Computational Algebra

December 17, 2025

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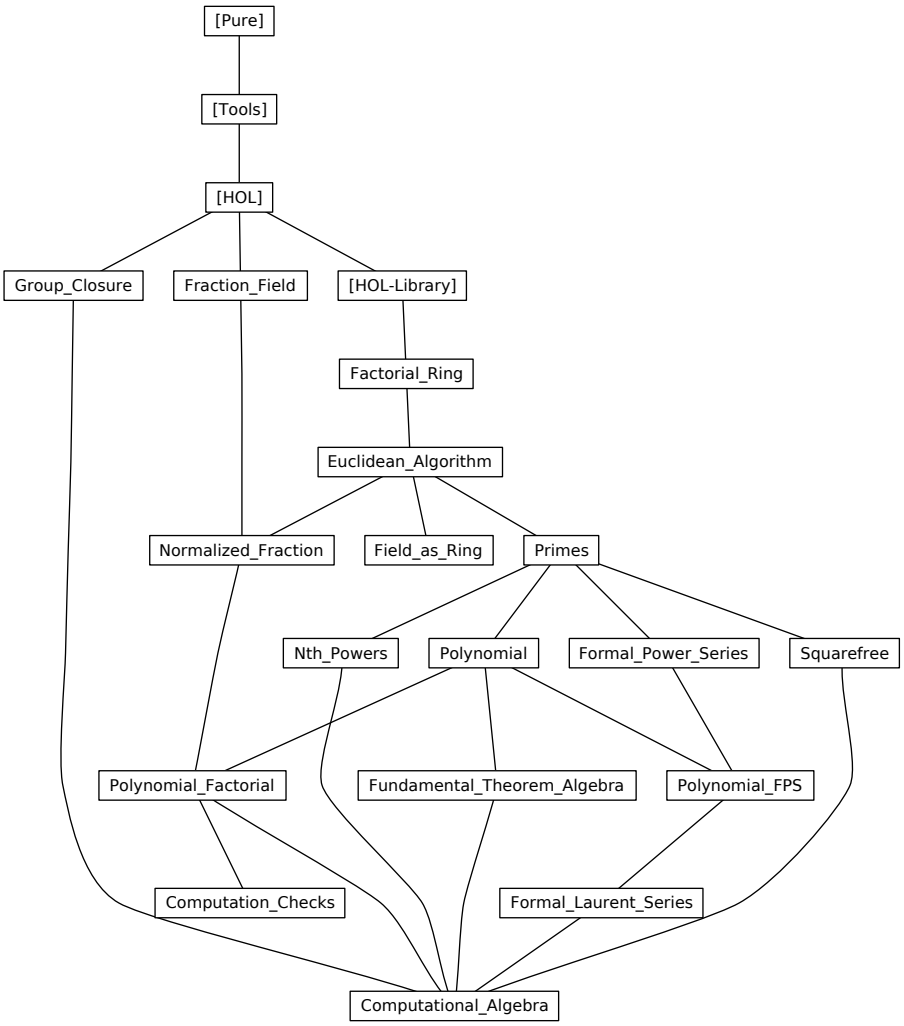

1 Factorial (semi)rings

theory *Factorial-Ring*
imports
Main
HOL−Library.Multiset
begin

unbundle *multiset.lifting*

1.1 Irreducible and prime elements

context *comm-semiring-1*
begin

definition *irreducible* :: *'a* ⇒ *bool* **where**
irreducible p ⟷ *p* ≠ *0* ∧ ¬*p dvd 1* ∧ (∀ *a b*. *p* = *a* ∗ *b* ⟶ *a dvd 1* ∨ *b dvd 1*)

lemma *not-irreducible-zero* [*simp*]: ¬*irreducible 0*
⟨*proof*⟩

lemma *irreducible-not-unit*: *irreducible p* ⟹ ¬*p dvd 1*
⟨*proof*⟩

lemma *not-irreducible-one* [*simp*]: ¬*irreducible 1*
⟨*proof*⟩

lemma *irreducibleI*:
p ≠ *0* ⟹ ¬*p dvd 1* ⟹ (⋀*a b*. *p* = *a* ∗ *b* ⟹ *a dvd 1* ∨ *b dvd 1*) ⟹ *irreducible p*
⟨*proof*⟩

lemma *irreducibleD*: *irreducible p* ⟹ *p* = *a* ∗ *b* ⟹ *a dvd 1* ∨ *b dvd 1*
⟨*proof*⟩

lemma *irreducible-mono*:
assumes *irr*: *irreducible b* **and** *a dvd b* ¬*a dvd 1*
shows *irreducible a*
⟨*proof*⟩

lemma *irreducible-multD*:
assumes *l*: *irreducible* (*a*∗*b*)
shows *a dvd 1* ∧ *irreducible b* ∨ *b dvd 1* ∧ *irreducible a*
⟨*proof*⟩

lemma *irreducible-power-iff* [*simp*]:
irreducible (*p* ^ *n*) ⟷ *irreducible p* ∧ *n* = *1*
⟨*proof*⟩

definition *prime-elem* :: *'a* ⇒ *bool* **where**
prime-elem p ⟷ *p* ≠ *0* ∧ ¬*p dvd 1* ∧ (∀ *a b*. *p dvd* (*a* ∗ *b*) ⟶ *p dvd a* ∨ *p dvd b*)

lemma *not-prime-elem-zero* [*simp*]: ¬*prime-elem 0*
⟨*proof*⟩

lemma *prime-elem-not-unit*: *prime-elem p* ⟹ ¬*p dvd 1*
⟨*proof*⟩

lemma *prime-elemI*:
p ≠ *0* ⟹ ¬*p dvd 1* ⟹ (⋀*a b*. *p dvd* (*a* ∗ *b*) ⟹ *p dvd a* ∨ *p dvd b*) ⟹ *prime-elem p*
⟨*proof*⟩

lemma *prime-elem-dvd-multD*:
prime-elem p ⟹ *p dvd* (*a* ∗ *b*) ⟹ *p dvd a* ∨ *p dvd b*
⟨*proof*⟩

lemma *prime-elem-dvd-mult-iff*:
prime-elem p ⟹ *p dvd* (*a* ∗ *b*) ⟷ *p dvd a* ∨ *p dvd b*
⟨*proof*⟩

lemma *not-prime-elem-one* [*simp*]:
¬ *prime-elem 1*
⟨*proof*⟩

lemma *prime-elem-not-zeroI*:
assumes *prime-elem p*
shows *p* ≠ *0*
⟨*proof*⟩

lemma *prime-elem-dvd-power*:
prime-elem p ⟹ *p dvd x* ^ *n* ⟹ *p dvd x*
⟨*proof*⟩

lemma *prime-elem-dvd-power-iff*:
prime-elem p ⟹ *n* > *0* ⟹ *p dvd x* ^ *n* ⟷ *p dvd x*
⟨*proof*⟩

lemma *prime-elem-imp-nonzero* [*simp*]:
ASSUMPTION (*prime-elem x*) ⟹ *x* ≠ *0*
⟨*proof*⟩

lemma *prime-elem-imp-not-one* [*simp*]:
ASSUMPTION (*prime-elem x*) ⟹ *x* ≠ *1*
⟨*proof*⟩

end

lemma (**in** *normalization-semidom*) *irreducible-cong*:
assumes *normalize a* = *normalize b*
shows *irreducible a* ⟷ *irreducible b*
⟨*proof*⟩

lemma (**in** *normalization-semidom*) *associatedE1*:
assumes *normalize a* = *normalize b*
obtains *u* **where** *is-unit u a* = *u* ∗ *b*
⟨*proof*⟩

lemma (**in** *normalization-semidom*) *associatedE2*:
assumes *normalize a* = *normalize b*
obtains *u* **where** *is-unit u b* = *u* ∗ *a*
⟨*proof*⟩

lemma (**in** *normalization-semidom*) *normalize-power-normalize*:
normalize (*normalize x* ^ *n*) = *normalize* (*x* ^ *n*)
⟨*proof*⟩

context *algebraic-semidom*
begin

lemma *prime-elem-imp-irreducible*:
assumes *prime-elem p*
shows *irreducible p*
⟨*proof*⟩

lemma (**in** *algebraic-semidom*) *unit-imp-no-irreducible-divisors*:
assumes *is-unit x irreducible p*
shows ¬*p dvd x*
⟨*proof*⟩

lemma *unit-imp-no-prime-divisors*:
assumes *is-unit x prime-elem p*
shows ¬*p dvd x*
⟨*proof*⟩

lemma *prime-elem-mono*:
assumes *prime-elem p* ¬*q dvd 1 q dvd p*
shows *prime-elem q*
⟨*proof*⟩

lemma *irreducibleD′*:
assumes *irreducible a b dvd a*
shows *a dvd b* ∨ *is-unit b*

⟨*proof*⟩

lemma *irreducibleI'*:
assumes $a \neq 0$ $\neg$*is-unit* a $\bigwedge b.\ b$ *dvd* $a \Longrightarrow a$ *dvd* $b \vee$ *is-unit* b
shows *irreducible* a
⟨*proof*⟩

lemma *irreducible-altdef*:
irreducible $x \longleftrightarrow x \neq 0 \wedge \neg$*is-unit* $x \wedge (\forall b.\ b$ *dvd* $x \longrightarrow x$ *dvd* $b \vee$ *is-unit* $b)$
⟨*proof*⟩

lemma *prime-elem-multD*:
assumes *prime-elem* $(a * b)$
shows *is-unit* $a \vee$ *is-unit* b
⟨*proof*⟩

lemma *prime-elemD2*:
assumes *prime-elem* p **and** a *dvd* p **and** $\neg$ *is-unit* a
shows p *dvd* a
⟨*proof*⟩

lemma *prime-elem-dvd-prod-msetE*:
assumes *prime-elem* p
assumes *dvd*: p *dvd prod-mset* A
obtains a **where** $a \in\# A$ **and** p *dvd* a
⟨*proof*⟩

context
begin

lemma *prime-elem-powerD*:
assumes *prime-elem* $(p \mathbin{\hat{}} n)$
shows *prime-elem* $p \wedge n = 1$
⟨*proof*⟩

lemma *prime-elem-power-iff*:
prime-elem $(p \mathbin{\hat{}} n) \longleftrightarrow$ *prime-elem* $p \wedge n = 1$
⟨*proof*⟩

end

lemma *irreducible-mult-unit-left*:
is-unit $a \Longrightarrow$ *irreducible* $(a * p) \longleftrightarrow$ *irreducible* p
⟨*proof*⟩

lemma *prime-elem-mult-unit-left*:
is-unit $a \Longrightarrow$ *prime-elem* $(a * p) \longleftrightarrow$ *prime-elem* p
⟨*proof*⟩

lemma *prime-elem-dvd-cases*:
assumes *pk*: $p*k$ *dvd* $m*n$ **and** *p*: *prime-elem p*
shows $(\exists\, x.\ k\ dvd\ x*n \wedge m = p*x) \vee (\exists\, y.\ k\ dvd\ m*y \wedge n = p*y)$
⟨*proof*⟩

lemma *prime-elem-power-dvd-prod*:
assumes *pc*: $p\hat{}c$ *dvd* $m*n$ **and** *p*: *prime-elem p*
shows $\exists\, a\ b.\ a+b = c \wedge p\hat{}a\ dvd\ m \wedge p\hat{}b\ dvd\ n$
⟨*proof*⟩

lemma *prime-elem-power-dvd-cases*:
assumes $p \hat{} c$ *dvd* $m * n$ **and** $a + b = Suc\ c$ **and** *prime-elem p*
shows $p \hat{} a$ *dvd* $m \vee p \hat{} b$ *dvd n*
⟨*proof*⟩

lemma *prime-elem-not-unit'* [*simp*]:
ASSUMPTION (*prime-elem x*) $\Longrightarrow \neg$*is-unit x*
⟨*proof*⟩

lemma *prime-elem-dvd-power-iff*:
assumes *prime-elem p*
shows $p\ dvd\ a \hat{} n \longleftrightarrow p\ dvd\ a \wedge n > 0$
⟨*proof*⟩

lemma *prime-power-dvd-multD*:
assumes *prime-elem p*
assumes $p \hat{} n\ dvd\ a * b$ **and** $n > 0$ **and** $\neg\ p\ dvd\ a$
shows $p \hat{} n\ dvd\ b$
⟨*proof*⟩

end

1.2 Generalized primes: normalized prime elements

context *normalization-semidom*
begin

lemma *irreducible-normalized-divisors*:
assumes *irreducible x y dvd x normalize y = y*
shows $y = 1 \vee y =$ *normalize x*
⟨*proof*⟩

lemma *irreducible-normalize-iff* [*simp*]: *irreducible* (*normalize x*) = *irreducible x*
⟨*proof*⟩

lemma *prime-elem-normalize-iff* [*simp*]: *prime-elem* (*normalize x*) = *prime-elem x*
⟨*proof*⟩

lemma *prime-elem-associated*:
assumes *prime-elem p* **and** *prime-elem q* **and** *q dvd p*
shows *normalize q = normalize p*
⟨*proof*⟩

definition *prime* :: *'a ⇒ bool* **where**
prime p ⟷ prime-elem p ∧ normalize p = p

lemma *not-prime-0* [*simp*]: *¬prime 0* ⟨*proof*⟩

lemma *not-prime-unit*: *is-unit x ⟹ ¬prime x*
⟨*proof*⟩

lemma *not-prime-1* [*simp*]: *¬prime 1* ⟨*proof*⟩

lemma *primeI*: *prime-elem x ⟹ normalize x = x ⟹ prime x*
⟨*proof*⟩

lemma *prime-imp-prime-elem* [*dest*]: *prime p ⟹ prime-elem p*
⟨*proof*⟩

lemma *normalize-prime*: *prime p ⟹ normalize p = p*
⟨*proof*⟩

lemma *prime-normalize-iff* [*simp*]: *prime (normalize p) ⟷ prime-elem p*
⟨*proof*⟩

lemma *prime-power-iff*:
prime (p ^ n) ⟷ prime p ∧ n = 1
⟨*proof*⟩

lemma *prime-imp-nonzero* [*simp*]:
ASSUMPTION (prime x) ⟹ x ≠ 0
⟨*proof*⟩

lemma *prime-imp-not-one* [*simp*]:
ASSUMPTION (prime x) ⟹ x ≠ 1
⟨*proof*⟩

lemma *prime-not-unit'* [*simp*]:
ASSUMPTION (prime x) ⟹ ¬is-unit x
⟨*proof*⟩

lemma *prime-normalize'* [*simp*]: *ASSUMPTION (prime x) ⟹ normalize x = x*
⟨*proof*⟩

lemma *unit-factor-prime*: *prime x ⟹ unit-factor x = 1*
⟨*proof*⟩

lemma *unit-factor-prime'* [*simp*]: *ASSUMPTION* (*prime x*) $\Longrightarrow$ *unit-factor x = 1*
⟨*proof*⟩

lemma *prime-imp-prime-elem'* [*simp*]: *ASSUMPTION* (*prime x*) $\Longrightarrow$ *prime-elem x*
⟨*proof*⟩

lemma *prime-dvd-multD*: *prime p* $\Longrightarrow$ *p dvd a * b* $\Longrightarrow$ *p dvd a* $\vee$ *p dvd b*
⟨*proof*⟩

lemma *prime-dvd-mult-iff*: *prime p* $\Longrightarrow$ *p dvd a * b* $\longleftrightarrow$ *p dvd a* $\vee$ *p dvd b*
⟨*proof*⟩

lemma *prime-dvd-power*:
prime p $\Longrightarrow$ *p dvd x ^ n* $\Longrightarrow$ *p dvd x*
⟨*proof*⟩

lemma *prime-dvd-power-iff*:
prime p $\Longrightarrow$ *n > 0* $\Longrightarrow$ *p dvd x ^ n* $\longleftrightarrow$ *p dvd x*
⟨*proof*⟩

lemma *prime-dvd-prod-mset-iff*: *prime p* $\Longrightarrow$ *p dvd prod-mset A* $\longleftrightarrow$ ($\exists$ *x*. *x* $\in\#$ *A* $\wedge$ *p dvd x*)
⟨*proof*⟩

lemma *prime-dvd-prod-iff*: *finite A* $\Longrightarrow$ *prime p* $\Longrightarrow$ *p dvd prod f A* $\longleftrightarrow$ ($\exists$ *x*$\in$*A*. *p dvd f x*)
⟨*proof*⟩

lemma *primes-dvd-imp-eq*:
assumes *prime p prime q p dvd q*
shows *p = q*
⟨*proof*⟩

lemma *prime-dvd-prod-mset-primes-iff*:
assumes *prime p* $\bigwedge$*q*. *q* $\in\#$ *A* $\Longrightarrow$ *prime q*
shows *p dvd prod-mset A* $\longleftrightarrow$ *p* $\in\#$ *A*
⟨*proof*⟩

lemma *prod-mset-primes-dvd-imp-subset*:
assumes *prod-mset A dvd prod-mset B* $\bigwedge$*p*. *p* $\in\#$ *A* $\Longrightarrow$ *prime p* $\bigwedge$*p*. *p* $\in\#$ *B* $\Longrightarrow$ *prime p*
shows *A* $\subseteq\#$ *B*
⟨*proof*⟩

lemma *prod-mset-dvd-prod-mset-primes-iff*:
assumes $\bigwedge$*x*. *x* $\in\#$ *A* $\Longrightarrow$ *prime x* $\bigwedge$*x*. *x* $\in\#$ *B* $\Longrightarrow$ *prime x*
shows *prod-mset A dvd prod-mset B* $\longleftrightarrow$ *A* $\subseteq\#$ *B*

⟨*proof*⟩

lemma *is-unit-prod-mset-primes-iff*:
assumes $\bigwedge x.\ x \in\# A \Longrightarrow$ *prime* x
shows *is-unit* (*prod-mset* A) $\longleftrightarrow A = \{\#\}$
⟨*proof*⟩

lemma *prod-mset-primes-irreducible-imp-prime*:
assumes *irred*: *irreducible* (*prod-mset* A)
assumes A: $\bigwedge x.\ x \in\# A \Longrightarrow$ *prime* x
assumes B: $\bigwedge x.\ x \in\# B \Longrightarrow$ *prime* x
assumes C: $\bigwedge x.\ x \in\# C \Longrightarrow$ *prime* x
assumes *dvd*: *prod-mset* A *dvd prod-mset* $B * $ *prod-mset* C
shows *prod-mset* A *dvd prod-mset* $B \vee$ *prod-mset* A *dvd prod-mset* C
⟨*proof*⟩

lemma *prod-mset-primes-finite-divisor-powers*:
assumes A: $\bigwedge x.\ x \in\# A \Longrightarrow$ *prime* x
assumes B: $\bigwedge x.\ x \in\# B \Longrightarrow$ *prime* x
assumes $A \neq \{\#\}$
shows *finite* $\{n.$ *prod-mset* $A \mathbin{\hat{}} n$ *dvd prod-mset* $B\}$
⟨*proof*⟩

end

1.3 In a semiring with GCD, each irreducible element is a prime element

context *semiring-gcd*
begin

lemma *irreducible-imp-prime-elem-gcd*:
assumes *irreducible* x
shows *prime-elem* x
⟨*proof*⟩

lemma *prime-elem-imp-coprime*:
assumes *prime-elem* p $\neg p$ *dvd* n
shows *coprime* p n
⟨*proof*⟩

lemma *prime-imp-coprime*:
assumes *prime* p $\neg p$ *dvd* n
shows *coprime* p n
⟨*proof*⟩

lemma *prime-elem-imp-power-coprime*:
prime-elem $p \Longrightarrow \neg$ p *dvd* $a \Longrightarrow$ *coprime* a $(p \mathbin{\hat{}} m)$
⟨*proof*⟩

lemma *prime-imp-power-coprime*:
prime p $\Longrightarrow$ $\neg$ *p dvd a* $\Longrightarrow$ *coprime a* (*p* $\hat{}$ *m*)
⟨*proof*⟩

lemma *prime-elem-divprod-pow*:
assumes *p*: *prime-elem p* **and** *ab*: *coprime a b* **and** *pab*: *p*$\hat{}$*n dvd a* $*$ *b*
shows *p*$\hat{}$*n dvd a* $\vee$ *p*$\hat{}$*n dvd b*
⟨*proof*⟩

lemma *primes-coprime*:
prime p $\Longrightarrow$ *prime q* $\Longrightarrow$ *p* $\neq$ *q* $\Longrightarrow$ *coprime p q*
⟨*proof*⟩

end

1.4 Factorial semirings: algebraic structures with unique prime factorizations

class *factorial-semiring* = *normalization-semidom* +
assumes *prime-factorization-exists*:
x $\neq$ *0* $\Longrightarrow$ $\exists$ *A*. ($\forall$ *x*. *x* $\in\#$ *A* $\longrightarrow$ *prime-elem x*) $\wedge$ *normalize* (*prod-mset A*) = *normalize x*

Alternative characterization

lemma (**in** *normalization-semidom*) *factorial-semiring-altI-aux*:
assumes *finite-divisors*: $\bigwedge$*x*. *x* $\neq$ *0* $\Longrightarrow$ *finite* {*y*. *y dvd x* $\wedge$ *normalize y* = *y*}
assumes *irreducible-imp-prime-elem*: $\bigwedge$*x*. *irreducible x* $\Longrightarrow$ *prime-elem x*
assumes *x* $\neq$ *0*
shows $\exists$ *A*. ($\forall$ *x*. *x* $\in\#$ *A* $\longrightarrow$ *prime-elem x*) $\wedge$ *normalize* (*prod-mset A*) = *normalize x*
⟨*proof*⟩

lemma *factorial-semiring-altI*:
assumes *finite-divisors*: $\bigwedge$*x*::'*a*. *x* $\neq$ *0* $\Longrightarrow$ *finite* {*y*. *y dvd x* $\wedge$ *normalize y* = *y*}
assumes *irreducible-imp-prime*: $\bigwedge$*x*::'*a*. *irreducible x* $\Longrightarrow$ *prime-elem x*
shows *OFCLASS*('*a* :: *normalization-semidom*, *factorial-semiring-class*)
⟨*proof*⟩

Properties

context *factorial-semiring*
begin

lemma *prime-factorization-exists'*:
assumes *x* $\neq$ *0*
obtains *A* **where** $\bigwedge$*x*. *x* $\in\#$ *A* $\Longrightarrow$ *prime x normalize* (*prod-mset A*) = *normalize x*
⟨*proof*⟩

lemma *irreducible-imp-prime-elem*:
assumes *irreducible x*
shows *prime-elem x*
⟨*proof*⟩

lemma *finite-divisor-powers*:
assumes $y \neq 0$ $\neg$*is-unit x*
shows *finite* $\{n.\ x \mathbin{\hat{}} n\ dvd\ y\}$
⟨*proof*⟩

lemma *finite-prime-divisors*:
assumes $x \neq 0$
shows *finite* $\{p.\ prime\ p \wedge p\ dvd\ x\}$
⟨*proof*⟩

lemma *infinite-unit-divisor-powers*:
assumes $y \neq 0$
assumes *is-unit x*
shows *infinite* $\{n.\ x \hat{} n\ dvd\ y\}$
⟨*proof*⟩

corollary *is-unit-iff-infinite-divisor-powers*:
assumes $y \neq 0$
shows *is-unit x* $\longleftrightarrow$ *infinite* $\{n.\ x \hat{} n\ dvd\ y\}$
⟨*proof*⟩

lemma *prime-elem-iff-irreducible*: *prime-elem x* $\longleftrightarrow$ *irreducible x*
⟨*proof*⟩

lemma *prime-divisor-exists*:
assumes $a \neq 0$ $\neg$*is-unit a*
shows $\exists\, b.\ b\ dvd\ a \wedge prime\ b$
⟨*proof*⟩

lemma *prime-divisors-induct* [*case-names zero unit factor*]:
assumes *P 0* $\bigwedge x.$ *is-unit x* $\Longrightarrow$ *P x* $\bigwedge p\ x.$ *prime p* $\Longrightarrow$ *P x* $\Longrightarrow$ *P* $(p * x)$
shows *P x*
⟨*proof*⟩

lemma *no-prime-divisors-imp-unit*:
assumes $a \neq 0$ $\bigwedge b.\ b\ dvd\ a \Longrightarrow normalize\ b = b \Longrightarrow \neg$ *prime-elem b*
shows *is-unit a*
⟨*proof*⟩

lemma *prime-divisorE*:
assumes $a \neq 0$ **and** $\neg$ *is-unit a*
obtains *p* **where** *prime p* **and** *p dvd a*
⟨*proof*⟩

definition *multiplicity* :: *'a* ⇒ *'a* ⇒ *nat* **where**
multiplicity p x = (*if finite* {*n*. *p* ^ *n dvd x*} *then Max* {*n*. *p* ^ *n dvd x*} *else 0*)

lemma *multiplicity-dvd*: *p* ^ *multiplicity p x dvd x*
⟨*proof*⟩

lemma *multiplicity-dvd'*: *n* ≤ *multiplicity p x* ⟹ *p* ^ *n dvd x*
⟨*proof*⟩

context
fixes *x p* :: *'a*
assumes *xp*: *x* ≠ *0* ¬*is-unit p*
begin

lemma *multiplicity-eq-Max*: *multiplicity p x* = *Max* {*n*. *p* ^ *n dvd x*}
⟨*proof*⟩

lemma *multiplicity-geI*:
assumes *p* ^ *n dvd x*
shows *multiplicity p x* ≥ *n*
⟨*proof*⟩

lemma *multiplicity-lessI*:
assumes ¬*p* ^ *n dvd x*
shows *multiplicity p x* < *n*
⟨*proof*⟩

lemma *power-dvd-iff-le-multiplicity*:
p ^ *n dvd x* ⟷ *n* ≤ *multiplicity p x*
⟨*proof*⟩

lemma *multiplicity-eq-zero-iff*:
shows *multiplicity p x* = *0* ⟷ ¬*p dvd x*
⟨*proof*⟩

lemma *multiplicity-gt-zero-iff*:
shows *multiplicity p x* > *0* ⟷ *p dvd x*
⟨*proof*⟩

lemma *multiplicity-decompose*:
¬*p dvd* (*x div p* ^ *multiplicity p x*)
⟨*proof*⟩

lemma *multiplicity-decompose'*:
obtains *y* **where** *x* = *p* ^ *multiplicity p x* ∗ *y* ¬*p dvd y*
⟨*proof*⟩

end

lemma *multiplicity-zero* [*simp*]: *multiplicity p 0 = 0*
⟨*proof*⟩

lemma *prime-elem-multiplicity-eq-zero-iff*:
prime-elem p ⟹ x ≠ 0 ⟹ multiplicity p x = 0 ⟷ ¬p dvd x
⟨*proof*⟩

lemma *prime-multiplicity-other*:
assumes *prime p prime q p ≠ q*
shows *multiplicity p q = 0*
⟨*proof*⟩

lemma *prime-multiplicity-gt-zero-iff*:
prime-elem p ⟹ x ≠ 0 ⟹ multiplicity p x > 0 ⟷ p dvd x
⟨*proof*⟩

lemma *multiplicity-unit-left*: *is-unit p ⟹ multiplicity p x = 0*
⟨*proof*⟩

lemma *multiplicity-unit-right*:
assumes *is-unit x*
shows *multiplicity p x = 0*
⟨*proof*⟩

lemma *multiplicity-one* [*simp*]: *multiplicity p 1 = 0*
⟨*proof*⟩

lemma *multiplicity-eqI*:
assumes *p ^ n dvd x ¬p ^ Suc n dvd x*
shows *multiplicity p x = n*
⟨*proof*⟩

context
fixes *x p* :: *'a*
assumes *xp*: *x ≠ 0 ¬is-unit p*
begin

lemma *multiplicity-times-same*:
assumes *p ≠ 0*
shows *multiplicity p (p * x) = Suc (multiplicity p x)*
⟨*proof*⟩

end

lemma *multiplicity-same-power'*: *multiplicity p (p ^ n) = (if p = 0 ∨ is-unit p then 0 else n)*
⟨*proof*⟩

lemma *multiplicity-same-power*:
  $p \neq 0 \Longrightarrow \neg$*is-unit* $p \Longrightarrow$ *multiplicity* $p\ (p\ \widehat{}\ n) = n$
  ⟨*proof*⟩

lemma *multiplicity-prime-elem-times-other*:
  assumes *prime-elem* p $\neg p$ *dvd* q
  shows *multiplicity* $p\ (q * x) =$ *multiplicity* $p\ x$
⟨*proof*⟩

lemma *multiplicity-self*:
  assumes $p \neq 0$ $\neg$*is-unit* p
  shows *multiplicity* $p\ p = 1$
⟨*proof*⟩

lemma *multiplicity-times-unit-left*:
  assumes *is-unit* c
  shows *multiplicity* $(c * p)\ x =$ *multiplicity* $p\ x$
⟨*proof*⟩

lemma *multiplicity-times-unit-right*:
  assumes *is-unit* c
  shows *multiplicity* $p\ (c * x) =$ *multiplicity* $p\ x$
⟨*proof*⟩

lemma *multiplicity-normalize-left* [*simp*]:
  multiplicity (*normalize* p) $x =$ *multiplicity* $p\ x$
⟨*proof*⟩

lemma *multiplicity-normalize-right* [*simp*]:
  multiplicity p (*normalize* x) = *multiplicity* $p\ x$
⟨*proof*⟩

lemma *multiplicity-prime* [*simp*]: *prime-elem* $p \Longrightarrow$ *multiplicity* $p\ p = 1$
  ⟨*proof*⟩

lemma *multiplicity-prime-power* [*simp*]: *prime-elem* $p \Longrightarrow$ *multiplicity* $p\ (p\ \widehat{}\ n)$
$= n$
  ⟨*proof*⟩

lift-definition *prime-factorization* :: $'a \Rightarrow\ 'a$ *multiset* **is**
  $\lambda x\ p.$ *if prime* p *then multiplicity* $p\ x$ *else* 0
⟨*proof*⟩

abbreviation *prime-factors* :: $'a \Rightarrow\ 'a$ *set* **where**
  prime-factors $a \equiv$ *set-mset* (*prime-factorization* a)

lemma *count-prime-factorization-nonprime*:
  $\neg$*prime* $p \Longrightarrow$ *count* (*prime-factorization* x) $p = 0$

⟨*proof*⟩

lemma *count-prime-factorization-prime*:
prime p ⟹ *count* (*prime-factorization x*) *p* = *multiplicity p x*
⟨*proof*⟩

lemma *count-prime-factorization*:
count (*prime-factorization x*) *p* = (*if prime p then multiplicity p x else 0*)
⟨*proof*⟩

lemma *dvd-imp-multiplicity-le*:
assumes *a dvd b b* ≠ *0*
shows *multiplicity p a* ≤ *multiplicity p b*
⟨*proof*⟩

lemma *prime-power-inj*:
assumes *prime a a* ^ *m* = *a* ^ *n*
shows *m* = *n*
⟨*proof*⟩

lemma *prime-power-inj′*:
assumes *prime p prime q*
assumes *p* ^ *m* = *q* ^ *n m* > *0 n* > *0*
shows *p* = *q m* = *n*
⟨*proof*⟩

lemma *prime-power-eq-one-iff* [*simp*]: *prime p* ⟹ *p* ^ *n* = *1* ⟷ *n* = *0*
⟨*proof*⟩

lemma *one-eq-prime-power-iff* [*simp*]: *prime p* ⟹ *1* = *p* ^ *n* ⟷ *n* = *0*
⟨*proof*⟩

lemma *prime-power-inj″*:
assumes *prime p prime q*
shows *p* ^ *m* = *q* ^ *n* ⟷ (*m* = *0* ∧ *n* = *0*) ∨ (*p* = *q* ∧ *m* = *n*)
⟨*proof*⟩

lemma *prime-factorization-0* [*simp*]: *prime-factorization 0* = {#}
⟨*proof*⟩

lemma *prime-factorization-empty-iff*:
prime-factorization x = {#} ⟷ *x* = *0* ∨ *is-unit x*
⟨*proof*⟩

lemma *prime-factorization-unit*:
assumes *is-unit x*
shows *prime-factorization x* = {#}
⟨*proof*⟩

lemma *prime-factorization-1* [*simp*]: *prime-factorization 1 = {#}*
⟨*proof*⟩

lemma *prime-factorization-times-prime*:
assumes *x ≠ 0 prime p*
shows *prime-factorization (p * x) = {#p#} + prime-factorization x*
⟨*proof*⟩

lemma *prod-mset-prime-factorization-weak*:
assumes *x ≠ 0*
shows *normalize (prod-mset (prime-factorization x)) = normalize x*
⟨*proof*⟩

lemma *in-prime-factors-iff*:
p ∈ prime-factors x ⟷ x ≠ 0 ∧ p dvd x ∧ prime p
⟨*proof*⟩

lemma *in-prime-factors-imp-prime* [*intro*]:
p ∈ prime-factors x ⟹ prime p
⟨*proof*⟩

lemma *in-prime-factors-imp-dvd* [*dest*]:
p ∈ prime-factors x ⟹ p dvd x
⟨*proof*⟩

lemma *prime-factorsI*:
x ≠ 0 ⟹ prime p ⟹ p dvd x ⟹ p ∈ prime-factors x
⟨*proof*⟩

lemma *prime-factors-dvd*:
x ≠ 0 ⟹ prime-factors x = {p. prime p ∧ p dvd x}
⟨*proof*⟩

lemma *prime-factors-multiplicity*:
prime-factors n = {p. prime p ∧ multiplicity p n > 0}
⟨*proof*⟩

lemma *prime-factorization-prime*:
assumes *prime p*
shows *prime-factorization p = {#p#}*
⟨*proof*⟩

lemma *prime-factorization-prod-mset-primes*:
assumes *⋀p. p ∈# A ⟹ prime p*
shows *prime-factorization (prod-mset A) = A*
⟨*proof*⟩

lemma *prime-factorization-cong*:
normalize x = normalize y ⟹ prime-factorization x = prime-factorization y

⟨*proof*⟩

lemma *prime-factorization-unique*:
assumes $x \neq 0$ $y \neq 0$
shows *prime-factorization* x = *prime-factorization* $y \longleftrightarrow$ *normalize* x = *normalize* y
⟨*proof*⟩

lemma *prime-factorization-normalize* [*simp*]:
prime-factorization (*normalize* x) = *prime-factorization* x
⟨*proof*⟩

lemma *prime-factorization-eqI-strong*:
assumes $\bigwedge p.\ p \in\# P \Longrightarrow$ *prime* p *prod-mset* $P = n$
shows *prime-factorization* $n = P$
⟨*proof*⟩

lemma *prime-factorization-eqI*:
assumes $\bigwedge p.\ p \in\# P \Longrightarrow$ *prime* p *normalize* (*prod-mset* P) = *normalize* n
shows *prime-factorization* $n = P$
⟨*proof*⟩

lemma *prime-factorization-mult*:
assumes $x \neq 0$ $y \neq 0$
shows *prime-factorization* $(x * y)$ = *prime-factorization* x + *prime-factorization* y
⟨*proof*⟩

lemma *prime-factorization-prod*:
assumes *finite* A $\bigwedge x.\ x \in A \Longrightarrow f\ x \neq 0$
shows *prime-factorization* (*prod* f A) = $(\sum n{\in}A.$ *prime-factorization* $(f\ n))$
⟨*proof*⟩

lemma *prime-elem-multiplicity-mult-distrib*:
assumes *prime-elem* p $x \neq 0$ $y \neq 0$
shows *multiplicity* p $(x * y)$ = *multiplicity* p x + *multiplicity* p y
⟨*proof*⟩

lemma *prime-elem-multiplicity-prod-mset-distrib*:
assumes *prime-elem* p $0 \notin\# A$
shows *multiplicity* p (*prod-mset* A) = *sum-mset* (*image-mset* (*multiplicity* p) A)
⟨*proof*⟩

lemma *prime-elem-multiplicity-power-distrib*:
assumes *prime-elem* p $x \neq 0$
shows *multiplicity* p $(x \mathbin{\widehat{}} n)$ = n * *multiplicity* p x
⟨*proof*⟩

lemma *prime-elem-multiplicity-prod-distrib*:
assumes *prime-elem p 0* $\notin$ *f ' A finite A*
shows *multiplicity p* (*prod f A*) = ($\sum$ *x*$\in$*A*. *multiplicity p* (*f x*))
⟨*proof*⟩

lemma *multiplicity-distinct-prime-power*:
prime p $\Longrightarrow$ *prime q* $\Longrightarrow$ *p* $\neq$ *q* $\Longrightarrow$ *multiplicity p* (*q* ^ *n*) = *0*
⟨*proof*⟩

lemma *prime-factorization-prime-power*:
prime p $\Longrightarrow$ *prime-factorization* (*p* ^ *n*) = *replicate-mset n p*
⟨*proof*⟩

lemma *prime-factorization-subset-iff-dvd*:
assumes [*simp*]: *x* $\neq$ *0 y* $\neq$ *0*
shows *prime-factorization x* $\subseteq$# *prime-factorization y* $\longleftrightarrow$ *x dvd y*
⟨*proof*⟩

lemma *prime-factorization-subset-imp-dvd*:
x $\neq$ *0* $\Longrightarrow$ (*prime-factorization x* $\subseteq$# *prime-factorization y*) $\Longrightarrow$ *x dvd y*
⟨*proof*⟩

lemma *prime-factorization-divide*:
assumes *b dvd a*
shows *prime-factorization* (*a div b*) = *prime-factorization a* − *prime-factorization b*
⟨*proof*⟩

lemma *zero-not-in-prime-factors* [*simp*]: *0* $\notin$ *prime-factors x*
⟨*proof*⟩

lemma *prime-prime-factors*:
prime p $\Longrightarrow$ *prime-factors p* = {*p*}
⟨*proof*⟩

lemma *prime-factors-product*:
x $\neq$ *0* $\Longrightarrow$ *y* $\neq$ *0* $\Longrightarrow$ *prime-factors* (*x* ∗ *y*) = *prime-factors x* ∪ *prime-factors y*
⟨*proof*⟩

lemma *dvd-prime-factors* [*intro*]:
y $\neq$ *0* $\Longrightarrow$ *x dvd y* $\Longrightarrow$ *prime-factors x* $\subseteq$ *prime-factors y*
⟨*proof*⟩

lemma *multiplicity-le-imp-dvd*:
assumes *x* $\neq$ *0* $\bigwedge$*p*. *prime p* $\Longrightarrow$ *multiplicity p x* $\leq$ *multiplicity p y*
shows *x dvd y*
⟨*proof*⟩

lemma *dvd-multiplicity-eq*:
x ≠ *0* ⟹ *y* ≠ *0* ⟹ *x dvd y* ⟷ (∀ *p*. *multiplicity p x* ≤ *multiplicity p y*)
⟨*proof*⟩

lemma *multiplicity-eq-imp-eq*:
assumes *x* ≠ *0 y* ≠ *0*
assumes ⋀*p*. *prime p* ⟹ *multiplicity p x* = *multiplicity p y*
shows *normalize x* = *normalize y*
⟨*proof*⟩

lemma *prime-factorization-unique′*:
assumes ∀ *p* ∈# *M*. *prime p* ∀ *p* ∈# *N*. *prime p* (∏ *i* ∈# *M*. *i*) = (∏ *i* ∈# *N*. *i*)
shows *M* = *N*
⟨*proof*⟩

lemma *prime-factorization-unique″*:
assumes ∀ *p* ∈# *M*. *prime p* ∀ *p* ∈# *N*. *prime p normalize* (∏ *i* ∈# *M*. *i*) = *normalize* (∏ *i* ∈# *N*. *i*)
shows *M* = *N*
⟨*proof*⟩

lemma *multiplicity-cong*:
(⋀*r*. *p* ^ *r dvd a* ⟷ *p* ^ *r dvd b*) ⟹ *multiplicity p a* = *multiplicity p b*
⟨*proof*⟩

lemma *not-dvd-imp-multiplicity-0*:
assumes ¬*p dvd x*
shows *multiplicity p x* = *0*
⟨*proof*⟩

lemma *multiplicity-zero-left* [*simp*]: *multiplicity 0 x* = *0*
⟨*proof*⟩

lemma *inj-on-Prod-primes*:
assumes ⋀*P p*. *P* ∈ *A* ⟹ *p* ∈ *P* ⟹ *prime p*
assumes ⋀*P*. *P* ∈ *A* ⟹ *finite P*
shows *inj-on Prod A*
⟨*proof*⟩

lemma *divides-primepow-weak*:
assumes *prime p* **and** *a dvd p* ^ *n*
obtains *m* **where** *m* ≤ *n* **and** *normalize a* = *normalize* (*p* ^ *m*)
⟨*proof*⟩

lemma *divide-out-primepow-ex*:
assumes *n* ≠ *0* ∃ *p*∈*prime-factors n*. *P p*
obtains *p k n′* **where** *P p prime p p dvd n* ¬*p dvd n′ k* > *0 n* = *p* ^ *k* ∗ *n′*
⟨*proof*⟩

lemma *divide-out-primepow*:
assumes $n \neq 0$ $\neg$*is-unit* n
obtains p k n' **where** *prime* p p *dvd* n $\neg p$ *dvd* n' $k > 0$ $n = p \mathbin{\hat{}} k * n'$
⟨*proof*⟩

1.5 GCD and LCM computation with unique factorizations

definition *gcd-factorial* a b = (*if* $a = 0$ *then normalize* b
else if $b = 0$ *then normalize* a
else normalize (*prod-mset* (*prime-factorization* a ∩# *prime-factorization* b)))

definition *lcm-factorial* a b = (*if* $a = 0 \vee b = 0$ *then* 0
else normalize (*prod-mset* (*prime-factorization* a ∪# *prime-factorization* b)))

definition *Gcd-factorial* A =
(*if* $A \subseteq \{0\}$ *then* 0 *else normalize* (*prod-mset* (*Inf* (*prime-factorization* ‘ (A − {0})))))

definition *Lcm-factorial* A =
(*if* $A = \{\}$ *then* 1
else if $0 \notin A \wedge$ *subset-mset.bdd-above* (*prime-factorization* ‘ (A − {0})) *then*
normalize (*prod-mset* (*Sup* (*prime-factorization* ‘ A)))
else
0)

lemma *prime-factorization-gcd-factorial*:
assumes [*simp*]: $a \neq 0$ $b \neq 0$
shows *prime-factorization* (*gcd-factorial* a b) = *prime-factorization* a ∩# *prime-factorization* b
⟨*proof*⟩

lemma *prime-factorization-lcm-factorial*:
assumes [*simp*]: $a \neq 0$ $b \neq 0$
shows *prime-factorization* (*lcm-factorial* a b) = *prime-factorization* a ∪# *prime-factorization* b
⟨*proof*⟩

lemma *prime-factorization-Gcd-factorial*:
assumes $\neg A \subseteq \{0\}$
shows *prime-factorization* (*Gcd-factorial* A) = *Inf* (*prime-factorization* ‘ (A − {0}))
⟨*proof*⟩

lemma *prime-factorization-Lcm-factorial*:
assumes $0 \notin A$ *subset-mset.bdd-above* (*prime-factorization* ‘ A)
shows *prime-factorization* (*Lcm-factorial* A) = *Sup* (*prime-factorization* ‘ A)
⟨*proof*⟩

lemma *gcd-factorial-commute*: *gcd-factorial a b* = *gcd-factorial b a*
⟨*proof*⟩

lemma *gcd-factorial-dvd1*: *gcd-factorial a b dvd a*
⟨*proof*⟩

lemma *gcd-factorial-dvd2*: *gcd-factorial a b dvd b*
⟨*proof*⟩

lemma *normalize-gcd-factorial* [*simp*]: *normalize* (*gcd-factorial a b*) = *gcd-factorial a b*
⟨*proof*⟩

lemma *normalize-lcm-factorial* [*simp*]: *normalize* (*lcm-factorial a b*) = *lcm-factorial a b*
⟨*proof*⟩

lemma *gcd-factorial-greatest*: *c dvd gcd-factorial a b* **if** *c dvd a c dvd b* **for** *a b c*
⟨*proof*⟩

lemma *lcm-factorial-gcd-factorial*:
lcm-factorial a b = *normalize* (*a* ∗ *b div gcd-factorial a b*) **for** *a b*
⟨*proof*⟩

lemma *normalize-Gcd-factorial*:
normalize (*Gcd-factorial A*) = *Gcd-factorial A*
⟨*proof*⟩

lemma *Gcd-factorial-eq-0-iff*:
Gcd-factorial A = *0* ⟷ *A* ⊆ {*0*}
⟨*proof*⟩

lemma *Gcd-factorial-dvd*:
assumes *x* ∈ *A*
shows *Gcd-factorial A dvd x*
⟨*proof*⟩

lemma *Gcd-factorial-greatest*:
assumes ⋀*y*. *y* ∈ *A* ⟹ *x dvd y*
shows *x dvd Gcd-factorial A*
⟨*proof*⟩

lemma *normalize-Lcm-factorial*:
normalize (*Lcm-factorial A*) = *Lcm-factorial A*
⟨*proof*⟩

lemma *Lcm-factorial-eq-0-iff*:
Lcm-factorial A = *0* ⟷ *0* ∈ *A* ∨ ¬*subset-mset.bdd-above* (*prime-factorization* ‘ *A*)

