⟨n0,n1⟩"
    noti: "[⟨b,sigma⟩ -b-> w] ⇒ ⟨noti(b),sigma⟩ -b-> not(w)"
    andi: "[⟨b0,sigma⟩ -b-> w0; ⟨b1,sigma⟩ -b-> w1]
           ⇒ ⟨b0 andi b1,sigma⟩ -b-> (w0 and w1)"

```

```

ori:    "[⟨b0,sigma⟩ -b-> w0; ⟨b1,sigma⟩ -b-> w1]
        ⇒ <b0 ori b1,sigma> -b-> (w0 or w1)"
type_intros  bexp.intros
            apply_funtype and_type or_type bool_1I bool_0I not_type
type_elims   evala.dom_subset [THEN subsetD, elim_format]

```

1.3 Commands

```

consts com :: i
datatype com =
  skip                                (<skip> [])
| assignment ("x ∈ loc", "a ∈ aexp") (infixl <:=> 60)
| semicolon ("c0 ∈ com", "c1 ∈ com")  (<;_> [60, 60] 10)
| while ("b ∈ bexp", "c ∈ com")        (<while _ do _> 60)
| "if" ("b ∈ bexp", "c0 ∈ com", "c1 ∈ com") (<if _ then _ else _> 60)

```

```

consts evalc :: i

```

```

abbreviation
  evalc_syntax :: "[i, i] ⇒ o"    (infixl <-c->> 50)
  where "p -c-> s ≡ ⟨p,s⟩ ∈ evalc"

```

inductive

```

domains "evalc" ⊆ "(com × (loc -> nat)) × (loc -> nat)"

```

intros

```

skip:    "[sigma ∈ loc -> nat] ⇒ <skip,sigma> -c-> sigma"

assign:  "[m ∈ nat; x ∈ loc; ⟨a,sigma⟩ -a-> m]
          ⇒ <x := a,sigma> -c-> sigma(x:=m)"

semi:    "[⟨c0,sigma⟩ -c-> sigma2; ⟨c1,sigma2⟩ -c-> sigma1]
          ⇒ <c0; c1, sigma> -c-> sigma1"

if1:     "[b ∈ bexp; c1 ∈ com; sigma ∈ loc->nat;
          ⟨b,sigma⟩ -b-> 1; ⟨c0,sigma⟩ -c-> sigma1]
          ⇒ <if b then c0 else c1, sigma> -c-> sigma1"

if0:     "[b ∈ bexp; c0 ∈ com; sigma ∈ loc->nat;
          ⟨b,sigma⟩ -b-> 0; ⟨c1,sigma⟩ -c-> sigma1]
          ⇒ <if b then c0 else c1, sigma> -c-> sigma1"

while0:  "[c ∈ com; ⟨b, sigma⟩ -b-> 0]
          ⇒ <while b do c,sigma> -c-> sigma"

while1:  "[c ∈ com; ⟨b,sigma⟩ -b-> 1; ⟨c,sigma⟩ -c-> sigma2;
          <while b do c, sigma2> -c-> sigma1]
          ⇒ <while b do c, sigma> -c-> sigma1"

```

```

type_intros  com.intros update_type

```

```

type_elims   evala.dom_subset [THEN subsetD, elim_format]
              evalb.dom_subset [THEN subsetD, elim_format]

```

1.4 Misc lemmas

```

lemmas evala_1 [simp] = evala.dom_subset [THEN subsetD, THEN SigmaD1, THEN SigmaD1]
  and evala_2 [simp] = evala.dom_subset [THEN subsetD, THEN SigmaD1, THEN SigmaD2]
  and evala_3 [simp] = evala.dom_subset [THEN subsetD, THEN SigmaD2]

```

```

lemmas evalb_1 [simp] = evalb.dom_subset [THEN subsetD, THEN SigmaD1, THEN SigmaD1]
  and evalb_2 [simp] = evalb.dom_subset [THEN subsetD, THEN SigmaD1, THEN SigmaD2]
  and evalb_3 [simp] = evalb.dom_subset [THEN subsetD, THEN SigmaD2]

```

```

lemmas evalc_1 [simp] = evalc.dom_subset [THEN subsetD, THEN SigmaD1, THEN SigmaD1]
  and evalc_2 [simp] = evalc.dom_subset [THEN subsetD, THEN SigmaD1, THEN SigmaD2]
  and evalc_3 [simp] = evalc.dom_subset [THEN subsetD, THEN SigmaD2]

```

```

inductive_cases
  evala_N_E [elim!]: "<N(n),sigma> -a-> i"
  and evala_X_E [elim!]: "<X(x),sigma> -a-> i"
  and evala_Op1_E [elim!]: "<Op1(f,e),sigma> -a-> i"
  and evala_Op2_E [elim!]: "<Op2(f,a1,a2),sigma> -a-> i"

```

end

2 Denotational semantics of expressions and commands

theory Denotation imports Com begin

2.1 Definitions

consts

```

A    :: "i ⇒ i ⇒ i"
B    :: "i ⇒ i ⇒ i"
C    :: "i ⇒ i"

```

definition

```

Gamma :: "[i,i,i] ⇒ i"  (<Γ>) where
  "Γ(b,cden) ≡
    (λphi. {io ∈ (phi O cden). B(b,fst(io))=1} ∪
      {io ∈ id(loc->nat). B(b,fst(io))=0})"

```

primrec