⟨*proof*⟩

lemma *dvd-Lcm-factorial*:
assumes $x \in A$
shows *x dvd Lcm-factorial A*
⟨*proof*⟩

lemma *Lcm-factorial-least*:
assumes $\bigwedge y.\ y \in A \Longrightarrow y\ dvd\ x$
shows *Lcm-factorial A dvd x*
⟨*proof*⟩

lemmas *gcd-lcm-factorial* =
gcd-factorial-dvd1 gcd-factorial-dvd2 gcd-factorial-greatest
normalize-gcd-factorial lcm-factorial-gcd-factorial
normalize-Gcd-factorial Gcd-factorial-dvd Gcd-factorial-greatest
normalize-Lcm-factorial dvd-Lcm-factorial Lcm-factorial-least

end

class *factorial-semiring-gcd* = *factorial-semiring* + *gcd* + *Gcd* +
assumes *gcd-eq-gcd-factorial*: *gcd a b* = *gcd-factorial a b*
and *lcm-eq-lcm-factorial*: *lcm a b* = *lcm-factorial a b*
and *Gcd-eq-Gcd-factorial*: *Gcd A* = *Gcd-factorial A*
and *Lcm-eq-Lcm-factorial*: *Lcm A* = *Lcm-factorial A*
begin

lemma *prime-factorization-gcd*:
assumes [*simp*]: $a \neq 0\ b \neq 0$
shows *prime-factorization* (*gcd a b*) = *prime-factorization a* ∩# *prime-factorization b*
⟨*proof*⟩

lemma *prime-factorization-lcm*:
assumes [*simp*]: $a \neq 0\ b \neq 0$
shows *prime-factorization* (*lcm a b*) = *prime-factorization a* ∪# *prime-factorization b*
⟨*proof*⟩

lemma *prime-factorization-Gcd*:
assumes $Gcd\ A \neq 0$
shows *prime-factorization* (*Gcd A*) = *Inf* (*prime-factorization* ‘ (*A* − {*0*}))
⟨*proof*⟩

lemma *prime-factorization-Lcm*:
assumes $Lcm\ A \neq 0$
shows *prime-factorization* (*Lcm A*) = *Sup* (*prime-factorization* ‘ *A*)
⟨*proof*⟩

lemma *prime-factors-gcd* [*simp*]:
  $a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow$ *prime-factors* (*gcd a b*) =
    prime-factors a ∩ *prime-factors b*
  ⟨*proof*⟩

lemma *prime-factors-lcm* [*simp*]:
  $a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow$ *prime-factors* (*lcm a b*) =
    prime-factors a ∪ *prime-factors b*
  ⟨*proof*⟩

subclass *semiring-gcd*
  ⟨*proof*⟩

subclass *semiring-Gcd*
  ⟨*proof*⟩

lemma
  assumes $x \neq 0$ $y \neq 0$
  shows *gcd-eq-factorial′*:
      gcd x y = *normalize* (∏ *p* ∈ *prime-factors x* ∩ *prime-factors y*.
          p ^ *min* (*multiplicity p x*) (*multiplicity p y*)) (**is** - = *?rhs1*)
    and *lcm-eq-factorial′*:
      lcm x y = *normalize* (∏ *p* ∈ *prime-factors x* ∪ *prime-factors y*.
          p ^ *max* (*multiplicity p x*) (*multiplicity p y*)) (**is** - = *?rhs2*)
⟨*proof*⟩

lemma
  assumes $x \neq 0$ $y \neq 0$ *prime p*
  shows *multiplicity-gcd*: *multiplicity p* (*gcd x y*) = *min* (*multiplicity p x*)
(*multiplicity p y*)
    and *multiplicity-lcm*: *multiplicity p* (*lcm x y*) = *max* (*multiplicity p x*)
(*multiplicity p y*)
⟨*proof*⟩

lemma *gcd-lcm-distrib*:
  gcd x (*lcm y z*) = *lcm* (*gcd x y*) (*gcd x z*)
⟨*proof*⟩

lemma *lcm-gcd-distrib*:
  lcm x (*gcd y z*) = *gcd* (*lcm x y*) (*lcm x z*)
⟨*proof*⟩

end

class *factorial-ring-gcd* = *factorial-semiring-gcd* + *idom*
begin

subclass *ring-gcd* ⟨*proof*⟩

subclass *idom-divide* ⟨*proof*⟩

end

class *factorial-semiring-multiplicative* =
factorial-semiring + *normalization-semidom-multiplicative*
begin

lemma *normalize-prod-mset-primes*:
(⋀*p*. *p* ∈# *A* ⟹ *prime p*) ⟹ *normalize* (*prod-mset A*) = *prod-mset A*
⟨*proof*⟩

lemma *prod-mset-prime-factorization*:
assumes *x* ≠ *0*
shows *prod-mset* (*prime-factorization x*) = *normalize x*
⟨*proof*⟩

lemma *prime-decomposition*: *unit-factor x* ∗ *prod-mset* (*prime-factorization x*) = *x*
⟨*proof*⟩

lemma *prod-prime-factors*:
assumes *x* ≠ *0*
shows (∏ *p* ∈ *prime-factors x*. *p* ^ *multiplicity p x*) = *normalize x*
⟨*proof*⟩

lemma *prime-factorization-unique''*:
assumes *S-eq*: *S* = {*p*. *0* < *f p*}
and *finite S*
and *S*: ∀ *p*∈*S*. *prime p normalize n* = (∏ *p*∈*S*. *p* ^ *f p*)
shows *S* = *prime-factors n* ∧ (∀ *p*. *prime p* ⟶ *f p* = *multiplicity p n*)
⟨*proof*⟩

lemma *divides-primepow*:
assumes *prime p* **and** *a dvd p* ^ *n*
obtains *m* **where** *m* ≤ *n* **and** *normalize a* = *p* ^ *m*
⟨*proof*⟩

lemma *Ex-other-prime-factor*:
assumes *n* ≠ *0* **and** ¬(∃ *k*. *normalize n* = *p* ^ *k*) *prime p*
shows ∃ *q*∈*prime-factors n*. *q* ≠ *p*
⟨*proof*⟩

Now a string of results due to Maya Kdzioka

lemma *multiplicity-dvd-iff-dvd*:
assumes *x* ≠ *0*
shows *p*^*k dvd x* ⟷ *p*^*k dvd p*^*multiplicity p x*
⟨*proof*⟩

lemma *multiplicity-decomposeI*:
assumes $x = p\hat{}k * x'$ **and** $\neg\ p\ dvd\ x'$ **and** $p \neq 0$
shows *multiplicity p x* = *k*
⟨*proof*⟩

lemma *multiplicity-sum-lt*:
assumes *multiplicity p a* < *multiplicity p b a* ≠ *0 b* ≠ *0*
shows *multiplicity p* (*a* + *b*) = *multiplicity p a*
⟨*proof*⟩

corollary *multiplicity-sum-min*:
assumes *multiplicity p a* ≠ *multiplicity p b a* ≠ *0 b* ≠ *0*
shows *multiplicity p* (*a* + *b*) = *min* (*multiplicity p a*) (*multiplicity p b*)
⟨*proof*⟩

end

lifting-update *multiset.lifting*
lifting-forget *multiset.lifting*

end

2 Abstract euclidean algorithm in euclidean (semi)rings

theory *Euclidean-Algorithm*
imports *Factorial-Ring*
begin

2.1 Generic construction of the (simple) euclidean algorithm

class *normalization-euclidean-semiring* = *euclidean-semiring* + *normalization-semidom*
begin

lemma *euclidean-size-normalize* [*simp*]:
euclidean-size (*normalize a*) = *euclidean-size a*
⟨*proof*⟩

context
begin

qualified function *gcd* :: *'a* ⇒ *'a* ⇒ *'a*
where *gcd a b* = (*if b* = *0 then normalize a else gcd b* (*a mod b*))
⟨*proof*⟩
termination
⟨*proof*⟩

declare *gcd.simps* [*simp del*]

lemma *eucl-induct* [*case-names zero mod*]:
assumes *H1*: $\bigwedge b.\ P\ b\ 0$
and *H2*: $\bigwedge a\ b.\ b \neq 0 \Longrightarrow P\ b\ (a\ mod\ b) \Longrightarrow P\ a\ b$
shows $P\ a\ b$
⟨*proof*⟩ **lemma** *gcd-0*:
$gcd\ a\ 0 = normalize\ a$
⟨*proof*⟩ **lemma** *gcd-mod*:
$a \neq 0 \Longrightarrow gcd\ a\ (b\ mod\ a) = gcd\ b\ a$
⟨*proof*⟩ **definition** $lcm :: 'a \Rightarrow 'a \Rightarrow 'a$
where $lcm\ a\ b = normalize\ (a * b\ div\ gcd\ a\ b)$

qualified definition $Lcm :: 'a\ set \Rightarrow 'a$ — Somewhat complicated definition of Lcm that has the advantage of working for infinite sets as well
where
[*code del*]: $Lcm\ A = (if\ \exists l.\ l \neq 0 \wedge (\forall a{\in}A.\ a\ dvd\ l)\ then$
$let\ l = SOME\ l.\ l \neq 0 \wedge (\forall a{\in}A.\ a\ dvd\ l) \wedge euclidean\text{-}size\ l =$
$(LEAST\ n.\ \exists l.\ l \neq 0 \wedge (\forall a{\in}A.\ a\ dvd\ l) \wedge euclidean\text{-}size\ l = n)$
$in\ normalize\ l$
$else\ 0)$

qualified definition $Gcd :: 'a\ set \Rightarrow 'a$
where [*code del*]: $Gcd\ A = Lcm\ \{d.\ \forall a{\in}A.\ d\ dvd\ a\}$

lemma *semiring-gcd*:
class.semiring-gcd one zero times gcd lcm
divide plus minus unit-factor normalize
⟨*proof*⟩

interpretation *semiring-gcd one zero times gcd lcm*
divide plus minus unit-factor normalize
⟨*proof*⟩

lemma *semiring-Gcd*:
class.semiring-Gcd one zero times gcd lcm Gcd Lcm
divide plus minus unit-factor normalize
⟨*proof*⟩

end

interpretation *semiring-Gcd one zero times*
Euclidean-Algorithm.gcd Euclidean-Algorithm.lcm Euclidean-Algorithm.Gcd Euclidean-Algorithm.Lcm
divide plus minus unit-factor normalize
⟨*proof*⟩

subclass *factorial-semiring*
⟨*proof*⟩

lemma *Gcd-eucl-set* [*code*]:

Euclidean-Algorithm.Gcd (*set xs*) = *fold Euclidean-Algorithm.gcd xs 0*
⟨*proof*⟩

lemma *Lcm-eucl-set* [*code*]:
Euclidean-Algorithm.Lcm (*set xs*) = *fold Euclidean-Algorithm.lcm xs 1*
⟨*proof*⟩

end

lemma *prime-elem-int-abs-iff* [*simp*]:
fixes *p* :: *int*
shows *prime-elem* |*p*| ⟷ *prime-elem p*
⟨*proof*⟩

lemma *prime-elem-int-minus-iff* [*simp*]:
fixes *p* :: *int*
shows *prime-elem* (− *p*) ⟷ *prime-elem p*
⟨*proof*⟩

lemma *prime-int-iff*:
fixes *p* :: *int*
shows *prime p* ⟷ *p* > *0* ∧ *prime-elem p*
⟨*proof*⟩

2.2 The (simple) euclidean algorithm as gcd computation

class *euclidean-semiring-gcd* = *normalization-euclidean-semiring* + *gcd* + *Gcd* +
assumes *gcd-eucl*: *Euclidean-Algorithm.gcd* = *GCD.gcd*
and *lcm-eucl*: *Euclidean-Algorithm.lcm* = *GCD.lcm*
assumes *Gcd-eucl*: *Euclidean-Algorithm.Gcd* = *GCD.Gcd*
and *Lcm-eucl*: *Euclidean-Algorithm.Lcm* = *GCD.Lcm*
begin

subclass *semiring-gcd*
⟨*proof*⟩

subclass *semiring-Gcd*
⟨*proof*⟩

subclass *factorial-semiring-gcd*
⟨*proof*⟩

lemma *gcd-mod-right* [*simp*]:
a ≠ *0* ⟹ *gcd a* (*b mod a*) = *gcd a b*
⟨*proof*⟩

lemma *gcd-mod-left* [*simp*]:
b ≠ *0* ⟹ *gcd* (*a mod b*) *b* = *gcd a b*
⟨*proof*⟩

lemma *euclidean-size-gcd-le1* [*simp*]:
assumes $a \neq 0$
shows *euclidean-size* (*gcd a b*) $\leq$ *euclidean-size a*
⟨*proof*⟩

lemma *euclidean-size-gcd-le2* [*simp*]:
$b \neq 0 \Longrightarrow$ *euclidean-size* (*gcd a b*) $\leq$ *euclidean-size b*
⟨*proof*⟩

lemma *euclidean-size-gcd-less1*:
assumes $a \neq 0$ **and** $\neg$ *a dvd b*
shows *euclidean-size* (*gcd a b*) $<$ *euclidean-size a*
⟨*proof*⟩

lemma *euclidean-size-gcd-less2*:
assumes $b \neq 0$ **and** $\neg$ *b dvd a*
shows *euclidean-size* (*gcd a b*) $<$ *euclidean-size b*
⟨*proof*⟩

lemma *euclidean-size-lcm-le1*:
assumes $a \neq 0$ **and** $b \neq 0$
shows *euclidean-size a* $\leq$ *euclidean-size* (*lcm a b*)
⟨*proof*⟩

lemma *euclidean-size-lcm-le2*:
$a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow$ *euclidean-size b* $\leq$ *euclidean-size* (*lcm a b*)
⟨*proof*⟩

lemma *euclidean-size-lcm-less1*:
assumes $b \neq 0$ **and** $\neg$ *b dvd a*
shows *euclidean-size a* $<$ *euclidean-size* (*lcm a b*)
⟨*proof*⟩

lemma *euclidean-size-lcm-less2*:
assumes $a \neq 0$ **and** $\neg$ *a dvd b*
shows *euclidean-size b* $<$ *euclidean-size* (*lcm a b*)
⟨*proof*⟩

end

lemma *factorial-euclidean-semiring-gcdI*:
OFCLASS('*a*::{*factorial-semiring-gcd*, *normalization-euclidean-semiring*}, *euclidean-semiring-gcd-class*)
⟨*proof*⟩

2.3 The extended euclidean algorithm

class *euclidean-ring-gcd* = *euclidean-semiring-gcd* + *idom*
begin

subclass *euclidean-ring* ⟨*proof*⟩
subclass *ring-gcd* ⟨*proof*⟩
subclass *factorial-ring-gcd* ⟨*proof*⟩

function *euclid-ext-aux* :: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow ('a \times 'a) \times 'a$
  where *euclid-ext-aux* $s'\ s\ t'\ t\ r'\ r =$ (
    if $r = 0$ *then let* $c = 1\ div$ *unit-factor* r' *in* $((s' * c,\ t' * c),$ *normalize* $r')$
    else let $q = r'\ div\ r$
      in euclid-ext-aux $s\ (s' - q * s)\ t\ (t' - q * t)\ r\ (r'\ mod\ r)$)
  ⟨*proof*⟩
termination
  ⟨*proof*⟩

abbreviation (*input*) *euclid-ext* :: $'a \Rightarrow 'a \Rightarrow ('a \times 'a) \times 'a$
  where *euclid-ext* $\equiv$ *euclid-ext-aux* 1 0 0 1

lemma
  assumes *gcd* $r'\ r = gcd\ a\ b$
  assumes $s' * a + t' * b = r'$
  assumes $s * a + t * b = r$
  assumes *euclid-ext-aux* $s'\ s\ t'\ t\ r'\ r = ((x,\ y),\ c)$
  shows *euclid-ext-aux-eq-gcd*: $c = gcd\ a\ b$
    and *euclid-ext-aux-bezout*: $x * a + y * b = gcd\ a\ b$
⟨*proof*⟩

declare *euclid-ext-aux.simps* [*simp del*]

definition *bezout-coefficients* :: $'a \Rightarrow 'a \Rightarrow 'a \times 'a$
  where [*code*]: *bezout-coefficients* $a\ b =$ *fst* (*euclid-ext* $a\ b$)

lemma *bezout-coefficients-0*:
  bezout-coefficients $a\ 0 = (1\ div$ *unit-factor* $a,\ 0)$
  ⟨*proof*⟩

lemma *bezout-coefficients-left-0*:
  bezout-coefficients $0\ a = (0,\ 1\ div$ *unit-factor* $a)$
  ⟨*proof*⟩

lemma *bezout-coefficients*:
  assumes *bezout-coefficients* $a\ b = (x,\ y)$
  shows $x * a + y * b = gcd\ a\ b$
  ⟨*proof*⟩

lemma *bezout-coefficients-fst-snd*:
  fst (*bezout-coefficients* $a\ b) * a +$ *snd* (*bezout-coefficients* $a\ b) * b = gcd\ a\ b$
  ⟨*proof*⟩

lemma *euclid-ext-eq* [*simp*]:

euclid-ext a b = (*bezout-coefficients a b*, *gcd a b*) (**is** *?p* = *?q*)
⟨*proof*⟩

declare *euclid-ext-eq* [*symmetric*, *code-unfold*]

end

class *normalization-euclidean-semiring-multiplicative* =
normalization-euclidean-semiring + *normalization-semidom-multiplicative*
begin

subclass *factorial-semiring-multiplicative* ⟨*proof*⟩

end

class *field-gcd* =
field + *unique-euclidean-ring* + *euclidean-ring-gcd* + *normalization-semidom-multiplicative*
begin

subclass *normalization-euclidean-semiring-multiplicative* ⟨*proof*⟩

subclass *normalization-euclidean-semiring* ⟨*proof*⟩

subclass *semiring-gcd-mult-normalize* ⟨*proof*⟩

end

2.4 Typical instances

instance *nat* :: *normalization-euclidean-semiring* ⟨*proof*⟩

instance *nat* :: *euclidean-semiring-gcd*
⟨*proof*⟩

instance *nat* :: *normalization-euclidean-semiring-multiplicative* ⟨*proof*⟩

lemma *prime-factorization-Suc-0* [*simp*]: *prime-factorization* (*Suc 0*) = {#}
⟨*proof*⟩

instance *int* :: *normalization-euclidean-semiring* ⟨*proof*⟩

instance *int* :: *euclidean-ring-gcd*
⟨*proof*⟩

instance *int* :: *normalization-euclidean-semiring-multiplicative* ⟨*proof*⟩

lemma (**in** *idom*) *prime-CHAR-semidom*:
assumes *CHAR*('*a*) > *0*
shows *prime CHAR*('*a*)

⟨*proof*⟩

end

3 Primes

theory *Primes*
imports *Euclidean-Algorithm*
begin

3.1 Primes on *nat* and *int*

lemma *Suc-0-not-prime-nat* [*simp*]: ¬ *prime* (*Suc 0*)
⟨*proof*⟩

lemma *prime-ge-2-nat*:
$p \geq 2$ **if** *prime p* **for** *p* :: *nat*
⟨*proof*⟩

lemma *prime-ge-2-int*:
$p \geq 2$ **if** *prime p* **for** *p* :: *int*
⟨*proof*⟩

lemma *prime-ge-0-int*: *prime p* $\Longrightarrow$ $p \geq (0::int)$
⟨*proof*⟩

lemma *prime-gt-0-nat*: *prime p* $\Longrightarrow$ $p > (0::nat)$
⟨*proof*⟩

lemma *prime-gt-0-int*: *prime p* $\Longrightarrow$ $p > (0::int)$
⟨*proof*⟩

lemma *prime-ge-1-nat*: *prime p* $\Longrightarrow$ $p \geq (1::nat)$
⟨*proof*⟩

lemma *prime-ge-Suc-0-nat*: *prime p* $\Longrightarrow$ $p \geq$ *Suc 0*
⟨*proof*⟩

lemma *prime-ge-1-int*: *prime p* $\Longrightarrow$ $p \geq (1::int)$
⟨*proof*⟩

lemma *prime-gt-1-nat*: *prime p* $\Longrightarrow$ $p > (1::nat)$
⟨*proof*⟩

lemma *prime-gt-Suc-0-nat*: *prime p* $\Longrightarrow$ $p >$ *Suc 0*
⟨*proof*⟩

lemma *prime-gt-1-int*: *prime p* $\Longrightarrow$ *p* > (*1*::*int*)
⟨*proof*⟩

lemma *prime-natI*:
prime p **if** *p* ≥ *2* **and** ⋀*m n*. *p dvd m* * *n* $\Longrightarrow$ *p dvd m* ∨ *p dvd n* **for** *p* :: *nat*
⟨*proof*⟩

lemma *prime-intI*:
prime p **if** *p* ≥ *2* **and** ⋀*m n*. *p dvd m* * *n* $\Longrightarrow$ *p dvd m* ∨ *p dvd n* **for** *p* :: *int*
⟨*proof*⟩

lemma *prime-elem-nat-iff* [*simp*]:
prime-elem n ⟷ *prime n* **for** *n* :: *nat*
⟨*proof*⟩

lemma *prime-elem-iff-prime-abs* [*simp*]:
prime-elem k ⟷ *prime* |*k*| **for** *k* :: *int*
⟨*proof*⟩

lemma *prime-nat-int-transfer* [*simp*]:
prime (*int n*) ⟷ *prime n* (**is** *?P* ⟷ *?Q*)
⟨*proof*⟩

lemma *prime-nat-iff-prime* [*simp*]:
prime (*nat k*) ⟷ *prime k*
⟨*proof*⟩

lemma *prime-int-nat-transfer*:
prime k ⟷ *k* ≥ *0* ∧ *prime* (*nat k*)
⟨*proof*⟩

lemma *prime-nat-naiveI*:
prime p **if** *p* ≥ *2* **and** *dvd*: ⋀*n*. *n dvd p* $\Longrightarrow$ *n* = *1* ∨ *n* = *p* **for** *p* :: *nat*
⟨*proof*⟩

lemma *prime-int-naiveI*:
prime p **if** *p* ≥ *2* **and** *dvd*: ⋀*k*. *k dvd p* $\Longrightarrow$ |*k*| = *1* ∨ |*k*| = *p* **for** *p* :: *int*
⟨*proof*⟩

lemma *prime-nat-iff*:
prime (*n* :: *nat*) ⟷ (*1* < *n* ∧ (∀ *m*. *m dvd n* ⟶ *m* = *1* ∨ *m* = *n*))
⟨*proof*⟩

lemma *prime-nat-iff'*:
prime (*p* :: *nat*) ⟷ *p* > *1* ∧ (∀ *n* ∈ {*2*..<*p*}. ¬ *n dvd p*)
⟨*proof*⟩

lemma *prime-int-iff*:
prime (*n*::*int*) ⟷ (*1* < *n* ∧ (∀ *m*. *m* ≥ *0* ∧ *m dvd n* ⟶ *m* = *1* ∨ *m* = *n*))

⟨*proof*⟩

lemma *prime-int-iff'*:
  prime (*p* :: *int*) ⟷ *p* > *1* ∧ (∀ *n* ∈ {*2*..<*p*}. ¬ *n dvd p*) (**is** *?P* ⟷ *?Q*)
⟨*proof*⟩

lemma *prime-nat-not-dvd*:
  assumes *prime p p* > *n n* ≠ (*1*::*nat*)
  shows ¬*n dvd p*
⟨*proof*⟩

lemma *prime-int-not-dvd*:
  assumes *prime p p* > *n n* > (*1*::*int*)
  shows ¬*n dvd p*
⟨*proof*⟩

lemma *prime-odd-nat*: *prime p* ⟹ *p* > (*2*::*nat*) ⟹ *odd p*
  ⟨*proof*⟩

lemma *prime-odd-int*: *prime p* ⟹ *p* > (*2*::*int*) ⟹ *odd p*
  ⟨*proof*⟩

lemma *prime-int-altdef*:
  prime p = (*1* < *p* ∧ (∀ *m*::*int*. *m* ≥ *0* ⟶ *m dvd p* ⟶
    m = *1* ∨ *m* = *p*))
  ⟨*proof*⟩

lemma *not-prime-eq-prod-nat*:
  assumes *m* > *1* ¬ *prime* (*m*::*nat*)
  shows ∃ *n k*. *n* = *m* ∗ *k* ∧ *1* < *m* ∧ *m* < *n* ∧ *1* < *k* ∧ *k* < *n*
  ⟨*proof*⟩

3.2 Make prime naively executable

lemma *prime-int-numeral-eq* [*simp*]:
  prime (*numeral m* :: *int*) ⟷ *prime* (*numeral m* :: *nat*)
  ⟨*proof*⟩

class *check-prime-by-range* = *normalization-semidom* + *discrete-linordered-semidom*
+
  assumes *prime-iff*: ‹*prime a* ⟷ *1* < *a* ∧ (∀ *d*∈{*2*..*a div 2*}. ¬ *d dvd a*)›
begin

lemma *two-is-prime* [*simp*]:
  ‹*prime 2*›
  ⟨*proof*⟩

end

lemma *divisor-less-eq-half-nat*:
  ‹$m \leq n$ *div* 2› **if** ‹m *dvd* n› ‹$m < n$› **for** m n :: *nat*
  ⟨*proof*⟩

instance *nat* :: *check-prime-by-range*
  ⟨*proof*⟩

lemma *two-is-prime-nat* [*simp*]:
  ‹*prime* (2::*nat*)›
  ⟨*proof*⟩

lemma *divisor-less-eq-half-int*:
  ‹$k \leq l$ *div* 2› **if** ‹k *dvd* l› ‹$k < l$› ‹$l \geq 0$› ‹$k \geq 0$› **for** k l :: *int*
⟨*proof*⟩

instance *int* :: *check-prime-by-range*
  ⟨*proof*⟩

lemma *prime-nat-numeral-eq* [*simp*]: — TODO Sieve Of Erathosthenes might speed this up
  prime (*numeral* m :: *nat*) ⟷
    (1::*nat*) < *numeral* m ∧
    (∀ n::*nat* ∈ *set* [2..<*Suc* (*numeral* m *div* 2)]. ¬ n *dvd* *numeral* m)
  ⟨*proof*⟩

context *check-prime-by-range*
begin

definition *check-divisors* :: ‹'a ⇒ 'a ⇒ 'a ⇒ *bool*›
  where ‹*check-divisors* l u a ⟷ (∀ d∈{l..u}. ¬ d *dvd* a)›

lemma *check-divisors-rec* [*code*]:
  ‹*check-divisors* l u a ⟷ $u < l$ ∨ (¬ l *dvd* a ∧ *check-divisors* (l + 1) u a)›
  ⟨*proof*⟩

lemma *prime-eq-check-divisors* [*code*]:
  ‹*prime* a ⟷ $a > 1$ ∧ *check-divisors* 2 (a *div* 2) a›
  ⟨*proof*⟩

end

3.3 Largest exponent of a prime factor

lemma *prime-factor-nat*:
  $n \neq$ (1::*nat*) ⟹ ∃ p. *prime* p ∧ p *dvd* n
  ⟨*proof*⟩

lemma *prime-factor-int*:
  fixes k :: *int*

assumes $|k| \neq 1$
obtains p **where** *prime p p dvd k*
⟨*proof*⟩

Possibly duplicates other material, but avoid the complexities of multisets.

lemma *prime-power-cancel-less*:
assumes *prime p* **and** *eq*: $m * (p \hat{} k) = m' * (p \hat{} k')$ **and** *less*: $k < k'$ **and** ¬ *p dvd m*
shows *False*
⟨*proof*⟩

lemma *prime-power-cancel*:
assumes *prime p* **and** *eq*: $m * (p \hat{} k) = m' * (p \hat{} k')$ **and** ¬ *p dvd m* ¬ *p dvd m'*
shows $k = k'$
⟨*proof*⟩

lemma *prime-power-cancel2*:
assumes *prime p* $m * (p \hat{} k) = m' * (p \hat{} k')$ ¬ *p dvd m* ¬ *p dvd m'*
obtains $m = m'$ $k = k'$
⟨*proof*⟩

lemma *prime-power-canonical*:
fixes m :: *nat*
assumes *prime p* $m > 0$
shows ∃ *k n*. ¬ *p dvd n* ∧ $m = n * p \hat{} k$
⟨*proof*⟩

3.4 Infinitely many primes

lemma *next-prime-bound*: ∃ *p*::*nat*. *prime p* ∧ $n < p$ ∧ $p \leq fact\ n + 1$
⟨*proof*⟩

lemma *bigger-prime*: ∃ *p*. *prime p* ∧ $p > (n::nat)$
⟨*proof*⟩

lemma *primes-infinite*: ¬ (*finite* {(*p*::*nat*). *prime p*})
⟨*proof*⟩

3.5 Powers of Primes

Versions for type nat only

lemma *prime-product*:
fixes *p*::*nat*
assumes *prime* $(p * q)$
shows $p = 1 \vee q = 1$
⟨*proof*⟩

lemma *prime-power-mult-nat*:
fixes $p :: nat$
assumes p: *prime* p **and** xy: $x * y = p \hat{} k$
shows $\exists i\, j.\ x = p \hat{} i \wedge y = p \hat{} j$
⟨*proof*⟩

lemma *prime-power-exp-nat*:
fixes p::*nat*
assumes p: *prime* p **and** n: $n \neq 0$
and xn: $x \hat{} n = p \hat{} k$ **shows** $\exists i.\ x = p \hat{} i$
⟨*proof*⟩

lemma *divides-primepow-nat*:
fixes $p :: nat$
assumes p: *prime* p
shows d *dvd* $p \hat{} k \longleftrightarrow (\exists i \leq k.\ d = p \hat{} i)$
⟨*proof*⟩

lemma *gcd-prime-int*:
assumes *prime* $(p :: int)$
shows *gcd p k* = (*if p dvd k then p else 1*)
⟨*proof*⟩

3.6 Chinese Remainder Theorem Variants

lemma *bezout-gcd-nat*:
fixes a::*nat* **shows** $\exists x\, y.\ a * x - b * y = gcd\ a\ b \vee b * x - a * y = gcd\ a\ b$
⟨*proof*⟩

lemma *gcd-bezout-sum-nat*:
fixes a::*nat*
assumes $a * x + b * y = d$
shows *gcd a b dvd d*
⟨*proof*⟩

A binary form of the Chinese Remainder Theorem.

lemma *chinese-remainder*:
fixes a::*nat* **assumes** *ab*: *coprime a b* **and** a: $a \neq 0$ **and** b: $b \neq 0$
shows $\exists x\ q1\ q2.\ x = u + q1 * a \wedge x = v + q2 * b$
⟨*proof*⟩

Primality

lemma *coprime-bezout-strong*:
fixes a::*nat* **assumes** *coprime a b* $b \neq 1$
shows $\exists x\, y.\ a * x = b * y + 1$
⟨*proof*⟩

lemma *bezout-prime*:
assumes p: *prime p* **and** *pa*: ¬ *p dvd a*

shows $\exists\, x\ y.\ a*x = Suc\ (p*y)$
⟨*proof*⟩

3.7 Multiplicity and primality for natural numbers and integers

lemma *prime-factors-gt-0-nat*:
$p \in$ *prime-factors* $x \Longrightarrow p > (0::nat)$
⟨*proof*⟩

lemma *prime-factors-gt-0-int*:
$p \in$ *prime-factors* $x \Longrightarrow p > (0::int)$
⟨*proof*⟩

lemma *prime-factors-ge-0-int* [*elim*]:
fixes $n :: int$
shows $p \in$ *prime-factors* $n \Longrightarrow p \geq 0$
⟨*proof*⟩

lemma *prod-mset-prime-factorization-int*:
fixes $n :: int$
assumes $n > 0$
shows *prod-mset* (*prime-factorization* n) $= n$
⟨*proof*⟩

lemma *prime-factorization-exists-nat*:
$n > 0 \Longrightarrow (\exists\, M.\ (\forall\, p::nat \in$ *set-mset* $M.$ *prime* $p) \wedge n = (\prod i \in\# M.\ i))$
⟨*proof*⟩

lemma *prod-mset-prime-factorization-nat* [*simp*]:
$(n::nat) > 0 \Longrightarrow$ *prod-mset* (*prime-factorization* n) $= n$
⟨*proof*⟩

lemma *prime-factorization-nat*:
$n > (0::nat) \Longrightarrow n = (\prod p \in$ *prime-factors* $n.\ p\ \hat{}$ *multiplicity* $p\ n)$
⟨*proof*⟩

lemma *prime-factorization-int*:
$n > (0::int) \Longrightarrow n = (\prod p \in$ *prime-factors* $n.\ p\ \hat{}$ *multiplicity* $p\ n)$
⟨*proof*⟩

lemma *prime-factorization-unique-nat*:
fixes $f :: nat \Rightarrow$ -
assumes *S-eq*: $S = \{p.\ 0 < f\ p\}$
and *finite* S
and S: $\forall\, p{\in}S.$ *prime* $p\ n = (\prod p{\in}S.\ p\ \hat{}\ f\ p)$
shows $S =$ *prime-factors* $n \wedge (\forall\, p.$ *prime* $p \longrightarrow f\ p =$ *multiplicity* $p\ n)$
⟨*proof*⟩

lemma *prime-factorization-unique-int*:
fixes $f :: int \Rightarrow -$
assumes *S-eq*: $S = \{p.\ 0 < f\ p\}$
and *finite* S
and S: $\forall\, p{\in}S.\ prime\ p\ abs\ n = (\prod p{\in}S.\ p\ \hat{}\ f\ p)$
shows $S = prime\text{-}factors\ n \wedge (\forall\, p.\ prime\ p \longrightarrow f\ p = multiplicity\ p\ n)$
⟨*proof*⟩

lemma *prime-factors-characterization-nat*:
$S = \{p.\ 0 < f\ (p{::}nat)\} \Longrightarrow$
finite $S \Longrightarrow \forall\, p{\in}S.\ prime\ p \Longrightarrow n = (\prod p{\in}S.\ p\ \hat{}\ f\ p) \Longrightarrow prime\text{-}factors\ n = S$
⟨*proof*⟩

lemma *prime-factors-characterization′-nat*:
finite $\{p.\ 0 < f\ (p{::}nat)\} \Longrightarrow$
$(\forall\, p.\ 0 < f\ p \longrightarrow prime\ p) \Longrightarrow$
$prime\text{-}factors\ (\prod p \mid 0 < f\ p.\ p\ \hat{}\ f\ p) = \{p.\ 0 < f\ p\}$
⟨*proof*⟩

lemma *prime-factors-characterization-int*:
$S = \{p.\ 0 < f\ (p{::}int)\} \Longrightarrow$ *finite* $S \Longrightarrow$
$\forall\, p{\in}S.\ prime\ p \Longrightarrow abs\ n = (\prod p{\in}S.\ p\ \hat{}\ f\ p) \Longrightarrow prime\text{-}factors\ n = S$
⟨*proof*⟩

lemma *abs-prod*: $abs\ (prod\ f\ A :: 'a :: linordered\text{-}idom) = prod\ (\lambda x.\ abs\ (f\ x))\ A$
⟨*proof*⟩

lemma *primes-characterization′-int* [*rule-format*]:
finite $\{p.\ p \geq 0 \wedge 0 < f\ (p{::}int)\} \Longrightarrow \forall\, p.\ 0 < f\ p \longrightarrow prime\ p \Longrightarrow$
$prime\text{-}factors\ (\prod p \mid p \geq 0 \wedge 0 < f\ p.\ p\ \hat{}\ f\ p) = \{p.\ p \geq 0 \wedge 0 < f\ p\}$
⟨*proof*⟩

lemma *multiplicity-characterization-nat*:
$S = \{p.\ 0 < f\ (p{::}nat)\} \Longrightarrow$ *finite* $S \Longrightarrow \forall\, p{\in}S.\ prime\ p \Longrightarrow prime\ p \Longrightarrow$
$n = (\prod p{\in}S.\ p\ \hat{}\ f\ p) \Longrightarrow multiplicity\ p\ n = f\ p$
⟨*proof*⟩

lemma *multiplicity-characterization′-nat*: *finite* $\{p.\ 0 < f\ (p{::}nat)\} \longrightarrow$
$(\forall\, p.\ 0 < f\ p \longrightarrow prime\ p) \longrightarrow prime\ p \longrightarrow$
$multiplicity\ p\ (\prod p \mid 0 < f\ p.\ p\ \hat{}\ f\ p) = f\ p$
⟨*proof*⟩

lemma *multiplicity-characterization-int*: $S = \{p.\ 0 < f\ (p{::}int)\} \Longrightarrow$
finite $S \Longrightarrow \forall\, p{\in}S.\ prime\ p \Longrightarrow prime\ p \Longrightarrow n = (\prod p{\in}S.\ p\ \hat{}\ f\ p) \Longrightarrow$
$multiplicity\ p\ n = f\ p$
⟨*proof*⟩

lemma *multiplicity-characterization′-int* [*rule-format*]:

finite $\{p.\ p \geq 0 \wedge 0 < f\ (p::int)\} \Longrightarrow$
$(\forall p.\ 0 < f\ p \longrightarrow prime\ p) \Longrightarrow prime\ p \Longrightarrow$
$multiplicity\ p\ (\prod p \mid p \geq 0 \wedge 0 < f\ p.\ p\ \hat{}\ f\ p) = f\ p$
⟨*proof*⟩

lemma *multiplicity-one-nat* [*simp*]: $multiplicity\ p\ (Suc\ 0) = 0$
⟨*proof*⟩

lemma *multiplicity-eq-nat*:
fixes x **and** $y::nat$
assumes $x > 0\ y > 0\ \bigwedge p.\ prime\ p \Longrightarrow multiplicity\ p\ x = multiplicity\ p\ y$
shows $x = y$
⟨*proof*⟩

lemma *multiplicity-eq-int*:
fixes $x\ y :: int$
assumes $x > 0\ y > 0\ \bigwedge p.\ prime\ p \Longrightarrow multiplicity\ p\ x = multiplicity\ p\ y$
shows $x = y$
⟨*proof*⟩

lemma *multiplicity-prod-prime-powers*:
assumes *finite* $S\ \bigwedge x.\ x \in S \Longrightarrow prime\ x\ prime\ p$
shows $multiplicity\ p\ (\prod p \in S.\ p\ \hat{}\ f\ p) = (if\ p \in S\ then\ f\ p\ else\ 0)$
⟨*proof*⟩

lemma *prime-factorization-prod-mset*:
assumes $0 \notin\# A$
shows $prime\text{-}factorization\ (prod\text{-}mset\ A) = \sum_{\#}(image\text{-}mset\ prime\text{-}factorization\ A)$
⟨*proof*⟩

lemma *prime-factors-prod*:
assumes *finite* A **and** $0 \notin f\ `\ A$
shows $prime\text{-}factors\ (prod\ f\ A) = \bigcup((prime\text{-}factors \circ f)\ `\ A)$
⟨*proof*⟩

lemma *prime-factors-fact*:
$prime\text{-}factors\ (fact\ n) = \{p \in \{2..n\}.\ prime\ p\}$ (**is** *?M* = *?N*)
⟨*proof*⟩

lemma *prime-dvd-fact-iff*:
assumes *prime p*
shows $p\ dvd\ fact\ n \longleftrightarrow p \leq n$
⟨*proof*⟩

lemma *dvd-choose-prime*:
assumes *kn*: $k < n$ **and** *k*: $k \neq 0$ **and** *n*: $n \neq 0$ **and** *prime-n*: *prime n*
shows $n\ dvd\ (n\ choose\ k)$
⟨*proof*⟩

lemma (**in** *ring-1*) *minus-power-prime-CHAR*:
  assumes $p = CHAR('a)$ *prime* p
  shows $(-x :: 'a) \mathbin{\hat{}} p = -(x \mathbin{\hat{}} p)$
⟨*proof*⟩

3.8 Rings and fields with prime characteristic

We introduce some type classes for rings and fields with prime characteristic.

class *semiring-prime-char* = *semiring-1* +
  assumes *prime-char-aux*: $\exists n.$ *prime* $n \wedge$ *of-nat* $n = (0 :: 'a)$
begin

lemma *CHAR-pos* [*intro*, *simp*]: $CHAR('a) > 0$
  ⟨*proof*⟩

lemma *CHAR-nonzero* [*simp*]: $CHAR('a) \neq 0$
  ⟨*proof*⟩

lemma *CHAR-prime* [*intro*, *simp*]: *prime* $CHAR('a)$
  ⟨*proof*⟩

end

lemma *semiring-prime-charI* [*intro?*]:
  prime $CHAR('a :: semiring\text{-}1) \Longrightarrow OFCLASS('a, semiring\text{-}prime\text{-}char\text{-}class)$
  ⟨*proof*⟩

lemma *idom-prime-charI* [*intro?*]:
  assumes $CHAR('a :: idom) > 0$
  shows $OFCLASS('a, semiring\text{-}prime\text{-}char\text{-}class)$
⟨*proof*⟩

class *comm-semiring-prime-char* = *comm-semiring-1* + *semiring-prime-char*
class *comm-ring-prime-char* = *comm-ring-1* + *semiring-prime-char*
begin
subclass *comm-semiring-prime-char* ⟨*proof*⟩
end
class *idom-prime-char* = *idom* + *semiring-prime-char*
begin
subclass *comm-ring-prime-char* ⟨*proof*⟩
end

class *field-prime-char* = *field* +
  assumes *pos-char-exists*: $\exists n{>}0.$ *of-nat* $n = (0 :: 'a)$
begin
subclass *idom-prime-char*
  ⟨*proof*⟩
end

lemma *field-prime-charI* [*intro?*]:
n > *0* ⟹ *of-nat n* = (*0* :: *'a* :: *field*) ⟹ *OFCLASS*(*'a*, *field-prime-char-class*)
⟨*proof*⟩

lemma *field-prime-charI'* [*intro?*]:
CHAR(*'a* :: *field*) > *0* ⟹ *OFCLASS*(*'a*, *field-prime-char-class*)
⟨*proof*⟩

3.9 Finite fields

class *finite-field* = *field-prime-char* + *finite*

lemma *finite-fieldI* [*intro?*]:
assumes *finite* (*UNIV* :: *'a* :: *field set*)
shows *OFCLASS*(*'a*, *finite-field-class*)
⟨*proof*⟩

On a finite field with n elements, taking the n-th power of an element is the identity. This is an obvious consequence of the fact that the multiplicative group of the field is a finite group of order $n - 1$, so $x\hat{}n = 1$ for any non-zero x.

Note that this result is sharp in the sense that the multiplicative group of a finite field is cyclic, i.e. it contains an element of order $n - 1$. (We don't prove this here.)

lemma *finite-field-power-card-eq-same*:
fixes *x* :: *'a* :: *finite-field*
shows *x* ^ *card* (*UNIV* :: *'a set*) = *x*
⟨*proof*⟩

lemma *finite-field-power-card-power-eq-same*:
fixes *x* :: *'a* :: *finite-field*
assumes *m* = *card* (*UNIV* :: *'a set*) ^ *n*
shows *x* ^ *m* = *x*
⟨*proof*⟩

class *enum-finite-field* = *finite-field* +
fixes *enum-finite-field* :: *nat* ⇒ *'a*
assumes *enum-finite-field*: *enum-finite-field* ` {..<*card* (*UNIV* :: *'a set*)} = *UNIV*
begin

lemma *inj-on-enum-finite-field*: *inj-on enum-finite-field* {..<*card* (*UNIV* :: *'a set*)}
⟨*proof*⟩

end

To get rid of the pending sort hypotheses, we prove that the field with 2 elements is indeed a finite field.

typedef *gf2* = {*0*, *1* :: *nat*}
⟨*proof*⟩

setup-lifting *type-definition-gf2*

instantiation *gf2* :: *field*
begin
lift-definition *zero-gf2* :: *gf2* **is** *0* ⟨*proof*⟩
lift-definition *one-gf2* :: *gf2* **is** *1* ⟨*proof*⟩
lift-definition *uminus-gf2* :: *gf2* ⇒ *gf2* **is** λ*x*. *x* ⟨*proof*⟩
lift-definition *plus-gf2* :: *gf2* ⇒ *gf2* ⇒ *gf2* **is** λ*x y*. *if x* = *y then 0 else 1* ⟨*proof*⟩
lift-definition *minus-gf2* :: *gf2* ⇒ *gf2* ⇒ *gf2* **is** λ*x y*. *if x* = *y then 0 else 1* ⟨*proof*⟩
lift-definition *times-gf2* :: *gf2* ⇒ *gf2* ⇒ *gf2* **is** λ*x y*. *x* ∗ *y* ⟨*proof*⟩
lift-definition *inverse-gf2* :: *gf2* ⇒ *gf2* **is** λ*x*. *x* ⟨*proof*⟩
lift-definition *divide-gf2* :: *gf2* ⇒ *gf2* ⇒ *gf2* **is** λ*x y*. *x* ∗ *y* ⟨*proof*⟩

instance
⟨*proof*⟩

end

instance *gf2* :: *finite-field*
⟨*proof*⟩

3.10 The Freshman's Dream in rings of prime characteristic

lemma (**in** *comm-semiring-1*) *freshmans-dream*:
fixes *x y* :: *'a* **and** *n* :: *nat*
assumes *prime CHAR*(*'a*)
assumes *n-def*: *n* = *CHAR*(*'a*)
shows (*x* + *y*) ^ *n* = *x* ^ *n* + *y* ^ *n*
⟨*proof*⟩

lemma (**in** *comm-semiring-1*) *freshmans-dream'*:
assumes [*simp*]: *prime CHAR*(*'a*) **and** *m* = *CHAR*(*'a*) ^ *n*
shows (*x* + *y* :: *'a*) ^ *m* = *x* ^ *m* + *y* ^ *m*
⟨*proof*⟩

lemma (**in** *comm-semiring-1*) *freshmans-dream-sum*:
fixes *f* :: *'b* ⇒ *'a*
assumes *prime CHAR*(*'a*) **and** *n* = *CHAR*(*'a*)
shows *sum f A* ^ *n* = *sum* (λ*i*. *f i* ^ *n*) *A*
⟨*proof*⟩

lemma (**in** *comm-semiring-1*) *freshmans-dream-sum'*:
fixes *f* :: *'b* ⇒ *'a*
assumes *prime CHAR*(*'a*) *m* = *CHAR*(*'a*) ^ *n*
shows *sum f A* ^ *m* = *sum* (λ*i*. *f i* ^ *m*) *A*
⟨*proof*⟩

lemmas *prime-imp-coprime-nat* = *prime-imp-coprime*[**where** *?'a* = *nat*]
lemmas *prime-imp-coprime-int* = *prime-imp-coprime*[**where** *?'a* = *int*]
lemmas *prime-dvd-mult-nat* = *prime-dvd-mult-iff*[**where** *?'a* = *nat*]
lemmas *prime-dvd-mult-int* = *prime-dvd-mult-iff*[**where** *?'a* = *int*]
lemmas *prime-dvd-mult-eq-nat* = *prime-dvd-mult-iff*[**where** *?'a* = *nat*]
lemmas *prime-dvd-mult-eq-int* = *prime-dvd-mult-iff*[**where** *?'a* = *int*]
lemmas *prime-dvd-power-nat* = *prime-dvd-power*[**where** *?'a* = *nat*]
lemmas *prime-dvd-power-int* = *prime-dvd-power*[**where** *?'a* = *int*]
lemmas *prime-dvd-power-nat-iff* = *prime-dvd-power-iff*[**where** *?'a* = *nat*]
lemmas *prime-dvd-power-int-iff* = *prime-dvd-power-iff*[**where** *?'a* = *int*]
lemmas *prime-imp-power-coprime-nat* = *prime-imp-power-coprime*[**where** *?'a* = *nat*]
lemmas *prime-imp-power-coprime-int* = *prime-imp-power-coprime*[**where** *?'a* = *int*]
lemmas *primes-coprime-nat* = *primes-coprime*[**where** *?'a* = *nat*]
lemmas *primes-coprime-int* = *primes-coprime*[**where** *?'a* = *nat*]
lemmas *prime-divprod-pow-nat* = *prime-elem-divprod-pow*[**where** *?'a* = *nat*]
lemmas *prime-exp* = *prime-elem-power-iff*[**where** *?'a* = *nat*]

end

4 Polynomials as type over a ring structure

theory *Polynomial*
imports
Complex-Main
HOL−Library.More-List
HOL−Library.Infinite-Set
Primes
begin

context *semidom-modulo*
begin

lemma *not-dvd-imp-mod-neq-0*:
‹*a mod b* ≠ *0*› **if** ‹¬ *b dvd a*›
⟨*proof*⟩

end

4.1 Auxiliary: operations for lists (later) representing coefficients

definition *cCons* :: *'a*::*zero* ⇒ *'a list* ⇒ *'a list* (**infixr** ‹##› *65*)
where *x ## xs* = (*if xs* = [] ∧ *x* = *0 then* [] *else x # xs*)

lemma *cCons-0-Nil-eq* [*simp*]: *0 ## [] = []*
⟨*proof*⟩

lemma *cCons-Cons-eq* [*simp*]: *x ## y # ys = x # y # ys*
⟨*proof*⟩

lemma *cCons-append-Cons-eq* [*simp*]: *x ## xs @ y # ys = x # xs @ y # ys*
⟨*proof*⟩

lemma *cCons-not-0-eq* [*simp*]: *x ≠ 0 ⟹ x ## xs = x # xs*
⟨*proof*⟩

lemma *strip-while-not-0-Cons-eq* [*simp*]:
strip-while (λ*x*. *x = 0*) (*x # xs*) = *x ## strip-while* (λ*x*. *x = 0*) *xs*
⟨*proof*⟩

lemma *tl-cCons* [*simp*]: *tl* (*x ## xs*) = *xs*
⟨*proof*⟩

4.2 Definition of type *poly*

typedef (**overloaded**) *'a poly* = {*f* :: *nat* ⇒ *'a*::*zero*. ∀ ∞ *n*. *f n = 0*}
morphisms *coeff Abs-poly*
⟨*proof*⟩

setup-lifting *type-definition-poly*

lemma *poly-eq-iff*: *p = q* ⟷ (∀ *n*. *coeff p n = coeff q n*)
⟨*proof*⟩

lemma *poly-eqI*: (⋀*n*. *coeff p n = coeff q n*) ⟹ *p = q*
⟨*proof*⟩

lemma *MOST-coeff-eq-0*: ∀ ∞ *n*. *coeff p n = 0*
⟨*proof*⟩

lemma *coeff-Abs-poly*:
assumes ⋀*i*. *i > n ⟹ f i = 0*
shows *coeff* (*Abs-poly f*) = *f*
⟨*proof*⟩

4.3 Degree of a polynomial

definition *degree* :: *'a*::*zero poly* ⇒ *nat*
where *degree p* = (*LEAST n*. ∀ *i*>*n*. *coeff p i = 0*)

lemma *degree-cong*:
assumes ⋀*i*. *coeff p i = 0* ⟷ *coeff q i = 0*
shows *degree p = degree q*

⟨*proof*⟩

lemma *coeff-Abs-poly-If-le*:
  coeff (*Abs-poly* (λ*i*. *if i* ≤ *n then f i else 0*)) = (λ*i*. *if i* ≤ *n then f i else 0*)
⟨*proof*⟩

lemma *coeff-eq-0*:
  assumes *degree p* < *n*
  shows *coeff p n* = *0*
⟨*proof*⟩

lemma *le-degree*: *coeff p n* ≠ *0* ⟹ *n* ≤ *degree p*
  ⟨*proof*⟩

lemma *degree-le*: ∀ *i*>*n*. *coeff p i* = *0* ⟹ *degree p* ≤ *n*
  ⟨*proof*⟩

lemma *less-degree-imp*: *n* < *degree p* ⟹ ∃ *i*>*n*. *coeff p i* ≠ *0*
  ⟨*proof*⟩

lemma *poly-eqI2*:
  assumes *degree p* = *degree q* **and** ⋀*i*. *i* ≤ *degree p* ⟹ *coeff p i* = *coeff q i*
  shows *p* = *q*
  ⟨*proof*⟩

4.4 The zero polynomial

instantiation *poly* :: (*zero*) *zero*
begin

lift-definition *zero-poly* :: *'a poly*
  is λ-. *0*
  ⟨*proof*⟩

instance ⟨*proof*⟩

end

lemma *coeff-0* [*simp*]: *coeff 0 n* = *0*
  ⟨*proof*⟩

lemma *degree-0* [*simp*]: *degree 0* = *0*
  ⟨*proof*⟩

lemma *leading-coeff-neq-0*:
  assumes *p* ≠ *0*
  shows *coeff p* (*degree p*) ≠ *0*
⟨*proof*⟩

lemma *leading-coeff-0-iff* [*simp*]: *coeff p* (*degree p*) = *0* ⟷ *p* = *0*
⟨*proof*⟩

lemma *degree-lessI*:
assumes *p* ≠ *0* ∨ *n* > *0* ∀ *k*≥*n*. *coeff p k* = *0*
shows *degree p* < *n*
⟨*proof*⟩

lemma *eq-zero-or-degree-less*:
assumes *degree p* ≤ *n* **and** *coeff p n* = *0*
shows *p* = *0* ∨ *degree p* < *n*
⟨*proof*⟩

lemma *coeff-0-degree-minus-1*: *coeff rrr dr* = *0* ⟹ *degree rrr* ≤ *dr* ⟹ *degree rrr* ≤ *dr* − *1*
⟨*proof*⟩

4.5 List-style constructor for polynomials

lift-definition *pCons* :: *'a*::*zero* ⇒ *'a poly* ⇒ *'a poly*
is λ*a p*. *case-nat a* (*coeff p*)
⟨*proof*⟩

lemmas *coeff-pCons* = *pCons.rep-eq*

lemma *coeff-pCons'*: *poly.coeff* (*pCons c p*) *n* = (*if n* = *0 then c else poly.coeff p* (*n* − *1*))
⟨*proof*⟩

lemma *coeff-pCons-0* [*simp*]: *coeff* (*pCons a p*) *0* = *a*
⟨*proof*⟩

lemma *coeff-pCons-Suc* [*simp*]: *coeff* (*pCons a p*) (*Suc n*) = *coeff p n*
⟨*proof*⟩

lemma *degree-pCons-le*: *degree* (*pCons a p*) ≤ *Suc* (*degree p*)
⟨*proof*⟩

lemma *degree-pCons-eq*: *p* ≠ *0* ⟹ *degree* (*pCons a p*) = *Suc* (*degree p*)
⟨*proof*⟩

lemma *degree-pCons-0*: *degree* (*pCons a 0*) = *0*
⟨*proof*⟩

lemma *degree-pCons-eq-if* [*simp*]: *degree* (*pCons a p*) = (*if p* = *0 then 0 else Suc* (*degree p*))
⟨*proof*⟩

lemma *pCons-0-0* [*simp*]: *pCons 0 0* = *0*

⟨*proof*⟩

lemma *pCons-eq-iff* [*simp*]: *pCons a p* = *pCons b q* ⟷ *a* = *b* ∧ *p* = *q*
⟨*proof*⟩

lemma *pCons-eq-0-iff* [*simp*]: *pCons a p* = *0* ⟷ *a* = *0* ∧ *p* = *0*
⟨*proof*⟩

lemma *pCons-cases* [*cases type*: *poly*]:
obtains (*pCons*) *a q* **where** *p* = *pCons a q*
⟨*proof*⟩

lemma *pCons-induct* [*case-names 0 pCons*, *induct type*: *poly*]:
assumes *zero*: *P 0*
assumes *pCons*: ⋀*a p*. *a* ≠ *0* ∨ *p* ≠ *0* ⟹ *P p* ⟹ *P* (*pCons a p*)
shows *P p*
⟨*proof*⟩

lemma *degree-eq-zeroE*:
fixes *p* :: *'a*::*zero poly*
assumes *degree p* = *0*
obtains *a* **where** *p* = *pCons a 0*
⟨*proof*⟩

4.6 Quickcheck generator for polynomials

quickcheck-generator *poly constructors*: *0* :: - *poly*, *pCons*

4.7 List-style syntax for polynomials

syntax
-poly :: *args* ⇒ *'a poly* (‹(‹*indent=2 notation=‹mixfix polynomial enumeration›*›[:-:])›)
syntax-consts
-poly ⇌ *pCons*
translations
[:*x*, *xs*:] ⇌ *CONST pCons x* [:*xs*:]
[:*x*:] ⇌ *CONST pCons x 0*

lemma *degree-0-id*:
assumes *degree p* = *0*
shows [: *coeff p 0* :] = *p*
⟨*proof*⟩

lemma *degree0-coeffs*: *degree p* = *0* ⟹ ∃ *a*. *p* = [: *a* :]
⟨*proof*⟩

lemma *degree1-coeffs*:
fixes *p* :: *'a*::*zero poly*
assumes *degree p* = *1*

obtains a b **where** $p = [:\, b,\, a\, :]$ $a \neq 0$
⟨*proof*⟩

lemma *degree2-coeffs*:
fixes p :: *'a::zero poly*
assumes *degree* $p = 2$
obtains a b c **where** $p = [:\, c,\, b,\, a\, :]$ $a \neq 0$
⟨*proof*⟩

4.8 Representation of polynomials by lists of coefficients

primrec *Poly* :: *'a::zero list* ⇒ *'a poly*
where
  [*code-post*]: *Poly* [] = *0*
| [*code-post*]: *Poly* (*a* # *as*) = *pCons a* (*Poly as*)

lemma *Poly-replicate-0* [*simp*]: *Poly* (*replicate n 0*) = *0*
⟨*proof*⟩

lemma *Poly-eq-0*: *Poly as* = *0* ⟷ (∃ *n*. *as* = *replicate n 0*)
⟨*proof*⟩

lemma *Poly-append-replicate-zero* [*simp*]: *Poly* (*as* @ *replicate n 0*) = *Poly as*
⟨*proof*⟩

lemma *Poly-snoc-zero* [*simp*]: *Poly* (*as* @ [*0*]) = *Poly as*
⟨*proof*⟩

lemma *Poly-cCons-eq-pCons-Poly* [*simp*]: *Poly* (*a* ## *p*) = *pCons a* (*Poly p*)
⟨*proof*⟩

lemma *Poly-on-rev-starting-with-0* [*simp*]: *hd as* = *0* ⟹ *Poly* (*rev* (*tl as*)) = *Poly* (*rev as*)
⟨*proof*⟩

lemma *degree-Poly*: *degree* (*Poly xs*) ≤ *length xs*
⟨*proof*⟩

lemma *coeff-Poly-eq* [*simp*]: *coeff* (*Poly xs*) = *nth-default 0 xs*
⟨*proof*⟩

definition *coeffs* :: *'a poly* ⇒ *'a::zero list*
where *coeffs p* = (*if p* = *0 then* [] *else map* (λ*i*. *coeff p i*) [*0* ..< *Suc* (*degree p*)])

lemma *coeffs-eq-Nil* [*simp*]: *coeffs p* = [] ⟷ *p* = *0*
⟨*proof*⟩

lemma *not-0-coeffs-not-Nil*: *p* ≠ *0* ⟹ *coeffs p* ≠ []
⟨*proof*⟩

lemma *coeffs-0-eq-Nil* [*simp*]: *coeffs 0 = []*
⟨*proof*⟩

lemma *coeffs-pCons-eq-cCons* [*simp*]: *coeffs (pCons a p) = a ## coeffs p*
⟨*proof*⟩

lemma *length-coeffs*: *p ≠ 0 ⟹ length (coeffs p) = degree p + 1*
⟨*proof*⟩

lemma *coeffs-nth*: *p ≠ 0 ⟹ n ≤ degree p ⟹ coeffs p ! n = coeff p n*
⟨*proof*⟩

lemma *coeff-in-coeffs*: *p ≠ 0 ⟹ n ≤ degree p ⟹ coeff p n ∈ set (coeffs p)*
⟨*proof*⟩

lemma *not-0-cCons-eq* [*simp*]: *p ≠ 0 ⟹ a ## coeffs p = a # coeffs p*
⟨*proof*⟩

lemma *Poly-coeffs* [*simp*, *code abstype*]: *Poly (coeffs p) = p*
⟨*proof*⟩

lemma *coeffs-Poly* [*simp*]: *coeffs (Poly as) = strip-while (HOL.eq 0) as*
⟨*proof*⟩

lemma *no-trailing-coeffs* [*simp*]:
no-trailing (HOL.eq 0) (coeffs p)
⟨*proof*⟩

lemma *strip-while-coeffs* [*simp*]:
strip-while (HOL.eq 0) (coeffs p) = coeffs p
⟨*proof*⟩

lemma *coeffs-eq-iff*: *p = q ⟷ coeffs p = coeffs q*
(**is** *?P ⟷ ?Q*)
⟨*proof*⟩

lemma *nth-default-coeffs-eq*: *nth-default 0 (coeffs p) = coeff p*
⟨*proof*⟩

lemma *range-coeff*: *range (coeff p) = insert 0 (set (coeffs p))*
⟨*proof*⟩

lemma [*code*]: *coeff p = nth-default 0 (coeffs p)*
⟨*proof*⟩

lemma *coeffs-eqI*:
assumes *coeff*: *⋀n. coeff p n = nth-default 0 xs n*
assumes *zero*: *no-trailing (HOL.eq 0) xs*

shows *coeffs p = xs*
⟨*proof*⟩

lemma *degree-eq-length-coeffs* [*code*]: *degree p = length* (*coeffs p*) − *1*
⟨*proof*⟩

lemma *length-coeffs-degree*: *p ≠ 0 ⟹ length* (*coeffs p*) = *Suc* (*degree p*)
⟨*proof*⟩

lemma [*code abstract*]: *coeffs 0* = []
⟨*proof*⟩

lemma [*code abstract*]: *coeffs* (*pCons a p*) = *a* ## *coeffs p*
⟨*proof*⟩

lemma *set-coeffs-subset-singleton-0-iff* [*simp*]:
set (*coeffs p*) ⊆ {*0*} ⟷ *p = 0*
⟨*proof*⟩

lemma *set-coeffs-not-only-0* [*simp*]:
set (*coeffs p*) ≠ {*0*}
⟨*proof*⟩

lemma *forall-coeffs-conv*:
(∀ *n*. *P* (*coeff p n*)) ⟷ (∀ *c* ∈ *set* (*coeffs p*). *P c*) **if** *P 0*
⟨*proof*⟩

instantiation *poly* :: ({*zero*, *equal*}) *equal*
begin

definition [*code*]: *HOL.equal* (*p*::*'a poly*) *q* ⟷ *HOL.equal* (*coeffs p*) (*coeffs q*)

instance
⟨*proof*⟩

end

lemma [*code nbe*]: *HOL.equal* (*p* :: - *poly*) *p* ⟷ *True*
⟨*proof*⟩

definition *is-zero* :: *'a*::*zero poly* ⇒ *bool*
where [*code*]: *is-zero p* ⟷ *List.null* (*coeffs p*)

lemma *is-zero-null* [*code-abbrev*]: *is-zero p* ⟷ *p = 0*
⟨*proof*⟩

Reconstructing the polynomial from the list

definition *poly-of-list* :: *'a*::*comm-monoid-add list* ⇒ *'a poly*
where [*simp*]: *poly-of-list* = *Poly*

lemma *poly-of-list-impl* [*code abstract*]: *coeffs* (*poly-of-list as*) = *strip-while* (*HOL.eq 0*) *as*
⟨*proof*⟩

4.9 Fold combinator for polynomials

definition *fold-coeffs* :: ('*a*::*zero* ⇒ '*b* ⇒ '*b*) ⇒ '*a poly* ⇒ '*b* ⇒ '*b*
where *fold-coeffs f p* = *foldr f* (*coeffs p*)

lemma *fold-coeffs-0-eq* [*simp*]: *fold-coeffs f 0* = *id*
⟨*proof*⟩

lemma *fold-coeffs-pCons-eq* [*simp*]: *f 0* = *id* ⟹ *fold-coeffs f* (*pCons a p*) = *f a* ∘ *fold-coeffs f p*
⟨*proof*⟩

lemma *fold-coeffs-pCons-0-0-eq* [*simp*]: *fold-coeffs f* (*pCons 0 0*) = *id*
⟨*proof*⟩

lemma *fold-coeffs-pCons-coeff-not-0-eq* [*simp*]:
a ≠ *0* ⟹ *fold-coeffs f* (*pCons a p*) = *f a* ∘ *fold-coeffs f p*
⟨*proof*⟩

lemma *fold-coeffs-pCons-not-0-0-eq* [*simp*]:
p ≠ *0* ⟹ *fold-coeffs f* (*pCons a p*) = *f a* ∘ *fold-coeffs f p*
⟨*proof*⟩

4.10 Canonical morphism on polynomials – evaluation

definition *poly* :: ‹'*a*::*comm-semiring-0 poly* ⇒ '*a* ⇒ '*a*›
where ‹*poly p a* = *horner-sum id a* (*coeffs p*)›

lemma *poly-eq-fold-coeffs*:
‹*poly p* = *fold-coeffs* (λ*a f x*. *a* + *x* ∗ *f x*) *p* (λ*x*. *0*)›
⟨*proof*⟩

lemma *poly-0* [*simp*]: *poly 0 x* = *0*
⟨*proof*⟩

lemma *poly-pCons* [*simp*]: *poly* (*pCons a p*) *x* = *a* + *x* ∗ *poly p x*
⟨*proof*⟩

lemma *poly-altdef*: *poly p x* = (∑ *i*≤*degree p*. *coeff p i* ∗ *x* ^ *i*)
for *x* :: '*a*::{*comm-semiring-0*,*semiring-1*}
⟨*proof*⟩

lemma *poly-0-coeff-0*: *poly p 0* = *coeff p 0*
⟨*proof*⟩

lemma *poly-zero*:
fixes $p :: 'a :: comm\text{-}ring\text{-}1\ poly$
assumes x: $poly\ p\ x = 0$ **shows** $p = 0 \longleftrightarrow degree\ p = 0$
⟨*proof*⟩

4.11 Monomials

lift-definition $monom :: 'a \Rightarrow nat \Rightarrow 'a::zero\ poly$
is $\lambda a\ m\ n.\ if\ m = n\ then\ a\ else\ 0$
⟨*proof*⟩

lemma *coeff-monom* [*simp*]: $coeff\ (monom\ a\ m)\ n = (if\ m = n\ then\ a\ else\ 0)$
⟨*proof*⟩

lemma *monom-0*: $monom\ a\ 0 = [:a:]$
⟨*proof*⟩

lemma *monom-Suc*: $monom\ a\ (Suc\ n) = pCons\ 0\ (monom\ a\ n)$
⟨*proof*⟩

lemma *monom-eq-0* [*simp*]: $monom\ 0\ n = 0$
⟨*proof*⟩

lemma *monom-eq-0-iff* [*simp*]: $monom\ a\ n = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *monom-eq-iff* [*simp*]: $monom\ a\ n = monom\ b\ n \longleftrightarrow a = b$
⟨*proof*⟩

lemma *degree-monom-le*: $degree\ (monom\ a\ n) \leq n$
⟨*proof*⟩

lemma *degree-monom-eq*: $a \neq 0 \Longrightarrow degree\ (monom\ a\ n) = n$
⟨*proof*⟩

lemma *coeffs-monom* [*code abstract*]:
$coeffs\ (monom\ a\ n) = (if\ a = 0\ then\ []\ else\ replicate\ n\ 0\ @\ [a])$
⟨*proof*⟩

lemma *fold-coeffs-monom* [*simp*]: $a \neq 0 \Longrightarrow fold\text{-}coeffs\ f\ (monom\ a\ n) = f\ 0\ \widehat{\ }\widehat{\ }\ n \circ f\ a$
⟨*proof*⟩

lemma *poly-monom*: $poly\ (monom\ a\ n)\ x = a * x\ \widehat{\ }\ n$
for $a\ x :: 'a::comm\text{-}semiring\text{-}1$
⟨*proof*⟩

lemma *monom-eq-iff'*: $monom\ c\ n = monom\ d\ m \longleftrightarrow\ c = d \wedge (c = 0 \vee n = m)$

⟨*proof*⟩

lemma *monom-eq-const-iff*: *monom c n* = [:*d*:] ⟷ *c* = *d* ∧ (*c* = *0* ∨ *n* = *0*)
⟨*proof*⟩

4.12 Leading coefficient

abbreviation *lead-coeff*:: *'a*::*zero poly* ⇒ *'a*
where *lead-coeff p* ≡ *coeff p* (*degree p*)

lemma *lead-coeff-pCons*[*simp*]:
p ≠ *0* ⟹ *lead-coeff* (*pCons a p*) = *lead-coeff p*
p = *0* ⟹ *lead-coeff* (*pCons a p*) = *a*
⟨*proof*⟩

lemma *lead-coeff-monom* [*simp*]: *lead-coeff* (*monom c n*) = *c*
⟨*proof*⟩

lemma *last-coeffs-eq-coeff-degree*:
last (*coeffs p*) = *lead-coeff p* **if** *p* ≠ *0*
⟨*proof*⟩

lemma *lead-coeff-list-def*:
lead-coeff p = (*if coeffs p*=[] *then 0 else last* (*coeffs p*))
⟨*proof*⟩

4.13 Addition and subtraction

instantiation *poly* :: (*comm-monoid-add*) *comm-monoid-add*
begin

lift-definition *plus-poly* :: *'a poly* ⇒ *'a poly* ⇒ *'a poly*
is λ*p q n*. *coeff p n* + *coeff q n*
⟨*proof*⟩

lemma *coeff-add* [*simp*]: *coeff* (*p* + *q*) *n* = *coeff p n* + *coeff q n*
⟨*proof*⟩

instance
⟨*proof*⟩

end

instantiation *poly* :: (*cancel-comm-monoid-add*) *cancel-comm-monoid-add*
begin

lift-definition *minus-poly* :: *'a poly* ⇒ *'a poly* ⇒ *'a poly*
is λ*p q n*. *coeff p n* − *coeff q n*
⟨*proof*⟩

lemma *coeff-diff* [*simp*]: *coeff* (*p* − *q*) *n* = *coeff p n* − *coeff q n*
⟨*proof*⟩

instance
⟨*proof*⟩

end

instantiation *poly* :: (*ab-group-add*) *ab-group-add*
begin

lift-definition *uminus-poly* :: *'a poly* ⇒ *'a poly*
is λ*p n*. − *coeff p n*
⟨*proof*⟩

lemma *coeff-minus* [*simp*]: *coeff* (− *p*) *n* = − *coeff p n*
⟨*proof*⟩

instance
⟨*proof*⟩

end

lemma *add-pCons* [*simp*]: *pCons a p* + *pCons b q* = *pCons* (*a* + *b*) (*p* + *q*)
⟨*proof*⟩

lemma *minus-pCons* [*simp*]: − *pCons a p* = *pCons* (− *a*) (− *p*)
⟨*proof*⟩

lemma *diff-pCons* [*simp*]: *pCons a p* − *pCons b q* = *pCons* (*a* − *b*) (*p* − *q*)
⟨*proof*⟩

lemma *degree-add-le-max*: *degree* (*p* + *q*) ≤ *max* (*degree p*) (*degree q*)
⟨*proof*⟩

lemma *degree-add-le*: *degree p* ≤ *n* ⟹ *degree q* ≤ *n* ⟹ *degree* (*p* + *q*) ≤ *n*
⟨*proof*⟩

lemma *degree-add-less*: *degree p* < *n* ⟹ *degree q* < *n* ⟹ *degree* (*p* + *q*) < *n*
⟨*proof*⟩

lemma *degree-add-eq-right*: **assumes** *degree p* < *degree q* **shows** *degree* (*p* + *q*)
= *degree q*
⟨*proof*⟩

lemma *degree-add-eq-left*: *degree q* < *degree p* ⟹ *degree* (*p* + *q*) = *degree p*
⟨*proof*⟩

lemma *degree-minus* [*simp*]: *degree* (− *p*) = *degree p*

⟨*proof*⟩

lemma *lead-coeff-add-le*: *degree p* < *degree q* ⟹ *lead-coeff* (*p* + *q*) = *lead-coeff q*
⟨*proof*⟩

lemma *lead-coeff-minus*: *lead-coeff* (− *p*) = − *lead-coeff p*
⟨*proof*⟩

lemma *degree-diff-le-max*: *degree* (*p* − *q*) ≤ *max* (*degree p*) (*degree q*)
for *p q* :: *'a*::*ab-group-add poly*
⟨*proof*⟩

lemma *degree-diff-le*: *degree p* ≤ *n* ⟹ *degree q* ≤ *n* ⟹ *degree* (*p* − *q*) ≤ *n*
for *p q* :: *'a*::*ab-group-add poly*
⟨*proof*⟩

lemma *degree-diff-less*: *degree p* < *n* ⟹ *degree q* < *n* ⟹ *degree* (*p* − *q*) < *n*
for *p q* :: *'a*::*ab-group-add poly*
⟨*proof*⟩

lemma *add-monom*: *monom a n* + *monom b n* = *monom* (*a* + *b*) *n*
⟨*proof*⟩

lemma *diff-monom*: *monom a n* − *monom b n* = *monom* (*a* − *b*) *n*
⟨*proof*⟩

lemma *minus-monom*: − *monom a n* = *monom* (− *a*) *n*
⟨*proof*⟩

lemma *coeff-sum*: *coeff* (∑ *x*∈*A*. *p x*) *i* = (∑ *x*∈*A*. *coeff* (*p x*) *i*)
⟨*proof*⟩

lemma *monom-sum*: *monom* (∑ *x*∈*A*. *a x*) *n* = (∑ *x*∈*A*. *monom* (*a x*) *n*)
⟨*proof*⟩

fun *plus-coeffs* :: *'a*::*comm-monoid-add list* ⇒ *'a list* ⇒ *'a list*
where
plus-coeffs xs [] = *xs*
| *plus-coeffs* [] *ys* = *ys*
| *plus-coeffs* (*x* # *xs*) (*y* # *ys*) = (*x* + *y*) ## *plus-coeffs xs ys*

lemma *coeffs-plus-eq-plus-coeffs* [*code abstract*]:
coeffs (*p* + *q*) = *plus-coeffs* (*coeffs p*) (*coeffs q*)
⟨*proof*⟩

lemma *coeffs-uminus* [*code abstract*]:
coeffs (− *p*) = *map uminus* (*coeffs p*)
⟨*proof*⟩

lemma [*code*]: $p - q = p + - q$
  for p q :: $'a$::*ab-group-add poly*
  ⟨*proof*⟩

lemma *poly-add* [*simp*]: *poly* $(p + q)$ x = *poly* p x + *poly* q x
⟨*proof*⟩

lemma *poly-minus* [*simp*]: *poly* $(- p)$ x = $-$ *poly* p x
  for x :: $'a$::*comm-ring*
  ⟨*proof*⟩

lemma *poly-diff* [*simp*]: *poly* $(p - q)$ x = *poly* p x $-$ *poly* q x
  for x :: $'a$::*comm-ring*
  ⟨*proof*⟩

lemma *poly-sum*: *poly* $(\sum k{\in}A.\ p\ k)$ x = $(\sum k{\in}A.$ *poly* $(p\ k)$ $x)$
  ⟨*proof*⟩

lemma *poly-sum-list*: *poly* $(\sum p{\leftarrow}ps.\ p)$ y = $(\sum p{\leftarrow}ps.$ *poly* p $y)$
  ⟨*proof*⟩

lemma *poly-sum-mset*: *poly* $(\sum x{\in}\#A.\ p\ x)$ y = $(\sum x{\in}\#A.$ *poly* $(p\ x)$ $y)$
  ⟨*proof*⟩

lemma *degree-sum-le*: *finite* $S \Longrightarrow (\bigwedge p.\ p \in S \Longrightarrow$ *degree* $(f\ p) \leq n) \Longrightarrow$ *degree* (*sum* f S) $\leq n$
⟨*proof*⟩

lemma *degree-sum-less*:
  assumes $\bigwedge x.\ x \in A \Longrightarrow$ *degree* $(f\ x) < n$ $n > 0$
  shows *degree* (*sum* f A) $< n$
  ⟨*proof*⟩

lemma *poly-as-sum-of-monoms'*:
  assumes *degree* $p \leq n$
  shows $(\sum i{\leq}n.$ *monom* (*coeff* p i) $i) = p$
⟨*proof*⟩

lemma *poly-as-sum-of-monoms*: $(\sum i{\leq}$*degree* $p.$ *monom* (*coeff* p i) $i) = p$
  ⟨*proof*⟩

lemma *Poly-snoc*: *Poly* (*xs* @ [x]) = *Poly* *xs* + *monom* x (*length* *xs*)
  ⟨*proof*⟩

4.14 Multiplication by a constant, polynomial multiplication and the unit polynomial

lift-definition *smult* :: $'a$::*comm-semiring-0* ⇒ $'a$ *poly* ⇒ $'a$ *poly*
  is $\lambda a\ p\ n.\ a * coeff\ p\ n$

⟨*proof*⟩

lemma *coeff-smult* [*simp*]: *coeff* (*smult a p*) *n* = *a* ∗ *coeff p n*
⟨*proof*⟩

lemma *degree-smult-le*: *degree* (*smult a p*) ≤ *degree p*
⟨*proof*⟩

lemma *smult-smult* [*simp*]: *smult a* (*smult b p*) = *smult* (*a* ∗ *b*) *p*
⟨*proof*⟩

lemma *smult-0-right* [*simp*]: *smult a 0* = *0*
⟨*proof*⟩

lemma *smult-0-left* [*simp*]: *smult 0 p* = *0*
⟨*proof*⟩

lemma *smult-1-left* [*simp*]: *smult* (*1*::*'a*::*comm-semiring-1*) *p* = *p*
⟨*proof*⟩

lemma *smult-add-right*: *smult a* (*p* + *q*) = *smult a p* + *smult a q*
⟨*proof*⟩

lemma *smult-add-left*: *smult* (*a* + *b*) *p* = *smult a p* + *smult b p*
⟨*proof*⟩

lemma *smult-minus-right* [*simp*]: *smult a* (− *p*) = − *smult a p*
for *a* :: *'a*::*comm-ring*
⟨*proof*⟩

lemma *smult-minus-left* [*simp*]: *smult* (− *a*) *p* = − *smult a p*
for *a* :: *'a*::*comm-ring*
⟨*proof*⟩

lemma *smult-diff-right*: *smult a* (*p* − *q*) = *smult a p* − *smult a q*
for *a* :: *'a*::*comm-ring*
⟨*proof*⟩

lemma *smult-diff-left*: *smult* (*a* − *b*) *p* = *smult a p* − *smult b p*
for *a b* :: *'a*::*comm-ring*
⟨*proof*⟩

lemmas *smult-distribs* =
smult-add-left smult-add-right
smult-diff-left smult-diff-right

lemma *smult-pCons* [*simp*]: *smult a* (*pCons b p*) = *pCons* (*a* ∗ *b*) (*smult a p*)
⟨*proof*⟩

lemma *smult-monom*: *smult a* (*monom b n*) = *monom* (*a* ∗ *b*) *n*
⟨*proof*⟩

lemma *smult-Poly*: *smult c* (*Poly xs*) = *Poly* (*map* ((∗) *c*) *xs*)
⟨*proof*⟩

lemma *degree-smult-eq* [*simp*]: *degree* (*smult a p*) = (*if a* = *0 then 0 else degree p*)
for *a* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*}
⟨*proof*⟩

lemma *smult-eq-0-iff* [*simp*]: *smult a p* = *0* ⟷ *a* = *0* ∨ *p* = *0*
for *a* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*}
⟨*proof*⟩

lemma *coeffs-smult* [*code abstract*]:
coeffs (*smult a p*) = (*if a* = *0 then* [] *else map* (*Groups.times a*) (*coeffs p*))
for *p* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
⟨*proof*⟩

lemma *smult-eq-iff*:
fixes *b* :: *'a* :: *field*
assumes *b* ≠ *0*
shows *smult a p* = *smult b q* ⟷ *smult* (*a* / *b*) *p* = *q*
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *smult-cancel*:
fixes *p*::*'a*::*idom poly*
assumes *c*≠*0* **and** *smult*: *smult c p* = *smult c q*
shows *p*=*q*
⟨*proof*⟩

instantiation *poly* :: (*comm-semiring-0*) *comm-semiring-0*
begin

definition *p* ∗ *q* = *fold-coeffs* (λ*a p*. *smult a q* + *pCons 0 p*) *p 0*

lemma *mult-poly-0-left*: (*0*::*'a poly*) ∗ *q* = *0*
⟨*proof*⟩

lemma *mult-pCons-left* [*simp*]: *pCons a p* ∗ *q* = *smult a q* + *pCons 0* (*p* ∗ *q*)
⟨*proof*⟩

lemma *mult-poly-0-right*: *p* ∗ (*0*::*'a poly*) = *0*
⟨*proof*⟩

lemma *mult-pCons-right* [*simp*]: *p* ∗ *pCons a q* = *smult a p* + *pCons 0* (*p* ∗ *q*)
⟨*proof*⟩

lemmas *mult-poly-0 = mult-poly-0-left mult-poly-0-right*

lemma *mult-smult-left* [*simp*]: *smult a p * q = smult a (p * q)*
⟨*proof*⟩

lemma *mult-smult-right* [*simp*]: *p * smult a q = smult a (p * q)*
⟨*proof*⟩

lemma *mult-poly-add-left*: *(p + q) * r = p * r + q * r*
for *p q r :: 'a poly*
⟨*proof*⟩

instance
⟨*proof*⟩

end

lemma *coeff-mult-degree-sum*:
*coeff (p * q) (degree p + degree q) = coeff p (degree p) * coeff q (degree q)*
⟨*proof*⟩

instance *poly :: ({comm-semiring-0,semiring-no-zero-divisors}) semiring-no-zero-divisors*
⟨*proof*⟩

instance *poly :: (comm-semiring-0-cancel) comm-semiring-0-cancel* ⟨*proof*⟩

lemma *coeff-mult*: *coeff (p * q) n = (∑ i≤n. coeff p i * coeff q (n−i))*
⟨*proof*⟩

lemma *coeff-mult-0*: *coeff (p * q) 0 = coeff p 0 * coeff q 0*
⟨*proof*⟩

lemma *degree-mult-le*: *degree (p * q) ≤ degree p + degree q*
⟨*proof*⟩

lemma *mult-monom*: *monom a m * monom b n = monom (a * b) (m + n)*
⟨*proof*⟩

instantiation *poly :: (comm-semiring-1) comm-semiring-1*
begin

lift-definition *one-poly :: 'a poly*
is *λn. of-bool (n = 0)*
⟨*proof*⟩

lemma *coeff-1* [*simp*]:
coeff 1 n = of-bool (n = 0)
⟨*proof*⟩

lemma *one-pCons*:
$1 = [:1:]$
⟨*proof*⟩

lemma *pCons-one*:
$[:1:] = 1$
⟨*proof*⟩

instance
⟨*proof*⟩

end

lemma *poly-1* [*simp*]:
poly $1\ x = 1$
⟨*proof*⟩

lemma *one-poly-eq-simps* [*simp*]:
$1 = [:1:] \longleftrightarrow \mathit{True}$
$[:1:] = 1 \longleftrightarrow \mathit{True}$
⟨*proof*⟩

lemma *degree-1* [*simp*]:
degree $1 = 0$
⟨*proof*⟩

lemma *coeffs-1-eq* [*simp*, *code abstract*]:
coeffs $1 = [1]$
⟨*proof*⟩

lemma *smult-one* [*simp*]:
smult $c\ 1 = [:c:]$
⟨*proof*⟩

lemma *smult-sum*: *smult* $(\sum i \in S.\ f\ i)\ p = (\sum i \in S.\ \mathit{smult}\ (f\ i)\ p)$
⟨*proof*⟩

lemma *smult-power*: $(\mathit{smult}\ a\ p)\ \hat{}\ n = \mathit{smult}\ (a\ \hat{}\ n)\ (p\ \hat{}\ n)$
⟨*proof*⟩

lemma *monom-eq-1* [*simp*]:
monom $1\ 0 = 1$
⟨*proof*⟩

lemma *monom-eq-1-iff*:
monom $c\ n = 1 \longleftrightarrow c = 1 \wedge n = 0$
⟨*proof*⟩

lemma *monom-altdef*:

monom c n = smult c ([:0, 1:] ^ n)
⟨*proof*⟩

lemma *degree-sum-list-le*: (⋀ *p* . *p* ∈ *set ps* ⟹ *degree p* ≤ *n*)
⟹ *degree* (*sum-list ps*) ≤ *n*
⟨*proof*⟩

lemma *degree-prod-list-le*: *degree* (*prod-list ps*) ≤ *sum-list* (*map degree ps*)
⟨*proof*⟩

instance *poly* :: ({*comm-semiring-1*,*semiring-1-no-zero-divisors*}) *semiring-1-no-zero-divisors*
⟨*proof*⟩
instance *poly* :: (*comm-ring*) *comm-ring* ⟨*proof*⟩
instance *poly* :: (*comm-ring-1*) *comm-ring-1* ⟨*proof*⟩
instance *poly* :: (*comm-ring-1*) *comm-semiring-1-cancel* ⟨*proof*⟩

lemma *prod-smult*: (∏ *x*∈*A*. *smult* (*c x*) (*p x*)) = *smult* (*prod c A*) (*prod p A*)
⟨*proof*⟩

lemma *degree-power-le*: *degree* (*p* ^ *n*) ≤ *degree p* ∗ *n*
⟨*proof*⟩

lemma *coeff-0-power*: *coeff* (*p* ^ *n*) *0* = *coeff p 0* ^ *n*
⟨*proof*⟩

lemma *poly-smult* [*simp*]: *poly* (*smult a p*) *x* = *a* ∗ *poly p x*
⟨*proof*⟩

lemma *poly-mult* [*simp*]: *poly* (*p* ∗ *q*) *x* = *poly p x* ∗ *poly q x*
⟨*proof*⟩

lemma *poly-power* [*simp*]: *poly* (*p* ^ *n*) *x* = *poly p x* ^ *n*
for *p* :: *'a*::*comm-semiring-1 poly*
⟨*proof*⟩

lemma *poly-prod*: *poly* (∏ *k*∈*A*. *p k*) *x* = (∏ *k*∈*A*. *poly* (*p k*) *x*)
⟨*proof*⟩

lemma *poly-prod-list*: *poly* (∏ *p*←*ps*. *p*) *y* = (∏ *p*←*ps*. *poly p y*)
⟨*proof*⟩

lemma *poly-prod-mset*: *poly* (∏ *x*∈#*A*. *p x*) *y* = (∏ *x*∈#*A*. *poly* (*p x*) *y*)
⟨*proof*⟩

lemma *poly-const-pow*: [: *c* :] ^ *n* = [: *c* ^ *n* :]
⟨*proof*⟩

lemma *monom-power*: *monom c n* ^ *k* = *monom* (*c* ^ *k*) (*n* ∗ *k*)
⟨*proof*⟩

lemma *degree-prod-sum-le*: *finite S* $\Longrightarrow$ *degree* (*prod f S*) $\leq$ *sum* (*degree* $\circ$ *f*) *S*
⟨*proof*⟩

lemma *coeff-0-prod-list*: *coeff* (*prod-list xs*) *0* = *prod-list* (*map* (λ*p*. *coeff p 0*) *xs*)
⟨*proof*⟩

lemma *coeff-monom-mult*: *coeff* (*monom c n* ∗ *p*) *k* = (*if k* < *n then 0 else c* ∗ *coeff p* (*k* − *n*))
⟨*proof*⟩

lemma *coeff-monom-Suc*: *coeff* (*monom a* (*Suc d*) ∗ *p*) (*Suc i*) = *coeff* (*monom a d* ∗ *p*) *i*
⟨*proof*⟩

lemma *monom-1-dvd-iff'*: *monom 1 n dvd p* ⟷ (∀ *k*<*n*. *coeff p k* = *0*)
⟨*proof*⟩

lemma *coeff-sum-monom*:
assumes *n*: *n* ≤ *d*
shows *coeff* (∑ *i*≤*d*. *monom* (*f i*) *i*) *n* = *f n* (**is** *?l* = -)
⟨*proof*⟩

4.15 Mapping polynomials

definition *map-poly* :: ('*a* :: *zero* ⇒ '*b* :: *zero*) ⇒ '*a poly* ⇒ '*b poly*
where *map-poly f p* = *Poly* (*map f* (*coeffs p*))

lemma *map-poly-0* [*simp*]: *map-poly f 0* = *0*
⟨*proof*⟩

lemma *map-poly-1*: *map-poly f 1* = [:*f 1*:]
⟨*proof*⟩

lemma *map-poly-1'* [*simp*]: *f 1* = *1* ⟹ *map-poly f 1* = *1*
⟨*proof*⟩

lemma *coeff-map-poly*:
assumes *f 0* = *0*
shows *coeff* (*map-poly f p*) *n* = *f* (*coeff p n*)
⟨*proof*⟩

lemma *lead-coeff-map-poly-nz*:
assumes *f* (*lead-coeff p*) ≠ *0 f 0* = *0*
shows *lead-coeff* (*map-poly f p*) = *f* (*lead-coeff p*)
⟨*proof*⟩

lemma *coeffs-map-poly* [*code abstract*]:
coeffs (*map-poly f p*) = *strip-while* ((=) *0*) (*map f* (*coeffs p*))

⟨*proof*⟩

lemma *coeffs-map-poly'*:
assumes $\bigwedge x.\ x \neq 0 \Longrightarrow f\ x \neq 0$
shows *coeffs* (*map-poly f p*) = *map f* (*coeffs p*)
⟨*proof*⟩

lemma *set-coeffs-map-poly*:
$(\bigwedge x.\ f\ x = 0 \longleftrightarrow x = 0) \Longrightarrow$ *set* (*coeffs* (*map-poly f p*)) = *f* ‘ *set* (*coeffs p*)
⟨*proof*⟩

lemma *degree-map-poly*:
assumes $\bigwedge x.\ x \neq 0 \Longrightarrow f\ x \neq 0$
shows *degree* (*map-poly f p*) = *degree p*
⟨*proof*⟩

lemma *map-poly-eq-0-iff*:
assumes $f\ 0 = 0\ \bigwedge x.\ x \in$ *set* (*coeffs p*) $\Longrightarrow x \neq 0 \Longrightarrow f\ x \neq 0$
shows *map-poly f p* = 0 $\longleftrightarrow$ *p* = 0
⟨*proof*⟩

lemma *map-poly-smult*:
assumes $f\ 0 = 0 \bigwedge c\ x.\ f\ (c * x) = f\ c * f\ x$
shows *map-poly f* (*smult c p*) = *smult* (*f c*) (*map-poly f p*)
⟨*proof*⟩

lemma *map-poly-pCons*:
assumes *f* 0 = 0
shows *map-poly f* (*pCons c p*) = *pCons* (*f c*) (*map-poly f p*)
⟨*proof*⟩

lemma *map-poly-map-poly*:
assumes *f* 0 = 0 *g* 0 = 0
shows *map-poly f* (*map-poly g p*) = *map-poly* (*f* ∘ *g*) *p*
⟨*proof*⟩

lemma *map-poly-id* [*simp*]: *map-poly id p* = *p*
⟨*proof*⟩

lemma *map-poly-id'* [*simp*]: *map-poly* (λ*x*. *x*) *p* = *p*
⟨*proof*⟩

lemma *map-poly-cong*:
assumes $(\bigwedge x.\ x \in$ *set* (*coeffs p*) $\Longrightarrow f\ x = g\ x)$
shows *map-poly f p* = *map-poly g p*
⟨*proof*⟩

lemma *map-poly-monom*: $f\ 0 = 0 \Longrightarrow$ *map-poly f* (*monom c n*) = *monom* (*f c*) *n*
⟨*proof*⟩

lemma *map-poly-idI*:
assumes $\bigwedge x.\ x \in set\ (coeffs\ p) \Longrightarrow f\ x = x$
shows *map-poly f p* = *p*
⟨*proof*⟩

lemma *map-poly-idI'*:
assumes $\bigwedge x.\ x \in set\ (coeffs\ p) \Longrightarrow f\ x = x$
shows *p* = *map-poly f p*
⟨*proof*⟩

lemma *smult-conv-map-poly*: *smult c p* = *map-poly* (λ*x*. *c* ∗ *x*) *p*
⟨*proof*⟩

lemma *poly-cnj*: *cnj* (*poly p z*) = *poly* (*map-poly cnj p*) (*cnj z*)
⟨*proof*⟩

lemma *poly-cnj-real*:
assumes $\bigwedge n.\ poly.coeff\ p\ n \in \mathbb{R}$
shows *cnj* (*poly p z*) = *poly p* (*cnj z*)
⟨*proof*⟩

lemma *real-poly-cnj-root-iff*:
assumes $\bigwedge n.\ poly.coeff\ p\ n \in \mathbb{R}$
shows *poly p* (*cnj z*) = *0* ⟷ *poly p z* = *0*
⟨*proof*⟩

lemma *sum-to-poly*: $(\sum x{\in}A.\ [:f\ x:]) = [:\sum x{\in}A.\ f\ x:]$
⟨*proof*⟩

lemma *diff-to-poly*: [:*c*:] − [:*d*:] = [:*c* − *d*:]
⟨*proof*⟩

lemma *mult-to-poly*: [:*c*:] ∗ [:*d*:] = [:*c* ∗ *d*:]
⟨*proof*⟩

lemma *prod-to-poly*: $(\prod x{\in}A.\ [:f\ x:]) = [:\prod x{\in}A.\ f\ x:]$
⟨*proof*⟩

lemma *poly-map-poly-cnj* [*simp*]: *poly* (*map-poly cnj p*) *x* = *cnj* (*poly p* (*cnj x*))
⟨*proof*⟩

lemma *map-poly-degree-eq*:
assumes *f* (*lead-coeff p*) ≠ *0*
shows *degree* (*map-poly f p*) = *degree p*
⟨*proof*⟩

lemma *map-poly-degree-less*:
assumes *f* (*lead-coeff p*) =*0 degree p*≠*0*

shows *degree* (*map-poly f p*) < *degree p*
⟨*proof*⟩

lemma *map-poly-degree-leq*:
shows *degree* (*map-poly f p*) ≤ *degree p*
⟨*proof*⟩

4.16 Conversions

lemma *of-nat-poly*: *of-nat n* = [:*of-nat n*:]
⟨*proof*⟩

lemma *of-nat-monom*: *of-nat n* = *monom* (*of-nat n*) *0*
⟨*proof*⟩

lemma *degree-of-nat* [*simp*]: *degree* (*of-nat n*) = *0*
⟨*proof*⟩

lemma *lead-coeff-of-nat* [*simp*]: *lead-coeff* (*of-nat n*) = *of-nat n*
⟨*proof*⟩

lemma *of-int-poly*: *of-int k* = [:*of-int k*:]
⟨*proof*⟩

lemma *of-int-monom*: *of-int k* = *monom* (*of-int k*) *0*
⟨*proof*⟩

lemma *degree-of-int* [*simp*]: *degree* (*of-int k*) = *0*
⟨*proof*⟩

lemma *lead-coeff-of-int* [*simp*]: *lead-coeff* (*of-int k*) = *of-int k*
⟨*proof*⟩

lemma *poly-of-nat* [*simp*]: *poly* (*of-nat n*) *x* = *of-nat n*
⟨*proof*⟩

lemma *poly-of-int* [*simp*]: *poly* (*of-int n*) *x* = *of-int n*
⟨*proof*⟩

lemma *poly-numeral* [*simp*]: *poly* (*numeral n*) *x* = *numeral n*
⟨*proof*⟩

lemma *numeral-poly*: *numeral n* = [:*numeral n*:]
⟨*proof*⟩

lemma *numeral-monom*:
numeral n = *monom* (*numeral n*) *0*
⟨*proof*⟩

lemma *degree-numeral* [*simp*]:
degree (*numeral n*) = *0*
⟨*proof*⟩

lemma *lead-coeff-numeral* [*simp*]:
lead-coeff (*numeral n*) = *numeral n*
⟨*proof*⟩

lemma *coeff-linear-poly-power*:
fixes *c* :: *'a* :: *semiring-1*
assumes *i* ≤ *n*
shows *coeff* ([:*a*, *b*:] ^ *n*) *i* = *of-nat* (*n choose i*) * *b* ^ *i* * *a* ^ (*n* − *i*)
⟨*proof*⟩