```

"A(N(n), sigma) = n"
"A(X(x), sigma) = sigma`x"
"A(Op1(f,a), sigma) = f`A(a,sigma)"
"A(Op2(f,a0,a1), sigma) = f`<A(a0,sigma),A(a1,sigma)>"

```

primrec

```

"B(true, sigma) = 1"
"B(false, sigma) = 0"
"B(ROp(f,a0,a1), sigma) = f'<A(a0,sigma),A(a1,sigma)>"
"B(noti(b), sigma) = not(B(b,sigma))"
"B(b0 andi b1, sigma) = B(b0,sigma) and B(b1,sigma)"
"B(b0 ori b1, sigma) = B(b0,sigma) or B(b1,sigma)"

```

primrec

```

"C(skip) = id(loc->nat)"
"C(x := a) =
  {io ∈ (loc->nat) × (loc->nat). snd(io) = fst(io)(x := A(a,fst(io)))}"
"C(c0; c1) = C(c1) ∪ C(c0)"
"C(if b then c0 else c1) =
  {io ∈ C(c0). B(b,fst(io)) = 1} ∪ {io ∈ C(c1). B(b,fst(io)) = 0}"
"C(while b do c) = lfp((loc->nat) × (loc->nat), Γ(b,C(c)))"

```

2.2 Misc lemmas

lemma *A_type [TC]*: " $\llbracket a \in \text{aexp}; \text{sigma} \in \text{loc} \rightarrow \text{nat} \rrbracket \implies A(a, \text{sigma}) \in \text{nat}$ "
 $\langle \text{proof} \rangle$

lemma *B_type [TC]*: " $\llbracket b \in \text{bexp}; \text{sigma} \in \text{loc} \rightarrow \text{nat} \rrbracket \implies B(b, \text{sigma}) \in \text{bool}$ "
 $\langle \text{proof} \rangle$

lemma *C_subset*: " $c \in \text{com} \implies C(c) \subseteq (\text{loc} \rightarrow \text{nat}) \times (\text{loc} \rightarrow \text{nat})$ "
 $\langle \text{proof} \rangle$

lemma *C_type_D [dest]*:
 $\llbracket \langle x, y \rangle \in C(c); c \in \text{com} \rrbracket \implies x \in \text{loc} \rightarrow \text{nat} \wedge y \in \text{loc} \rightarrow \text{nat}$
 $\langle \text{proof} \rangle$

lemma *C_type_fst [dest]*: " $\llbracket x \in C(c); c \in \text{com} \rrbracket \implies \text{fst}(x) \in \text{loc} \rightarrow \text{nat}$ "
 $\langle \text{proof} \rangle$

lemma *Gamma_bnd_mono*:
 $\text{cden} \subseteq (\text{loc} \rightarrow \text{nat}) \times (\text{loc} \rightarrow \text{nat})$
 $\implies \text{bnd_mono } ((\text{loc} \rightarrow \text{nat}) \times (\text{loc} \rightarrow \text{nat}), \Gamma(b, \text{cdn}))$
 $\langle \text{proof} \rangle$

end

3 Equivalence

theory *Equiv* **imports** *Denotation Com* **begin**

lemma *aexp_iff [rule_format]*:
 $\llbracket a \in \text{aexp}; \text{sigma}: \text{loc} \rightarrow \text{nat} \rrbracket$
 $\implies \forall n. \langle a, \text{sigma} \rangle \text{-a-} n \iff A(a, \text{sigma}) = n$
 $\langle \text{proof} \rangle$

```

declare aexp_iff [THEN iffD1, simp]
        aexp_iff [THEN iffD2, intro!]

inductive_cases [elim!]:
  "<true,sigma> -b-> x"
  "<false,sigma> -b-> x"
  "<ROp(f,a0,a1),sigma> -b-> x"
  "<noti(b),sigma> -b-> x"
  "<b0 andi b1,sigma> -b-> x"
  "<b0 ori b1,sigma> -b-> x"

lemma bexp_iff [rule_format]:
  "[b ∈ bexp; sigma: loc -> nat]
   ⇒ ∀ w. <b,sigma> -b-> w ⇔ B(b,sigma) = w"
  <proof>

declare bexp_iff [THEN iffD1, simp]
        bexp_iff [THEN iffD2, intro!]

lemma com1: "<c,sigma> -c-> sigma' ⇒ <sigma,sigma'> ∈ C(c)"
  <proof>

declare B_type [intro!] A_type [intro!]
declare evalc.intros [intro]

lemma com2 [rule_format]: "c ∈ com ⇒ ∀ x ∈ C(c). <c,fst(x)> -c-> snd(x)"
  <proof>

3.1 Main theorem

theorem com_equivalence:
  "c ∈ com ⇒ C(c) = {io ∈ (loc->nat) × (loc->nat). <c,fst(io)> -c-> snd(io)}"
  <proof>

end

```

References

- [1] Glynn Winskel. *The Formal Semantics of Programming Languages*. 1993.