4.17 Lemmas about divisibility

lemma *dvd-smult*:
assumes *p dvd q*
shows *p dvd smult a q*
⟨*proof*⟩

lemma *dvd-smult-cancel*: *p dvd smult a q* ⟹ *a* ≠ *0* ⟹ *p dvd q*
for *a* :: *'a*::*field*
⟨*proof*⟩

lemma *dvd-smult-iff*: *a* ≠ *0* ⟹ *p dvd smult a q* ⟷ *p dvd q*
for *a* :: *'a*::*field*
⟨*proof*⟩

lemma *smult-dvd-cancel*:
assumes *smult a p dvd q*
shows *p dvd q*
⟨*proof*⟩

lemma *smult-dvd*: *p dvd q* ⟹ *a* ≠ *0* ⟹ *smult a p dvd q*
for *a* :: *'a*::*field*
⟨*proof*⟩

lemma *smult-dvd-iff*: *smult a p dvd q* ⟷ (*if a* = *0 then q* = *0 else p dvd q*)
for *a* :: *'a*::*field*
⟨*proof*⟩

lemma *is-unit-smult-iff*: *smult c p dvd 1* ⟷ *c dvd 1* ∧ *p dvd 1*
⟨*proof*⟩

4.18 Polynomials form an integral domain

instance *poly* :: (*idom*) *idom* ⟨*proof*⟩

instance *poly* :: ({*ring-char-0*, *comm-ring-1*}) *ring-char-0*

⟨*proof*⟩

lemma *semiring-char-poly* [*simp*]: $CHAR('a :: comm\text{-}semiring\text{-}1\ poly) = CHAR('a)$
⟨*proof*⟩

instance *poly* :: ({*semiring-prime-char*,*comm-semiring-1*}) *semiring-prime-char*
⟨*proof*⟩
instance *poly* :: ({*comm-semiring-prime-char*,*comm-semiring-1*}) *comm-semiring-prime-char*
⟨*proof*⟩
instance *poly* :: ({*comm-ring-prime-char*,*comm-semiring-1*}) *comm-ring-prime-char*
⟨*proof*⟩
instance *poly* :: ({*idom-prime-char*,*comm-semiring-1*}) *idom-prime-char*
⟨*proof*⟩

lemma *degree-mult-eq*: $p \neq 0 \Longrightarrow q \neq 0 \Longrightarrow degree\ (p * q) = degree\ p + degree\ q$
for $p\ q :: 'a$::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
⟨*proof*⟩

lemma *degree-prod-sum-eq*:
$(\bigwedge x.\ x \in A \Longrightarrow f\ x \neq 0) \Longrightarrow$
degree (*prod* $f\ A :: 'a ::$ *idom poly*) = $(\sum x{\in}A.\ degree\ (f\ x))$
⟨*proof*⟩

lemma *dvd-imp-degree*:
‹*degree* $x \leq$ *degree* y› **if** ‹x *dvd* y› ‹$x \neq 0$› ‹$y \neq 0$›
for $x\ y :: ‹'a$::{*comm-semiring-1*,*semiring-no-zero-divisors*} *poly*›
⟨*proof*⟩

lemma *degree-prod-eq-sum-degree*:
fixes $A :: 'a\ set$
and $f :: 'a \Rightarrow 'b$::*idom poly*
assumes *f0*: $\forall i{\in}A.\ f\ i \neq 0$
shows *degree* $(\prod i{\in}A.\ (f\ i)) = (\sum i{\in}A.\ degree\ (f\ i))$
⟨*proof*⟩

lemma *degree-mult-eq-0*:
degree $(p * q) = 0 \longleftrightarrow p = 0 \vee q = 0 \vee (p \neq 0 \wedge q \neq 0 \wedge degree\ p = 0 \wedge$
$degree\ q = 0)$
for $p\ q :: 'a$::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
⟨*proof*⟩

lemma *degree-power-eq*: $p \neq 0 \Longrightarrow degree\ ((p :: 'a :: idom\ poly)\ \hat{}\ n) = n * degree$
p
⟨*proof*⟩

lemma *degree-mult-right-le*:
fixes $p\ q :: 'a$::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
assumes $q \neq 0$
shows *degree* $p \leq$ *degree* $(p * q)$

⟨*proof*⟩

lemma *coeff-degree-mult*: *coeff* (*p* ∗ *q*) (*degree* (*p* ∗ *q*)) = *coeff q* (*degree q*) ∗ *coeff p* (*degree p*)
for *p q* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
⟨*proof*⟩

lemma *dvd-imp-degree-le*: *p dvd q* ⟹ *q* ≠ *0* ⟹ *degree p* ≤ *degree q*
for *p q* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*} *poly*
⟨*proof*⟩

lemma *divides-degree*:
fixes *p q* :: *'a* ::{*comm-semiring-1*,*semiring-no-zero-divisors*} *poly*
assumes *p dvd q*
shows *degree p* ≤ *degree q* ∨ *q* = *0*
⟨*proof*⟩

lemma *const-poly-dvd-iff*:
fixes *c* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*}
shows [:*c*:] *dvd p* ⟷ (∀ *n*. *c dvd coeff p n*)
⟨*proof*⟩

lemma *const-poly-dvd-const-poly-iff* [*simp*]: [:*a*:] *dvd* [:*b*:] ⟷ *a dvd b*
for *a b* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*}
⟨*proof*⟩

lemma *lead-coeff-mult*: *lead-coeff* (*p* ∗ *q*) = *lead-coeff p* ∗ *lead-coeff q*
for *p q* :: *'a*::{*comm-semiring-0*, *semiring-no-zero-divisors*} *poly*
⟨*proof*⟩

lemma *lead-coeff-prod*: *lead-coeff* (*prod f A*) = (∏ *x*∈*A*. *lead-coeff* (*f x*))
for *f* :: *'a* ⇒ *'b*::{*comm-semiring-1*, *semiring-no-zero-divisors*} *poly*
⟨*proof*⟩

lemma *lead-coeff-smult*: *lead-coeff* (*smult c p*) = *c* ∗ *lead-coeff p*
for *p* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
⟨*proof*⟩

lemma *lead-coeff-1* [*simp*]: *lead-coeff 1* = *1*
⟨*proof*⟩

lemma *lead-coeff-power*: *lead-coeff* (*p* ^ *n*) = *lead-coeff p* ^ *n*
for *p* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*} *poly*
⟨*proof*⟩

4.19 Polynomials form an ordered integral domain

definition *pos-poly* :: *'a*::*linordered-semidom poly* ⇒ *bool*
where *pos-poly p* ⟷ *0* < *coeff p* (*degree p*)

lemma *pos-poly-pCons*: *pos-poly* (*pCons a p*) ⟷ *pos-poly p* ∨ (*p* = *0* ∧ *0* < *a*)
⟨*proof*⟩

lemma *not-pos-poly-0* [*simp*]: ¬ *pos-poly 0*
⟨*proof*⟩

lemma *pos-poly-add*: *pos-poly p* ⟹ *pos-poly q* ⟹ *pos-poly* (*p* + *q*)
⟨*proof*⟩

lemma *pos-poly-mult*: *pos-poly p* ⟹ *pos-poly q* ⟹ *pos-poly* (*p* ∗ *q*)
⟨*proof*⟩

lemma *pos-poly-total*: *p* = *0* ∨ *pos-poly p* ∨ *pos-poly* (− *p*)
for *p* :: *'a*::*linordered-idom poly*
⟨*proof*⟩

lemma *pos-poly-coeffs* [*code*]: *pos-poly p* ⟷ (*let as* = *coeffs p in as* ≠ [] ∧ *last as* > *0*)
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

instantiation *poly* :: (*linordered-idom*) *linordered-idom*
begin

definition *x* < *y* ⟷ *pos-poly* (*y* − *x*)

definition *x* ≤ *y* ⟷ *x* = *y* ∨ *pos-poly* (*y* − *x*)

definition |*x*::*'a poly*| = (*if x* < *0 then* − *x else x*)

definition *sgn* (*x*::*'a poly*) = (*if x* = *0 then 0 else if 0* < *x then 1 else* − *1*)

instance
⟨*proof*⟩

end

TODO: Simplification rules for comparisons

4.20 Synthetic division and polynomial roots

4.20.1 Synthetic division

Synthetic division is simply division by the linear polynomial *x* − *c*.

definition *synthetic-divmod* :: *'a*::*comm-semiring-0 poly* ⇒ *'a* ⇒ *'a poly* × *'a*
where *synthetic-divmod p c* = *fold-coeffs* (λ*a* (*q*, *r*). (*pCons r q*, *a* + *c* ∗ *r*)) *p* (*0*, *0*)

definition *synthetic-div* :: *'a::comm-semiring-0 poly* ⇒ *'a* ⇒ *'a poly*
where *synthetic-div p c* = *fst* (*synthetic-divmod p c*)

lemma *synthetic-divmod-0* [*simp*]: *synthetic-divmod 0 c* = (*0*, *0*)
⟨*proof*⟩

lemma *synthetic-divmod-pCons* [*simp*]:
synthetic-divmod (*pCons a p*) *c* = (λ(*q*, *r*). (*pCons r q*, *a* + *c* ∗ *r*)) (*synthetic-divmod p c*)
⟨*proof*⟩

lemma *synthetic-div-0* [*simp*]: *synthetic-div 0 c* = *0*
⟨*proof*⟩

lemma *synthetic-div-unique-lemma*: *smult c p* = *pCons a p* ⟹ *p* = *0*
⟨*proof*⟩

lemma *snd-synthetic-divmod*: *snd* (*synthetic-divmod p c*) = *poly p c*
⟨*proof*⟩

lemma *synthetic-div-pCons* [*simp*]:
synthetic-div (*pCons a p*) *c* = *pCons* (*poly p c*) (*synthetic-div p c*)
⟨*proof*⟩

lemma *synthetic-div-eq-0-iff*: *synthetic-div p c* = *0* ⟷ *degree p* = *0*
⟨*proof*⟩

lemma *degree-synthetic-div*: *degree* (*synthetic-div p c*) = *degree p* − *1*
⟨*proof*⟩

lemma *synthetic-div-correct*:
p + *smult c* (*synthetic-div p c*) = *pCons* (*poly p c*) (*synthetic-div p c*)
⟨*proof*⟩

lemma *synthetic-div-unique*: *p* + *smult c q* = *pCons r q* ⟹ *r* = *poly p c* ∧ *q* = *synthetic-div p c*
⟨*proof*⟩

lemma *synthetic-div-correct'*: [:−*c*, *1*:] ∗ *synthetic-div p c* + [:*poly p c*:] = *p*
for *c* :: *'a::comm-ring-1*
⟨*proof*⟩

4.20.2 Polynomial roots

lemma *poly-eq-0-iff-dvd*: *poly p c* = *0* ⟷ [:− *c*, *1*:] *dvd p*
(**is** *?lhs* ⟷ *?rhs*)
for *c* :: *'a::comm-ring-1*
⟨*proof*⟩

lemma *dvd-iff-poly-eq-0*: [:*c*, *1*:] *dvd p* ⟷ *poly p* (− *c*) = *0*
for *c* :: *'a*::*comm-ring-1*
⟨*proof*⟩

lemma *poly-roots-finite*: *p* ≠ *0* ⟹ *finite* {*x*. *poly p x* = *0*}
for *p* :: *'a*::{*comm-ring-1*,*ring-no-zero-divisors*} *poly*
⟨*proof*⟩

lemma *poly-eq-poly-eq-iff*: *poly p* = *poly q* ⟷ *p* = *q*
(**is** *?lhs* ⟷ *?rhs*)
for *p q* :: *'a*::{*comm-ring-1*,*ring-no-zero-divisors*,*ring-char-0*} *poly*
⟨*proof*⟩

A nice extension rule for polynomials.

lemma *poly-ext*:
fixes *p q* :: *'a* :: {*ring-char-0*, *idom*} *poly*
assumes ⋀*x*. *poly p x* = *poly q x* **shows** *p* = *q*
⟨*proof*⟩

Copied from non-negative variants.

lemma *coeff-linear-power-neg*[*simp*]:
fixes *a* :: *'a*::*comm-ring-1*
shows *coeff* ([:*a*, −*1*:] ^ *n*) *n* = (−*1*)^*n*
⟨*proof*⟩

lemma *degree-linear-power-neg*[*simp*]:
fixes *a* :: *'a*::{*idom*,*comm-ring-1*}
shows *degree* ([:*a*, −*1*:] ^ *n*) = *n*
⟨*proof*⟩

lemma *poly-all-0-iff-0*: (∀ *x*. *poly p x* = *0*) ⟷ *p* = *0*
for *p* :: *'a*::{*ring-char-0*,*comm-ring-1*,*ring-no-zero-divisors*} *poly*
⟨*proof*⟩

lemma *card-poly-roots-bound*:
fixes *p* :: *'a*::{*comm-ring-1*,*ring-no-zero-divisors*} *poly*
assumes *p* ≠ *0*
shows *card* {*x*. *poly p x* = *0*} ≤ *degree p*
⟨*proof*⟩

lemma *poly-eqI-degree*:
fixes *p q* :: *'a* :: {*comm-ring-1*, *ring-no-zero-divisors*} *poly*
assumes ⋀*x*. *x* ∈ *A* ⟹ *poly p x* = *poly q x*
assumes *card A* > *degree p card A* > *degree q*
shows *p* = *q*
⟨*proof*⟩

lemma *poly-eqI-degree-lead-coeff*:
fixes *p q* :: *'a* :: {*comm-ring-1*, *ring-no-zero-divisors*} *poly*

assumes *poly.coeff p n = poly.coeff q n card A ≥ n degree p ≤ n degree q ≤ n*
assumes $\bigwedge z.\ z \in A \Longrightarrow poly\ p\ z = poly\ q\ z$
shows *p = q*
⟨*proof*⟩

4.20.3 Order of polynomial roots

definition *order* :: *'a::idom ⇒ 'a poly ⇒ nat*
where *order a p = (LEAST n. ¬ [:−a, 1:] ^ Suc n dvd p)*

lemma *coeff-linear-power*: *coeff ([:a, 1:] ^ n) n = 1*
for *a* :: *'a::comm-semiring-1*
⟨*proof*⟩

lemma *degree-linear-power*: *degree ([:a, 1:] ^ n) = n*
for *a* :: *'a::comm-semiring-1*
⟨*proof*⟩

lemma *order-1*: *[:−a, 1:] ^ order a p dvd p*
⟨*proof*⟩

lemma *order-2*:
assumes *p ≠ 0*
shows *¬ [:−a, 1:] ^ Suc (order a p) dvd p*
⟨*proof*⟩

lemma *order*: *p ≠ 0 ⟹ [:−a, 1:] ^ order a p dvd p ∧ ¬ [:−a, 1:] ^ Suc (order a p) dvd p*
⟨*proof*⟩

lemma *order-degree*:
assumes *p*: *p ≠ 0*
shows *order a p ≤ degree p*
⟨*proof*⟩

lemma *order-root*: *poly p a = 0 ⟷ p = 0 ∨ order a p ≠ 0* (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *order-0I*: *poly p a ≠ 0 ⟹ order a p = 0*
⟨*proof*⟩

lemma *order-unique-lemma*:
fixes *p* :: *'a::idom poly*
assumes *[:−a, 1:] ^ n dvd p ¬ [:−a, 1:] ^ Suc n dvd p*
shows *order a p = n*
⟨*proof*⟩

lemma *order-mult*:
assumes *p ∗ q ≠ 0* **shows** *order a (p ∗ q) = order a p + order a q*

⟨*proof*⟩

lemma *order-smult*:
  assumes $c \neq 0$
  shows *order x* (*smult c p*) = *order x p*
⟨*proof*⟩

lemma *order-gt-0-iff*: $p \neq 0 \Longrightarrow$ *order x p* > 0 $\longleftrightarrow$ *poly p x* = 0
  ⟨*proof*⟩

lemma *order-eq-0-iff*: $p \neq 0 \Longrightarrow$ *order x p* = 0 $\longleftrightarrow$ *poly p x* $\neq$ 0
  ⟨*proof*⟩

Next three lemmas contributed by Wenda Li

lemma *order-1-eq-0* [*simp*]:*order x 1* = 0
  ⟨*proof*⟩

lemma *order-uminus*[*simp*]: *order x* (−*p*) = *order x p*
  ⟨*proof*⟩

lemma *order-power-n-n*: *order a* ([:−*a*,*1*:]^*n*)=*n*
⟨*proof*⟩

lemma *order-0-monom* [*simp*]: $c \neq 0 \Longrightarrow$ *order 0* (*monom c n*) = *n*
  ⟨*proof*⟩

lemma *dvd-imp-order-le*: $q \neq 0 \Longrightarrow$ *p dvd q* $\Longrightarrow$ *Polynomial.order a p* ≤ *Polynomial.order a q*
  ⟨*proof*⟩

Now justify the standard squarefree decomposition, i.e. *f* / *gcd f f'*.

lemma *order-divides*: [:−*a*, *1*:] ^ *n dvd p* $\longleftrightarrow$ *p* = 0 ∨ *n* ≤ *order a p*
  ⟨*proof*⟩

lemma *order-decomp*:
  assumes $p \neq 0$
  shows ∃ *q*. *p* = [:− *a*, *1*:] ^ *order a p* ∗ *q* ∧ ¬ [:− *a*, *1*:] *dvd q*
⟨*proof*⟩

lemma *monom-1-dvd-iff*: $p \neq 0 \Longrightarrow$ *monom 1 n dvd p* $\longleftrightarrow$ *n* ≤ *order 0 p*
  ⟨*proof*⟩

lemma *poly-root-order-induct* [*case-names 0 no-roots root*]:
  fixes *p* :: *'a* :: *idom poly*
  assumes *P 0* $\bigwedge p.\ (\bigwedge x.\ poly\ p\ x \neq 0) \Longrightarrow P\ p$
    $\bigwedge p\ x\ n.\ n > 0 \Longrightarrow poly\ p\ x \neq 0 \Longrightarrow P\ p \Longrightarrow P\ ([:-x,\ 1:] \hat{}\ n * p)$
  shows *P p*
⟨*proof*⟩

context
includes *multiset.lifting*
begin

lift-definition *proots* :: ('*a* :: *idom*) *poly* ⇒ '*a multiset* **is**
λ(*p* :: '*a poly*) (*x* :: '*a*). *if p* = *0 then 0 else order x p*
⟨*proof*⟩

lemma *proots-0* [*simp*]: *proots* (*0* :: '*a* :: *idom poly*) = {#}
⟨*proof*⟩

lemma *proots-1* [*simp*]: *proots* (*1* :: '*a* :: *idom poly*) = {#}
⟨*proof*⟩

lemma *proots-const* [*simp*]: *proots* [: *x* :] = *0*
⟨*proof*⟩

lemma *proots-numeral* [*simp*]: *proots* (*numeral n*) = *0*
⟨*proof*⟩

lemma *count-proots* [*simp*]:
p ≠ *0* ⟹ *count* (*proots p*) *a* = *order a p*
⟨*proof*⟩

lemma *set-count-proots* [*simp*]:
p ≠ *0* ⟹ *set-mset* (*proots p*) = {*x*. *poly p x* = *0*}
⟨*proof*⟩

lemma *proots-uminus* [*simp*]: *proots* (−*p*) = *proots p*
⟨*proof*⟩

lemma *proots-smult* [*simp*]: *c* ≠ *0* ⟹ *proots* (*smult c p*) = *proots p*
⟨*proof*⟩

lemma *proots-mult*:
assumes *p* ≠ *0 q* ≠ *0*
shows *proots* (*p* ∗ *q*) = *proots p* + *proots q*
⟨*proof*⟩

lemma *proots-prod*:
assumes ⋀*x*. *x* ∈ *A* ⟹ *f x* ≠ *0*
shows *proots* (∏ *x*∈*A*. *f x*) = (∑ *x*∈*A*. *proots* (*f x*))
⟨*proof*⟩

lemma *proots-prod-mset*:
assumes *0* ∉# *A*
shows *proots* (∏ *p*∈#*A*. *p*) = (∑ *p*∈#*A*. *proots p*)

⟨*proof*⟩

lemma *proots-prod-list*:
assumes *0* ∉ *set ps*
shows *proots* (∏ *p*←*ps*. *p*) = (∑ *p*←*ps*. *proots p*)
⟨*proof*⟩

lemma *proots-power*: *proots* (*p* ^ *n*) = *repeat-mset n* (*proots p*)
⟨*proof*⟩

lemma *proots-linear-factor* [*simp*]: *proots* [:*x*, *1*:] = {#−*x*#}
⟨*proof*⟩

lemma *size-proots-le*: *size* (*proots p*) ≤ *degree p*
⟨*proof*⟩

end

4.21 Additional induction rules on polynomials

An induction rule for induction over the roots of a polynomial with a certain property. (e.g. all positive roots)

lemma *poly-root-induct* [*case-names 0 no-roots root*]:
fixes *p* :: *'a* :: *idom poly*
assumes *Q 0*
and ⋀*p*. (⋀*a*. *P a* ⟹ *poly p a* ≠ *0*) ⟹ *Q p*
and ⋀*a p*. *P a* ⟹ *Q p* ⟹ *Q* ([:*a*, −*1*:] ∗ *p*)
shows *Q p*
⟨*proof*⟩

lemma *dropWhile-replicate-append*:
dropWhile ((=) *a*) (*replicate n a* @ *ys*) = *dropWhile* ((=) *a*) *ys*
⟨*proof*⟩

lemma *Poly-append-replicate-0*: *Poly* (*xs* @ *replicate n 0*) = *Poly xs*
⟨*proof*⟩

An induction rule for simultaneous induction over two polynomials, prepending one coefficient in each step.

lemma *poly-induct2* [*case-names 0 pCons*]:
assumes *P 0 0* ⋀*a p b q*. *P p q* ⟹ *P* (*pCons a p*) (*pCons b q*)
shows *P p q*
⟨*proof*⟩

4.22 Composition of polynomials

definition *pcompose* :: *'a*::*comm-semiring-0 poly* ⇒ *'a poly* ⇒ *'a poly*
where *pcompose p q* = *fold-coeffs* (λ*a c*. [:*a*:] + *q* ∗ *c*) *p 0*

notation *pcompose* (**infixl** ‹$\circ_p$› *71*)

lemma *pcompose-0* [*simp*]: *pcompose 0 q = 0*
⟨*proof*⟩

lemma *pcompose-pCons*: *pcompose* (*pCons a p*) *q* = [:*a*:] + *q* * *pcompose p q*
⟨*proof*⟩

lemma *pcompose-altdef*: *pcompose p q* = *poly* (*map-poly* (λ*x*. [:*x*:]) *p*) *q*
⟨*proof*⟩

lemma *coeff-pcompose-0* [*simp*]:
coeff (*pcompose p q*) *0* = *poly p* (*coeff q 0*)
⟨*proof*⟩

lemma *pcompose-1*: *pcompose 1 p = 1*
for *p* :: *'a*::*comm-semiring-1 poly*
⟨*proof*⟩

lemma *poly-pcompose*: *poly* (*pcompose p q*) *x* = *poly p* (*poly q x*)
⟨*proof*⟩

lemma *degree-pcompose-le*: *degree* (*pcompose p q*) ≤ *degree p* * *degree q*
⟨*proof*⟩

lemma *pcompose-add*: *pcompose* (*p* + *q*) *r* = *pcompose p r* + *pcompose q r*
for *p q r* :: *'a*::{*comm-semiring-0*, *ab-semigroup-add*} *poly*
⟨*proof*⟩

lemma *pcompose-uminus*: *pcompose* (−*p*) *r* = −*pcompose p r*
for *p r* :: *'a*::*comm-ring poly*
⟨*proof*⟩

lemma *pcompose-diff*: *pcompose* (*p* − *q*) *r* = *pcompose p r* − *pcompose q r*
for *p q r* :: *'a*::*comm-ring poly*
⟨*proof*⟩

lemma *pcompose-smult*: *pcompose* (*smult a p*) *r* = *smult a* (*pcompose p r*)
for *p r* :: *'a*::*comm-semiring-0 poly*
⟨*proof*⟩

lemma *pcompose-mult*: *pcompose* (*p* * *q*) *r* = *pcompose p r* * *pcompose q r*
for *p q r* :: *'a*::*comm-semiring-0 poly*
⟨*proof*⟩

lemma *pcompose-assoc*: *pcompose p* (*pcompose q r*) = *pcompose* (*pcompose p q*) *r*
for *p q r* :: *'a*::*comm-semiring-0 poly*
⟨*proof*⟩

lemma *pcompose-idR*[*simp*]: *pcompose p* [: *0*, *1* :] = *p*
for *p* :: *'a*::*comm-semiring-1 poly*
⟨*proof*⟩

lemma *pcompose-sum*: *pcompose* (*sum f A*) *p* = *sum* (λ*i*. *pcompose* (*f i*) *p*) *A*
⟨*proof*⟩

lemma *pcompose-prod*: *pcompose* (*prod f A*) *p* = *prod* (λ*i*. *pcompose* (*f i*) *p*) *A*
⟨*proof*⟩

lemma *pcompose-const* [*simp*]: *pcompose* [:*a*:] *q* = [:*a*:]
⟨*proof*⟩

lemma *pcompose-0'*: *pcompose p 0* = [:*coeff p 0*:]
⟨*proof*⟩

lemma *pcompose-coeff-0*:
coeff (*pcompose p q*) *0* = *poly p* (*coeff q 0*)
⟨*proof*⟩

lemma *pcompose-pCons-0*: *pcompose p* [:*a*:] = [:*poly p a*:]
⟨*proof*⟩

lemma *degree-pcompose*: *degree* (*pcompose p q*) = *degree p* ∗ *degree q*
for *p q* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
⟨*proof*⟩

lemma *pcompose-eq-0*:
fixes *p q* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
assumes *pcompose p q* = *0 degree q* > *0*
shows *p* = *0*
⟨*proof*⟩

lemma *pcompose-eq-0-iff*:
fixes *p q* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
assumes *degree q* > *0*
shows *pcompose p q* = *0* ⟷ *p* = *0*
⟨*proof*⟩

lemma *coeff-pcompose-linear*:
coeff (*pcompose p* [:*0*, *a* :: *'a* :: *comm-semiring-1*:]) *i* = *a* ^ *i* ∗ *coeff p i*
⟨*proof*⟩

lemma *lead-coeff-comp*:
fixes *p q* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*} *poly*
assumes *degree q* > *0*
shows *lead-coeff* (*pcompose p q*) = *lead-coeff p* ∗ *lead-coeff q* ^ (*degree p*)
⟨*proof*⟩

lemma *coeff-pcompose-monom-linear* [*simp*]:
fixes $p :: 'a :: comm\text{-}ring\text{-}1\ poly$
shows $coeff\ (pcompose\ p\ (monom\ c\ (Suc\ 0)))\ k = c\ \widehat{}\ k * coeff\ p\ k$
⟨*proof*⟩

lemma *of-nat-mult-conv-smult*: $of\text{-}nat\ n * P = smult\ (of\text{-}nat\ n)\ P$
⟨*proof*⟩

lemma *numeral-mult-conv-smult*: $numeral\ n * P = smult\ (numeral\ n)\ P$
⟨*proof*⟩

lemma *sum-order-le-degree*:
assumes $p \neq 0$
shows $(\sum x \mid poly\ p\ x = 0.\ order\ x\ p) \leq degree\ p$
⟨*proof*⟩

4.23 Closure properties of coefficients

context
fixes $R :: 'a :: comm\text{-}semiring\text{-}1\ set$
assumes *R-0*: $0 \in R$
assumes *R-plus*: $\bigwedge x\ y.\ x \in R \Longrightarrow y \in R \Longrightarrow x + y \in R$
assumes *R-mult*: $\bigwedge x\ y.\ x \in R \Longrightarrow y \in R \Longrightarrow x * y \in R$
begin

lemma *coeff-mult-semiring-closed*:
assumes $\bigwedge i.\ coeff\ p\ i \in R\ \bigwedge i.\ coeff\ q\ i \in R$
shows $coeff\ (p * q)\ i \in R$
⟨*proof*⟩

lemma *coeff-pcompose-semiring-closed*:
assumes $\bigwedge i.\ coeff\ p\ i \in R\ \bigwedge i.\ coeff\ q\ i \in R$
shows $coeff\ (pcompose\ p\ q)\ i \in R$
⟨*proof*⟩

end

4.24 Shifting polynomials

definition $poly\text{-}shift :: nat \Rightarrow 'a::zero\ poly \Rightarrow 'a\ poly$
where $poly\text{-}shift\ n\ p = Abs\text{-}poly\ (\lambda i.\ coeff\ p\ (i + n))$

lemma *nth-default-drop*: $nth\text{-}default\ x\ (drop\ n\ xs)\ m = nth\text{-}default\ x\ xs\ (m + n)$
⟨*proof*⟩

lemma *nth-default-take*: $nth\text{-}default\ x\ (take\ n\ xs)\ m = (if\ m < n\ then\ nth\text{-}default\ x\ xs\ m\ else\ x)$
⟨*proof*⟩

lemma *coeff-poly-shift*: *coeff* (*poly-shift n p*) *i* = *coeff p* (*i* + *n*)
⟨*proof*⟩

lemma *poly-shift-id* [*simp*]: *poly-shift 0* = (λ*x*. *x*)
⟨*proof*⟩

lemma *poly-shift-0* [*simp*]: *poly-shift n 0* = *0*
⟨*proof*⟩

lemma *poly-shift-1*: *poly-shift n 1* = (*if n* = *0 then 1 else 0*)
⟨*proof*⟩

lemma *poly-shift-monom*: *poly-shift n* (*monom c m*) = (*if m* ≥ *n then monom c* (*m* − *n*) *else 0*)
⟨*proof*⟩

lemma *coeffs-shift-poly* [*code abstract*]:
coeffs (*poly-shift n p*) = *drop n* (*coeffs p*)
⟨*proof*⟩

4.25 Truncating polynomials

definition *poly-cutoff*
where *poly-cutoff n p* = *Abs-poly* (λ*k*. *if k* < *n then coeff p k else 0*)

lemma *coeff-poly-cutoff*: *coeff* (*poly-cutoff n p*) *k* = (*if k* < *n then coeff p k else 0*)
⟨*proof*⟩

lemma *poly-cutoff-0* [*simp*]: *poly-cutoff n 0* = *0*
⟨*proof*⟩

lemma *poly-cutoff-1* [*simp*]: *poly-cutoff n 1* = (*if n* = *0 then 0 else 1*)
⟨*proof*⟩

lemma *coeffs-poly-cutoff* [*code abstract*]:
coeffs (*poly-cutoff n p*) = *strip-while* ((=) *0*) (*take n* (*coeffs p*))
⟨*proof*⟩

4.26 Reflecting polynomials

definition *reflect-poly* :: *'a*::*zero poly* ⇒ *'a poly*
where *reflect-poly p* = *Poly* (*rev* (*coeffs p*))

lemma *coeffs-reflect-poly* [*code abstract*]:
coeffs (*reflect-poly p*) = *rev* (*dropWhile* ((=) *0*) (*coeffs p*))
⟨*proof*⟩

lemma *reflect-poly-0* [*simp*]: *reflect-poly 0* = *0*
⟨*proof*⟩

lemma *reflect-poly-1* [*simp*]: *reflect-poly 1 = 1*
⟨*proof*⟩

lemma *coeff-reflect-poly*:
coeff (*reflect-poly p*) *n* = (*if n* > *degree p then 0 else coeff p* (*degree p* − *n*))
⟨*proof*⟩

lemma *coeff-0-reflect-poly-0-iff* [*simp*]: *coeff* (*reflect-poly p*) *0* = *0* ⟷ *p* = *0*
⟨*proof*⟩

lemma *reflect-poly-at-0-eq-0-iff* [*simp*]: *poly* (*reflect-poly p*) *0* = *0* ⟷ *p* = *0*
⟨*proof*⟩

lemma *reflect-poly-pCons'*:
p ≠ *0* ⟹ *reflect-poly* (*pCons c p*) = *reflect-poly p* + *monom c* (*Suc* (*degree p*))
⟨*proof*⟩

lemma *reflect-poly-const* [*simp*]: *reflect-poly* [:*a*:] = [:*a*:]
⟨*proof*⟩

lemma *poly-reflect-poly-nz*:
x ≠ *0* ⟹ *poly* (*reflect-poly p*) *x* = *x* ^ *degree p* ∗ *poly p* (*inverse x*)
for *x* :: *'a*::*field*
⟨*proof*⟩

lemma *coeff-0-reflect-poly* [*simp*]: *coeff* (*reflect-poly p*) *0* = *lead-coeff p*
⟨*proof*⟩

lemma *poly-reflect-poly-0* [*simp*]: *poly* (*reflect-poly p*) *0* = *lead-coeff p*
⟨*proof*⟩

lemma *reflect-poly-reflect-poly* [*simp*]: *coeff p 0* ≠ *0* ⟹ *reflect-poly* (*reflect-poly p*) = *p*
⟨*proof*⟩

lemma *degree-reflect-poly-le*: *degree* (*reflect-poly p*) ≤ *degree p*
⟨*proof*⟩

lemma *reflect-poly-pCons*: *a* ≠ *0* ⟹ *reflect-poly* (*pCons a p*) = *Poly* (*rev* (*a* # *coeffs p*))
⟨*proof*⟩

lemma *degree-reflect-poly-eq* [*simp*]: *coeff p 0* ≠ *0* ⟹ *degree* (*reflect-poly p*) = *degree p*
⟨*proof*⟩

lemma *reflect-poly-eq-0-iff* [*simp*]: *reflect-poly p* = *0* ⟷ *p* = *0*
⟨*proof*⟩

lemma *reflect-poly-mult*: *reflect-poly* (*p* ∗ *q*) = *reflect-poly p* ∗ *reflect-poly q*
for *p q* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
⟨*proof*⟩

lemma *reflect-poly-smult*: *reflect-poly* (*smult c p*) = *smult c* (*reflect-poly p*)
for *p* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
⟨*proof*⟩

lemma *reflect-poly-power*: *reflect-poly* (*p* ^ *n*) = *reflect-poly p* ^ *n*
for *p* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*} *poly*
⟨*proof*⟩

lemma *reflect-poly-prod*: *reflect-poly* (*prod f A*) = *prod* (λ*x*. *reflect-poly* (*f x*)) *A*
for *f* :: - ⇒ -::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
⟨*proof*⟩

lemma *reflect-poly-prod-list*: *reflect-poly* (*prod-list xs*) = *prod-list* (*map reflect-poly xs*)
for *xs* :: -::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly list*
⟨*proof*⟩

lemma *reflect-poly-Poly-nz*:
no-trailing (*HOL.eq 0*) *xs* ⟹ *reflect-poly* (*Poly xs*) = *Poly* (*rev xs*)
⟨*proof*⟩

lemmas *reflect-poly-simps* =
reflect-poly-0 reflect-poly-1 reflect-poly-const reflect-poly-smult reflect-poly-mult
reflect-poly-power reflect-poly-prod reflect-poly-prod-list

4.27 Derivatives

function *pderiv* :: (*'a* :: {*comm-semiring-1*,*semiring-no-zero-divisors*}) *poly* ⇒ *'a poly*
where *pderiv* (*pCons a p*) = (*if p* = *0 then 0 else p* + *pCons 0* (*pderiv p*))
⟨*proof*⟩

termination *pderiv*
⟨*proof*⟩

declare *pderiv.simps*[*simp del*]

lemma *pderiv-0* [*simp*]: *pderiv 0* = *0*
⟨*proof*⟩

lemma *pderiv-pCons*: *pderiv* (*pCons a p*) = *p* + *pCons 0* (*pderiv p*)
⟨*proof*⟩

lemma *pderiv-1* [*simp*]: *pderiv 1 = 0*
⟨*proof*⟩

lemma *pderiv-of-nat* [*simp*]: *pderiv* (*of-nat n*) = *0*
and *pderiv-numeral* [*simp*]: *pderiv* (*numeral m*) = *0*
⟨*proof*⟩

lemma *coeff-pderiv*: *coeff* (*pderiv p*) *n* = *of-nat* (*Suc n*) ∗ *coeff p* (*Suc n*)
⟨*proof*⟩

fun *pderiv-coeffs-code* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*} ⇒ *'a list* ⇒ *'a list*
where
pderiv-coeffs-code f (*x* # *xs*) = *cCons* (*f* ∗ *x*) (*pderiv-coeffs-code* (*f*+*1*) *xs*)
| *pderiv-coeffs-code f* [] = []

definition *pderiv-coeffs* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*} *list* ⇒ *'a list*
where *pderiv-coeffs xs* = *pderiv-coeffs-code 1* (*tl xs*)

lemma *pderiv-coeffs-code*:
nth-default 0 (*pderiv-coeffs-code f xs*) *n* = (*f* + *of-nat n*) ∗ *nth-default 0 xs n*
⟨*proof*⟩

lemma *coeffs-pderiv-code* [*code abstract*]: *coeffs* (*pderiv p*) = *pderiv-coeffs* (*coeffs p*)
⟨*proof*⟩

lemma *pderiv-eq-0-iff*: *pderiv p* = *0* ⟷ *degree p* = *0*
for *p* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*,*semiring-char-0*} *poly*
⟨*proof*⟩

lemma *degree-pderiv*: *degree* (*pderiv p*) = *degree p* − *1*
for *p* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*,*semiring-char-0*} *poly*
⟨*proof*⟩

lemma *not-dvd-pderiv*:
fixes *p* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*,*semiring-char-0*} *poly*
assumes *degree p* ≠ *0*
shows ¬ *p dvd pderiv p*
⟨*proof*⟩

lemma *dvd-pderiv-iff* [*simp*]: *p dvd pderiv p* ⟷ *degree p* = *0*
for *p* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*,*semiring-char-0*} *poly*
⟨*proof*⟩

lemma *pderiv-singleton* [*simp*]: *pderiv* [:*a*:] = *0*
⟨*proof*⟩

lemma *pderiv-add*: *pderiv* (*p* + *q*) = *pderiv p* + *pderiv q*
⟨*proof*⟩

lemma *pderiv-minus*: *pderiv* (− *p* :: *'a* :: *idom poly*) = − *pderiv p*
⟨*proof*⟩

lemma *pderiv-diff*: *pderiv* ((*p* :: - :: *idom poly*) − *q*) = *pderiv p* − *pderiv q*
⟨*proof*⟩

lemma *pderiv-smult*: *pderiv* (*smult a p*) = *smult a* (*pderiv p*)
⟨*proof*⟩

lemma *pderiv-mult*: *pderiv* (*p* ∗ *q*) = *p* ∗ *pderiv q* + *q* ∗ *pderiv p*
⟨*proof*⟩

lemma *pderiv-power-Suc*: *pderiv* (*p* ^ *Suc n*) = *smult* (*of-nat* (*Suc n*)) (*p* ^ *n*) ∗ *pderiv p*
⟨*proof*⟩

lemma *pderiv-power*:
pderiv (*p* ^ *n*) = *smult* (*of-nat n*) (*p* ^ (*n* − *1*) ∗ *pderiv p*)
⟨*proof*⟩

lemma *pderiv-monom*:
pderiv (*monom c n*) = *monom* (*of-nat n* ∗ *c*) (*n* − *1*)
⟨*proof*⟩

lemma *pderiv-pcompose*: *pderiv* (*pcompose p q*) = *pcompose* (*pderiv p*) *q* ∗ *pderiv q*
⟨*proof*⟩

lemma *pderiv-prod*: *pderiv* (*prod f* (*as*)) = (∑ *a*∈*as*. *prod f* (*as* − {*a*}) ∗ *pderiv* (*f a*))
⟨*proof*⟩

lemma *coeff-higher-pderiv*:
coeff ((*pderiv* ^^ *m*) *f*) *n* = *pochhammer* (*of-nat* (*Suc n*)) *m* ∗ *coeff f* (*n* + *m*)
⟨*proof*⟩

lemma *higher-pderiv-0* [*simp*]: (*pderiv* ^^ *n*) *0* = *0*
⟨*proof*⟩

lemma *higher-pderiv-add*: (*pderiv* ^^ *n*) (*p* + *q*) = (*pderiv* ^^ *n*) *p* + (*pderiv* ^^ *n*) *q*
⟨*proof*⟩

lemma *higher-pderiv-smult*: (*pderiv* ^^ *n*) (*smult c p*) = *smult c* ((*pderiv* ^^ *n*) *p*)
⟨*proof*⟩

lemma *higher-pderiv-monom*:
$m \leq n + 1 \Longrightarrow (pderiv \mathbin{\frown\frown} m)\ (monom\ c\ n) = monom\ (pochhammer\ (int\ n - int\ m + 1)\ m * c)\ (n - m)$
⟨*proof*⟩

lemma *higher-pderiv-monom-eq-zero*:
$m > n + 1 \Longrightarrow (pderiv \mathbin{\frown\frown} m)\ (monom\ c\ n) = 0$
⟨*proof*⟩

lemma *higher-pderiv-sum*: $(pderiv \mathbin{\frown\frown} n)\ (sum\ f\ A) = (\sum x{\in}A.\ (pderiv \mathbin{\frown\frown} n)\ (f\ x))$
⟨*proof*⟩

lemma *higher-pderiv-sum-mset*: $(pderiv \mathbin{\frown\frown} n)\ (sum\text{-}mset\ A) = (\sum p{\in}\#A.\ (pderiv \mathbin{\frown\frown} n)\ p)$
⟨*proof*⟩

lemma *higher-pderiv-sum-list*: $(pderiv \mathbin{\frown\frown} n)\ (sum\text{-}list\ ps) = (\sum p{\leftarrow}ps.\ (pderiv \mathbin{\frown\frown} n)\ p)$
⟨*proof*⟩

lemma *degree-higher-pderiv*: $Polynomial.degree\ ((pderiv \mathbin{\frown\frown} n)\ p) = Polynomial.degree\ p - n$
for $p :: 'a{::}\{comm\text{-}semiring\text{-}1, semiring\text{-}no\text{-}zero\text{-}divisors, semiring\text{-}char\text{-}0\}\ poly$
⟨*proof*⟩

lemma *DERIV-pow2*: $DERIV\ (\lambda x.\ x \mathbin{\hat{}} Suc\ n)\ x :> real\ (Suc\ n) * (x \mathbin{\hat{}} n)$
⟨*proof*⟩
declare *DERIV-pow2* [*simp*] *DERIV-pow* [*simp*]

lemma *DERIV-add-const*: $DERIV\ f\ x :> D \Longrightarrow DERIV\ (\lambda x.\ a + f\ x :: 'a{::}real\text{-}normed\text{-}field)\ x :> D$
⟨*proof*⟩

lemma *poly-DERIV* [*simp*]: $DERIV\ (\lambda x.\ poly\ p\ x)\ x :> poly\ (pderiv\ p)\ x$
⟨*proof*⟩

lemma *poly-isCont*[*simp*]:
fixes $x{::}'a{::}real\text{-}normed\text{-}field$
shows $isCont\ (\lambda x.\ poly\ p\ x)\ x$
⟨*proof*⟩

lemma *tendsto-poly* [*tendsto-intros*]: $(f \longrightarrow a)\ F \Longrightarrow ((\lambda x.\ poly\ p\ (f\ x)) \longrightarrow poly\ p\ a)\ F$
for $f :: - \Rightarrow 'a{::}real\text{-}normed\text{-}field$
⟨*proof*⟩

lemma *continuous-within-poly*: *continuous* (*at z within s*) (*poly p*)
for *z* :: *'a*::{*real-normed-field*}
⟨*proof*⟩

lemma *continuous-poly* [*continuous-intros*]: *continuous F f* ⟹ *continuous F* (λ*x*. *poly p* (*f x*))
for *f* :: - ⇒ *'a*::*real-normed-field*
⟨*proof*⟩

lemma *continuous-on-poly* [*continuous-intros*]:
fixes *p* :: *'a* :: {*real-normed-field*} *poly*
assumes *continuous-on A f*
shows *continuous-on A* (λ*x*. *poly p* (*f x*))
⟨*proof*⟩

Consequences of the derivative theorem above.

lemma *poly-differentiable*[*simp*]: (λ*x*. *poly p x*) *differentiable* (*at x*)
for *x* :: *real*
⟨*proof*⟩

lemma *poly-IVT-pos*: *a* < *b* ⟹ *poly p a* < *0* ⟹ *0* < *poly p b* ⟹ ∃ *x*. *a* < *x* ∧ *x* < *b* ∧ *poly p x* = *0*
for *a b* :: *real*
⟨*proof*⟩

lemma *poly-IVT-neg*: *a* < *b* ⟹ *0* < *poly p a* ⟹ *poly p b* < *0* ⟹ ∃ *x*. *a* < *x* ∧ *x* < *b* ∧ *poly p x* = *0*
for *a b* :: *real*
⟨*proof*⟩

lemma *poly-IVT*: *a* < *b* ⟹ *poly p a* ∗ *poly p b* < *0* ⟹ ∃ *x*>*a*. *x* < *b* ∧ *poly p x* = *0*
for *p* :: *real poly*
⟨*proof*⟩

lemma *poly-MVT*: *a* < *b* ⟹ ∃ *x*. *a* < *x* ∧ *x* < *b* ∧ *poly p b* − *poly p a* = (*b* − *a*) ∗ *poly* (*pderiv p*) *x*
for *a b* :: *real*
⟨*proof*⟩

lemma *poly-MVT'*:
fixes *a b* :: *real*
assumes {*min a b*..*max a b*} ⊆ *A*
shows ∃ *x*∈*A*. *poly p b* − *poly p a* = (*b* − *a*) ∗ *poly* (*pderiv p*) *x*
⟨*proof*⟩

lemma *poly-pinfty-gt-lc*:
fixes *p* :: *real poly*
assumes *lead-coeff p* > *0*

shows $\exists n.\ \forall\ x \geq n.\ poly\ p\ x \geq lead\text{-}coeff\ p$
⟨*proof*⟩

lemma *dvd-monic*:
fixes $p\ q$:: $'a$ :: *idom poly*
assumes *monic*:*lead-coeff p*=*1* **and** *p dvd* (*smult c q*) **and** *c*≠*0*
shows *p dvd q* ⟨*proof*⟩

lemma *lemma-order-pderiv1*:
pderiv ([:− *a*, *1*:] ^ *Suc n* ∗ *q*)
= [:− *a*, *1*:] ^ *Suc n* ∗ *pderiv q* + *smult* (*of-nat* (*Suc n*)) (*q* ∗ [:− *a*, *1*:] ^ *n*)
⟨*proof*⟩

lemma *order-pderiv*:
fixes *p*::$'a$::{*idom*,*semiring-char-0*} *poly*
assumes *p*≠*0 poly p x* = *0*
shows *order x p* = *Suc* (*order x* (*pderiv p*)) ⟨*proof*⟩

lemma *lemma-order-pderiv*:
fixes *p* :: $'a$ :: *field-char-0 poly*
assumes *n*: *0* < *n*
and *pd*: *pderiv p* ≠ *0*
and *pe*: *p* = [:− *a*, *1*:] ^ *n* ∗ *q*
and *nd*: ¬ [:− *a*, *1*:] *dvd q*
shows *n* = *Suc* (*order a* (*pderiv p*))
⟨*proof*⟩

lemma *poly-squarefree-decomp-order*:
fixes *p* :: $'a$::*field-char-0 poly*
assumes *pderiv p* ≠ *0*
and *p*: *p* = *q* ∗ *d*
and *p'*: *pderiv p* = *e* ∗ *d*
and *d*: *d* = *r* ∗ *p* + *s* ∗ *pderiv p*
shows *order a q* = (*if order a p* = *0 then 0 else 1*)
⟨*proof*⟩

lemma *poly-squarefree-decomp-order2*:
pderiv p ≠ *0* ⟹ *p* = *q* ∗ *d* ⟹ *pderiv p* = *e* ∗ *d* ⟹
d = *r* ∗ *p* + *s* ∗ *pderiv p* ⟹ ∀ *a*. *order a q* = (*if order a p* = *0 then 0 else 1*)
for *p* :: $'a$::*field-char-0 poly*
⟨*proof*⟩

lemma *order-pderiv2*:
pderiv p ≠ *0* ⟹ *order a p* ≠ *0* ⟹ *order a* (*pderiv p*) = *n* ⟷ *order a p* = *Suc n*
for *p* :: $'a$::*field-char-0 poly*
⟨*proof*⟩

definition *rsquarefree* :: $'a$::*idom poly* ⇒ *bool*

where *rsquarefree p* $\longleftrightarrow$ *p* $\neq$ *0* $\wedge$ ($\forall$ *a. order a p = 0* $\vee$ *order a p = 1*)

lemma *pderiv-iszero*: *pderiv p = 0* $\Longrightarrow$ $\exists$ *h. p =* [:*h*:]
for *p* :: *'a*::{*semidom*,*semiring-char-0*} *poly*
⟨*proof*⟩

lemma *rsquarefree-roots*: *rsquarefree p* $\longleftrightarrow$ ($\forall$ *a.* $\neg$ (*poly p a = 0* $\wedge$ *poly* (*pderiv p*) *a = 0*))
for *p* :: *'a*::*field-char-0 poly*
⟨*proof*⟩

lemma *rsquarefree-root-order*:
assumes *rsquarefree p poly p z = 0 p* $\neq$ *0*
shows *order z p = 1*
⟨*proof*⟩

lemma *poly-squarefree-decomp*:
fixes *p* :: *'a*::*field-char-0 poly*
assumes *pderiv p* $\neq$ *0*
and *p = q* $*$ *d*
and *pderiv p = e* $*$ *d*
and *d = r* $*$ *p* + *s* $*$ *pderiv p*
shows *rsquarefree q* $\wedge$ ($\forall$ *a. poly q a = 0* $\longleftrightarrow$ *poly p a = 0*)
⟨*proof*⟩

lemma *has-field-derivative-poly* [*derivative-intros*]:
assumes (*f has-field-derivative f'*) (*at x within A*)
shows ((λ*x. poly p* (*f x*)) *has-field-derivative*
(*f'* $*$ *poly* (*pderiv p*) (*f x*))) (*at x within A*)
⟨*proof*⟩

4.28 Algebraic numbers

lemma *intpolyE*:
assumes $\bigwedge$*i. poly.coeff p i* $\in$ $\mathbb{Z}$
obtains *q* **where** *p = map-poly of-int q*
⟨*proof*⟩

lemma *ratpolyE*:
assumes $\bigwedge$*i. poly.coeff p i* $\in$ $\mathbb{Q}$
obtains *q* **where** *p = map-poly of-rat q*
⟨*proof*⟩

Algebraic numbers can be defined in two equivalent ways: all real numbers that are roots of rational polynomials or of integer polynomials. The Algebraic-Numbers AFP entry uses the rational definition, but we need the integer definition.

The equivalence is obvious since any rational polynomial can be multiplied with the LCM of its coefficients, yielding an integer polynomial with the

same roots.

definition *algebraic* :: *'a* :: *field-char-0* ⇒ *bool*
  where *algebraic x* ⟷ (∃ *p*. (∀ *i*. *coeff p i* ∈ ℤ) ∧ *p* ≠ *0* ∧ *poly p x* = *0*)

lemma *algebraicI*: (⋀*i*. *coeff p i* ∈ ℤ) ⟹ *p* ≠ *0* ⟹ *poly p x* = *0* ⟹ *algebraic x*
  ⟨*proof*⟩

lemma *algebraicE*:
  assumes *algebraic x*
  obtains *p* **where** ⋀*i*. *coeff p i* ∈ ℤ *p* ≠ *0 poly p x* = *0*
  ⟨*proof*⟩

lemma *algebraic-altdef*: *algebraic x* ⟷ (∃ *p*. (∀ *i*. *coeff p i* ∈ ℚ) ∧ *p* ≠ *0* ∧ *poly p x* = *0*)
  for *p* :: *'a*::*field-char-0 poly*
⟨*proof*⟩

lemma *algebraicI'*: (⋀*i*. *coeff p i* ∈ ℚ) ⟹ *p* ≠ *0* ⟹ *poly p x* = *0* ⟹ *algebraic x*
  ⟨*proof*⟩

lemma *algebraicE'*:
  assumes *algebraic* (*x* :: *'a* :: *field-char-0*)
  obtains *p* **where** *p* ≠ *0 poly* (*map-poly of-int p*) *x* = *0*
⟨*proof*⟩

lemma *algebraicE'-nonzero*:
  assumes *algebraic* (*x* :: *'a* :: *field-char-0*) *x* ≠ *0*
  obtains *p* **where** *p* ≠ *0 coeff p 0* ≠ *0 poly* (*map-poly of-int p*) *x* = *0*
⟨*proof*⟩

lemma *rat-imp-algebraic*: *x* ∈ ℚ ⟹ *algebraic x*
⟨*proof*⟩

lemma *algebraic-0* [*simp*, *intro*]: *algebraic 0*
  and *algebraic-1* [*simp*, *intro*]: *algebraic 1*
  and *algebraic-numeral* [*simp*, *intro*]: *algebraic* (*numeral n*)
  and *algebraic-of-nat* [*simp*, *intro*]: *algebraic* (*of-nat k*)
  and *algebraic-of-int* [*simp*, *intro*]: *algebraic* (*of-int m*)
  ⟨*proof*⟩

lemma *algebraic-ii* [*simp*, *intro*]: *algebraic* i
⟨*proof*⟩

lemma *algebraic-minus* [*intro*]:
  assumes *algebraic x*
  shows *algebraic* (−*x*)
⟨*proof*⟩

lemma *algebraic-minus-iff* [*simp*]:
  algebraic $(-x) \longleftrightarrow$ *algebraic* $(x :: 'a :: \textit{field-char-0})$
  ⟨*proof*⟩

lemma *algebraic-inverse* [*intro*]:
  assumes *algebraic x*
  shows *algebraic* (*inverse x*)
⟨*proof*⟩

lemma *algebraic-root*:
  assumes *algebraic y*
    and *poly p x* = *y* **and** $\forall i.$ *coeff p i* $\in \mathbb{Z}$ **and** *lead-coeff p* = *1* **and** *degree p* > *0*
  shows *algebraic x*
⟨*proof*⟩

lemma *algebraic-abs-real* [*simp*]:
  algebraic $|x :: \textit{real}| \longleftrightarrow$ *algebraic x*
  ⟨*proof*⟩

lemma *algebraic-nth-root-real* [*intro*]:
  assumes *algebraic x*
  shows *algebraic* (*root n x*)
⟨*proof*⟩

lemma *algebraic-sqrt* [*intro*]: *algebraic x* $\Longrightarrow$ *algebraic* (*sqrt x*)
  ⟨*proof*⟩

lemma *algebraic-csqrt* [*intro*]: *algebraic x* $\Longrightarrow$ *algebraic* (*csqrt x*)
  ⟨*proof*⟩

lemma *algebraic-cnj* [*intro*]:
  assumes *algebraic x*
  shows *algebraic* (*cnj x*)
⟨*proof*⟩

lemma *algebraic-cnj-iff* [*simp*]: *algebraic* (*cnj x*) $\longleftrightarrow$ *algebraic x*
  ⟨*proof*⟩

lemma *algebraic-of-real* [*intro*]:
  assumes *algebraic x*
  shows *algebraic* (*of-real x*)
⟨*proof*⟩

lemma *algebraic-of-real-iff* [*simp*]:
  algebraic $(\textit{of-real } x :: 'a :: \{\textit{real-algebra-1}, \textit{field-char-0}\}) \longleftrightarrow$ *algebraic x*
⟨*proof*⟩

4.29 Algebraic integers

inductive *algebraic-int* :: *'a* :: *field* ⇒ *bool* **where**
⟦*lead-coeff p = 1*; ∀ *i*. *coeff p i* ∈ ℤ; *poly p x = 0*⟧ ⟹ *algebraic-int x*

lemma *algebraic-int-altdef-ipoly*:
fixes *x* :: *'a* :: *field-char-0*
shows *algebraic-int x* ⟷ (∃ *p*. *poly* (*map-poly of-int p*) *x = 0* ∧ *lead-coeff p = 1*)
⟨*proof*⟩

theorem *rational-algebraic-int-is-int*:
assumes *algebraic-int x* **and** *x* ∈ ℚ
shows *x* ∈ ℤ
⟨*proof*⟩

lemma *algebraic-int-imp-algebraic* [*dest*]: *algebraic-int x* ⟹ *algebraic x*
⟨*proof*⟩

lemma *int-imp-algebraic-int*:
assumes *x* ∈ ℤ
shows *algebraic-int x*
⟨*proof*⟩

lemma *algebraic-int-0* [*simp*, *intro*]: *algebraic-int 0*
and *algebraic-int-1* [*simp*, *intro*]: *algebraic-int 1*
and *algebraic-int-numeral* [*simp*, *intro*]: *algebraic-int* (*numeral n*)
and *algebraic-int-of-nat* [*simp*, *intro*]: *algebraic-int* (*of-nat k*)
and *algebraic-int-of-int* [*simp*, *intro*]: *algebraic-int* (*of-int m*)
⟨*proof*⟩

lemma *algebraic-int-ii* [*simp*, *intro*]: *algebraic-int* i
⟨*proof*⟩

lemma *algebraic-int-minus* [*intro*]:
assumes *algebraic-int x*
shows *algebraic-int* (−*x*)
⟨*proof*⟩

lemma *algebraic-int-minus-iff* [*simp*]:
algebraic-int (−*x*) ⟷ *algebraic-int* (*x* :: *'a* :: *field-char-0*)
⟨*proof*⟩

lemma *algebraic-int-inverse* [*intro*]:
assumes *poly p x = 0* **and** ∀ *i*. *coeff p i* ∈ ℤ **and** *coeff p 0 = 1*
shows *algebraic-int* (*inverse x*)
⟨*proof*⟩

lemma *algebraic-int-root*:
assumes *algebraic-int y*

and *poly p x = y* **and** $\forall i$. *coeff p i* $\in \mathbb{Z}$ **and** *lead-coeff p = 1* **and** *degree p > 0*
shows *algebraic-int x*
⟨*proof*⟩

lemma *algebraic-int-abs-real* [*simp*]:
algebraic-int |*x* :: *real*| ⟷ *algebraic-int x*
⟨*proof*⟩

lemma *algebraic-int-nth-root-real* [*intro*]:
assumes *algebraic-int x*
shows *algebraic-int* (*root n x*)
⟨*proof*⟩

lemma *algebraic-int-sqrt* [*intro*]: *algebraic-int x* ⟹ *algebraic-int* (*sqrt x*)
⟨*proof*⟩

lemma *algebraic-int-csqrt* [*intro*]: *algebraic-int x* ⟹ *algebraic-int* (*csqrt x*)
⟨*proof*⟩

lemma *algebraic-int-cnj* [*intro*]:
assumes *algebraic-int x*
shows *algebraic-int* (*cnj x*)
⟨*proof*⟩

lemma *algebraic-int-cnj-iff* [*simp*]: *algebraic-int* (*cnj x*) ⟷ *algebraic-int x*
⟨*proof*⟩

lemma *algebraic-int-of-real* [*intro*]:
assumes *algebraic-int x*
shows *algebraic-int* (*of-real x*)
⟨*proof*⟩

lemma *algebraic-int-of-real-iff* [*simp*]:
algebraic-int (*of-real x* :: *'a* :: {*field-char-0*, *real-algebra-1*}) ⟷ *algebraic-int x*
⟨*proof*⟩

4.30 Division of polynomials

4.30.1 Division in general

instantiation *poly* :: (*idom-divide*) *idom-divide*
begin

fun *divide-poly-main* :: *'a* ⇒ *'a poly* ⇒ *'a poly* ⇒ *'a poly* ⇒ *nat* ⇒ *nat* ⇒ *'a poly*
where
divide-poly-main lc q r d dr (*Suc n*) =
(*let cr* = *coeff r dr*; *a* = *cr div lc*; *mon* = *monom a n in*
if False ∨ *a* ∗ *lc* = *cr then* — *False* ∨ is only because of problem in function-package

```
        divide-poly-main
          lc
          (q + mon)
          (r − mon ∗ d)
          d (dr − 1) n else 0)
  | divide-poly-main lc q r d dr 0 = q

definition divide-poly :: 'a poly ⇒ 'a poly ⇒ 'a poly
  where divide-poly f g =
    (if g = 0 then 0
     else
      divide-poly-main (coeff g (degree g)) 0 f g (degree f)
        (1 + length (coeffs f) − length (coeffs g)))

lemma divide-poly-main:
  assumes d: d ≠ 0 lc = coeff d (degree d)
    and degree (d ∗ r) ≤ dr divide-poly-main lc q (d ∗ r) d dr n = q'
    and n = 1 + dr − degree d ∨ dr = 0 ∧ n = 0 ∧ d ∗ r = 0
  shows q' = q + r
  ⟨proof⟩

lemma divide-poly-main-0: divide-poly-main 0 0 r d dr n = 0
⟨proof⟩

lemma divide-poly:
  assumes g: g ≠ 0
  shows (f ∗ g) div g = (f :: 'a poly)
⟨proof⟩

lemma divide-poly-0: f div 0 = 0
  for f :: 'a poly
  ⟨proof⟩

instance
  ⟨proof⟩

end

instance poly :: (idom-divide) algebraic-semidom ⟨proof⟩

lemma div-const-poly-conv-map-poly:
  assumes [:c:] dvd p
  shows p div [:c:] = map-poly (λx. x div c) p
⟨proof⟩

lemma is-unit-monom-0:
  fixes a :: 'a::field
  assumes a ≠ 0
  shows is-unit (monom a 0)
```

⟨*proof*⟩

lemma *is-unit-triv*: $a \neq 0 \Longrightarrow$ *is-unit* [:*a*:]
  for a :: *'a*::*field*
  ⟨*proof*⟩

lemma *is-unit-iff-degree*:
  fixes p :: *'a*::*field poly*
  assumes $p \neq 0$
  shows *is-unit* $p \longleftrightarrow$ *degree* $p = 0$
    (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *is-unit-pCons-iff*: *is-unit* (*pCons a p*) $\longleftrightarrow$ $p = 0 \wedge a \neq 0$
  for p :: *'a*::*field poly*
  ⟨*proof*⟩

lemma *is-unit-monom-trivial*: *is-unit* $p \Longrightarrow$ *monom* (*coeff p* (*degree p*)) *0* $= p$
  for p :: *'a*::*field poly*
  ⟨*proof*⟩

lemma *is-unit-const-poly-iff*: [:*c*:] *dvd 1* $\longleftrightarrow$ *c dvd 1*
  for c :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*}
  ⟨*proof*⟩

lemma *is-unit-polyE*:
  fixes p :: *'a* :: {*comm-semiring-1*,*semiring-no-zero-divisors*} *poly*
  assumes *p dvd 1*
  obtains *c* **where** $p =$ [:*c*:] *c dvd 1*
⟨*proof*⟩

lemma *is-unit-polyE'*:
  fixes p :: *'a*::*field poly*
  assumes *is-unit p*
  obtains *a* **where** $p =$ *monom a 0* **and** $a \neq 0$
⟨*proof*⟩

lemma *is-unit-poly-iff*: *p dvd 1* $\longleftrightarrow$ $(\exists\, c.\ p =$ [:*c*:] $\wedge$ *c dvd 1*)
  for p :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*} *poly*
  ⟨*proof*⟩

lemma *coprime-poly-0*:
  poly p x $\neq 0 \vee$ *poly q x* $\neq 0$ **if** *coprime p q*
  for x :: *'a* :: *field*
⟨*proof*⟩

lemma *root-imp-reducible-poly*:
  fixes x :: *'a* :: *field*
  assumes *poly p x* $= 0$ **and** *degree p* > 1

shows $\neg$*irreducible p*
⟨*proof*⟩

lemma *reducible-polyI*:
fixes *p* :: *'a* :: *field poly*
assumes *p* = *q* * *r degree q* > *0 degree r* > *0*
shows ¬*irreducible p*
⟨*proof*⟩

4.30.2 Pseudo-Division

This part is by René Thiemann and Akihisa Yamada.

fun *pseudo-divmod-main* ::
'a :: *comm-ring-1* ⇒ *'a poly* ⇒ *'a poly* ⇒ *'a poly* ⇒ *nat* ⇒ *nat* ⇒ *'a poly* × *'a poly*
where
pseudo-divmod-main lc q r d dr (*Suc n*) =
(*let*
rr = *smult lc r*;
qq = *coeff r dr*;
rrr = *rr* − *monom qq n* * *d*;
qqq = *smult lc q* + *monom qq n*
in pseudo-divmod-main lc qqq rrr d (*dr* − *1*) *n*)
| *pseudo-divmod-main lc q r d dr 0* = (*q*,*r*)

definition *pseudo-divmod* :: *'a* :: *comm-ring-1 poly* ⇒ *'a poly* ⇒ *'a poly* × *'a poly*
where *pseudo-divmod p q* ≡
if q = *0 then* (*0*, *p*)
else
pseudo-divmod-main (*coeff q* (*degree q*)) *0 p q* (*degree p*)
(*1* + *length* (*coeffs p*) − *length* (*coeffs q*))

lemma *pseudo-divmod-main*:
assumes *d*: *d* ≠ *0 lc* = *coeff d* (*degree d*)
and *degree r* ≤ *dr pseudo-divmod-main lc q r d dr n* = (*q'*,*r'*)
and *n* = *1* + *dr* − *degree d* ∨ *dr* = *0* ∧ *n* = *0* ∧ *r* = *0*
shows (*r'* = *0* ∨ *degree r'* < *degree d*) ∧ *smult* (*lc*^*n*) (*d* * *q* + *r*) = *d* * *q'* + *r'*
⟨*proof*⟩

lemma *pseudo-divmod*:
assumes *g*: *g* ≠ *0*
and *: *pseudo-divmod f g* = (*q*,*r*)
shows *smult* (*coeff g* (*degree g*) ^(*Suc* (*degree f*) − *degree g*)) *f* = *g* * *q* + *r* (**is** *?A*)
and *r* = *0* ∨ *degree r* < *degree g* (**is** *?B*)
⟨*proof*⟩

definition *pseudo-mod-main lc r d dr n* = *snd* (*pseudo-divmod-main lc 0 r d dr n*)

lemma *snd-pseudo-divmod-main*:
snd (*pseudo-divmod-main lc q r d dr n*) = *snd* (*pseudo-divmod-main lc q' r d dr n*)
⟨*proof*⟩

definition *pseudo-mod* :: *'a*::{*comm-ring-1*,*semiring-1-no-zero-divisors*} *poly* ⇒ *'a poly* ⇒ *'a poly*
where *pseudo-mod f g* = *snd* (*pseudo-divmod f g*)

lemma *pseudo-mod*:
fixes *f g* :: *'a*::{*comm-ring-1*,*semiring-1-no-zero-divisors*} *poly*
defines *r* ≡ *pseudo-mod f g*
assumes *g*: *g* ≠ *0*
shows ∃ *a q*. *a* ≠ *0* ∧ *smult a f* = *g* ∗ *q* + *r r* = *0* ∨ *degree r* < *degree g*
⟨*proof*⟩

lemma *fst-pseudo-divmod-main-as-divide-poly-main*:
assumes *d*: *d* ≠ *0*
defines *lc*: *lc* ≡ *coeff d* (*degree d*)
shows *fst* (*pseudo-divmod-main lc q r d dr n*) =
divide-poly-main lc (*smult* (*lc*^*n*) *q*) (*smult* (*lc*^*n*) *r*) *d dr n*
⟨*proof*⟩

4.30.3 Division in polynomials over fields

lemma *pseudo-divmod-field*:
fixes *g* :: *'a*::*field poly*
assumes *g*: *g* ≠ *0*
and ∗: *pseudo-divmod f g* = (*q*,*r*)
defines *c* ≡ *coeff g* (*degree g*) ^ (*Suc* (*degree f*) − *degree g*)
shows *f* = *g* ∗ *smult* (*1*/*c*) *q* + *smult* (*1*/*c*) *r*
⟨*proof*⟩

lemma *divide-poly-main-field*:
fixes *d* :: *'a*::*field poly*
assumes *d*: *d* ≠ *0*
defines *lc*: *lc* ≡ *coeff d* (*degree d*)
shows *divide-poly-main lc q r d dr n* =
fst (*pseudo-divmod-main lc* (*smult* ((*1* / *lc*)^*n*) *q*) (*smult* ((*1* / *lc*)^*n*) *r*) *d dr n*)
⟨*proof*⟩

lemma *divide-poly-field*:
fixes *f g* :: *'a*::*field poly*
defines *f'* ≡ *smult* ((*1* / *coeff g* (*degree g*)) ^ (*Suc* (*degree f*) − *degree g*)) *f*
shows *f div g* = *fst* (*pseudo-divmod f' g*)
⟨*proof*⟩

instantiation *poly* :: ({*semidom-divide-unit-factor*,*idom-divide*}) *normalization-semidom*
begin

definition *unit-factor-poly* :: *'a poly* ⇒ *'a poly*
  where *unit-factor-poly p* = [:*unit-factor* (*lead-coeff p*):]

definition *normalize-poly* :: *'a poly* ⇒ *'a poly*
  where *normalize p* = *p div* [:*unit-factor* (*lead-coeff p*):]

instance
⟨*proof*⟩

end

instance *poly* :: ({*semidom-divide-unit-factor*,*idom-divide*,*normalization-semidom-multiplicative*})
  normalization-semidom-multiplicative
  ⟨*proof*⟩

lemma *normalize-poly-eq-map-poly*: *normalize p* = *map-poly* (λ*x*. *x div unit-factor*
(*lead-coeff p*)) *p*
⟨*proof*⟩

lemma *coeff-normalize* [*simp*]:
  coeff (*normalize p*) *n* = *coeff p n div unit-factor* (*lead-coeff p*)
  ⟨*proof*⟩

class *field-unit-factor* = *field* + *unit-factor* +
  assumes *unit-factor-field* [*simp*]: *unit-factor* = *id*
begin

subclass *semidom-divide-unit-factor*
⟨*proof*⟩

end

lemma *unit-factor-pCons*:
  unit-factor (*pCons a p*) = (*if p* = *0 then* [:*unit-factor a*:] *else unit-factor p*)
  ⟨*proof*⟩

lemma *normalize-monom* [*simp*]: *normalize* (*monom a n*) = *monom* (*normalize*
a) *n*
  ⟨*proof*⟩

lemma *unit-factor-monom* [*simp*]: *unit-factor* (*monom a n*) = [:*unit-factor a*:]
  ⟨*proof*⟩

lemma *normalize-const-poly*: *normalize* [:*c*:] = [:*normalize c*:]
  ⟨*proof*⟩

lemma *normalize-smult*:
  fixes *c* :: *'a* :: {*normalization-semidom-multiplicative*, *idom-divide*}
  shows *normalize* (*smult c p*) = *smult* (*normalize c*) (*normalize p*)
⟨*proof*⟩

instantiation *poly* :: (*field*) *idom-modulo*
begin

definition *modulo-poly* :: *'a poly* ⇒ *'a poly* ⇒ *'a poly*
  where *mod-poly-def*: *f mod g* =
    (*if g* = *0 then f else pseudo-mod* (*smult* ((*1* / *lead-coeff g*) ^ (*Suc* (*degree f*) − *degree g*)) *f*) *g*)

instance
⟨*proof*⟩

end

lemma *pseudo-divmod-eq-div-mod*:
  ‹*pseudo-divmod f g* = (*f div g*, *f mod g*)› **if** ‹*lead-coeff g* = *1*›
  ⟨*proof*⟩

lemma *degree-mod-less-degree*:
  ‹*degree* (*x mod y*) < *degree y*› **if** ‹*y* ≠ *0*› ‹¬ *y dvd x*›
⟨*proof*⟩

instantiation *poly* :: (*field*) *unique-euclidean-ring*
begin

definition *euclidean-size-poly* :: *'a poly* ⇒ *nat*
  where *euclidean-size-poly p* = (*if p* = *0 then 0 else 2* ^ *degree p*)

definition *division-segment-poly* :: *'a poly* ⇒ *'a poly*
  where [*simp*]: *division-segment-poly p* = *1*

instance ⟨*proof*⟩

end

lemma *euclidean-relation-polyI* [*case-names by0 divides euclidean-relation*]:
  ‹(*x div y*, *x mod y*) = (*q*, *r*)›
    if *by0*: ‹*y* = *0* ⟹ *q* = *0* ∧ *r* = *x*›
    and *divides*: ‹*y* ≠ *0* ⟹ *y dvd x* ⟹ *r* = *0* ∧ *x* = *q* ∗ *y*›
    and *euclidean-relation*: ‹*y* ≠ *0* ⟹ ¬ *y dvd x* ⟹ *degree r* < *degree y* ∧ *x* = *q* ∗ *y* + *r*›
  ⟨*proof*⟩

lemma *div-poly-eq-0-iff*:
  ‹*x div y* = *0* ⟷ *x* = *0* ∨ *y* = *0* ∨ *degree x* < *degree y*› **for** *x y* :: ‹*'a*::*field poly*›

⟨*proof*⟩

lemma *div-poly-less*:
‹*x div y = 0*› **if** ‹*degree x < degree y*› **for** *x y* :: ‹*'a*::*field poly*›
⟨*proof*⟩

lemma *mod-poly-less*:
‹*x mod y = x*› **if** ‹*degree x < degree y*›
⟨*proof*⟩

lemma *degree-div-less*:
‹*degree* (*x div y*) < *degree x*›
if ‹*degree x > 0*› ‹*degree y > 0*›
for *x y* :: ‹*'a*::*field poly*›
⟨*proof*⟩

lemma *degree-mod-less'*: *b ≠ 0* ⟹ *a mod b ≠ 0* ⟹ *degree* (*a mod b*) < *degree b*
⟨*proof*⟩

lemma *degree-mod-less*: *y ≠ 0* ⟹ *x mod y = 0* ∨ *degree* (*x mod y*) < *degree y*
⟨*proof*⟩

lemma *div-smult-left*: ‹*smult a x div y = smult a* (*x div y*)› (**is** *?Q*)
and *mod-smult-left*: ‹*smult a x mod y = smult a* (*x mod y*)› (**is** *?R*)
for *x y* :: ‹*'a*::*field poly*›
⟨*proof*⟩

lemma *poly-div-minus-left* [*simp*]: (− *x*) *div y* = − (*x div y*)
for *x y* :: *'a*::*field poly*
⟨*proof*⟩

lemma *poly-mod-minus-left* [*simp*]: (− *x*) *mod y* = − (*x mod y*)
for *x y* :: *'a*::*field poly*
⟨*proof*⟩

lemma *poly-div-add-left*: ‹(*x* + *y*) *div z = x div z* + *y div z*› (**is** *?Q*)
and *poly-mod-add-left*: ‹(*x* + *y*) *mod z = x mod z* + *y mod z*› (**is** *?R*)
for *x y z* :: ‹*'a*::*field poly*›
⟨*proof*⟩

lemma *poly-div-diff-left*: (*x* − *y*) *div z = x div z* − *y div z*
for *x y z* :: *'a*::*field poly*
⟨*proof*⟩

lemma *poly-mod-diff-left*: (*x* − *y*) *mod z = x mod z* − *y mod z*
for *x y z* :: *'a*::*field poly*
⟨*proof*⟩

lemma *div-smult-right*: ‹*x div smult a y = smult* (*inverse a*) (*x div y*)› (**is** *?Q*)

and *mod-smult-right*: ‹*x mod smult a y* = (*if a* = *0 then x else x mod y*)› (**is** *?R*)
⟨*proof*⟩

lemma *mod-mult-unit-eq*:
‹*x mod* (*z* ∗ *y*) = *x mod y*›
if ‹*is-unit z*›
for *x y z* :: ‹'*a*::*field poly*›
⟨*proof*⟩

lemma *poly-div-minus-right* [*simp*]: *x div* (− *y*) = − (*x div y*)
for *x y* :: '*a*::*field poly*
⟨*proof*⟩

lemma *poly-mod-minus-right* [*simp*]: *x mod* (− *y*) = *x mod y*
for *x y* :: '*a*::*field poly*
⟨*proof*⟩

lemma *poly-div-mult-right*: ‹*x div* (*y* ∗ *z*) = (*x div y*) *div z*› (**is** *?Q*)
and *poly-mod-mult-right*: ‹*x mod* (*y* ∗ *z*) = *y* ∗ (*x div y mod z*) + *x mod y*› (**is** *?R*)
for *x y z* :: ‹'*a*::*field poly*›
⟨*proof*⟩

lemma *dvd-pCons-imp-dvd-pCons-mod*:
‹*y dvd pCons a* (*x mod y*)› **if** ‹*y dvd pCons a x*›
⟨*proof*⟩

lemma *degree-less-if-less-eqI*:
‹*degree x* < *degree y*› **if** ‹*degree x* ≤ *degree y*› ‹*coeff x* (*degree y*) = *0*› ‹*x* ≠ *0*›
⟨*proof*⟩

lemma *div-pCons-eq*:
‹*pCons a p div q* = (*if q* = *0 then 0 else pCons* (*coeff* (*pCons a* (*p mod q*)) (*degree q*) / *lead-coeff q*) (*p div q*))› (**is** *?Q*)
and *mod-pCons-eq*:
‹*pCons a p mod q* = (*if q* = *0 then pCons a p else pCons a* (*p mod q*) − *smult* (*coeff* (*pCons a* (*p mod q*)) (*degree q*) / *lead-coeff q*) *q*)› (**is** *?R*)
for *x y* :: ‹'*a*::*field poly*›
⟨*proof*⟩

lemma *div-mod-fold-coeffs*:
(*p div q*, *p mod q*) =
(*if q* = *0 then* (*0*, *p*)
else
fold-coeffs
(λ*a* (*s*, *r*).
let b = *coeff* (*pCons a r*) (*degree q*) / *coeff q* (*degree q*)
in (*pCons b s*, *pCons a r* − *smult b q*)) *p* (*0*, *0*))
⟨*proof*⟩

lemma *mod-pCons*:
fixes *a* :: *'a*::*field*
and *x y* :: *'a*::*field poly*
assumes *y*: *y* ≠ *0*
defines *b* ≡ *coeff* (*pCons a* (*x mod y*)) (*degree y*) / *coeff y* (*degree y*)
shows (*pCons a x*) *mod y* = *pCons a* (*x mod y*) − *smult b y*
⟨*proof*⟩

4.30.4 List-based versions for fast implementation

fun *minus-poly-rev-list* :: *'a* :: *group-add list* ⇒ *'a list* ⇒ *'a list*
where
minus-poly-rev-list (*x* # *xs*) (*y* # *ys*) = (*x* − *y*) # (*minus-poly-rev-list xs ys*)
| *minus-poly-rev-list xs* [] = *xs*
| *minus-poly-rev-list* [] (*y* # *ys*) = []

fun *pseudo-divmod-main-list* ::
'a::*comm-ring-1* ⇒ *'a list* ⇒ *'a list* ⇒ *'a list* ⇒ *nat* ⇒ *'a list* × *'a list*
where
pseudo-divmod-main-list lc q r d (*Suc n*) =
(*let*
rr = *map* ((∗) *lc*) *r*;
a = *hd r*;
qqq = *cCons a* (*map* ((∗) *lc*) *q*);
rrr = *tl* (*if a* = *0 then rr else minus-poly-rev-list rr* (*map* ((∗) *a*) *d*))
in pseudo-divmod-main-list lc qqq rrr d n)
| *pseudo-divmod-main-list lc q r d 0* = (*q*, *r*)

fun *pseudo-mod-main-list* :: *'a*::*comm-ring-1* ⇒ *'a list* ⇒ *'a list* ⇒ *nat* ⇒ *'a list*
where
pseudo-mod-main-list lc r d (*Suc n*) =
(*let*
rr = *map* ((∗) *lc*) *r*;
a = *hd r*;
rrr = *tl* (*if a* = *0 then rr else minus-poly-rev-list rr* (*map* ((∗) *a*) *d*))
in pseudo-mod-main-list lc rrr d n)
| *pseudo-mod-main-list lc r d 0* = *r*

fun *divmod-poly-one-main-list* ::
'a::*comm-ring-1 list* ⇒ *'a list* ⇒ *'a list* ⇒ *nat* ⇒ *'a list* × *'a list*
where
divmod-poly-one-main-list q r d (*Suc n*) =
(*let*
a = *hd r*;
qqq = *cCons a q*;
rr = *tl* (*if a* = *0 then r else minus-poly-rev-list r* (*map* ((∗) *a*) *d*))
in divmod-poly-one-main-list qqq rr d n)

```
| divmod-poly-one-main-list q r d 0 = (q, r)

fun mod-poly-one-main-list :: 'a::comm-ring-1 list ⇒ 'a list ⇒ nat ⇒ 'a list
  where
    mod-poly-one-main-list r d (Suc n) =
      (let
        a = hd r;
        rr = tl (if a = 0 then r else minus-poly-rev-list r (map ((*) a) d))
       in mod-poly-one-main-list rr d n)
  | mod-poly-one-main-list r d 0 = r

definition pseudo-divmod-list :: 'a::comm-ring-1 list ⇒ 'a list ⇒ 'a list × 'a list
  where pseudo-divmod-list p q =
    (if q = [] then ([], p)
     else
      (let rq = rev q;
          (qu,re) = pseudo-divmod-main-list (hd rq) [] (rev p) rq (1 + length p −
length q)
       in (qu, rev re)))

definition pseudo-mod-list :: 'a::comm-ring-1 list ⇒ 'a list ⇒ 'a list
  where pseudo-mod-list p q =
    (if q = [] then p
     else
      (let
        rq = rev q;
        re = pseudo-mod-main-list (hd rq) (rev p) rq (1 + length p − length q)
       in rev re))

lemma minus-zero-does-nothing: minus-poly-rev-list x (map ((*) 0) y) = x
  for x :: 'a::ring list
  ⟨proof⟩

lemma length-minus-poly-rev-list [simp]: length (minus-poly-rev-list xs ys) = length
xs
  ⟨proof⟩

lemma if-0-minus-poly-rev-list:
  (if a = 0 then x else minus-poly-rev-list x (map ((*) a) y)) =
    minus-poly-rev-list x (map ((*) a) y)
  for a :: 'a::ring
  ⟨proof⟩

lemma Poly-append: Poly (a @ b) = Poly a + monom 1 (length a) * Poly b
  for a :: 'a::comm-semiring-1 list
  ⟨proof⟩

lemma minus-poly-rev-list: length p ≥ length q ⟹
  Poly (rev (minus-poly-rev-list (rev p) (rev q))) =
```

Poly p − monom 1 (length p − length q) ∗ Poly q
for *p q* :: *'a* :: *comm-ring-1 list*
⟨*proof*⟩

lemma *smult-monom-mult*: *smult a (monom b n ∗ f) = monom (a ∗ b) n ∗ f*
⟨*proof*⟩

lemma *head-minus-poly-rev-list*:
length d ≤ length r ⟹ d ≠ [] ⟹
hd (minus-poly-rev-list (map ((∗) (last d)) r) (map ((∗) (hd r)) (rev d))) = 0
for *d r* :: *'a*::*comm-ring list*
⟨*proof*⟩

lemma *Poly-map*: *Poly (map ((∗) a) p) = smult a (Poly p)*
⟨*proof*⟩

lemma *last-coeff-is-hd*: *xs ≠ [] ⟹ coeff (Poly xs) (length xs − 1) = hd (rev xs)*
⟨*proof*⟩

lemma *pseudo-divmod-main-list-invar*:
assumes *leading-nonzero*: *last d ≠ 0*
and *lc*: *last d = lc*
and *d ≠ []*
and *pseudo-divmod-main-list lc q (rev r) (rev d) n = (q', rev r')*
and *n = 1 + length r − length d*
shows *pseudo-divmod-main lc (monom 1 n ∗ Poly q) (Poly r) (Poly d) (length r − 1) n =*
(Poly q', Poly r')
⟨*proof*⟩

lemma *pseudo-divmod-impl* [*code*]:
pseudo-divmod f g = map-prod poly-of-list poly-of-list (pseudo-divmod-list (coeffs f) (coeffs g))
for *f g* :: *'a*::*comm-ring-1 poly*
⟨*proof*⟩

lemma *pseudo-mod-main-list*:
snd (pseudo-divmod-main-list l q xs ys n) = pseudo-mod-main-list l xs ys n
⟨*proof*⟩

lemma *pseudo-mod-impl*[*code*]: *pseudo-mod f g = poly-of-list (pseudo-mod-list (coeffs f) (coeffs g))*
⟨*proof*⟩

4.30.5 Improved Code-Equations for Polynomial (Pseudo) Division

lemma *pdivmod-via-pseudo-divmod*:
‹*(f div g, f mod g) =*

```
    (if g = 0 then (0, f)
     else
      let
        ilc = inverse (lead-coeff g);
        h = smult ilc g;
        (q,r) = pseudo-divmod f h
      in (smult ilc q, r))›
  (is ‹?l = ?r›)
⟨proof⟩

lemma pdivmod-via-pseudo-divmod-list:
  (f div g, f mod g) =
    (let cg = coeffs g in
      if cg = [] then (0, f)
      else
        let
          cf = coeffs f;
          ilc = inverse (last cg);
          ch = map ((*) ilc) cg;
           (q, r) = pseudo-divmod-main-list 1 [] (rev cf) (rev ch) (1 + length cf −
length cg)
        in (poly-of-list (map ((*) ilc) q), poly-of-list (rev r)))
⟨proof⟩

lemma pseudo-divmod-main-list-1: pseudo-divmod-main-list 1 = divmod-poly-one-main-list
⟨proof⟩

fun divide-poly-main-list :: 'a::idom-divide ⇒ 'a list ⇒ 'a list ⇒ 'a list ⇒ nat ⇒
'a list
  where
    divide-poly-main-list lc q r d (Suc n) =
      (let
        cr = hd r
        in if cr = 0 then divide-poly-main-list lc (cCons cr q) (tl r) d n else let
        a = cr div lc;
        qq = cCons a q;
        rr = minus-poly-rev-list r (map ((*) a) d)
       in if hd rr = 0 then divide-poly-main-list lc qq (tl rr) d n else [])
  | divide-poly-main-list lc q r d 0 = q

lemma divide-poly-main-list-simp [simp]:
  divide-poly-main-list lc q r d (Suc n) =
    (let
      cr = hd r;
      a = cr div lc;
      qq = cCons a q;
      rr = minus-poly-rev-list r (map ((*) a) d)
     in if hd rr = 0 then divide-poly-main-list lc qq (tl rr) d n else [])
  ⟨proof⟩
```

declare *divide-poly-main-list.simps*(*1*)[*simp del*]

definition *divide-poly-list* :: *'a*::*idom-divide poly* ⇒ *'a poly* ⇒ *'a poly*
  where *divide-poly-list f g* =
    (*let cg* = *coeffs g in*
      if cg = [] *then g*
      else
        let
          cf = *coeffs f*;
          cgr = *rev cg*
         in poly-of-list (*divide-poly-main-list* (*hd cgr*) [] (*rev cf*) *cgr* (*1* + *length cf*
− *length cg*)))

lemmas *pdivmod-via-divmod-list* = *pdivmod-via-pseudo-divmod-list*[*unfolded pseudo-divmod-main-list-1*]

lemma *mod-poly-one-main-list*: *snd* (*divmod-poly-one-main-list q r d n*) = *mod-poly-one-main-list*
r d n
  ⟨*proof*⟩

lemma *mod-poly-code* [*code*]:
  f mod g =
    (*let cg* = *coeffs g in*
      if cg = [] *then f*
      else
        let
          cf = *coeffs f*;
          ilc = *inverse* (*last cg*);
          ch = *map* ((∗) *ilc*) *cg*;
          r = *mod-poly-one-main-list* (*rev cf*) (*rev ch*) (*1* + *length cf* − *length cg*)
        in poly-of-list (*rev r*))
  (**is** - = *?rhs*)
⟨*proof*⟩

definition *div-field-poly-impl* :: *'a* :: *field poly* ⇒ *'a poly* ⇒ *'a poly*
  where *div-field-poly-impl f g* =
    (*let cg* = *coeffs g in*
      if cg = [] *then 0*
      else
        let
          cf = *coeffs f*;
          ilc = *inverse* (*last cg*);
          ch = *map* ((∗) *ilc*) *cg*;
           q = *fst* (*divmod-poly-one-main-list* [] (*rev cf*) (*rev ch*) (*1* + *length cf* −
length cg))
        in poly-of-list ((*map* ((∗) *ilc*) *q*)))

We do not declare the following lemma as code equation, since then polynomial division on non-fields will no longer be executable. However, a code-

unfold is possible, since *div-field-poly-impl* is a bit more efficient than the generic polynomial division.

lemma *div-field-poly-impl*[*code-unfold*]: (*div*) = *div-field-poly-impl*
⟨*proof*⟩

lemma *divide-poly-main-list*:
  assumes *lc0*: *lc* ≠ *0*
    and *lc*: *last d* = *lc*
    and *d*: *d* ≠ []
    and *n* = (*1* + *length r* − *length d*)
  shows *Poly* (*divide-poly-main-list lc q* (*rev r*) (*rev d*) *n*) =
    divide-poly-main lc (*monom 1 n* ∗ *Poly q*) (*Poly r*) (*Poly d*) (*length r* − *1*) *n*
  ⟨*proof*⟩

lemma *divide-poly-list*[*code*]: *f div g* = *divide-poly-list f g*
⟨*proof*⟩

lemma *poly-mod*:
  poly (*p mod q*) *x* = *poly p x* **if** *poly q x* = *0*
⟨*proof*⟩

4.31 Primality and irreducibility in polynomial rings

lemma *prod-mset-const-poly*: (∏ *x*∈#*A*. [:*f x*:]) = [:*prod-mset* (*image-mset f A*):]
  ⟨*proof*⟩

lemma *irreducible-const-poly-iff*:
  fixes *c* :: *'a* :: {*comm-semiring-1*,*semiring-no-zero-divisors*}
  shows *irreducible* [:*c*:] ⟷ *irreducible c*
⟨*proof*⟩

lemma *lift-prime-elem-poly*:
  assumes *prime-elem* (*c* :: *'a* :: *semidom*)
  shows *prime-elem* [:*c*:]
⟨*proof*⟩

lemma *prime-elem-const-poly-iff*:
  fixes *c* :: *'a* :: *semidom*
  shows *prime-elem* [:*c*:] ⟷ *prime-elem c*
⟨*proof*⟩

4.32 Content and primitive part of a polynomial

definition *content* :: *'a*::*semiring-gcd poly* ⇒ *'a*
  where *content p* = *gcd-list* (*coeffs p*)

lemma *content-eq-fold-coeffs* [*code*]: *content p* = *fold-coeffs gcd p 0*
  ⟨*proof*⟩

lemma *content-0* [*simp*]: *content 0 = 0*
⟨*proof*⟩

lemma *content-1* [*simp*]: *content 1 = 1*
⟨*proof*⟩

lemma *content-const* [*simp*]: *content* [:*c*:] = *normalize c*
⟨*proof*⟩

lemma *const-poly-dvd-iff-dvd-content*: [:*c*:] *dvd p* ⟷ *c dvd content p*
for *c* :: *'a*::*semiring-gcd*
⟨*proof*⟩

lemma *content-dvd* [*simp*]: [:*content p*:] *dvd p*
⟨*proof*⟩

lemma *content-dvd-coeff* [*simp*]: *content p dvd coeff p n*
⟨*proof*⟩

lemma *content-dvd-coeffs*: *c* ∈ *set* (*coeffs p*) ⟹ *content p dvd c*
⟨*proof*⟩

lemma *normalize-content* [*simp*]: *normalize* (*content p*) = *content p*
⟨*proof*⟩

lemma *is-unit-content-iff* [*simp*]: *is-unit* (*content p*) ⟷ *content p = 1*
⟨*proof*⟩

lemma *content-smult* [*simp*]:
fixes *c* :: *'a* :: {*normalization-semidom-multiplicative*, *semiring-gcd*}
shows *content* (*smult c p*) = *normalize c* ∗ *content p*
⟨*proof*⟩

lemma *content-eq-zero-iff* [*simp*]: *content p = 0* ⟷ *p = 0*
⟨*proof*⟩

definition *primitive-part* :: *'a* :: *semiring-gcd poly* ⇒ *'a poly*
where *primitive-part p* = *map-poly* (λ*x*. *x div content p*) *p*

lemma *primitive-part-0* [*simp*]: *primitive-part 0 = 0*
⟨*proof*⟩

lemma *content-times-primitive-part* [*simp*]: *smult* (*content p*) (*primitive-part p*) = *p*
for *p* :: *'a* :: *semiring-gcd poly*
⟨*proof*⟩

lemma *primitive-part-eq-0-iff* [*simp*]: *primitive-part p = 0* ⟷ *p = 0*
⟨*proof*⟩

lemma *content-primitive-part* [*simp*]:
fixes $p :: 'a :: \{normalization\text{-}semidom\text{-}multiplicative, semiring\text{-}gcd\}\ poly$
assumes $p \neq 0$
shows $content\ (primitive\text{-}part\ p) = 1$
⟨*proof*⟩

lemma *content-decompose*:
obtains $p' :: 'a :: \{normalization\text{-}semidom\text{-}multiplicative, semiring\text{-}gcd\}\ poly$
where $p = smult\ (content\ p)\ p'\ content\ p' = 1$
⟨*proof*⟩

lemma *content-dvd-contentI* [*intro*]: $p\ dvd\ q \Longrightarrow content\ p\ dvd\ content\ q$
⟨*proof*⟩

lemma *primitive-part-const-poly* [*simp*]: $primitive\text{-}part\ [:x:] = [:unit\text{-}factor\ x:]$
⟨*proof*⟩

lemma *primitive-part-prim*: $content\ p = 1 \Longrightarrow primitive\text{-}part\ p = p$
⟨*proof*⟩

lemma *degree-primitive-part* [*simp*]: $degree\ (primitive\text{-}part\ p) = degree\ p$
⟨*proof*⟩

lemma *smult-content-normalize-primitive-part* [*simp*]:
fixes $p :: 'a :: \{normalization\text{-}semidom\text{-}multiplicative, semiring\text{-}gcd, idom\text{-}divide\}$
poly
shows $smult\ (content\ p)\ (normalize\ (primitive\text{-}part\ p)) = normalize\ p$
⟨*proof*⟩

context
begin

private

lemma *content-1-mult*:
fixes $f\ g :: 'a :: \{semiring\text{-}gcd, factorial\text{-}semiring\}\ poly$
assumes $content\ f = 1\ content\ g = 1$
shows $content\ (f * g) = 1$
⟨*proof*⟩

lemma *content-mult*:
fixes $p\ q :: 'a :: \{factorial\text{-}semiring, semiring\text{-}gcd, normalization\text{-}semidom\text{-}multiplicative\}$
poly
shows $content\ (p * q) = content\ p * content\ q$
⟨*proof*⟩

end

lemma *primitive-part-mult*:
fixes $p\ q :: 'a :: \{$*factorial-semiring*, *semiring-Gcd*, *ring-gcd*, *idom-divide*, *normalization-semidom-multiplicative*$\}$ *poly*
shows *primitive-part* $(p * q) =$ *primitive-part* $p *$ *primitive-part* q
⟨*proof*⟩

lemma *primitive-part-smult*:
fixes $p :: 'a :: \{$*factorial-semiring*, *semiring-Gcd*, *ring-gcd*, *idom-divide*, *normalization-semidom-multiplicative*$\}$ *poly*
shows *primitive-part* (*smult a p*) = *smult* (*unit-factor a*) (*primitive-part p*)
⟨*proof*⟩

lemma *primitive-part-dvd-primitive-partI* [*intro*]:
fixes $p\ q :: 'a :: \{$*factorial-semiring*, *semiring-Gcd*, *ring-gcd*, *idom-divide*, *normalization-semidom-multiplicative*$\}$ *poly*
shows p *dvd* $q \Longrightarrow$ *primitive-part* p *dvd primitive-part* q
⟨*proof*⟩

lemma *content-prod-mset*:
fixes $A :: 'a :: \{$*factorial-semiring*, *semiring-Gcd*, *normalization-semidom-multiplicative*$\}$ *poly multiset*
shows *content* (*prod-mset A*) = *prod-mset* (*image-mset content A*)
⟨*proof*⟩

lemma *content-prod-eq-1-iff*:
fixes $p\ q :: 'a :: \{$*factorial-semiring*, *semiring-Gcd*, *normalization-semidom-multiplicative*$\}$ *poly*
shows *content* $(p * q) = 1 \longleftrightarrow$ *content* $p = 1 \wedge$ *content* $q = 1$
⟨*proof*⟩

4.33 A typeclass for algebraically closed fields

Since the required sort constraints are not available inside the class, we have to resort to a somewhat awkward way of writing the definition of algebraically closed fields:

class *alg-closed-field* = *field* +
assumes *alg-closed*: $n > 0 \Longrightarrow f\ n \neq 0 \Longrightarrow \exists x.\ (\sum k \leq n.\ f\ k * x\ \hat{}\ k) = 0$

We can then however easily show the equivalence to the proper definition:

lemma *alg-closed-imp-poly-has-root*:
assumes *degree* $(p :: 'a ::$ *alg-closed-field poly*$) > 0$
shows $\exists x.$ *poly* $p\ x = 0$
⟨*proof*⟩

lemma *alg-closedI* [*Pure.intro*]:
assumes $\bigwedge p :: 'a$ *poly. degree* $p > 0 \Longrightarrow$ *lead-coeff* $p = 1 \Longrightarrow \exists x.$ *poly* $p\ x = 0$
shows *OFCLASS*$('a ::$ *field*, *alg-closed-field-class*$)$
⟨*proof*⟩

lemma (**in** *alg-closed-field*) *nth-root-exists*:
assumes $n > 0$
shows $\exists y.\ y \mathbin{\hat{}} n = (x :: 'a)$
⟨*proof*⟩

We can now prove by induction that every polynomial of degree n splits into a product of n linear factors:

lemma *alg-closed-imp-factorization*:
fixes $p :: 'a :: alg\text{-}closed\text{-}field\ poly$
assumes $p \neq 0$
shows $\exists A.\ size\ A = degree\ p \wedge p = smult\ (lead\text{-}coeff\ p)\ (\prod x \in\# A.\ [:-x,\ 1:])$
⟨*proof*⟩

As an alternative characterisation of algebraic closure, one can also say that any polynomial of degree at least 2 splits into non-constant factors:

lemma *alg-closed-imp-reducible*:
assumes $degree\ (p :: 'a :: alg\text{-}closed\text{-}field\ poly) > 1$
shows $\neg irreducible\ p$
⟨*proof*⟩

When proving algebraic closure through reducibility, we can assume w.l.o.g. that the polynomial is monic and has a non-zero constant coefficient:

lemma *alg-closedI-reducible*:
assumes $\bigwedge p :: 'a\ poly.\ degree\ p > 1 \Longrightarrow lead\text{-}coeff\ p = 1 \Longrightarrow coeff\ p\ 0 \neq 0 \Longrightarrow \neg irreducible\ p$
shows $OFCLASS('a :: field,\ alg\text{-}closed\text{-}field\text{-}class)$
⟨*proof*⟩

Using a clever Tschirnhausen transformation mentioned e.g. in the article by Nowak [1], we can also assume w.l.o.g. that the coefficient a_{n-1} is zero.

lemma *alg-closedI-reducible-coeff-deg-minus-one-eq-0*:
assumes $\bigwedge p :: 'a\ poly.\ degree\ p > 1 \Longrightarrow lead\text{-}coeff\ p = 1 \Longrightarrow coeff\ p\ (degree\ p - 1) = 0 \Longrightarrow coeff\ p\ 0 \neq 0 \Longrightarrow \neg irreducible\ p$
shows $OFCLASS('a :: field\text{-}char\text{-}0,\ alg\text{-}closed\text{-}field\text{-}class)$
⟨*proof*⟩

As a consequence of the full factorisation lemma proven above, we can also show that any polynomial with at least two different roots splits into two non-constant coprime factors:

lemma *alg-closed-imp-poly-splits-coprime*:
assumes $degree\ (p :: 'a :: \{alg\text{-}closed\text{-}field\}\ poly) > 1$
assumes $poly\ p\ x = 0\ poly\ p\ y = 0\ x \neq y$
obtains $r\ s$ **where** $degree\ r > 0\ degree\ s > 0\ coprime\ r\ s\ p = r * s$
⟨*proof*⟩

4.34 Polynomials and limits

lemma *filterlim-poly-at-infinity*:
fixes *p*::*'a*::*real-normed-field poly*
assumes *degree p*>*0*
shows *filterlim* (*poly p*) *at-infinity at-infinity*
⟨*proof*⟩

lemma *poly-divide-tendsto-aux*:
fixes *p*::*'a*::*real-normed-field poly*
shows ((λ*x*. *poly p x*/*x*^(*degree p*)) ⟶ *lead-coeff p*) *at-infinity*
⟨*proof*⟩

lemma *filterlim-power-at-infinity*:
assumes *n*≠*0*
shows *filterlim* (λ*x*::*'a*::*real-normed-field*. *x*^*n*) *at-infinity at-infinity*
⟨*proof*⟩

lemma *poly-divide-tendsto-0-at-infinity*:
fixes *p*::*'a*::*real-normed-field poly*
assumes *degree p* > *degree q*
shows ((λ*x*. *poly q x* / *poly p x*) ⟶ *0*) *at-infinity*
⟨*proof*⟩

lemma *poly-eventually-not-zero*:
fixes *p*::*real poly*
assumes *p*≠*0*
shows *eventually* (λ*x*. *poly p x* ≠ *0*) *at-infinity*
⟨*proof*⟩

no-notation *cCons* (**infixr** ‹##› *65*)

end

5 A formalization of formal power series

theory *Formal-Power-Series*
imports
Complex-Main
Euclidean-Algorithm
Primes
HOL−Library.FuncSet
HOL−Library.Multiset
begin

5.1 The type of formal power series

typedef *'a fps* = {*f* :: *nat* ⇒ *'a*. *True*}
morphisms *fps-nth Abs-fps*

⟨*proof*⟩

notation *fps-nth* (**infixl** ‹$› *75*)

lemma *expand-fps-eq*: $p = q \longleftrightarrow (\forall n.\ p\ \$\ n = q\ \$\ n)$
⟨*proof*⟩

lemmas *fps-eq-iff* = *expand-fps-eq*

lemma *fps-ext*: $(\bigwedge n.\ p\ \$\ n = q\ \$\ n) \Longrightarrow p = q$
⟨*proof*⟩

lemma *fps-nth-Abs-fps* [*simp*]: *Abs-fps f $ n = f n*
⟨*proof*⟩

Definition of the basic elements 0 and 1 and the basic operations of addition, negation and multiplication.

instantiation *fps* :: (*zero*) *zero*
begin
definition *fps-zero-def*: *0 = Abs-fps* (λ*n. 0*)
instance ⟨*proof*⟩
end

lemma *fps-zero-nth* [*simp*]: *0 $ n = 0*
⟨*proof*⟩

lemma *fps-nonzero-nth*: $f \neq 0 \longleftrightarrow (\exists\ n.\ f\ \$\ n \neq 0)$
⟨*proof*⟩

lemma *fps-nonzero-nth-minimal*: $f \neq 0 \longleftrightarrow (\exists n.\ f\ \$\ n \neq 0 \wedge (\forall m < n.\ f\ \$\ m = 0))$
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *fps-nonzeroI*: $f\$n \neq 0 \Longrightarrow f \neq 0$
⟨*proof*⟩

instantiation *fps* :: ({*one*, *zero*}) *one*
begin
definition *fps-one-def*: *1 = Abs-fps* (λ*n. if n = 0 then 1 else 0*)
instance ⟨*proof*⟩
end

lemma *fps-one-nth* [*simp*]: *1 $ n* = (*if n = 0 then 1 else 0*)
⟨*proof*⟩

instantiation *fps* :: (*plus*) *plus*
begin
definition *fps-plus-def*: (+) = (λ*f g. Abs-fps* (λ*n. f $ n + g $ n*))

instance ⟨*proof*⟩
end

lemma *fps-add-nth* [*simp*]: $(f + g)\ \$\ n = f\ \$\ n + g\ \$\ n$
⟨*proof*⟩

instantiation *fps* :: (*minus*) *minus*
begin
definition *fps-minus-def*: $(-) = (\lambda f\ g.\ \textit{Abs-fps}\ (\lambda n.\ f\ \$\ n - g\ \$\ n))$
instance ⟨*proof*⟩
end

lemma *fps-sub-nth* [*simp*]: $(f - g)\ \$\ n = f\ \$\ n - g\ \$\ n$
⟨*proof*⟩

instantiation *fps* :: (*uminus*) *uminus*
begin
definition *fps-uminus-def*: $\textit{uminus} = (\lambda f.\ \textit{Abs-fps}\ (\lambda n.\ -\ (f\ \$\ n)))$
instance ⟨*proof*⟩
end

lemma *fps-neg-nth* [*simp*]: $(-\ f)\ \$\ n = -\ (f\ \$\ n)$
⟨*proof*⟩

lemma *fps-neg-0* [*simp*]: $-(0::'a::\textit{group-add fps}) = 0$
⟨*proof*⟩

instantiation *fps* :: ({*comm-monoid-add*, *times*}) *times*
begin
definition *fps-times-def*: $(*) = (\lambda f\ g.\ \textit{Abs-fps}\ (\lambda n.\ \sum i{=}0..n.\ f\ \$\ i * g\ \$\ (n - i)))$
instance ⟨*proof*⟩
end

lemma *fps-mult-nth*: $(f * g)\ \$\ n = (\sum i{=}0..n.\ f\$i * g\$(n - i))$
⟨*proof*⟩

lemma *fps-mult-nth-0* [*simp*]: $(f * g)\ \$\ 0 = f\ \$\ 0 * g\ \$\ 0$
⟨*proof*⟩

lemma *fps-mult-nth-1*: $(f * g)\ \$\ 1 = f\$0 * g\$1 + f\$1 * g\$0$
⟨*proof*⟩

lemma *fps-mult-nth-1'* [*simp*]: $(f * g)\ \$\ \textit{Suc}\ 0 = f\$0 * g\$\textit{Suc}\ 0 + f\$\textit{Suc}\ 0 * g\$0$
⟨*proof*⟩

lemmas *mult-nth-0* = *fps-mult-nth-0*
lemmas *mult-nth-1* = *fps-mult-nth-1*

instance *fps* :: ({*comm-monoid-add*, *mult-zero*}) *mult-zero*
⟨*proof*⟩

declare *atLeastAtMost-iff* [*presburger*]
declare *Bex-def* [*presburger*]
declare *Ball-def* [*presburger*]

lemma *mult-delta-left*:
fixes *x y* :: *'a*::*mult-zero*
shows (*if b then x else 0*) ∗ *y* = (*if b then x* ∗ *y else 0*)
⟨*proof*⟩

lemma *mult-delta-right*:
fixes *x y* :: *'a*::*mult-zero*
shows *x* ∗ (*if b then y else 0*) = (*if b then x* ∗ *y else 0*)
⟨*proof*⟩

lemma *fps-one-mult*:
fixes *f* :: *'a*::{*comm-monoid-add*, *mult-zero*, *monoid-mult*} *fps*
shows *1* ∗ *f* = *f*
and *f* ∗ *1* = *f*
⟨*proof*⟩

5.2 Subdegrees

definition *subdegree* :: (*'a*::*zero*) *fps* ⇒ *nat* **where**
subdegree f = (*if f* = *0 then 0 else LEAST n*. *f*$*n* ≠ *0*)

lemma *subdegreeI*:
assumes *f* $ *d* ≠ *0* **and** ⋀*i*. *i* < *d* ⟹ *f* $ *i* = *0*
shows *subdegree f* = *d*
⟨*proof*⟩

lemma *nth-subdegree-nonzero* [*simp*,*intro*]: *f* ≠ *0* ⟹ *f* $ *subdegree f* ≠ *0*
⟨*proof*⟩

lemma *nth-less-subdegree-zero* [*dest*]: *n* < *subdegree f* ⟹ *f* $ *n* = *0*
⟨*proof*⟩

lemma *subdegree-geI*:
assumes *f* ≠ *0* ⋀*i*. *i* < *n* ⟹ *f*$*i* = *0*
shows *subdegree f* ≥ *n*
⟨*proof*⟩

lemma *subdegree-greaterI*:
assumes *f* ≠ *0* ⋀*i*. *i* ≤ *n* ⟹ *f*$*i* = *0*
shows *subdegree f* > *n*
⟨*proof*⟩

lemma *subdegree-leI*:
f $ n ≠ 0 ⟹ subdegree f ≤ n
⟨*proof*⟩

lemma *subdegree-0* [*simp*]: *subdegree 0 = 0*
⟨*proof*⟩

lemma *subdegree-1* [*simp*]: *subdegree 1 = 0*
⟨*proof*⟩

lemma *subdegree-eq-0-iff*: *subdegree f = 0 ⟷ f = 0 ∨ f $ 0 ≠ 0*
⟨*proof*⟩

lemma *subdegree-eq-0* [*simp*]: *f $ 0 ≠ 0 ⟹ subdegree f = 0*
⟨*proof*⟩

lemma *nth-subdegree-zero-iff* [*simp*]: *f $ subdegree f = 0 ⟷ f = 0*
⟨*proof*⟩

lemma *fps-nonzero-subdegree-nonzeroI*: *subdegree f > 0 ⟹ f ≠ 0*
⟨*proof*⟩

lemma *subdegree-uminus* [*simp*]:
subdegree (−(f::('a::group-add) fps)) = subdegree f
⟨*proof*⟩

lemma *subdegree-minus-commute* [*simp*]:
fixes *f* :: *'a::group-add fps*
shows *subdegree (f−g) = subdegree (g − f)*
⟨*proof*⟩

lemma *subdegree-add-ge'*:
fixes *f g* :: *'a::monoid-add fps*
assumes *f + g ≠ 0*
shows *subdegree (f + g) ≥ min (subdegree f) (subdegree g)*
⟨*proof*⟩

lemma *subdegree-add-ge*:
assumes *f ≠ −(g :: ('a :: group-add) fps)*
shows *subdegree (f + g) ≥ min (subdegree f) (subdegree g)*
⟨*proof*⟩

lemma *subdegree-add-eq1*:
assumes *f ≠ 0*
and *subdegree f < subdegree (g :: 'a::monoid-add fps)*
shows *subdegree (f + g) = subdegree f*
⟨*proof*⟩

lemma *subdegree-add-eq2*:

assumes $g \neq 0$
and *subdegree* $g <$ *subdegree* $(f :: 'a :: monoid\text{-}add\ fps)$
shows *subdegree* $(f + g) =$ *subdegree* g
⟨*proof*⟩

lemma *subdegree-diff-eq1*:
assumes $f \neq 0$
and *subdegree* $f <$ *subdegree* $(g :: 'a :: group\text{-}add\ fps)$
shows *subdegree* $(f - g) =$ *subdegree* f
⟨*proof*⟩

lemma *subdegree-diff-eq1-cancel*:
assumes $f \neq 0$
and *subdegree* $f <$ *subdegree* $(g :: 'a :: cancel\text{-}comm\text{-}monoid\text{-}add\ fps)$
shows *subdegree* $(f - g) =$ *subdegree* f
⟨*proof*⟩

lemma *subdegree-diff-eq2*:
assumes $g \neq 0$
and *subdegree* $g <$ *subdegree* $(f :: 'a :: group\text{-}add\ fps)$
shows *subdegree* $(f - g) =$ *subdegree* g
⟨*proof*⟩

lemma *subdegree-diff-ge* [*simp*]:
assumes $f \neq (g :: 'a :: group\text{-}add\ fps)$
shows *subdegree* $(f - g) \geq min$ (*subdegree* f) (*subdegree* g)
⟨*proof*⟩

lemma *subdegree-diff-ge'*:
fixes $f\ g :: 'a :: comm\text{-}monoid\text{-}diff\ fps$
assumes $f - g \neq 0$
shows *subdegree* $(f - g) \geq$ *subdegree* f
⟨*proof*⟩

lemma *nth-subdegree-mult-left* [*simp*]:
fixes $f\ g :: ('a :: \{mult\text{-}zero, comm\text{-}monoid\text{-}add\})\ fps$
shows $(f * g)\ \$\ (subdegree\ f) = f\ \$\ subdegree\ f * g\ \$\ 0$
⟨*proof*⟩

lemma *nth-subdegree-mult-right* [*simp*]:
fixes $f\ g :: ('a :: \{mult\text{-}zero, comm\text{-}monoid\text{-}add\})\ fps$
shows $(f * g)\ \$\ (subdegree\ g) = f\ \$\ 0 * g\ \$\ subdegree\ g$
⟨*proof*⟩

lemma *nth-subdegree-mult* [*simp*]:
fixes $f\ g :: ('a :: \{mult\text{-}zero, comm\text{-}monoid\text{-}add\})\ fps$
shows $(f * g)\ \$\ (subdegree\ f + subdegree\ g) = f\ \$\ subdegree\ f * g\ \$\ subdegree\ g$
⟨*proof*⟩

lemma *fps-mult-nth-eq0*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fps*
assumes *n* < *subdegree f* + *subdegree g*
shows (*f*∗*g*) $ *n* = *0*
⟨*proof*⟩

lemma *fps-mult-subdegree-ge*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fps*
assumes *f*∗*g* ≠ *0*
shows *subdegree* (*f*∗*g*) ≥ *subdegree f* + *subdegree g*
⟨*proof*⟩

lemma *subdegree-mult'*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fps*
assumes *f* $ *subdegree f* ∗ *g* $ *subdegree g* ≠ *0*
shows *subdegree* (*f*∗*g*) = *subdegree f* + *subdegree g*
⟨*proof*⟩

lemma *subdegree-mult* [*simp*]:
fixes *f g* :: *'a* :: {*semiring-no-zero-divisors*} *fps*
assumes *f* ≠ *0* *g* ≠ *0*
shows *subdegree* (*f* ∗ *g*) = *subdegree f* + *subdegree g*
⟨*proof*⟩

lemma *fps-mult-nth-conv-upto-subdegree-left*:
fixes *f g* :: (*'a* :: {*mult-zero*,*comm-monoid-add*}) *fps*
shows (*f* ∗ *g*) $ *n* = (∑ *i*=*subdegree f*..*n*. *f* $ *i* ∗ *g* $ (*n* − *i*))
⟨*proof*⟩

lemma *fps-mult-nth-conv-upto-subdegree-right*:
fixes *f g* :: (*'a* :: {*mult-zero*,*comm-monoid-add*}) *fps*
shows (*f* ∗ *g*) $ *n* = (∑ *i*=*0*..*n* − *subdegree g*. *f* $ *i* ∗ *g* $ (*n* − *i*))
⟨*proof*⟩

lemma *fps-mult-nth-conv-inside-subdegrees*:
fixes *f g* :: (*'a* :: {*mult-zero*,*comm-monoid-add*}) *fps*
shows (*f* ∗ *g*) $ *n* = (∑ *i*=*subdegree f*..*n* − *subdegree g*. *f* $ *i* ∗ *g* $ (*n* − *i*))
⟨*proof*⟩

lemma *fps-mult-nth-outside-subdegrees*:
fixes *f g* :: (*'a* :: {*mult-zero*,*comm-monoid-add*}) *fps*
shows *n* < *subdegree f* ⟹ (*f* ∗ *g*) $ *n* = *0*
and *n* < *subdegree g* ⟹ (*f* ∗ *g*) $ *n* = *0*
⟨*proof*⟩

5.3 Ring structure

instance *fps* :: (*semigroup-add*) *semigroup-add*
⟨*proof*⟩

instance *fps* :: (*ab-semigroup-add*) *ab-semigroup-add*
⟨*proof*⟩

instance *fps* :: (*monoid-add*) *monoid-add*
⟨*proof*⟩

instance *fps* :: (*comm-monoid-add*) *comm-monoid-add*
⟨*proof*⟩

instance *fps* :: (*cancel-semigroup-add*) *cancel-semigroup-add*
⟨*proof*⟩

instance *fps* :: (*cancel-ab-semigroup-add*) *cancel-ab-semigroup-add*
⟨*proof*⟩

instance *fps* :: (*cancel-comm-monoid-add*) *cancel-comm-monoid-add* ⟨*proof*⟩

instance *fps* :: (*group-add*) *group-add*
⟨*proof*⟩

instance *fps* :: (*ab-group-add*) *ab-group-add*
⟨*proof*⟩

instance *fps* :: (*zero-neq-one*) *zero-neq-one*
⟨*proof*⟩

lemma *fps-mult-assoc-lemma*:
fixes k :: *nat*
and f :: $nat \Rightarrow nat \Rightarrow nat \Rightarrow$ *'a*::*comm-monoid-add*
shows $(\sum j{=}0..k.\ \sum i{=}0..j.\ f\ i\ (j - i)\ (n - j)) =$
$(\sum j{=}0..k.\ \sum i{=}0..k - j.\ f\ j\ i\ (n - j - i))$
⟨*proof*⟩

instance *fps* :: (*semiring-0*) *semiring-0*
⟨*proof*⟩

instance *fps* :: (*semiring-0-cancel*) *semiring-0-cancel* ⟨*proof*⟩

lemma *fps-mult-commute-lemma*:
fixes n :: *nat*
and f :: $nat \Rightarrow nat \Rightarrow$ *'a*::*comm-monoid-add*
shows $(\sum i{=}0..n.\ f\ i\ (n - i)) = (\sum i{=}0..n.\ f\ (n - i)\ i)$
⟨*proof*⟩

instance *fps* :: (*comm-semiring-0*) *comm-semiring-0*
⟨*proof*⟩

instance *fps* :: (*comm-semiring-0-cancel*) *comm-semiring-0-cancel* ⟨*proof*⟩

instance *fps* :: (*semiring-1*) *semiring-1*
⟨*proof*⟩

instance *fps* :: (*comm-semiring-1*) *comm-semiring-1*
⟨*proof*⟩

instance *fps* :: (*semiring-1-cancel*) *semiring-1-cancel* ⟨*proof*⟩

lemma *fps-square-nth*: $(f\hat{}2)\ \$\ n = (\sum k{\leq}n.\ f\ \$\ k * f\ \$\ (n - k))$
⟨*proof*⟩

lemma *fps-sum-nth*: $sum\ f\ S\ \$\ n = sum\ (\lambda k.\ (f\ k)\ \$\ n)\ S$
⟨*proof*⟩

definition $fps\text{-}const\ c = Abs\text{-}fps\ (\lambda n.\ if\ n = 0\ then\ c\ else\ 0)$

lemma *fps-nth-fps-const* [*simp*]: $fps\text{-}const\ c\ \$\ n = (if\ n = 0\ then\ c\ else\ 0)$
⟨*proof*⟩

lemma *fps-const-0-eq-0* [*simp*]: $fps\text{-}const\ 0 = 0$
⟨*proof*⟩

lemma *fps-const-nonzero-eq-nonzero*: $c \neq 0 \Longrightarrow fps\text{-}const\ c \neq 0$
⟨*proof*⟩

lemma *fps-const-eq-0-iff* [*simp*]: $fps\text{-}const\ c = 0 \longleftrightarrow c = 0$
⟨*proof*⟩

lemma *fps-const-1-eq-1* [*simp*]: $fps\text{-}const\ 1 = 1$
⟨*proof*⟩

lemma *fps-const-eq-1-iff* [*simp*]: $fps\text{-}const\ c = 1 \longleftrightarrow c = 1$
⟨*proof*⟩

lemma *subdegree-fps-const* [*simp*]: $subdegree\ (fps\text{-}const\ c) = 0$
⟨*proof*⟩

lemma *fps-const-neg* [*simp*]: $-\ (fps\text{-}const\ (c::'a::group\text{-}add)) = fps\text{-}const\ (-\ c)$
⟨*proof*⟩

lemma *fps-const-add* [*simp*]: $fps\text{-}const\ (c::'a::monoid\text{-}add) + fps\text{-}const\ d = fps\text{-}const\ (c + d)$
⟨*proof*⟩

lemma *fps-const-add-left*: $fps\text{-}const\ (c::'a::monoid\text{-}add) + f = Abs\text{-}fps\ (\lambda n.\ if\ n = 0\ then\ c + f\$0\ else\ f\$n)$
⟨*proof*⟩

lemma *fps-const-add-right*: $f + fps\text{-}const\ (c::'a::monoid\text{-}add) =$
$Abs\text{-}fps\ (\lambda n.\ if\ n = 0\ then\ f\$0 + c\ else\ f\$n)$
⟨*proof*⟩

lemma *fps-const-sub* [*simp*]: $fps\text{-}const\ (c::'a::group\text{-}add) - fps\text{-}const\ d = fps\text{-}const\ (c - d)$
⟨*proof*⟩

lemmas *fps-const-minus* = *fps-const-sub*

lemma *fps-const-mult*[*simp*]:
fixes $c\ d :: 'a::\{comm\text{-}monoid\text{-}add,mult\text{-}zero\}$
shows $fps\text{-}const\ c * fps\text{-}const\ d = fps\text{-}const\ (c * d)$
⟨*proof*⟩

lemma *fps-const-mult-left*:
$fps\text{-}const\ (c::'a::\{comm\text{-}monoid\text{-}add,mult\text{-}zero\}) * f = Abs\text{-}fps\ (\lambda n.\ c * f\$n)$
⟨*proof*⟩

lemma *fps-const-mult-right*:
$f * fps\text{-}const\ (c::'a::\{comm\text{-}monoid\text{-}add,mult\text{-}zero\}) = Abs\text{-}fps\ (\lambda n.\ f\$n * c)$
⟨*proof*⟩

lemma *fps-mult-left-const-nth* [*simp*]:
$(fps\text{-}const\ (c::'a::\{comm\text{-}monoid\text{-}add,mult\text{-}zero\}) * f)\$n = c* f\$n$
⟨*proof*⟩

lemma *fps-mult-right-const-nth* [*simp*]:
$(f * fps\text{-}const\ (c::'a::\{comm\text{-}monoid\text{-}add,mult\text{-}zero\}))\$n = f\$n * c$
⟨*proof*⟩

lemma *fps-const-power* [*simp*]: $fps\text{-}const\ c\ \hat{}\ n = fps\text{-}const\ (c\hat{}n)$
⟨*proof*⟩

instance *fps* :: (*ring*) *ring* ⟨*proof*⟩

instance *fps* :: (*comm-ring*) *comm-ring* ⟨*proof*⟩

instance *fps* :: (*ring-1*) *ring-1* ⟨*proof*⟩

instance *fps* :: (*comm-ring-1*) *comm-ring-1* ⟨*proof*⟩

instance *fps* :: (*semiring-no-zero-divisors*) *semiring-no-zero-divisors*
⟨*proof*⟩

instance *fps* :: (*semiring-1-no-zero-divisors*) *semiring-1-no-zero-divisors* ⟨*proof*⟩

instance *fps* :: ({*cancel-semigroup-add*,*semiring-no-zero-divisors-cancel*})
semiring-no-zero-divisors-cancel
⟨*proof*⟩

instance *fps* :: (*ring-no-zero-divisors*) *ring-no-zero-divisors* ⟨*proof*⟩

instance *fps* :: (*ring-1-no-zero-divisors*) *ring-1-no-zero-divisors* ⟨*proof*⟩

instance *fps* :: (*idom*) *idom* ⟨*proof*⟩

lemma *fps-of-nat*: *fps-const* (*of-nat c*) = *of-nat c*
⟨*proof*⟩

lemma *fps-of-int*: *fps-const* (*of-int c*) = *of-int c*
⟨*proof*⟩

lemma *semiring-char-fps* [*simp*]: *CHAR*('*a* :: *comm-semiring-1 fps*) = *CHAR*('*a*)
⟨*proof*⟩

instance *fps* :: ({*semiring-prime-char*,*comm-semiring-1*}) *semiring-prime-char*
⟨*proof*⟩
instance *fps* :: ({*comm-semiring-prime-char*,*comm-semiring-1*}) *comm-semiring-prime-char*
⟨*proof*⟩
instance *fps* :: ({*comm-ring-prime-char*,*comm-semiring-1*}) *comm-ring-prime-char*
⟨*proof*⟩
instance *fps* :: ({*idom-prime-char*,*comm-semiring-1*}) *idom-prime-char*
⟨*proof*⟩

lemma *fps-numeral-fps-const*: *numeral k* = *fps-const* (*numeral k*)
⟨*proof*⟩

lemmas *numeral-fps-const* = *fps-numeral-fps-const*

lemma *neg-numeral-fps-const*:
(− *numeral k* :: '*a* :: *ring-1 fps*) = *fps-const* (− *numeral k*)
⟨*proof*⟩

lemma *fps-numeral-nth*: *numeral n* $ *i* = (*if i* = *0 then numeral n else 0*)
⟨*proof*⟩

lemma *fps-numeral-nth-0* [*simp*]: *numeral n* $ *0* = *numeral n*
⟨*proof*⟩

lemma *subdegree-numeral* [*simp*]: *subdegree* (*numeral n*) = *0*
⟨*proof*⟩

lemma *fps-nth-of-nat* [*simp*]:
(*of-nat c*) $ *n* = (*if n=0 then of-nat c else 0*)
⟨*proof*⟩

lemma *fps-nth-of-int* [*simp*]:
(*of-int c*) $ *n* = (*if n=0 then of-int c else 0*)
⟨*proof*⟩

lemma *fps-mult-of-nat-nth* [*simp*]:
shows (*of-nat k* ∗ *f*) $ *n* = *of-nat k* ∗ *f*$*n*
and (*f* ∗ *of-nat k*) $ *n* = *f*$*n* ∗ *of-nat k*
⟨*proof*⟩

lemma *fps-mult-of-int-nth* [*simp*]:
shows (*of-int k* ∗ *f*) $ *n* = *of-int k* ∗ *f*$*n*
and (*f* ∗ *of-int k*) $ *n* = *f*$*n* ∗ *of-int k*
⟨*proof*⟩

lemma *numeral-neq-fps-zero* [*simp*]: (*numeral f* :: *'a* :: *field-char-0 fps*) ≠ *0*
⟨*proof*⟩

instance *fps* :: (*semiring-char-0*) *semiring-char-0*
⟨*proof*⟩

lemma *subdegree-power-ge*:
f^*n* ≠ *0* ⟹ *subdegree* (*f*^*n*) ≥ *n* ∗ *subdegree f*
⟨*proof*⟩

lemma *fps-pow-nth-below-subdegree*:
k < *n* ∗ *subdegree f* ⟹ (*f*^*n*) $ *k* = *0*
⟨*proof*⟩

lemma *fps-pow-base* [*simp*]:
(*f* ^ *n*) $ (*n* ∗ *subdegree f*) = (*f* $ *subdegree f*) ^ *n*
⟨*proof*⟩

lemma *subdegree-power-eqI*:
fixes *f* :: *'a*::*semiring-1 fps*
shows (*f* $ *subdegree f*) ^ *n* ≠ *0* ⟹ *subdegree* (*f* ^ *n*) = *n* ∗ *subdegree f*
⟨*proof*⟩

lemma *subdegree-power* [*simp*]:
subdegree ((*f* :: (*'a* :: *semiring-1-no-zero-divisors*) *fps*) ^ *n*) = *n* ∗ *subdegree f*
⟨*proof*⟩

lemma *subdegree-prod*:
fixes *f* :: *'a* ⇒ *'b* :: *idom fps*
assumes ⋀*x*. *x* ∈ *A* ⟹ *f x* ≠ *0*
shows *subdegree* (∏ *x*∈*A*. *f x*) = (∑ *x*∈*A*. *subdegree* (*f x*))
⟨*proof*⟩

lemma *minus-one-power-iff*: $(-(1::'a::ring\text{-}1)) \hat{} n = (\textit{if even } n \textit{ then } 1 \textit{ else } -1)$
⟨*proof*⟩

definition *fps-X* = *Abs-fps* $(\lambda n.\ \textit{if } n = 1 \textit{ then } 1 \textit{ else } 0)$

lemma *subdegree-fps-X* [*simp*]: *subdegree* (*fps-X* :: ('*a* :: *zero-neq-one*) *fps*) = *1*
⟨*proof*⟩

lemma *fps-X-mult-nth* [*simp*]:
fixes *f* :: '*a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
shows (*fps-X* ∗ *f*) $ *n* = (*if n* = *0 then 0 else f* $ (*n* − *1*))
⟨*proof*⟩

lemma *fps-X-mult-right-nth* [*simp*]:
fixes *a* :: '*a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
shows (*a* ∗ *fps-X*) $ *n* = (*if n* = *0 then 0 else a* $ (*n* − *1*))
⟨*proof*⟩

lemma *fps-mult-fps-X-commute*:
fixes *a* :: '*a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
shows *fps-X* ∗ *a* = *a* ∗ *fps-X*
⟨*proof*⟩

lemma *fps-mult-fps-X-power-commute*: *fps-X* ^ *k* ∗ *a* = *a* ∗ *fps-X* ^ *k*
⟨*proof*⟩

lemma *fps-subdegree-mult-fps-X*:
fixes *f* :: '*a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
assumes *f* ≠ *0*
shows *subdegree* (*fps-X* ∗ *f*) = *subdegree f* + *1*
and *subdegree* (*f* ∗ *fps-X*) = *subdegree f* + *1*
⟨*proof*⟩

lemma *fps-mult-fps-X-nonzero*:
fixes *f* :: '*a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
assumes *f* ≠ *0*
shows *fps-X* ∗ *f* ≠ *0*
and *f* ∗ *fps-X* ≠ *0*
⟨*proof*⟩

lemma *fps-mult-fps-X-power-nonzero*:
assumes *f* ≠ *0*
shows *fps-X* ^ *n* ∗ *f* ≠ *0*
and *f* ∗ *fps-X* ^ *n* ≠ *0*
⟨*proof*⟩

lemma *fps-X-power-iff*: *fps-X* ^ *n* = *Abs-fps* $(\lambda m.\ \textit{if } m = n \textit{ then } 1 \textit{ else } 0)$
⟨*proof*⟩

lemma *fps-X-nth*[*simp*]: *fps-X*$*n* = (*if n* = *1 then 1 else 0*)
⟨*proof*⟩

lemma *fps-X-power-nth*[*simp*]: (*fps-X*^*k*) $*n* = (*if n* = *k then 1 else 0*)
⟨*proof*⟩

lemma *fps-X-power-subdegree*: *subdegree* (*fps-X*^*n*) = *n*
⟨*proof*⟩

lemma *fps-X-power-mult-nth*:
(*fps-X*^*k* ∗ *f*) $ *n* = (*if n* < *k then 0 else f* $ (*n* − *k*))
⟨*proof*⟩

lemma *fps-X-power-mult-right-nth*:
(*f* ∗ *fps-X*^*k*) $ *n* = (*if n* < *k then 0 else f* $ (*n* − *k*))
⟨*proof*⟩

lemma *fps-subdegree-mult-fps-X-power*:
assumes *f* ≠ *0*
shows *subdegree* (*fps-X* ^ *n* ∗ *f*) = *subdegree f* + *n*
and *subdegree* (*f* ∗ *fps-X* ^ *n*) = *subdegree f* + *n*
⟨*proof*⟩

lemma *fps-mult-fps-X-plus-1-nth*:
((*1*+*fps-X*)∗*a*) $*n* = (*if n* = *0 then* (*a*$*n* :: *'a*::*semiring-1*) *else a*$*n* + *a*$(*n* − *1*))
⟨*proof*⟩

lemma *fps-mult-right-fps-X-plus-1-nth*:
fixes *a* :: *'a* :: *semiring-1 fps*
shows (*a*∗(*1*+*fps-X*)) $ *n* = (*if n* = *0 then a*$*n else a*$*n* + *a*$(*n* − *1*))
⟨*proof*⟩

lemma *fps-X-neq-fps-const* [*simp*]: (*fps-X* :: *'a* :: *zero-neq-one fps*) ≠ *fps-const c*
⟨*proof*⟩

lemma *fps-X-neq-zero* [*simp*]: (*fps-X* :: *'a* :: *zero-neq-one fps*) ≠ *0*
⟨*proof*⟩

lemma *fps-X-neq-one* [*simp*]: (*fps-X* :: *'a* :: *zero-neq-one fps*) ≠ *1*
⟨*proof*⟩

lemma *fps-X-neq-numeral* [*simp*]: *fps-X* ≠ *numeral c*
⟨*proof*⟩

lemma *fps-X-pow-eq-fps-X-pow-iff* [*simp*]: *fps-X* ^ *m* = *fps-X* ^ *n* ⟷ *m* = *n*
⟨*proof*⟩

5.4 Shifting and slicing

definition *fps-shift* :: *nat* ⇒ *'a fps* ⇒ *'a fps* **where**
fps-shift n f = *Abs-fps* (λ*i*. *f* $ (*i* + *n*))

lemma *fps-shift-nth* [*simp*]: *fps-shift n f* $ *i* = *f* $ (*i* + *n*)
⟨*proof*⟩

lemma *fps-shift-0* [*simp*]: *fps-shift 0 f* = *f*
⟨*proof*⟩

lemma *fps-shift-zero* [*simp*]: *fps-shift n 0* = *0*
⟨*proof*⟩

lemma *fps-shift-one*: *fps-shift n 1* = (*if n* = *0 then 1 else 0*)
⟨*proof*⟩

lemma *fps-shift-fps-const*: *fps-shift n* (*fps-const c*) = (*if n* = *0 then fps-const c else 0*)
⟨*proof*⟩

lemma *fps-shift-numeral*: *fps-shift n* (*numeral c*) = (*if n* = *0 then numeral c else 0*)
⟨*proof*⟩

lemma *fps-shift-fps-X* [*simp*]:
n ≥ *1* ⟹ *fps-shift n fps-X* = (*if n* = *1 then 1 else 0*)
⟨*proof*⟩

lemma *fps-shift-fps-X-power* [*simp*]:
n ≤ *m* ⟹ *fps-shift n* (*fps-X* ^ *m*) = *fps-X* ^ (*m* − *n*)
⟨*proof*⟩

lemma *fps-shift-subdegree* [*simp*]:
n ≤ *subdegree f* ⟹ *subdegree* (*fps-shift n f*) = *subdegree f* − *n*
⟨*proof*⟩

lemma *fps-shift-fps-shift*:
fps-shift (*m* + *n*) *f* = *fps-shift m* (*fps-shift n f*)
⟨*proof*⟩

lemma *fps-shift-fps-shift-reorder*:
fps-shift m (*fps-shift n f*) = *fps-shift n* (*fps-shift m f*)
⟨*proof*⟩

lemma *fps-shift-rev-shift*:
m ≤ *n* ⟹ *fps-shift n* (*Abs-fps* (λ*k*. *if k*<*m then 0 else f* $ (*k*−*m*))) = *fps-shift* (*n*−*m*) *f*
m > *n* ⟹ *fps-shift n* (*Abs-fps* (λ*k*. *if k*<*m then 0 else f* $ (*k*−*m*))) =
Abs-fps (λ*k*. *if k*<*m*−*n then 0 else f* $ (*k*−(*m*−*n*)))

⟨*proof*⟩

lemma *fps-shift-add*:
fps-shift n (*f* + *g*) = *fps-shift n f* + *fps-shift n g*
⟨*proof*⟩

lemma *fps-shift-diff*:
fps-shift n (*f* − *g*) = *fps-shift n f* − *fps-shift n g*
⟨*proof*⟩

lemma *fps-shift-uminus*:
fps-shift n (−*f*) = − *fps-shift n f*
⟨*proof*⟩

lemma *fps-shift-mult*:
assumes *n* ≤ *subdegree* (*g* :: *'b* :: {*comm-monoid-add*, *mult-zero*} *fps*)
shows *fps-shift n* (*h*∗*g*) = *h* ∗ *fps-shift n g*
⟨*proof*⟩

lemma *fps-shift-mult-right-noncomm*:
assumes *n* ≤ *subdegree* (*g* :: *'b* :: {*comm-monoid-add*, *mult-zero*} *fps*)
shows *fps-shift n* (*g*∗*h*) = *fps-shift n g* ∗ *h*
⟨*proof*⟩

lemma *fps-shift-mult-right*:
assumes *n* ≤ *subdegree* (*g* :: *'b* :: *comm-semiring-0 fps*)
shows *fps-shift n* (*g*∗*h*) = *h* ∗ *fps-shift n g*
⟨*proof*⟩

lemma *fps-shift-mult-both*:
fixes *f g* :: *'a*::{*comm-monoid-add*, *mult-zero*} *fps*
assumes *m* ≤ *subdegree f n* ≤ *subdegree g*
shows *fps-shift m f* ∗ *fps-shift n g* = *fps-shift* (*m*+*n*) (*f*∗*g*)
⟨*proof*⟩

lemma *fps-shift-subdegree-zero-iff* [*simp*]:
fps-shift (*subdegree f*) *f* = *0* ⟷ *f* = *0*
⟨*proof*⟩

lemma *fps-shift-times-fps-X*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
shows *1* ≤ *subdegree f* ⟹ *fps-shift 1 f* ∗ *fps-X* = *f*
⟨*proof*⟩

lemma *fps-shift-times-fps-X'* [*simp*]:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
shows *fps-shift 1* (*f* ∗ *fps-X*) = *f*
⟨*proof*⟩

lemma *fps-shift-times-fps-X''*:
fixes $f :: 'a::\{comm\text{-}monoid\text{-}add,mult\text{-}zero,monoid\text{-}mult\}$ *fps*
shows $1 \leq n \Longrightarrow$ *fps-shift* n $(f * fps\text{-}X) =$ *fps-shift* $(n - 1)$ f
⟨*proof*⟩

lemma *fps-shift-times-fps-X-power*:
$n \leq$ *subdegree* $f \Longrightarrow$ *fps-shift* n $f *$ *fps-X* $\hat{}$ $n = f$
⟨*proof*⟩

lemma *fps-shift-times-fps-X-power'* [*simp*]:
fps-shift n $(f *$ *fps-X*$\hat{}n) = f$
⟨*proof*⟩

lemma *fps-shift-times-fps-X-power''*:
$m \leq n \Longrightarrow$ *fps-shift* n $(f *$ *fps-X*$\hat{}m) =$ *fps-shift* $(n - m)$ f
⟨*proof*⟩

lemma *fps-shift-times-fps-X-power'''*:
$m > n \Longrightarrow$ *fps-shift* n $(f *$ *fps-X*$\hat{}m) = f *$ *fps-X*$\hat{}(m - n)$
⟨*proof*⟩

lemma *subdegree-decompose*:
$f =$ *fps-shift* (*subdegree* f) $f *$ *fps-X* $\hat{}$ *subdegree* f
⟨*proof*⟩

lemma *subdegree-decompose'*:
$n \leq$ *subdegree* $f \Longrightarrow f =$ *fps-shift* n $f *$ *fps-X*$\hat{}n$
⟨*proof*⟩

instantiation *fps* :: (*zero*) *unit-factor*
begin
definition *fps-unit-factor-def* [*simp*]:
unit-factor $f =$ *fps-shift* (*subdegree* f) f
instance ⟨*proof*⟩
end

lemma *fps-unit-factor-zero-iff*: *unit-factor* $(f::'a::zero\ fps) = 0 \longleftrightarrow f = 0$
⟨*proof*⟩

lemma *fps-unit-factor-nth-0*: $f \neq 0 \Longrightarrow$ *unit-factor* f \$ $0 \neq 0$
⟨*proof*⟩

lemma *fps-X-unit-factor*: *unit-factor* (*fps-X* :: $'a$ :: *zero-neq-one fps*) $= 1$
⟨*proof*⟩

lemma *fps-X-power-unit-factor*: *unit-factor* (*fps-X* $\hat{}$ n) $= 1$
⟨*proof*⟩

lemma *fps-unit-factor-decompose*:

$f = \textit{unit-factor}\ f * \textit{fps-X} \mathbin{\hat{}} \textit{subdegree}\ f$
⟨*proof*⟩

lemma *fps-unit-factor-decompose'*:
$f = \textit{fps-X} \mathbin{\hat{}} \textit{subdegree}\ f * \textit{unit-factor}\ f$
⟨*proof*⟩

lemma *fps-unit-factor-uminus*:
$\textit{unit-factor}\ (-f) = -\ \textit{unit-factor}\ (f::'a::\textit{group-add fps})$
⟨*proof*⟩

lemma *fps-unit-factor-shift*:
assumes $n \leq \textit{subdegree}\ f$
shows $\textit{unit-factor}\ (\textit{fps-shift}\ n\ f) = \textit{unit-factor}\ f$
⟨*proof*⟩

lemma *fps-unit-factor-mult-fps-X*:
fixes $f :: 'a::\{\textit{comm-monoid-add},\textit{monoid-mult},\textit{mult-zero}\}\ \textit{fps}$
shows $\textit{unit-factor}\ (\textit{fps-X} * f) = \textit{unit-factor}\ f$
and $\textit{unit-factor}\ (f * \textit{fps-X}) = \textit{unit-factor}\ f$
⟨*proof*⟩

lemma *fps-unit-factor-mult-fps-X-power*:
shows $\textit{unit-factor}\ (\textit{fps-X} \mathbin{\hat{}} n * f) = \textit{unit-factor}\ f$
and $\textit{unit-factor}\ (f * \textit{fps-X} \mathbin{\hat{}} n) = \textit{unit-factor}\ f$
⟨*proof*⟩

lemma *fps-unit-factor-mult-unit-factor*:
fixes $f\ g :: 'a::\{\textit{comm-monoid-add},\textit{mult-zero}\}\ \textit{fps}$
shows $\textit{unit-factor}\ (f * \textit{unit-factor}\ g) = \textit{unit-factor}\ (f * g)$
and $\textit{unit-factor}\ (\textit{unit-factor}\ f * g) = \textit{unit-factor}\ (f * g)$
⟨*proof*⟩

lemma *fps-unit-factor-mult-both-unit-factor*:
fixes $f\ g :: 'a::\{\textit{comm-monoid-add},\textit{mult-zero}\}\ \textit{fps}$
shows $\textit{unit-factor}\ (\textit{unit-factor}\ f * \textit{unit-factor}\ g) = \textit{unit-factor}\ (f * g)$
⟨*proof*⟩

lemma *fps-unit-factor-mult'*:
fixes $f\ g :: 'a::\{\textit{comm-monoid-add},\textit{mult-zero}\}\ \textit{fps}$
assumes $f\ \$\ \textit{subdegree}\ f * g\ \$\ \textit{subdegree}\ g \neq 0$
shows $\textit{unit-factor}\ (f * g) = \textit{unit-factor}\ f * \textit{unit-factor}\ g$
⟨*proof*⟩

lemma *fps-unit-factor-mult*:
fixes $f\ g :: 'a::\textit{semiring-no-zero-divisors fps}$
shows $\textit{unit-factor}\ (f * g) = \textit{unit-factor}\ f * \textit{unit-factor}\ g$
⟨*proof*⟩

definition *fps-cutoff n f = Abs-fps* (λi. *if* $i < n$ *then* $f\$i$ *else 0*)

lemma *fps-cutoff-nth* [*simp*]: *fps-cutoff n f $ i* = (*if* $i < n$ *then* $f\$i$ *else 0*)
⟨*proof*⟩

lemma *fps-cutoff-zero-iff*: *fps-cutoff n f* = *0* $\longleftrightarrow$ ($f = 0 \vee n \leq$ *subdegree f*)
⟨*proof*⟩

lemma *fps-cutoff-0* [*simp*]: *fps-cutoff 0 f* = *0*
⟨*proof*⟩

lemma *fps-cutoff-zero* [*simp*]: *fps-cutoff n 0* = *0*
⟨*proof*⟩

lemma *fps-cutoff-one*: *fps-cutoff n 1* = (*if n* = *0 then 0 else 1*)
⟨*proof*⟩

lemma *fps-cutoff-fps-const*: *fps-cutoff n* (*fps-const c*) = (*if n* = *0 then 0 else fps-const c*)
⟨*proof*⟩

lemma *fps-cutoff-numeral*: *fps-cutoff n* (*numeral c*) = (*if n = 0 then 0 else numeral c*)
⟨*proof*⟩

lemma *fps-shift-cutoff*:
*fps-shift n f * fps-X^n + fps-cutoff n f = f*
⟨*proof*⟩

lemma *fps-shift-cutoff'*:
*fps-X^n * fps-shift n f + fps-cutoff n f = f*
⟨*proof*⟩

lemma *fps-cutoff-left-mult-nth*:
$k < n \Longrightarrow$ (*fps-cutoff n f * g*) $ *k* = (*f * g*) $ *k*
⟨*proof*⟩

lemma *fps-cutoff-add*: *fps-cutoff n* (*f + g* :: *'a* :: *monoid-add fps*) = *fps-cutoff n f + fps-cutoff n g*
⟨*proof*⟩

lemma *fps-cutoff-diff*: *fps-cutoff n* (*f − g* :: *'a* :: *group-add fps*) = *fps-cutoff n f − fps-cutoff n g*
⟨*proof*⟩

lemma *fps-cutoff-uminus*: *fps-cutoff n* (*−f* :: *'a* :: *group-add fps*) = *−fps-cutoff n f*
⟨*proof*⟩

lemma *fps-cutoff-right-mult-nth*:
assumes $k < n$
shows $(f * fps\text{-}cutoff\ n\ g)\ \$\ k = (f * g)\ \$\ k$
⟨*proof*⟩

lemma *fps-cutoff-eq-fps-cutoff-iff*:
$fps\text{-}cutoff\ n\ f = fps\text{-}cutoff\ n\ g \longleftrightarrow (\forall k{<}n.\ fps\text{-}nth\ f\ k = fps\text{-}nth\ g\ k)$
⟨*proof*⟩

lemma *fps-conv-fps-X-power-mult-fps-shift*:
assumes $f = 0 \vee subdegree\ f \geq n$
shows $f = fps\text{-}X\ \hat{}\ n * fps\text{-}shift\ n\ f$
⟨*proof*⟩

5.5 Metrizability

instantiation *fps* :: ({*minus*,*zero*}) *dist*
begin

definition
dist-fps-def: $dist\ (a :: 'a\ fps)\ b = (if\ a = b\ then\ 0\ else\ inverse\ (2\ \hat{}\ subdegree\ (a - b)))$

lemma *dist-fps-ge0*: $dist\ (a :: 'a\ fps)\ b \geq 0$
⟨*proof*⟩

instance ⟨*proof*⟩

end

instantiation *fps* :: (*group-add*) *metric-space*
begin

definition *uniformity-fps-def* [*code del*]:
$(uniformity :: ('a\ fps \times 'a\ fps)\ filter) = (INF\ e{\in}\{0 <..\}.\ principal\ \{(x,\ y).\ dist\ x\ y < e\})$

definition *open-fps-def'* [*code del*]:
$open\ (U :: 'a\ fps\ set) \longleftrightarrow (\forall x{\in}U.\ eventually\ (\lambda(x',\ y).\ x' = x \longrightarrow y \in U)\ uniformity)$

lemma *dist-fps-sym*: $dist\ (a :: 'a\ fps)\ b = dist\ b\ a$
⟨*proof*⟩

instance
⟨*proof*⟩

end

declare *uniformity-Abort*[**where** *'a='a :: group-add fps, code*]

lemma *open-fps-def*: *open* (*S* :: *'a::group-add fps set*) = (∀ *a* ∈ *S*. ∃ *r*. *r* >*0* ∧ {*y*. *dist y a* < *r*} ⊆ *S*)
⟨*proof*⟩

Topology

5.6 The topology of formal power series

A set of formal power series is open iff for any power series f in it, there exists some number n such that all power series that agree with f on the first n components are also in it.

lemma *open-fps-iff*:
open A ⟷ (∀ *F*∈*A*. ∃ *n*. {*G*. *fps-cutoff n G* = *fps-cutoff n F*} ⊆ *A*)
⟨*proof*⟩

lemma *open-fps-cutoff*: *open* {*H*. *fps-cutoff N H* = *fps-cutoff N G*}
⟨*proof*⟩

lemma *eventually-fps-nth-eq-nhds-fps-strong*:
eventually (λ*g*. ∀ *k*≤*n*. *fps-nth g k* = *fps-nth f k*) (*nhds f*)
⟨*proof*⟩

lemma *eventually-fps-nth-eq-nhds-fps*: *eventually* (λ*g*. *fps-nth g k* = *fps-nth f k*) (*nhds f*)
⟨*proof*⟩

A family of formal power series f_x tends to a limit series g at some filter F iff for any $N \geq 0$, the set of x for which f_x and G agree on the first N coefficients is in F.

For a sequence $(f_i)_{n\geq 0}$ this means that $f_i \longrightarrow G$ iff for any $N \geq 0$, f_x and G agree for all but finitely many x.

lemma *tendsto-fps-iff*:
filterlim f (*nhds* (*g* :: *'a* :: *group-add fps*)) *F* ⟷
(∀ *n*. *eventually* (λ*x*. *fps-nth* (*f x*) *n* = *fps-nth g n*) *F*)
⟨*proof*⟩

lemma *tendsto-fpsI*:
assumes ⋀*n*. *eventually* (λ*x*. *fps-nth* (*f x*) *n* = *fps-nth G n*) *F*
shows *filterlim f* (*nhds* (*G* :: *'a* :: *group-add fps*)) *F*
⟨*proof*⟩

The infinite sums and justification of the notation in textbooks.

lemma *reals-power-lt-ex*:
fixes *x y* :: *real*
assumes *xp*: *x* > *0*

and *y1*: $y > 1$
shows $\exists k{>}0.\ (1/y)\hat{}k < x$
⟨*proof*⟩

lemma *fps-sum-rep-nth*: $(sum\ (\lambda i.\ fps\text{-}const(a\$i){*}fps\text{-}X\hat{}i)\ \{0..m\})\$n = (if\ n \leq m\ then\ a\$n\ else\ 0)$
⟨*proof*⟩

lemma *fps-notation*: $(\lambda n.\ sum\ (\lambda i.\ fps\text{-}const(a\$i) * fps\text{-}X\hat{}i)\ \{0..n\}) \longrightarrow a$
(**is** $?s \longrightarrow a$)
⟨*proof*⟩

5.7 Division

declare *sum.cong*[*fundef-cong*]

fun *fps-left-inverse-constructor* ::
$'a$::{*comm-monoid-add*,*times*,*uminus*} *fps* ⇒ $'a$ ⇒ *nat* ⇒ $'a$
where
fps-left-inverse-constructor f a 0 = *a*
| *fps-left-inverse-constructor f a* (*Suc n*) =
− *sum* (λ*i*. *fps-left-inverse-constructor f a i* ∗ *f*$(*Suc n* − *i*)) {*0..n*} ∗ *a*

— This will construct a left inverse for f in case that $x * f\ \$\ 0 = 1$
abbreviation *fps-left-inverse* ≡ (λ*f x*. *Abs-fps* (*fps-left-inverse-constructor f x*))

fun *fps-right-inverse-constructor* ::
$'a$::{*comm-monoid-add*,*times*,*uminus*} *fps* ⇒ $'a$ ⇒ *nat* ⇒ $'a$
where
fps-right-inverse-constructor f a 0 = *a*
| *fps-right-inverse-constructor f a n* =
− *a* ∗ *sum* (λ*i*. *f*$*i* ∗ *fps-right-inverse-constructor f a* (*n* − *i*)) {*1..n*}

— This will construct a right inverse for f in case that $f\ \$\ 0 * y = 1$
abbreviation *fps-right-inverse* ≡ (λ*f y*. *Abs-fps* (*fps-right-inverse-constructor f y*))

instantiation *fps* :: ({*comm-monoid-add*,*inverse*,*times*,*uminus*}) *inverse*
begin

— For backwards compatibility.
abbreviation *natfun-inverse*:: $'a$ *fps* ⇒ *nat* ⇒ $'a$
where *natfun-inverse f* ≡ *fps-right-inverse-constructor f* (*inverse* (*f*$*0*))

definition *fps-inverse-def*: *inverse f* = *Abs-fps* (*natfun-inverse f*)
— With scalars from a (possibly non-commutative) ring, this defines a right inverse. Furthermore, if scalars are of class *mult-zero* and satisfy condition *inverse 0* = *0*, then this will evaluate to zero when the zeroth term is zero.

definition *fps-divide-def*: *f div g* = *fps-shift* (*subdegree g*) (*f* ∗ *inverse* (*unit-factor*

g))
— If scalars are of class *mult-zero* and satisfy condition *inverse 0 = 0*, then div by zero will equal zero.

instance ⟨*proof*⟩

end

lemma *fps-lr-inverse-0-iff*:
(*fps-left-inverse f x*) $ *0* = *0* ⟷ *x* = *0*
(*fps-right-inverse f x*) $ *0* = *0* ⟷ *x* = *0*
⟨*proof*⟩

lemma *fps-inverse-0-iff'*: (*inverse f*) $ *0* = *0* ⟷ *inverse* (*f* $ *0*) = *0*
⟨*proof*⟩

lemma *fps-inverse-0-iff*[*simp*]: (*inverse f*) $ *0* = (*0*::*'a*::*division-ring*) ⟷ *f* $ *0* = *0*
⟨*proof*⟩

lemma *fps-lr-inverse-nth-0*:
(*fps-left-inverse f x*) $ *0* = *x* (*fps-right-inverse f x*) $ *0* = *x*
⟨*proof*⟩

lemma *fps-inverse-nth-0* [*simp*]: (*inverse f*) $ *0* = *inverse* (*f* $ *0*)
⟨*proof*⟩

lemma *fps-lr-inverse-starting0*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*} *fps*
and *g* :: *'b*::{*ab-group-add*,*mult-zero*} *fps*
shows *fps-left-inverse f 0* = *0*
and *fps-right-inverse g 0* = *0*
⟨*proof*⟩

lemma *fps-lr-inverse-eq0-imp-starting0*:
fps-left-inverse f x = *0* ⟹ *x* = *0*
fps-right-inverse f x = *0* ⟹ *x* = *0*
⟨*proof*⟩

lemma *fps-lr-inverse-eq-0-iff*:
fixes *x* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*}
and *y* :: *'b*::{*ab-group-add*,*mult-zero*}
shows *fps-left-inverse f x* = *0* ⟷ *x* = *0*
and *fps-right-inverse g y* = *0* ⟷ *y* = *0*
⟨*proof*⟩

lemma *fps-inverse-eq-0-iff'*:
fixes *f* :: *'a*::{*ab-group-add*,*inverse*,*mult-zero*} *fps*
shows *inverse f* = *0* ⟷ *inverse* (*f* $ *0*) = *0*

⟨*proof*⟩

lemma *fps-inverse-eq-0-iff*[*simp*]: *inverse f* = (*0*:: ('*a*::*division-ring*) *fps*) ⟷ *f* $ *0* = *0*
⟨*proof*⟩

lemmas *fps-inverse-eq-0'* = *iffD2*[*OF fps-inverse-eq-0-iff'*]
lemmas *fps-inverse-eq-0* = *iffD2*[*OF fps-inverse-eq-0-iff*]

lemma *fps-const-lr-inverse*:
fixes *a* :: '*a*::{*ab-group-add*,*mult-zero*}
and *b* :: '*b*::{*comm-monoid-add*,*mult-zero*,*uminus*}
shows *fps-left-inverse* (*fps-const a*) *x* = *fps-const x*
and *fps-right-inverse* (*fps-const b*) *y* = *fps-const y*
⟨*proof*⟩

lemma *fps-const-inverse*:
fixes *a* :: '*a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*}
shows *inverse* (*fps-const a*) = *fps-const* (*inverse a*)
⟨*proof*⟩

lemma *fps-lr-inverse-zero*:
fixes *x* :: '*a*::{*ab-group-add*,*mult-zero*}
and *y* :: '*b*::{*comm-monoid-add*,*mult-zero*,*uminus*}
shows *fps-left-inverse 0 x* = *fps-const x*
and *fps-right-inverse 0 y* = *fps-const y*
⟨*proof*⟩

lemma *fps-inverse-zero-conv-fps-const*:
inverse (*0*::'*a*::{*comm-monoid-add*,*mult-zero*,*uminus*,*inverse*} *fps*) = *fps-const* (*inverse 0*)
⟨*proof*⟩

lemma *fps-inverse-zero'*:
assumes *inverse* (*0*::'*a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*}) = *0*
shows *inverse* (*0*::'*a fps*) = *0*
⟨*proof*⟩

lemma *fps-inverse-zero* [*simp*]:
inverse (*0*::'*a*::*division-ring fps*) = *0*
⟨*proof*⟩

lemma *fps-lr-inverse-one*:
fixes *x* :: '*a*::{*ab-group-add*,*mult-zero*,*one*}
and *y* :: '*b*::{*comm-monoid-add*,*mult-zero*,*uminus*,*one*}
shows *fps-left-inverse 1 x* = *fps-const x*
and *fps-right-inverse 1 y* = *fps-const y*
⟨*proof*⟩

lemma *fps-lr-inverse-one-one*:
fps-left-inverse 1 1 = (*1*::*'a*::{*ab-group-add*,*mult-zero*,*one*} *fps*)
fps-right-inverse 1 1 = (*1*::*'b*::{*comm-monoid-add*,*mult-zero*,*uminus*,*one*} *fps*)
⟨*proof*⟩

lemma *fps-inverse-one'*:
assumes *inverse* (*1*::*'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*one*}) = *1*
shows *inverse* (*1* :: *'a fps*) = *1*
⟨*proof*⟩

lemma *fps-inverse-one* [*simp*]: *inverse* (*1* :: *'a* :: *division-ring fps*) = *1*
⟨*proof*⟩

lemma *fps-lr-inverse-minus*:
fixes *f* :: *'a*::*ring-1 fps*
shows *fps-left-inverse* (−*f*) (−*x*) = − *fps-left-inverse f x*
and *fps-right-inverse* (−*f*) (−*x*) = − *fps-right-inverse f x*
⟨*proof*⟩

lemma *fps-inverse-minus* [*simp*]: *inverse* (−*f*) = −*inverse* (*f* :: *'a* :: *division-ring fps*)
⟨*proof*⟩

lemma *fps-left-inverse*:
fixes *f* :: *'a*::*ring-1 fps*
assumes *f0*: *x* ∗ *f*$*0* = *1*
shows *fps-left-inverse f x* ∗ *f* = *1*
⟨*proof*⟩

lemma *fps-right-inverse*:
fixes *f* :: *'a*::*ring-1 fps*
assumes *f0*: *f*$*0* ∗ *y* = *1*
shows *f* ∗ *fps-right-inverse f y* = *1*
⟨*proof*⟩

It is possible in a ring for an element to have a left inverse but not a right inverse, or vice versa. But when an element has both, they must be the same.

lemma *fps-left-inverse-eq-fps-right-inverse*:
fixes *f* :: *'a*::*ring-1 fps*
assumes *f0*: *x* ∗ *f*$*0* = *1 f* $ *0* ∗ *y* = *1*
— These assumptions imply that x equals y, but no need to assume that.
shows *fps-left-inverse f x* = *fps-right-inverse f y*
⟨*proof*⟩

lemma *fps-left-inverse-eq-fps-right-inverse-comm*:
fixes *f* :: *'a*::*comm-ring-1 fps*
assumes *f0*: *x* ∗ *f*$*0* = *1*
shows *fps-left-inverse f x* = *fps-right-inverse f x*

⟨*proof*⟩

lemma *fps-left-inverse'*:
fixes $f :: 'a::ring\text{-}1\ fps$
assumes $x * f\$0 = 1\ f\$0 * y = 1$
— These assumptions imply x equals y, but no need to assume that.
shows $fps\text{-}right\text{-}inverse\ f\ y * f = 1$
⟨*proof*⟩

lemma *fps-right-inverse'*:
fixes $f :: 'a::ring\text{-}1\ fps$
assumes $x * f\$0 = 1\ f\$0 * y = 1$
— These assumptions imply x equals y, but no need to assume that.
shows $f * fps\text{-}left\text{-}inverse\ f\ x = 1$
⟨*proof*⟩

lemma *inverse-mult-eq-1* [*intro*]:
assumes $f\$0 \neq (0::'a::division\text{-}ring)$
shows $inverse\ f * f = 1$
⟨*proof*⟩

lemma *inverse-mult-eq-1'*:
assumes $f\$0 \neq (0::'a::division\text{-}ring)$
shows $f * inverse\ f = 1$
⟨*proof*⟩

lemma *fps-mult-left-inverse-unit-factor*:
fixes $f :: 'a::ring\text{-}1\ fps$
assumes $x * f\ \$\ subdegree\ f = 1$
shows $fps\text{-}left\text{-}inverse\ (unit\text{-}factor\ f)\ x * f = fps\text{-}X\ \hat{}\ subdegree\ f$
⟨*proof*⟩

lemma *fps-mult-right-inverse-unit-factor*:
fixes $f :: 'a::ring\text{-}1\ fps$
assumes $f\ \$\ subdegree\ f * y = 1$
shows $f * fps\text{-}right\text{-}inverse\ (unit\text{-}factor\ f)\ y = fps\text{-}X\ \hat{}\ subdegree\ f$
⟨*proof*⟩

lemma *fps-mult-right-inverse-unit-factor-divring*:
$(f :: 'a::division\text{-}ring\ fps) \neq 0 \Longrightarrow f * inverse\ (unit\text{-}factor\ f) = fps\text{-}X\ \hat{}\ subdegree\ f$
⟨*proof*⟩

lemma *fps-left-inverse-idempotent-ring1*:
fixes $f :: 'a::ring\text{-}1\ fps$
assumes $x * f\$0 = 1\ y * x = 1$
— These assumptions imply y equals $f\$0$, but no need to assume that.
shows $fps\text{-}left\text{-}inverse\ (fps\text{-}left\text{-}inverse\ f\ x)\ y = f$
⟨*proof*⟩

lemma *fps-left-inverse-idempotent-comm-ring1*:
fixes $f :: 'a$::*comm-ring-1 fps*
assumes $x * f\$0 = 1$
shows *fps-left-inverse* (*fps-left-inverse f x*) ($f\$0$) $= f$
⟨*proof*⟩

lemma *fps-right-inverse-idempotent-ring1*:
fixes $f :: 'a$::*ring-1 fps*
assumes $f\$0 * x = 1\ x * y = 1$
— These assumptions imply y equals $f\$0$, but no need to assume that.
shows *fps-right-inverse* (*fps-right-inverse f x*) $y = f$
⟨*proof*⟩

lemma *fps-right-inverse-idempotent-comm-ring1*:
fixes $f :: 'a$::*comm-ring-1 fps*
assumes $f\$0 * x = 1$
shows *fps-right-inverse* (*fps-right-inverse f x*) ($f\$0$) $= f$
⟨*proof*⟩

lemma *fps-inverse-idempotent*[*intro*, *simp*]:
$f\$0 \neq (0::'a::$*division-ring*$) \Longrightarrow$ *inverse* (*inverse f*) $= f$
⟨*proof*⟩

lemma *fps-lr-inverse-unique-ring1*:
fixes $f\ g :: 'a ::$ *ring-1 fps*
assumes *fg*: $f * g = 1\ g\$0 * f\$0 = 1$
shows *fps-left-inverse g* ($f\$0$) $= f$
and *fps-right-inverse f* ($g\$0$) $= g$
⟨*proof*⟩

lemma *fps-lr-inverse-unique-divring*:
fixes $f\ g :: 'a ::$*division-ring fps*
assumes *fg*: $f * g = 1$
shows *fps-left-inverse g* ($f\$0$) $= f$
and *fps-right-inverse f* ($g\$0$) $= g$
⟨*proof*⟩

lemma *fps-inverse-unique*:
fixes $f\ g :: 'a ::$ *division-ring fps*
assumes *fg*: $f * g = 1$
shows *inverse f* $= g$
⟨*proof*⟩

lemma *inverse-fps-numeral*:
inverse (*numeral n* :: ($'a ::$ *field-char-0*) *fps*) $=$ *fps-const* (*inverse* (*numeral n*))
⟨*proof*⟩

lemma *inverse-fps-of-nat*:

inverse (*of-nat n* :: *'a* :: {*semiring-1*,*times*,*uminus*,*inverse*} *fps*) =
fps-const (*inverse* (*of-nat n*))
⟨*proof*⟩

lemma *fps-lr-inverse-mult-ring1*:
fixes *f g* :: *'a*::*ring-1 fps*
assumes *x*: *x* ∗ *f*$*0* = *1 f*$*0* ∗ *x* = *1*
and *y*: *y* ∗ *g*$*0* = *1 g*$*0* ∗ *y* = *1*
shows *fps-left-inverse* (*f* ∗ *g*) (*y*∗*x*) = *fps-left-inverse g y* ∗ *fps-left-inverse f x*
and *fps-right-inverse* (*f* ∗ *g*) (*y*∗*x*) = *fps-right-inverse g y* ∗ *fps-right-inverse f x*
⟨*proof*⟩

lemma *fps-lr-inverse-mult-divring*:
fixes *f g* :: *'a*::*division-ring fps*
shows *fps-left-inverse* (*f* ∗ *g*) (*inverse* ((*f*∗*g*)$*0*)) =
fps-left-inverse g (*inverse* (*g*$*0*)) ∗ *fps-left-inverse f* (*inverse* (*f*$*0*))
and *fps-right-inverse* (*f* ∗ *g*) (*inverse* ((*f*∗*g*)$*0*)) =
fps-right-inverse g (*inverse* (*g*$*0*)) ∗ *fps-right-inverse f* (*inverse* (*f*$*0*))
⟨*proof*⟩

lemma *fps-inverse-mult-divring*:
inverse (*f* ∗ *g*) = *inverse g* ∗ *inverse* (*f* :: *'a*::*division-ring fps*)
⟨*proof*⟩

lemma *fps-inverse-mult*: *inverse* (*f* ∗ *g* :: *'a*::*field fps*) = *inverse f* ∗ *inverse g*
⟨*proof*⟩

lemma *inverse-prod-fps*: *inverse* (*prod f A*) = (∏ *x*∈*A*. *inverse* (*f x*) :: *'a* :: *field fps*)
⟨*proof*⟩

lemma *fps-lr-inverse-gp-ring1*:
fixes *ones ones-inv* :: *'a* :: *ring-1 fps*
defines *ones* ≡ *Abs-fps* (λ*n*. *1*)
and *ones-inv* ≡ *Abs-fps* (λ*n*. *if n=0 then 1 else if n=1 then* − *1 else 0*)
shows *fps-left-inverse ones 1* = *ones-inv*
and *fps-right-inverse ones 1* = *ones-inv*
⟨*proof*⟩

lemma *fps-lr-inverse-gp-ring1'*:
fixes *ones* :: *'a* :: *ring-1 fps*
defines *ones* ≡ *Abs-fps* (λ*n*. *1*)
shows *fps-left-inverse ones 1* = *1* − *fps-X*
and *fps-right-inverse ones 1* = *1* − *fps-X*
⟨*proof*⟩

lemma *fps-inverse-gp*:
inverse (*Abs-fps*(λ*n*. (*1*::*'a*::*division-ring*))) =

Abs-fps (λ*n*. *if n*= *0 then 1 else if n*=*1 then* − *1 else 0*)
⟨*proof*⟩

lemma *fps-inverse-gp'*: *inverse* (*Abs-fps* (λ*n*. *1*::*'a*::*division-ring*)) = *1* − *fps-X*
⟨*proof*⟩

lemma *fps-lr-inverse-one-minus-fps-X*:
fixes *ones* :: *'a* :: *ring-1 fps*
defines *ones* ≡ *Abs-fps* (λ*n*. *1*)
shows *fps-left-inverse* (*1* − *fps-X*) *1* = *ones*
and *fps-right-inverse* (*1* − *fps-X*) *1* = *ones*
⟨*proof*⟩

lemma *fps-inverse-one-minus-fps-X*:
fixes *ones* :: *'a* :: *division-ring fps*
defines *ones* ≡ *Abs-fps* (λ*n*. *1*)
shows *inverse* (*1* − *fps-X*) = *ones*
⟨*proof*⟩

lemma *fps-lr-one-over-one-minus-fps-X-squared*:
shows *fps-left-inverse* ((*1* − *fps-X*)^*2*) (*1*::*'a*::*ring-1*) = *Abs-fps* (λ*n*. *of-nat* (*n*+*1*))
fps-right-inverse ((*1* − *fps-X*)^*2*) (*1*::*'a*) = *Abs-fps* (λ*n*. *of-nat* (*n*+*1*))
⟨*proof*⟩

lemma *fps-one-over-one-minus-fps-X-squared'*:
assumes *inverse* (*1*::*'a*::{*ring-1*,*inverse*}) = *1*
shows *inverse* ((*1* − *fps-X*)^*2* :: *'a fps*) = *Abs-fps* (λ*n*. *of-nat* (*n*+*1*))
⟨*proof*⟩

lemma *fps-one-over-one-minus-fps-X-squared*:
inverse ((*1* − *fps-X*)^*2* :: *'a* :: *division-ring fps*) = *Abs-fps* (λ*n*. *of-nat* (*n*+*1*))
⟨*proof*⟩

lemma *fps-lr-inverse-fps-X-plus1*:
fps-left-inverse (*1* + *fps-X*) (*1*::*'a*::*ring-1*) = *Abs-fps* (λ*n*. (−*1*)^*n*)
fps-right-inverse (*1* + *fps-X*) (*1*::*'a*) = *Abs-fps* (λ*n*. (−*1*)^*n*)
⟨*proof*⟩

lemma *fps-inverse-fps-X-plus1'*:
assumes *inverse* (*1*::*'a*::{*ring-1*,*inverse*}) = *1*
shows *inverse* (*1* + *fps-X*) = *Abs-fps* (λ*n*. (− (*1*::*'a*)) ^ *n*)
⟨*proof*⟩

lemma *fps-inverse-fps-X-plus1*:
inverse (*1* + *fps-X*) = *Abs-fps* (λ*n*. (− (*1*::*'a*::*division-ring*)) ^ *n*)
⟨*proof*⟩

lemma *subdegree-lr-inverse*:

fixes $x :: 'a::\{comm\text{-}monoid\text{-}add,mult\text{-}zero,uminus\}$
and $y :: 'b::\{ab\text{-}group\text{-}add,mult\text{-}zero\}$
shows $subdegree\ (fps\text{-}left\text{-}inverse\ f\ x) = 0$
and $subdegree\ (fps\text{-}right\text{-}inverse\ g\ y) = 0$
⟨*proof*⟩

lemma *subdegree-inverse* [*simp*]:
fixes $f :: 'a::\{ab\text{-}group\text{-}add,inverse,mult\text{-}zero\}\ fps$
shows $subdegree\ (inverse\ f) = 0$
⟨*proof*⟩

lemma *fps-right-inverse-constructor-rec*:
$n > 0 \Longrightarrow fps\text{-}right\text{-}inverse\text{-}constructor\ f\ a\ n =$
$-a * sum\ (\lambda i.\ fps\text{-}nth\ f\ i * fps\text{-}right\text{-}inverse\text{-}constructor\ f\ a\ (n - i))$
$\{1..n\}$
⟨*proof*⟩

lemma *fps-right-inverse-constructor-cong*:
assumes $\bigwedge k.\ k \leq n \Longrightarrow fps\text{-}nth\ f\ k = fps\text{-}nth\ g\ k$
shows $fps\text{-}right\text{-}inverse\text{-}constructor\ f\ c\ n = fps\text{-}right\text{-}inverse\text{-}constructor\ g\ c\ n$
⟨*proof*⟩

lemma *fps-cutoff-inverse*:
fixes $f :: 'a :: field\ fps$
assumes $fps\text{-}nth\ f\ 0 \neq 0$
shows $fps\text{-}cutoff\ n\ (inverse\ (fps\text{-}cutoff\ n\ f)) = fps\text{-}cutoff\ n\ (inverse\ f)$
⟨*proof*⟩

lemma *tendsto-inverse-fps-aux*:
fixes $f :: 'a :: field\ fps$
assumes $fps\text{-}nth\ f\ 0 \neq 0$
shows $((\lambda f.\ inverse\ f) \longrightarrow inverse\ f)\ (at\ f)$
⟨*proof*⟩

lemma *tendsto-inverse-fps* [*tendsto-intros*]:
fixes $g :: 'a :: field\ fps$
assumes $(f \longrightarrow g)\ F$
assumes $fps\text{-}nth\ g\ 0 \neq 0$
shows $((\lambda x.\ inverse\ (f\ x)) \longrightarrow inverse\ g)\ F$
⟨*proof*⟩

lemma *fps-div-zero* [*simp*]:
$0\ div\ (g :: 'a :: \{comm\text{-}monoid\text{-}add,inverse,mult\text{-}zero,uminus\}\ fps) = 0$
⟨*proof*⟩

lemma *fps-div-by-zero′*:
fixes $g :: 'a::\{comm\text{-}monoid\text{-}add,inverse,mult\text{-}zero,uminus\}\ fps$
assumes $inverse\ (0::'a) = 0$
shows $g\ div\ 0 = 0$

⟨*proof*⟩

lemma *fps-div-by-zero* [*simp*]: (*g*::'*a*::*division-ring fps*) *div* 0 = 0
⟨*proof*⟩

lemma *fps-divide-unit'*: *subdegree g* = 0 ⟹ *f div g* = *f* ∗ *inverse g*
⟨*proof*⟩

lemma *fps-divide-unit*: *g*$0 ≠ 0 ⟹ *f div g* = *f* ∗ *inverse g*
⟨*proof*⟩

lemma *fps-divide-nth-0'*:
subdegree (*g*::'*a*::*division-ring fps*) = 0 ⟹ (*f div g*) $ 0 = *f* $ 0 / (*g* $ 0)
⟨*proof*⟩

lemma *fps-divide-nth-0* [*simp*]:
g $ 0 ≠ 0 ⟹ (*f div g*) $ 0 = *f* $ 0 / (*g* $ 0 :: - :: *division-ring*)
⟨*proof*⟩

lemma *fps-divide-nth-below*:
fixes *f g* :: '*a*::{*comm-monoid-add*,*uminus*,*mult-zero*,*inverse*} *fps*
shows *n* < *subdegree f* − *subdegree g* ⟹ (*f div g*) $ *n* = 0
⟨*proof*⟩

lemma *fps-divide-nth-base*:
fixes *f g* :: '*a*::*division-ring fps*
assumes *subdegree g* ≤ *subdegree f*
shows (*f div g*) $ (*subdegree f* − *subdegree g*) = *f* $ *subdegree f* ∗ *inverse* (*g* $ *subdegree g*)
⟨*proof*⟩

lemma *fps-divide-subdegree-ge*:
fixes *f g* :: '*a*::{*comm-monoid-add*,*uminus*,*mult-zero*,*inverse*} *fps*
assumes *f* / *g* ≠ 0
shows *subdegree* (*f* / *g*) ≥ *subdegree f* − *subdegree g*
⟨*proof*⟩

lemma *fps-divide-subdegree*:
fixes *f g* :: '*a*::*division-ring fps*
assumes *f* ≠ 0 *g* ≠ 0 *subdegree g* ≤ *subdegree f*
shows *subdegree* (*f* / *g*) = *subdegree f* − *subdegree g*
⟨*proof*⟩

lemma *fps-divide-shift-numer*:
fixes *f g* :: '*a*::{*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*} *fps*
assumes *n* ≤ *subdegree f*
shows *fps-shift n f* / *g* = *fps-shift n* (*f*/*g*)
⟨*proof*⟩

lemma *fps-divide-shift-denom*:
fixes *f g* :: *'a*::{*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*} *fps*
assumes *n* ≤ *subdegree g subdegree g* ≤ *subdegree f*
shows *f* / *fps-shift n g* = *Abs-fps* (λ*k*. *if k*<*n then 0 else* (*f*/*g*) $ (*k*−*n*))
⟨*proof*⟩

lemma *fps-divide-unit-factor-numer*:
fixes *f g* :: *'a*::{*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*} *fps*
shows *unit-factor f* / *g* = *fps-shift* (*subdegree f*) (*f*/*g*)
⟨*proof*⟩

lemma *fps-divide-unit-factor-denom*:
fixes *f g* :: *'a*::{*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*} *fps*
assumes *subdegree g* ≤ *subdegree f*
shows
f / *unit-factor g* = *Abs-fps* (λ*k*. *if k*<*subdegree g then 0 else* (*f*/*g*) $ (*k*−*subdegree g*))
⟨*proof*⟩

lemma *fps-divide-unit-factor-both'*:
fixes *f g* :: *'a*::{*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*} *fps*
assumes *subdegree g* ≤ *subdegree f*
shows *unit-factor f* / *unit-factor g* = *fps-shift* (*subdegree f* − *subdegree g*) (*f* / *g*)
⟨*proof*⟩

lemma *fps-divide-unit-factor-both*:
fixes *f g* :: *'a*::*division-ring fps*
assumes *subdegree g* ≤ *subdegree f*
shows *unit-factor f* / *unit-factor g* = *unit-factor* (*f* / *g*)
⟨*proof*⟩

lemma *fps-divide-self*:
(*f*::*'a*::*division-ring fps*) ≠ *0* ⟹ *f* / *f* = *1*
⟨*proof*⟩

lemma *fps-divide-add*:
fixes *f g h* :: *'a*::{*semiring-0*,*inverse*,*uminus*} *fps*
shows (*f* + *g*) / *h* = *f* / *h* + *g* / *h*
⟨*proof*⟩

lemma *fps-divide-diff*:
fixes *f g h* :: *'a*::{*ring*,*inverse*} *fps*
shows (*f* − *g*) / *h* = *f* / *h* − *g* / *h*
⟨*proof*⟩

lemma *fps-divide-uminus*:
fixes *f g h* :: *'a*::{*ring*,*inverse*} *fps*
shows (− *f*) / *g* = − (*f* / *g*)

⟨*proof*⟩

lemma *fps-divide-uminus'*:
fixes *f g h* :: *'a*::*division-ring fps*
shows *f* / (− *g*) = − (*f* / *g*)
⟨*proof*⟩

lemma *fps-divide-times*:
fixes *f g h* :: *'a*::{*semiring-0*,*inverse*,*uminus*} *fps*
assumes *subdegree h* ≤ *subdegree g*
shows (*f* ∗ *g*) / *h* = *f* ∗ (*g* / *h*)
⟨*proof*⟩

lemma *fps-divide-times2*:
fixes *f g h* :: *'a*::{*comm-semiring-0*,*inverse*,*uminus*} *fps*
assumes *subdegree h* ≤ *subdegree f*
shows (*f* ∗ *g*) / *h* = (*f* / *h*) ∗ *g*
⟨*proof*⟩

lemma *fps-times-divide-eq*:
fixes *f g* :: *'a*::*field fps*
assumes *g* ≠ *0* **and** *subdegree f* ≥ *subdegree g*
shows *f div g* ∗ *g* = *f*
⟨*proof*⟩

lemma *fps-divide-times-eq*:
(*g* :: *'a*::*division-ring fps*) ≠ *0* ⟹ (*f* ∗ *g*) *div g* = *f*
⟨*proof*⟩

lemma *fps-divide-by-mult'*:
fixes *f g h* :: *'a* :: *division-ring fps*
assumes *subdegree h* ≤ *subdegree f*
shows *f* / (*g* ∗ *h*) = *f* / *h* / *g*
⟨*proof*⟩

lemma *fps-divide-by-mult*:
fixes *f g h* :: *'a* :: *field fps*
assumes *subdegree g* ≤ *subdegree f*
shows *f* / (*g* ∗ *h*) = *f* / *g* / *h*
⟨*proof*⟩

lemma *fps-divide-cancel*:
fixes *f g h* :: *'a* :: *division-ring fps*
shows *h* ≠ *0* ⟹ (*f* ∗ *h*) *div* (*g* ∗ *h*) = *f div g*
⟨*proof*⟩

lemma *fps-divide-1'*:
fixes *a* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*zero-neq-one*,*monoid-mult*} *fps*

assumes *inverse* (*1*::*'a*) = *1*
shows *a* / *1* = *a*
⟨*proof*⟩

lemma *fps-divide-1* [*simp*]: (*a* :: *'a*::*division-ring fps*) / *1* = *a*
⟨*proof*⟩

lemma *fps-divide-X'*:
fixes *f* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*zero-neq-one*,*monoid-mult*} *fps*
assumes *inverse* (*1*::*'a*) = *1*
shows *f* / *fps-X* = *fps-shift 1 f*
⟨*proof*⟩

lemma *fps-divide-X* [*simp*]: *a* / *fps-X* = *fps-shift 1* (*a*::*'a*::*division-ring fps*)
⟨*proof*⟩

lemma *fps-divide-X-power'*:
fixes *f* :: *'a*::{*semiring-1*,*inverse*,*uminus*} *fps*
assumes *inverse* (*1*::*'a*) = *1*
shows *f* / (*fps-X* ^ *n*) = *fps-shift n f*
⟨*proof*⟩

lemma *fps-divide-X-power* [*simp*]: *a* / (*fps-X* ^ *n*) = *fps-shift n* (*a*::*'a*::*division-ring fps*)
⟨*proof*⟩

lemma *fps-divide-shift-denom-conv-times-fps-X-power*:
fixes *f g* :: *'a*::{*semiring-1*,*inverse*,*uminus*} *fps*
assumes *n* ≤ *subdegree g subdegree g* ≤ *subdegree f*
shows *f* / *fps-shift n g* = *f* / *g* ∗ *fps-X* ^ *n*
⟨*proof*⟩

lemma *fps-divide-unit-factor-denom-conv-times-fps-X-power*:
fixes *f g* :: *'a*::{*semiring-1*,*inverse*,*uminus*} *fps*
assumes *subdegree g* ≤ *subdegree f*
shows *f* / *unit-factor g* = *f* / *g* ∗ *fps-X* ^ *subdegree g*
⟨*proof*⟩

lemma *fps-shift-altdef'*:
fixes *f* :: *'a*::{*semiring-1*,*inverse*,*uminus*} *fps*
assumes *inverse* (*1*::*'a*) = *1*
shows *fps-shift n f* = *f div fps-X*^*n*
⟨*proof*⟩

lemma *fps-shift-altdef*:
fps-shift n f = (*f* :: *'a* :: *division-ring fps*) *div fps-X*^*n*
⟨*proof*⟩

lemma *fps-div-fps-X-power-nth'*:
fixes $f :: 'a::\{semiring\text{-}1,inverse,uminus\}\ fps$
assumes $inverse\ (1::'a) = 1$
shows $(f\ div\ fps\text{-}X\hat{}n)\ \$\ k = f\ \$\ (k + n)$
⟨*proof*⟩

lemma *fps-div-fps-X-power-nth*: $((f :: 'a :: division\text{-}ring\ fps)\ div\ fps\text{-}X\hat{}n)\ \$\ k = f\ \$\ (k + n)$
⟨*proof*⟩

lemma *fps-div-fps-X-nth'*:
fixes $f :: 'a::\{semiring\text{-}1,inverse,uminus\}\ fps$
assumes $inverse\ (1::'a) = 1$
shows $(f\ div\ fps\text{-}X)\ \$\ k = f\ \$\ Suc\ k$
⟨*proof*⟩

lemma *fps-div-fps-X-nth*: $((f :: 'a :: division\text{-}ring\ fps)\ div\ fps\text{-}X)\ \$\ k = f\ \$\ Suc\ k$
⟨*proof*⟩

lemma *divide-fps-const'*:
fixes $c :: 'a :: \{inverse,comm\text{-}monoid\text{-}add,uminus,mult\text{-}zero\}$
shows $f\ /\ fps\text{-}const\ c = f * fps\text{-}const\ (inverse\ c)$
⟨*proof*⟩

lemma *divide-fps-const* [*simp*]:
fixes $c :: 'a :: \{comm\text{-}semiring\text{-}0,inverse,uminus\}$
shows $f\ /\ fps\text{-}const\ c = fps\text{-}const\ (inverse\ c) * f$
⟨*proof*⟩

lemma *fps-const-divide*: $fps\text{-}const\ (x :: - :: division\text{-}ring)\ /\ fps\text{-}const\ y = fps\text{-}const\ (x\ /\ y)$
⟨*proof*⟩

lemma *fps-numeral-divide-divide*:
$x\ /\ numeral\ b\ /\ numeral\ c = (x\ /\ numeral\ (b * c) :: 'a :: field\ fps)$
⟨*proof*⟩

lemma *fps-numeral-mult-divide*:
$numeral\ b * x\ /\ numeral\ c = (numeral\ b\ /\ numeral\ c * x :: 'a :: field\ fps)$
⟨*proof*⟩

lemmas *fps-numeral-simps* =
fps-numeral-divide-divide fps-numeral-mult-divide inverse-fps-numeral neg-numeral-fps-const

lemma *fps-is-left-unit-iff-zeroth-is-left-unit*:
fixes $f :: 'a :: ring\text{-}1\ fps$
shows $(\exists g.\ 1 = f * g) \longleftrightarrow (\exists k.\ 1 = f\$0 * k)$
⟨*proof*⟩

lemma *fps-is-right-unit-iff-zeroth-is-right-unit*:
fixes $f :: 'a :: ring\text{-}1\ fps$
shows $(\exists g.\ 1 = g * f) \longleftrightarrow (\exists k.\ 1 = k * f\$0)$
⟨*proof*⟩

lemma *fps-is-unit-iff* [*simp*]: $(f :: 'a :: field\ fps)\ dvd\ 1 \longleftrightarrow f\ \$\ 0 \neq 0$
⟨*proof*⟩

lemma *subdegree-eq-0-left*:
fixes $f :: 'a::\{comm\text{-}monoid\text{-}add,zero\text{-}neq\text{-}one,mult\text{-}zero\}\ fps$
assumes $\exists g.\ 1 = f * g$
shows $subdegree\ f = 0$
⟨*proof*⟩

lemma *subdegree-eq-0-right*:
fixes $f :: 'a::\{comm\text{-}monoid\text{-}add,zero\text{-}neq\text{-}one,mult\text{-}zero\}\ fps$
assumes $\exists g.\ 1 = g * f$
shows $subdegree\ f = 0$
⟨*proof*⟩

lemma *subdegree-eq-0'* [*simp*]: $(f :: 'a :: field\ fps)\ dvd\ 1 \Longrightarrow subdegree\ f = 0$
⟨*proof*⟩

lemma *fps-dvd1-left-trivial-unit-factor*:
fixes $f :: 'a::\{comm\text{-}monoid\text{-}add,\ zero\text{-}neq\text{-}one,\ mult\text{-}zero\}\ fps$
assumes $\exists g.\ 1 = f * g$
shows $unit\text{-}factor\ f = f$
⟨*proof*⟩

lemma *fps-dvd1-right-trivial-unit-factor*:
fixes $f :: 'a::\{comm\text{-}monoid\text{-}add,\ zero\text{-}neq\text{-}one,\ mult\text{-}zero\}\ fps$
assumes $\exists g.\ 1 = g * f$
shows $unit\text{-}factor\ f = f$
⟨*proof*⟩

lemma *fps-dvd1-trivial-unit-factor*:
$(f :: 'a::comm\text{-}semiring\text{-}1\ fps)\ dvd\ 1 \Longrightarrow unit\text{-}factor\ f = f$
⟨*proof*⟩

lemma *fps-unit-dvd-left*:
fixes $f :: 'a :: division\text{-}ring\ fps$
assumes $f\ \$\ 0 \neq 0$
shows $\exists g.\ 1 = f * g$
⟨*proof*⟩

lemma *fps-unit-dvd-right*:
fixes $f :: 'a :: division\text{-}ring\ fps$
assumes $f\ \$\ 0 \neq 0$
shows $\exists g.\ 1 = g * f$

⟨*proof*⟩

lemma *fps-unit-dvd* [*simp*]: $(f\ \$\ 0 :: 'a :: field) \neq 0 \Longrightarrow f\ dvd\ g$
⟨*proof*⟩

lemma *dvd-left-imp-subdegree-le*:
fixes $f\ g :: 'a::\{comm\text{-}monoid\text{-}add, mult\text{-}zero\}\ fps$
assumes $\exists k.\ g = f * k\ g \neq 0$
shows $subdegree\ f \leq subdegree\ g$
⟨*proof*⟩

lemma *dvd-right-imp-subdegree-le*:
fixes $f\ g :: 'a::\{comm\text{-}monoid\text{-}add, mult\text{-}zero\}\ fps$
assumes $\exists k.\ g = k * f\ g \neq 0$
shows $subdegree\ f \leq subdegree\ g$
⟨*proof*⟩

lemma *dvd-imp-subdegree-le*:
$f\ dvd\ g \Longrightarrow g \neq 0 \Longrightarrow subdegree\ f \leq subdegree\ g$
⟨*proof*⟩

lemma *subdegree-le-imp-dvd-left-ring1*:
fixes $f\ g :: 'a :: ring\text{-}1\ fps$
assumes $\exists y.\ f\ \$\ subdegree\ f * y = 1\ subdegree\ f \leq subdegree\ g$
shows $\exists k.\ g = f * k$
⟨*proof*⟩

lemma *subdegree-le-imp-dvd-left-divring*:
fixes $f\ g :: 'a :: division\text{-}ring\ fps$
assumes $f \neq 0\ subdegree\ f \leq subdegree\ g$
shows $\exists k.\ g = f * k$
⟨*proof*⟩

lemma *subdegree-le-imp-dvd-right-ring1*:
fixes $f\ g :: 'a :: ring\text{-}1\ fps$
assumes $\exists x.\ x * f\ \$\ subdegree\ f = 1\ subdegree\ f \leq subdegree\ g$
shows $\exists k.\ g = k * f$
⟨*proof*⟩

lemma *subdegree-le-imp-dvd-right-divring*:
fixes $f\ g :: 'a :: division\text{-}ring\ fps$
assumes $f \neq 0\ subdegree\ f \leq subdegree\ g$
shows $\exists k.\ g = k * f$
⟨*proof*⟩

lemma *fps-dvd-iff*:
assumes $(f :: 'a :: field\ fps) \neq 0\ g \neq 0$
shows $f\ dvd\ g \longleftrightarrow subdegree\ f \leq subdegree\ g$
⟨*proof*⟩

lemma *subdegree-div'*:
fixes *p q* :: *'a*::*division-ring fps*
assumes $\exists k.\ p = k * q$
shows *subdegree* (*p div q*) = *subdegree p* − *subdegree q*
⟨*proof*⟩

lemma *subdegree-div*:
fixes *p q* :: *'a* :: *field fps*
assumes *q dvd p*
shows *subdegree* (*p div q*) = *subdegree p* − *subdegree q*
⟨*proof*⟩

lemma *subdegree-div-unit'*:
fixes *p q* :: *'a* :: {*ab-group-add*,*mult-zero*,*inverse*} *fps*
assumes *q* $ *0* ≠ *0 p* $ *subdegree p* * *inverse* (*q* $ *0*) ≠ *0*
shows *subdegree* (*p div q*) = *subdegree p*
⟨*proof*⟩

lemma *subdegree-div-unit''*:
fixes *p q* :: *'a* :: {*ring-no-zero-divisors*,*inverse*} *fps*
assumes *q* $ *0* ≠ *0 inverse* (*q* $ *0*) ≠ *0*
shows *subdegree* (*p div q*) = *subdegree p*
⟨*proof*⟩

lemma *subdegree-div-unit*:
fixes *p q* :: *'a* :: *division-ring fps*
assumes *q* $ *0* ≠ *0*
shows *subdegree* (*p div q*) = *subdegree p*
⟨*proof*⟩

instantiation *fps* :: ({*comm-semiring-1*,*inverse*,*uminus*}) *modulo*
begin

definition *fps-mod-def*:
f mod g = (*if g* = *0 then f else*
let h = *unit-factor g in fps-cutoff* (*subdegree g*) (*f* * *inverse h*) * *h*)

instance ⟨*proof*⟩

end

lemma *fps-mod-zero* [*simp*]:
(*f*::*'a*::{*comm-semiring-1*,*inverse*,*uminus*} *fps*) *mod 0* = *f*
⟨*proof*⟩

lemma *fps-mod-eq-zero*:
assumes *g* ≠ *0* **and** *subdegree f* ≥ *subdegree g*
shows *f mod g* = *0*

⟨*proof*⟩

lemma *fps-mod-unit* [*simp*]: $g\$0 \neq 0 \Longrightarrow f \bmod g = 0$
⟨*proof*⟩

lemma *subdegree-mod*:
assumes *subdegree* (f::'a::*field fps*) < *subdegree* g
shows *subdegree* (f *mod* g) = *subdegree* f
⟨*proof*⟩

instance *fps* :: (*field*) *idom-modulo*
⟨*proof*⟩

instantiation *fps* :: (*field*) *normalization-semidom-multiplicative*
begin

definition *fps-normalize-def* [*simp*]:
normalize f = (*if* $f = 0$ *then* 0 *else fps-X* ^ *subdegree* f)

instance ⟨*proof*⟩

end

5.8 Computing reciprocals via Hensel lifting

lemma *inverse-fps-hensel-lifting*:
fixes F G :: 'a :: *field fps* **and** n :: *nat*
assumes *G-eq*: *fps-cutoff* n G = *fps-cutoff* n (*inverse* F)
assumes *unit*: *fps-nth* F 0 ≠ 0
shows *fps-cutoff* (2∗n) (*inverse* F) = *fps-cutoff* (2∗n) (G ∗ (2 − F ∗ G))
⟨*proof*⟩

lemma *inverse-fps-hensel-lifting'*:
fixes F G :: 'a :: *field fps* **and** n :: *nat*
assumes *G-eq*: *fps-cutoff* n G = *fps-cutoff* n (*inverse* F)
assumes *unit*: *fps-nth* F 0 ≠ 0
defines $P \equiv$ *fps-shift* n (F ∗ G − 1)
shows *fps-cutoff* (2∗n) (*inverse* F) = *fps-cutoff* (2∗n) (G ∗ (1 − *fps-X* ^ n ∗ P))
⟨*proof*⟩

5.9 Euclidean division

instantiation *fps* :: (*field*) *euclidean-ring-cancel*
begin

definition *fps-euclidean-size-def*:
euclidean-size f = (*if* $f = 0$ *then* 0 *else* 2 ^ *subdegree* f)

instance ⟨*proof*⟩

end

instance *fps* :: (*field*) *normalization-euclidean-semiring* ⟨*proof*⟩

instantiation *fps* :: (*field*) *euclidean-ring-gcd*
begin
definition *fps-gcd-def*: (*gcd* :: *'a fps* ⇒ -) = *Euclidean-Algorithm.gcd*
definition *fps-lcm-def*: (*lcm* :: *'a fps* ⇒ -) = *Euclidean-Algorithm.lcm*
definition *fps-Gcd-def*: (*Gcd* :: *'a fps set* ⇒ -) = *Euclidean-Algorithm.Gcd*
definition *fps-Lcm-def*: (*Lcm* :: *'a fps set* ⇒ -) = *Euclidean-Algorithm.Lcm*
instance ⟨*proof*⟩
end

lemma *fps-gcd*:
assumes [*simp*]: *f* ≠ *0 g* ≠ *0*
shows *gcd f g* = *fps-X* ^ *min* (*subdegree f*) (*subdegree g*)
⟨*proof*⟩

lemma *fps-gcd-altdef*: *gcd f g* =
(*if f* = *0* ∧ *g* = *0 then 0 else*
if f = *0 then fps-X* ^ *subdegree g else*
if g = *0 then fps-X* ^ *subdegree f else*
fps-X ^ *min* (*subdegree f*) (*subdegree g*))
⟨*proof*⟩

lemma *fps-lcm*:
assumes [*simp*]: *f* ≠ *0 g* ≠ *0*
shows *lcm f g* = *fps-X* ^ *max* (*subdegree f*) (*subdegree g*)
⟨*proof*⟩

lemma *fps-lcm-altdef*: *lcm f g* =
(*if f* = *0* ∨ *g* = *0 then 0 else fps-X* ^ *max* (*subdegree f*) (*subdegree g*))
⟨*proof*⟩

lemma *fps-Gcd*:
assumes *A* − {*0*} ≠ {}
shows *Gcd A* = *fps-X* ^ (*INF f*∈*A*−{*0*}. *subdegree f*)
⟨*proof*⟩

lemma *fps-Gcd-altdef*: *Gcd A* =
(*if A* ⊆ {*0*} *then 0 else fps-X* ^ (*INF f*∈*A*−{*0*}. *subdegree f*))
⟨*proof*⟩

lemma *fps-Lcm*:
assumes *A* ≠ {} *0* ∉ *A bdd-above* (*subdegree*‘*A*)
shows *Lcm A* = *fps-X* ^ (*SUP f*∈*A*. *subdegree f*)
⟨*proof*⟩

lemma *fps-Lcm-altdef*:
Lcm A =
(*if* 0 ∈ *A* ∨ ¬*bdd-above* (*subdegree*‘*A*) *then* 0 *else*
if A = {} *then* 1 *else fps-X* ^ (*SUP f*∈*A*. *subdegree f*))
⟨*proof*⟩

5.10 Formal Derivatives

definition *fps-deriv f* = *Abs-fps* (λ*n*. *of-nat* (*n* + 1) ∗ *f* $ (*n* + 1))

lemma *fps-deriv-nth*[*simp*]: *fps-deriv f* $ *n* = *of-nat* (*n* + 1) ∗ *f* $ (*n* + 1)
⟨*proof*⟩

lemma *fps-0th-higher-deriv*:
(*fps-deriv* ^^ *n*) *f* $ 0 = *fact n* ∗ *f* $ *n*
⟨*proof*⟩

lemma *fps-deriv-mult*[*simp*]:
fps-deriv (*f* ∗ *g*) = *f* ∗ *fps-deriv g* + *fps-deriv f* ∗ *g*
⟨*proof*⟩

lemma *fps-deriv-fps-X*[*simp*]: *fps-deriv fps-X* = 1
⟨*proof*⟩

lemma *fps-deriv-neg*[*simp*]:
fps-deriv (− (*f*:: ′*a*::*ring-1 fps*)) = − (*fps-deriv f*)
⟨*proof*⟩

lemma *fps-deriv-add*[*simp*]: *fps-deriv* (*f* + *g*) = *fps-deriv f* + *fps-deriv g*
⟨*proof*⟩

lemma *fps-deriv-sub*[*simp*]:
fps-deriv ((*f*:: ′*a*::*ring-1 fps*) − *g*) = *fps-deriv f* − *fps-deriv g*
⟨*proof*⟩

lemma *fps-deriv-const*[*simp*]: *fps-deriv* (*fps-const c*) = 0
⟨*proof*⟩

lemma *fps-deriv-of-nat* [*simp*]: *fps-deriv* (*of-nat n*) = 0
⟨*proof*⟩

lemma *fps-deriv-of-int* [*simp*]: *fps-deriv* (*of-int n*) = 0
⟨*proof*⟩

lemma *fps-deriv-numeral* [*simp*]: *fps-deriv* (*numeral n*) = 0
⟨*proof*⟩

lemma *fps-deriv-mult-const-left*[*simp*]:
fps-deriv (*fps-const c* ∗ *f*) = *fps-const c* ∗ *fps-deriv f*

⟨*proof*⟩

lemma *fps-deriv-linear*[*simp*]:
fps-deriv (*fps-const a* ∗ *f* + *fps-const b* ∗ *g*) =
fps-const a ∗ *fps-deriv f* + *fps-const b* ∗ *fps-deriv g*
⟨*proof*⟩

lemma *fps-deriv-0*[*simp*]: *fps-deriv 0* = *0*
⟨*proof*⟩

lemma *fps-deriv-1*[*simp*]: *fps-deriv 1* = *0*
⟨*proof*⟩

lemma *fps-deriv-mult-const-right*[*simp*]:
fps-deriv (*f* ∗ *fps-const c*) = *fps-deriv f* ∗ *fps-const c*
⟨*proof*⟩

lemma *fps-deriv-sum*:
fps-deriv (*sum f S*) = *sum* (λ*i*. *fps-deriv* (*f i*)) *S*
⟨*proof*⟩

lemma *fps-deriv-eq-0-iff* [*simp*]:
fps-deriv f = *0* ⟷ *f* = *fps-const* (*f*$*0* :: *'a*::{*semiring-no-zero-divisors*,*semiring-char-0*})
⟨*proof*⟩

lemma *fps-deriv-eq-iff*:
fixes *f g* :: *'a*::{*ring-1-no-zero-divisors*,*semiring-char-0*} *fps*
shows *fps-deriv f* = *fps-deriv g* ⟷ (*f* = *fps-const*(*f*$*0* − *g*$*0*) + *g*)
⟨*proof*⟩

lemma *fps-deriv-eq-iff-ex*:
fixes *f g* :: *'a*::{*ring-1-no-zero-divisors*,*semiring-char-0*} *fps*
shows (*fps-deriv f* = *fps-deriv g*) ⟷ (∃ *c*. *f* = *fps-const c* + *g*)
⟨*proof*⟩

fun *fps-nth-deriv* :: *nat* ⇒ *'a*::*semiring-1 fps* ⇒ *'a fps*
where
fps-nth-deriv 0 f = *f*
| *fps-nth-deriv* (*Suc n*) *f* = *fps-nth-deriv n* (*fps-deriv f*)

lemma *fps-nth-deriv-commute*: *fps-nth-deriv* (*Suc n*) *f* = *fps-deriv* (*fps-nth-deriv* *n f*)
⟨*proof*⟩

lemma *fps-nth-deriv-linear*[*simp*]:
fps-nth-deriv n (*fps-const a* ∗ *f* + *fps-const b* ∗ *g*) =
fps-const a ∗ *fps-nth-deriv n f* + *fps-const b* ∗ *fps-nth-deriv n g*
⟨*proof*⟩

lemma *fps-nth-deriv-neg*[*simp*]:
fps-nth-deriv n (− (*f* :: *'a*::*ring-1 fps*)) = − (*fps-nth-deriv n f*)
⟨*proof*⟩

lemma *fps-nth-deriv-add*[*simp*]:
fps-nth-deriv n ((*f* :: *'a*::*ring-1 fps*) + *g*) = *fps-nth-deriv n f* + *fps-nth-deriv n g*
⟨*proof*⟩

lemma *fps-nth-deriv-sub*[*simp*]:
fps-nth-deriv n ((*f* :: *'a*::*ring-1 fps*) − *g*) = *fps-nth-deriv n f* − *fps-nth-deriv n g*
⟨*proof*⟩

lemma *fps-nth-deriv-0*[*simp*]: *fps-nth-deriv n 0* = *0*
⟨*proof*⟩

lemma *fps-nth-deriv-1*[*simp*]: *fps-nth-deriv n 1* = (*if n* = *0 then 1 else 0*)
⟨*proof*⟩

lemma *fps-nth-deriv-const*[*simp*]:
fps-nth-deriv n (*fps-const c*) = (*if n* = *0 then fps-const c else 0*)
⟨*proof*⟩

lemma *fps-nth-deriv-mult-const-left*[*simp*]:
fps-nth-deriv n (*fps-const c* ∗ *f*) = *fps-const c* ∗ *fps-nth-deriv n f*
⟨*proof*⟩

lemma *fps-nth-deriv-mult-const-right*[*simp*]:
fps-nth-deriv n (*f* ∗ *fps-const c*) = *fps-nth-deriv n f* ∗ *fps-const c*
⟨*proof*⟩

lemma *fps-nth-deriv-sum*:
fps-nth-deriv n (*sum f S*) = *sum* (λ*i*. *fps-nth-deriv n* (*f i* :: *'a*::*ring-1 fps*)) *S*
⟨*proof*⟩

lemma *fps-deriv-maclauren-0*:
(*fps-nth-deriv k* (*f* :: *'a*::*comm-semiring-1 fps*)) $ *0* = *of-nat* (*fact k*) ∗ *f* $ *k*
⟨*proof*⟩

lemma *fps-deriv-lr-inverse*:
fixes *x y* :: *'a*::*ring-1*
assumes *x* ∗ *f*$*0* = *1 f*$*0* ∗ *y* = *1*
— These assumptions imply x equals y, but no need to assume that.
shows *fps-deriv* (*fps-left-inverse f x*) =
− *fps-left-inverse f x* ∗ *fps-deriv f* ∗ *fps-left-inverse f x*
and *fps-deriv* (*fps-right-inverse f y*) =
− *fps-right-inverse f y* ∗ *fps-deriv f* ∗ *fps-right-inverse f y*
⟨*proof*⟩

lemma *fps-deriv-lr-inverse-comm*:
fixes $x :: 'a$::*comm-ring-1*
assumes $x * f\$0 = 1$
shows *fps-deriv* (*fps-left-inverse f x*) = − *fps-deriv f* ∗ (*fps-left-inverse f x*)^2
and *fps-deriv* (*fps-right-inverse f x*) = − *fps-deriv f* ∗ (*fps-right-inverse f x*)^2
⟨*proof*⟩

lemma *fps-inverse-deriv-divring*:
fixes $a :: 'a$::*division-ring fps*
assumes $a\$0 \neq 0$
shows *fps-deriv* (*inverse a*) = − *inverse a* ∗ *fps-deriv a* ∗ *inverse a*
⟨*proof*⟩

lemma *fps-inverse-deriv*:
fixes $a :: 'a$::*field fps*
assumes $a\$0 \neq 0$
shows *fps-deriv* (*inverse a*) = − *fps-deriv a* ∗ (*inverse a*)^2
⟨*proof*⟩

lemma *fps-inverse-deriv'*:
fixes $a :: 'a$::*field fps*
assumes *a0*: $a \$ 0 \neq 0$
shows *fps-deriv* (*inverse a*) = − *fps-deriv a* / a^2
⟨*proof*⟩

lemma *fps-divide-deriv*:
assumes *b dvd* (*a* :: $'a$ :: *field fps*)
shows *fps-deriv* (*a* / *b*) = (*fps-deriv a* ∗ *b* − *a* ∗ *fps-deriv b*) / *b*^*2*
⟨*proof*⟩

lemma *fps-nth-deriv-fps-X*[*simp*]: *fps-nth-deriv n fps-X* = (*if n* = *0 then fps-X else if n*=*1 then 1 else 0*)
⟨*proof*⟩

5.11 Powers

lemma *fps-power-zeroth*: $(a\hat{}n) \$ 0 = (a\$0)\hat{}n$
⟨*proof*⟩

lemma *fps-power-zeroth-eq-one*: $a\$0 = 1 \Longrightarrow a\hat{}n \$ 0 = 1$
⟨*proof*⟩

lemma *fps-power-first*:
fixes $a :: 'a$::*comm-semiring-1 fps*
shows $(a\hat{}n) \$ 1$ = *of-nat n* ∗ $(a\$0)\hat{}(n-1)$ ∗ $a\$1$
⟨*proof*⟩

lemma *fps-power-first-eq*: $a \$ 0 = 1 \Longrightarrow a\hat{}n \$ 1$ = *of-nat n* ∗ $a\$1$

⟨*proof*⟩

lemma *fps-power-first-eq'*:
assumes $a \$ 1 = 1$
shows $a\hat{}n \$ 1 = of\text{-}nat\ n * (a\$0)\hat{}(n-1)$
⟨*proof*⟩

lemmas *startsby-one-power* = *fps-power-zeroth-eq-one*

lemma *startsby-zero-power*: $a \$ 0 = 0 \Longrightarrow n > 0 \Longrightarrow a\hat{}n \$0 = 0$
⟨*proof*⟩

lemma *startsby-power*: $a \$0 = v \Longrightarrow a\hat{}n \$0 = v\hat{}n$
⟨*proof*⟩

lemma *startsby-nonzero-power*:
fixes $a :: 'a$::*semiring-1-no-zero-divisors fps*
shows $a \$ 0 \neq 0 \Longrightarrow a\hat{}n \$ 0 \neq 0$
⟨*proof*⟩

lemma *startsby-zero-power-iff*[*simp*]:
$a\hat{}n \$0 = (0::'a::semiring\text{-}1\text{-}no\text{-}zero\text{-}divisors) \longleftrightarrow n \neq 0 \wedge a\$0 = 0$
⟨*proof*⟩

lemma *startsby-zero-power-prefix*:
assumes *a0*: $a \$ 0 = 0$
shows $\forall n < k.\ a\ \hat{}\ k \$ n = 0$
⟨*proof*⟩

lemma *startsby-zero-sum-depends*:
assumes *a0*: $a \$0 = 0$
and *kn*: $n \geq k$
shows $sum\ (\lambda i.\ (a\ \hat{}\ i)\$k)\ \{0\ ..\ n\} = sum\ (\lambda i.\ (a\ \hat{}\ i)\$k)\ \{0\ ..\ k\}$
⟨*proof*⟩

lemma *startsby-zero-power-nth-same*:
assumes *a0*: $a\$0 = 0$
shows $a\hat{}n \$ n = (a\$1)\ \hat{}\ n$
⟨*proof*⟩

lemma *fps-lr-inverse-power*:
fixes $a :: 'a$::*ring-1 fps*
assumes $x * a\$0 = 1\ a\$0 * x = 1$
shows *fps-left-inverse* $(a\hat{}n)\ (x\hat{}n) =$ *fps-left-inverse* $a\ x\ \hat{}\ n$
and *fps-right-inverse* $(a\hat{}n)\ (x\hat{}n) =$ *fps-right-inverse* $a\ x\ \hat{}\ n$
⟨*proof*⟩

lemma *fps-inverse-power*:
fixes $a :: 'a$::*division-ring fps*

shows *inverse* $(a \hat{} n) = inverse\ a \hat{} n$
⟨*proof*⟩

lemma *fps-deriv-power'*:
fixes $a :: 'a$::*comm-semiring-1 fps*
shows *fps-deriv* $(a \hat{} n) = (of\text{-}nat\ n) * fps\text{-}deriv\ a * a \hat{} (n - 1)$
⟨*proof*⟩

lemma *fps-deriv-power*:
fixes $a :: 'a$::*comm-semiring-1 fps*
shows *fps-deriv* $(a \hat{} n) = fps\text{-}const\ (of\text{-}nat\ n) * fps\text{-}deriv\ a * a \hat{} (n - 1)$
⟨*proof*⟩

5.12 Finite and infinite products

lemma *fps-prod-nth'*:
assumes *finite A*
shows *fps-nth* $(\prod x{\in}A.\ f\ x)\ n = (\sum X{\in}multisets\text{-}of\text{-}size\ A\ n.\ \prod x{\in}A.\ fps\text{-}nth\ (f\ x)\ (count\ X\ x))$
⟨*proof*⟩

theorem *tendsto-prod-fps*:
fixes $f :: nat \Rightarrow 'a :: \{idom,\ t2\text{-}space\}\ fps$
assumes [*simp*]: $\bigwedge k.\ f\ k \neq 0$
assumes *g*: $\bigwedge n\ k.\ k > g\ n \Longrightarrow subdegree\ (f\ k - 1) > n$
defines $P \equiv Abs\text{-}fps\ (\lambda n.\ (\sum X{\in}multisets\text{-}of\text{-}size\ \{..g\ n\}\ n.\ \prod i{\leq}g\ n.\ fps\text{-}nth\ (f\ i)\ (count\ X\ i)))$
shows $(\lambda n.\ \prod k{\leq}n.\ f\ k) \longrightarrow P$
⟨*proof*⟩

5.13 Integration

definition *fps-integral* :: $'a$::{*semiring-1*,*inverse*} *fps* ⇒ $'a$ ⇒ $'a$ *fps*
where *fps-integral a a0* =
Abs-fps $(\lambda n.\ if\ n{=}0\ then\ a0\ else\ inverse\ (of\text{-}nat\ n) * a\$(n - 1))$

abbreviation *fps-integral0 a* ≡ *fps-integral a 0*

lemma *fps-integral-nth-0-Suc* [*simp*]:
fixes $a :: 'a$::{*semiring-1*,*inverse*} *fps*
shows *fps-integral a a0* $ 0 = a0$
and *fps-integral a a0* $ *Suc n* = inverse\ (of\text{-}nat\ (Suc\ n)) * a\ \$\ n$
⟨*proof*⟩

lemma *fps-integral-conv-plus-const*:
fps-integral a a0 = *fps-integral a 0* + *fps-const a0*
⟨*proof*⟩

lemma *fps-deriv-fps-integral*:
fixes $a :: 'a$::{*division-ring*,*ring-char-0*} *fps*

shows *fps-deriv* (*fps-integral a a0*) = *a*
⟨*proof*⟩

lemma *fps-integral0-deriv*:
fixes *a* :: *'a*::{*division-ring*,*ring-char-0*} *fps*
shows *fps-integral0* (*fps-deriv a*) = *a* − *fps-const* (*a*$*0*)
⟨*proof*⟩

lemma *fps-integral-deriv*:
fixes *a* :: *'a*::{*division-ring*,*ring-char-0*} *fps*
shows *fps-integral* (*fps-deriv a*) (*a*$*0*) = *a*
⟨*proof*⟩

lemma *fps-integral0-zero*:
fps-integral0 (*0*::*'a*::{*semiring-1*,*inverse*} *fps*) = *0*
⟨*proof*⟩

lemma *fps-integral0-fps-const'*:
fixes *c* :: *'a*::{*semiring-1*,*inverse*}
assumes *inverse* (*1*::*'a*) = *1*
shows *fps-integral0* (*fps-const c*) = *fps-const c* ∗ *fps-X*
⟨*proof*⟩

lemma *fps-integral0-fps-const*:
fixes *c* :: *'a*::*division-ring*
shows *fps-integral0* (*fps-const c*) = *fps-const c* ∗ *fps-X*
⟨*proof*⟩

lemma *fps-integral0-one'*:
assumes *inverse* (*1*::*'a*::{*semiring-1*,*inverse*}) = *1*
shows *fps-integral0* (*1*::*'a fps*) = *fps-X*
⟨*proof*⟩

lemma *fps-integral0-one*:
fps-integral0 (*1*::*'a*::*division-ring fps*) = *fps-X*
⟨*proof*⟩

lemma *fps-integral0-fps-const-mult-left*:
fixes *a* :: *'a*::*division-ring fps*
shows *fps-integral0* (*fps-const c* ∗ *a*) = *fps-const c* ∗ *fps-integral0 a*
⟨*proof*⟩

lemma *fps-integral0-fps-const-mult-right*:
fixes *a* :: *'a*::{*semiring-1*,*inverse*} *fps*
shows *fps-integral0* (*a* ∗ *fps-const c*) = *fps-integral0 a* ∗ *fps-const c*
⟨*proof*⟩

lemma *fps-integral0-neg*:
fixes *a* :: *'a*::{*ring-1*,*inverse*} *fps*

shows *fps-integral0* $(-a) = -$ *fps-integral0 a*
⟨*proof*⟩

lemma *fps-integral0-add*:
fps-integral0 $(a+b)$ = *fps-integral0 a* + *fps-integral0 b*
⟨*proof*⟩

lemma *fps-integral0-linear*:
fixes *a b* :: *'a*::*division-ring*
shows *fps-integral0* (*fps-const a* ∗ *f* + *fps-const b* ∗ *g*) =
fps-const a ∗ *fps-integral0 f* + *fps-const b* ∗ *fps-integral0 g*
⟨*proof*⟩

lemma *fps-integral0-linear2*:
fps-integral0 (*f* ∗ *fps-const a* + *g* ∗ *fps-const b*) =
fps-integral0 f ∗ *fps-const a* + *fps-integral0 g* ∗ *fps-const b*
⟨*proof*⟩

lemma *fps-integral-linear*:
fixes *a b a0 b0* :: *'a*::*division-ring*
shows
fps-integral (*fps-const a* ∗ *f* + *fps-const b* ∗ *g*) (*a*∗*a0* + *b*∗*b0*) =
fps-const a ∗ *fps-integral f a0* + *fps-const b* ∗ *fps-integral g b0*
⟨*proof*⟩

lemma *fps-integral0-sub*:
fixes *a b* :: *'a*::{*ring-1*,*inverse*} *fps*
shows *fps-integral0* $(a-b)$ = *fps-integral0 a* − *fps-integral0 b*
⟨*proof*⟩

lemma *fps-integral0-of-nat*:
fps-integral0 (*of-nat n* :: *'a*::*division-ring fps*) = *of-nat n* ∗ *fps-X*
⟨*proof*⟩

lemma *fps-integral0-sum*:
fps-integral0 (*sum f S*) = *sum* (λ*i*. *fps-integral0* (*f i*)) *S*
⟨*proof*⟩

lemma *fps-integral0-by-parts*:
fixes *a b* :: *'a*::{*division-ring*,*ring-char-0*} *fps*
shows
fps-integral0 (*a* ∗ *b*) =
a ∗ *fps-integral0 b* − *fps-integral0* (*fps-deriv a* ∗ *fps-integral0 b*)
⟨*proof*⟩

lemma *fps-integral0-fps-X*:
fps-integral0 (*fps-X*::*'a*::{*semiring-1*,*inverse*} *fps*) =
fps-const (*inverse* (*of-nat 2*)) ∗ *fps-X*^{2}
⟨*proof*⟩

lemma *fps-integral0-fps-X-power*:
fps-integral0 ((*fps-X*::'*a*::{*semiring-1*,*inverse*} *fps*) ^ *n*) =
fps-const (*inverse* (*of-nat* (*Suc n*))) * *fps-X* ^ *Suc n*
⟨*proof*⟩

5.14 Composition

definition *fps-compose* :: '*a*::*semiring-1 fps* ⇒ '*a fps* ⇒ '*a fps* (**infixl** ‹*oo*› *55*)
where *a oo b* = *Abs-fps* (λ*n*. *sum* (λ*i*. *a*$*i* * (*b*^*i*$*n*)) {*0*..*n*})

lemma *fps-compose-nth*: (*a oo b*)$*n* = *sum* (λ*i*. *a*$*i* * (*b*^*i*$*n*)) {*0*..*n*}
⟨*proof*⟩

lemma *fps-compose-nth-0* [*simp*]: (*f oo g*) $ *0* = *f* $ *0*
⟨*proof*⟩

lemma *fps-compose-fps-X*[*simp*]: *a oo fps-X* = (*a* :: '*a*::*comm-ring-1 fps*)
⟨*proof*⟩

lemma *fps-const-compose*[*simp*]: *fps-const* (*a*::'*a*::*comm-ring-1*) *oo b* = *fps-const a*
⟨*proof*⟩

lemma *numeral-compose*[*simp*]: (*numeral k* :: '*a*::*comm-ring-1 fps*) *oo b* = *numeral k*
⟨*proof*⟩

lemma *neg-numeral-compose*[*simp*]: (− *numeral k* :: '*a*::*comm-ring-1 fps*) *oo b* = − *numeral k*
⟨*proof*⟩

lemma *fps-X-fps-compose-startby0*[*simp*]: *a*$*0* = *0* ⟹ *fps-X oo a* = (*a* :: '*a*::*comm-ring-1 fps*)
⟨*proof*⟩

5.15 Rules from Herbert Wilf's Generatingfunctionology

5.15.1 Rule 1

lemma *fps-power-mult-eq-shift*:
fps-X^*Suc k* * *Abs-fps* (λ*n*. *a* (*n* + *Suc k*)) =
Abs-fps a − *sum* (λ*i*. *fps-const* (*a i* :: '*a*::*comm-ring-1*) * *fps-X*^*i*) {*0* .. *k*}
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

5.15.2 Rule 2

definition *fps-XD* = (*) *fps-X* ∘ *fps-deriv*

lemma *fps-XD-add*[*simp*]:*fps-XD* (*a* + *b*) = *fps-XD a* + *fps-XD* (*b* :: *'a*::*comm-ring-1 fps*)
⟨*proof*⟩

lemma *fps-XD-mult-const*[*simp*]:*fps-XD* (*fps-const* (*c*::*'a*::*comm-ring-1*) ∗ *a*) = *fps-const c* ∗ *fps-XD a*
⟨*proof*⟩

lemma *fps-XD-linear*[*simp*]: *fps-XD* (*fps-const c* ∗ *a* + *fps-const d* ∗ *b*) =
fps-const c ∗ *fps-XD a* + *fps-const d* ∗ *fps-XD* (*b* :: *'a*::*comm-ring-1 fps*)
⟨*proof*⟩

lemma *fps-XDN-linear*:
(*fps-XD* ^^ *n*) (*fps-const c* ∗ *a* + *fps-const d* ∗ *b*) =
fps-const c ∗ (*fps-XD* ^^ *n*) *a* + *fps-const d* ∗ (*fps-XD* ^^ *n*) (*b* :: *'a*::*comm-ring-1 fps*)
⟨*proof*⟩

lemma *fps-mult-fps-X-deriv-shift*: *fps-X*∗ *fps-deriv a* = *Abs-fps* (λ*n*. *of-nat n*∗ *a*$*n*)
⟨*proof*⟩

lemma *fps-mult-fps-XD-shift*:
(*fps-XD* ^^ *k*) (*a* :: *'a*::*comm-ring-1 fps*) = *Abs-fps* (λ*n*. (*of-nat n* ^ *k*) ∗ *a*$*n*)
⟨*proof*⟩

5.15.3 Rule 3

Rule 3 is trivial and is given by `fps_times_def`.

5.15.4 Rule 5 — summation and "division" by $1 - X$

lemma *fps-divide-fps-X-minus1-sum-lemma*:
a = ((*1*::*'a*::*ring-1 fps*) − *fps-X*) ∗ *Abs-fps* (λ*n*. *sum* (λ*i*. *a* $ *i*) {*0*..*n*})
⟨*proof*⟩

lemma *fps-divide-fps-X-minus1-sum-ring1*:
assumes *inverse 1* = (*1*::*'a*::{*ring-1*,*inverse*})
shows *a* /((*1*::*'a fps*) − *fps-X*) = *Abs-fps* (λ*n*. *sum* (λ*i*. *a* $ *i*) {*0*..*n*})
⟨*proof*⟩

lemma *fps-divide-fps-X-minus1-sum*:
a /((*1*::*'a*::*division-ring fps*) − *fps-X*) = *Abs-fps* (λ*n*. *sum* (λ*i*. *a* $ *i*) {*0*..*n*})
⟨*proof*⟩

5.15.5 Rule 4 in its more general form

This generalizes Rule 3 for an arbitrary finite product of FPS, also the relevant instance of powers of a FPS.

definition *natpermute n k* = {*l* :: *nat list*. *length l* = *k* ∧ *sum-list l* = *n*}

lemma *natlist-trivial-1*: *natpermute n 1* = {[*n*]}
⟨*proof*⟩

lemma *natlist-trivial-Suc0* [*simp*]: *natpermute n* (*Suc 0*) = {[*n*]}
⟨*proof*⟩

lemma *append-natpermute-less-eq*:
assumes *xs* @ *ys* ∈ *natpermute n k*
shows *sum-list xs* ≤ *n*
and *sum-list ys* ≤ *n*
⟨*proof*⟩

lemma *natpermute-split*:
assumes *h* ≤ *k*
shows *natpermute n k* =
(⋃ *m* ∈{*0*..*n*}. {*l1* @ *l2* |*l1 l2*. *l1* ∈ *natpermute m h* ∧ *l2* ∈ *natpermute* (*n* − *m*) (*k* − *h*)})
(**is** *?L* = *?R* **is** - = (⋃ *m* ∈{*0*..*n*}. *?S m*))
⟨*proof*⟩

lemma *natpermute-0*: *natpermute n 0* = (*if n* = *0 then* {[]} *else* {})
⟨*proof*⟩

lemma *natpermute-0'*[*simp*]: *natpermute 0 k* = (*if k* = *0 then* {[]} *else* {*replicate k 0*})
⟨*proof*⟩

lemma *natpermute-finite*: *finite* (*natpermute n k*)
⟨*proof*⟩

lemma *natpermute-contain-maximal*:
{*xs* ∈ *natpermute n* (*k* + *1*). *n* ∈ *set xs*} = (⋃ *i*∈{*0* .. *k*}. {(*replicate* (*k* + *1*) *0*) [*i*:=*n*]})
(**is** *?A* = *?B*)
⟨*proof*⟩

The general form.

lemma *fps-prod-nth*:
fixes *m* :: *nat*
and *a* :: *nat* ⇒ *'a*::*comm-ring-1 fps*
shows (*prod a* {*0* .. *m*}) $ *n* =
sum (λ*v*. *prod* (λ*j*. (*a j*) $ (*v*!*j*)) {*0*..*m*}) (*natpermute n* (*m*+*1*))
(**is** *?P m n*)
⟨*proof*⟩

The special form for powers.

lemma *fps-power-nth-Suc*:

fixes $m :: nat$
and $a :: 'a::comm\text{-}ring\text{-}1\ fps$
shows $(a \hat{} Suc\ m)\$n = sum\ (\lambda v.\ prod\ (\lambda j.\ a\ \$\ (v!j))\ \{0..m\})\ (natpermute\ n\ (m+1))$
⟨*proof*⟩

lemma *fps-power-nth*:
fixes $m :: nat$
and $a :: 'a::comm\text{-}ring\text{-}1\ fps$
shows $(a \hat{} m)\$n =$
$(if\ m=0\ then\ 1\$n\ else\ sum\ (\lambda v.\ prod\ (\lambda j.\ a\ \$\ (v!j))\ \{0..m-1\})\ (natpermute\ n\ m))$
⟨*proof*⟩

lemmas *fps-nth-power-0 = fps-power-zeroth*

lemma *natpermute-max-card*:
assumes *n0*: $n \neq 0$
shows $card\ \{xs \in natpermute\ n\ (k+1).\ n \in set\ xs\} = k+1$
⟨*proof*⟩

lemma *fps-power-Suc-nth*:
fixes $f :: 'a :: comm\text{-}ring\text{-}1\ fps$
assumes *k*: $k > 0$
shows $(f \hat{} Suc\ m)\ \$\ k =$
$of\text{-}nat\ (Suc\ m) * (f\ \$\ k * (f\ \$\ 0) \hat{} m) +$
$(\sum v \in \{v \in natpermute\ k\ (m+1).\ k \notin set\ v\}.\ \prod j = 0..m.\ f\ \$\ v\ !\ j)$
⟨*proof*⟩

lemma *fps-power-Suc-eqD*:
fixes $f\ g :: 'a :: \{idom,semiring\text{-}char\text{-}0\}\ fps$
assumes $f \hat{} Suc\ m = g \hat{} Suc\ m\ f\ \$\ 0 = g\ \$\ 0\ f\ \$\ 0 \neq 0$
shows $f = g$
⟨*proof*⟩

lemma *fps-power-Suc-eqD'*:
fixes $f\ g :: 'a :: \{idom,semiring\text{-}char\text{-}0\}\ fps$
assumes $f \hat{} Suc\ m = g \hat{} Suc\ m\ f\ \$\ subdegree\ f = g\ \$\ subdegree\ g$
shows $f = g$
⟨*proof*⟩

lemma *fps-power-eqD'*:
fixes $f\ g :: 'a :: \{idom,semiring\text{-}char\text{-}0\}\ fps$
assumes $f \hat{} m = g \hat{} m\ f\ \$\ subdegree\ f = g\ \$\ subdegree\ g\ m > 0$
shows $f = g$
⟨*proof*⟩

lemma *fps-power-eqD*:
fixes $f\ g :: 'a :: \{idom,semiring\text{-}char\text{-}0\}\ fps$

assumes $f \hat{} m = g \hat{} m$ $f \$ 0 = g \$ 0$ $f \$ 0 \neq 0$ $m > 0$
shows $f = g$
⟨*proof*⟩

lemma *fps-compose-inj-right*:
assumes *a0*: $a\$0 = (0::'a::idom)$
and *a1*: $a\$1 \neq 0$
shows $(b \ oo \ a = c \ oo \ a) \longleftrightarrow b = c$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

5.16 Radicals

declare *prod.cong* [*fundef-cong*]

function *radical* :: $(nat \Rightarrow 'a \Rightarrow 'a) \Rightarrow nat \Rightarrow 'a::field\ fps \Rightarrow nat \Rightarrow 'a$
where
radical r 0 a 0 = *1*
| *radical r 0 a* (*Suc n*) = *0*
| *radical r* (*Suc k*) *a 0* = *r* (*Suc k*) (*a$0*)
| *radical r* (*Suc k*) *a* (*Suc n*) =
(*a$ Suc n − sum* (*λxs. prod* (*λj. radical r* (*Suc k*) *a* (*xs ! j*)) {*0..k*})
{*xs. xs ∈ natpermute* (*Suc n*) (*Suc k*) *∧ Suc n ∉ set xs*}) /
(*of-nat* (*Suc k*) *∗* (*radical r* (*Suc k*) *a 0*)^*k*)
⟨*proof*⟩

termination *radical*
⟨*proof*⟩

definition *fps-radical r n a* = *Abs-fps* (*radical r n a*)

lemma *radical-0* [*simp*]: $\bigwedge n.\ 0 < n \Longrightarrow$ *radical r 0 a n* = *0*
⟨*proof*⟩

lemma *fps-radical0*[*simp*]: *fps-radical r 0 a* = *1*
⟨*proof*⟩

lemma *fps-radical-nth-0*[*simp*]: *fps-radical r n a $ 0* = (*if n* = *0 then 1 else r n* (*a$0*))
⟨*proof*⟩

lemma *fps-radical-power-nth*[*simp*]:
assumes *r*: (*r k* (*a$0*)) ^ *k* = *a$0*
shows *fps-radical r k a* ^ *k $ 0* = (*if k* = *0 then 1 else a$0*)
⟨*proof*⟩

lemma *power-radical*:
fixes *a*:: *'a*::*field-char-0 fps*
assumes *a0*: *a$0 ≠ 0*

shows $(r\ (Suc\ k)\ (a\$0))\ \hat{}\ Suc\ k = a\$0 \longleftrightarrow (fps\text{-}radical\ r\ (Suc\ k)\ a)\ \hat{}\ (Suc\ k) = a$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *radical-unique*:
assumes *r0*: $(r\ (Suc\ k)\ (b\$0))\ \hat{}\ Suc\ k = b\$0$
and *a0*: $r\ (Suc\ k)\ (b\$0 :: 'a :: field\text{-}char\text{-}0) = a\$0$
and *b0*: $b\$0 \neq 0$
shows $a\hat{}(Suc\ k) = b \longleftrightarrow a = fps\text{-}radical\ r\ (Suc\ k)\ b$
(**is** *?lhs* $\longleftrightarrow$ *?rhs* **is** - $\longleftrightarrow$ $a = ?r$)
⟨*proof*⟩

lemma *radical-power*:
assumes *r0*: $r\ (Suc\ k)\ ((a\$0)\ \hat{}\ Suc\ k) = a\$0$
and *a0*: $(a\$0 :: 'a :: field\text{-}char\text{-}0) \neq 0$
shows $(fps\text{-}radical\ r\ (Suc\ k)\ (a\ \hat{}\ Suc\ k)) = a$
⟨*proof*⟩

lemma *fps-deriv-radical'*:
fixes $a :: 'a :: field\text{-}char\text{-}0\ fps$
assumes *r0*: $(r\ (Suc\ k)\ (a\$0))\ \hat{}\ Suc\ k = a\$0$
and *a0*: $a\$0 \neq 0$
shows $fps\text{-}deriv\ (fps\text{-}radical\ r\ (Suc\ k)\ a) =$
$fps\text{-}deriv\ a\ /\ ((of\text{-}nat\ (Suc\ k)) * (fps\text{-}radical\ r\ (Suc\ k)\ a)\ \hat{}\ k)$
⟨*proof*⟩

lemma *fps-deriv-radical*:
fixes $a :: 'a :: field\text{-}char\text{-}0\ fps$
assumes *r0*: $(r\ (Suc\ k)\ (a\$0))\ \hat{}\ Suc\ k = a\$0$
and *a0*: $a\$0 \neq 0$
shows $fps\text{-}deriv\ (fps\text{-}radical\ r\ (Suc\ k)\ a) =$
$fps\text{-}deriv\ a\ /\ (fps\text{-}const\ (of\text{-}nat\ (Suc\ k)) * (fps\text{-}radical\ r\ (Suc\ k)\ a)\ \hat{}\ k)$
⟨*proof*⟩

lemma *radical-mult-distrib*:
fixes $a :: 'a :: field\text{-}char\text{-}0\ fps$
assumes *k*: $k > 0$
and *ra0*: $r\ k\ (a\ \$\ 0)\ \hat{}\ k = a\ \$\ 0$
and *rb0*: $r\ k\ (b\ \$\ 0)\ \hat{}\ k = b\ \$\ 0$
and *a0*: $a\ \$\ 0 \neq 0$
and *b0*: $b\ \$\ 0 \neq 0$
shows $r\ k\ ((a * b)\ \$\ 0) = r\ k\ (a\ \$\ 0) * r\ k\ (b\ \$\ 0) \longleftrightarrow$
$fps\text{-}radical\ r\ k\ (a * b) = fps\text{-}radical\ r\ k\ a * fps\text{-}radical\ r\ k\ b$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *radical-divide*:
fixes *a* :: *'a*::*field-char-0 fps*
assumes *kp*: *k* > *0*
and *ra0*: (*r k* (*a* $ *0*)) ^ *k* = *a* $ *0*
and *rb0*: (*r k* (*b* $ *0*)) ^ *k* = *b* $ *0*
and *a0*: *a*$*0* ≠ *0*
and *b0*: *b*$*0* ≠ *0*
shows *r k* ((*a* $ *0*) / (*b*$*0*)) = *r k* (*a*$*0*) / *r k* (*b* $ *0*) ⟷
fps-radical r k (*a*/*b*) = *fps-radical r k a* / *fps-radical r k b*
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *radical-inverse*:
fixes *a* :: *'a*::*field-char-0 fps*
assumes *k*: *k* > *0*
and *ra0*: *r k* (*a* $ *0*) ^ *k* = *a* $ *0*
and *r1*: (*r k 1*)^*k* = *1*
and *a0*: *a*$*0* ≠ *0*
shows *r k* (*inverse* (*a* $ *0*)) = *r k 1* / (*r k* (*a* $ *0*)) ⟷
fps-radical r k (*inverse a*) = *fps-radical r k 1* / *fps-radical r k a*
⟨*proof*⟩

5.17 Chain rule

lemma *fps-compose-deriv*:
fixes *a* :: *'a*::*idom fps*
assumes *b0*: *b*$*0* = *0*
shows *fps-deriv* (*a oo b*) = ((*fps-deriv a*) *oo b*) ∗ *fps-deriv b*
⟨*proof*⟩

lemma *fps-poly-sum-fps-X*:
assumes ∀ *i* > *n*. *a*$*i* = *0*
shows *a* = *sum* (λ*i*. *fps-const* (*a*$*i*) ∗ *fps-X*^*i*) {*0*..*n*} (**is** *a* = *?r*)
⟨*proof*⟩

5.18 Compositional inverses

fun *compinv* :: *'a fps* ⇒ *nat* ⇒ *'a*::*field*
where
compinv a 0 = *fps-X*$*0*
| *compinv a* (*Suc n*) =
(*fps-X*$ *Suc n* − *sum* (λ*i*. (*compinv a i*) ∗ (*a*^*i*)$*Suc n*) {*0* .. *n*}) / (*a*$*1*) ^
Suc n

definition *fps-inv a* = *Abs-fps* (*compinv a*)

lemma *fps-inv*:
assumes *a0*: *a*$*0* = *0*
and *a1*: *a*$*1* ≠ *0*

shows *fps-inv a oo a = fps-X*
⟨*proof*⟩

fun *gcompinv* :: *'a fps ⇒ 'a fps ⇒ nat ⇒ 'a::field*
where
gcompinv b a 0 = b$0
| *gcompinv b a (Suc n) =*
(b$ Suc n − sum (λi. (gcompinv b a i) ∗ (a^i)$Suc n) {0 .. n}) / (a$1) ^ Suc n

definition *fps-ginv b a = Abs-fps (gcompinv b a)*

lemma *fps-ginv*:
assumes *a0*: *a$0 = 0*
and *a1*: *a$1 ≠ 0*
shows *fps-ginv b a oo a = b*
⟨*proof*⟩

lemma *fps-inv-ginv*: *fps-inv = fps-ginv fps-X*
⟨*proof*⟩

lemma *fps-compose-1*[*simp*]: *1 oo a = 1*
⟨*proof*⟩

lemma *fps-compose-0*[*simp*]: *0 oo a = 0*
⟨*proof*⟩

lemma *fps-compose-0-right*[*simp*]: *a oo 0 = fps-const (a $ 0)*
⟨*proof*⟩

lemma *fps-compose-add-distrib*: *(a + b) oo c = (a oo c) + (b oo c)*
⟨*proof*⟩

lemma *fps-compose-sum-distrib*: *(sum f S) oo a = sum (λi. f i oo a) S*
⟨*proof*⟩

lemma *convolution-eq*:
sum (λi. a (i :: nat) ∗ b (n − i)) {0 .. n} =
sum (λ(i,j). a i ∗ b j) {(i,j). i ≤ n ∧ j ≤ n ∧ i + j = n}
⟨*proof*⟩

lemma *product-composition-lemma*:
assumes *c0*: *c$0 = (0::'a::idom)*
and *d0*: *d$0 = 0*
shows *((a oo c) ∗ (b oo d))$n =*
sum (λ(k,m). a$k ∗ b$m ∗ (c^k ∗ d^m) $ n) {(k,m). k + m ≤ n} (**is** *?l = ?r*)
⟨*proof*⟩

lemma *sum-pair-less-iff*:
  sum (λ((*k*::*nat*),*m*). *a k* ∗ *b m* ∗ *c* (*k* + *m*)) {(*k*,*m*). *k* + *m* ≤ *n*} =
    sum (λ*s*. *sum* (λ*i*. *a i* ∗ *b* (*s* − *i*) ∗ *c s*) {*0*..*s*}) {*0*..*n*}
  (**is** *?l* = *?r*)
⟨*proof*⟩

lemma *fps-compose-mult-distrib-lemma*:
  assumes *c0*: *c*$*0* = (*0*::′*a*::*idom*)
  shows ((*a oo c*) ∗ (*b oo c*))$*n* = *sum* (λ*s*. *sum* (λ*i*. *a*$*i* ∗ *b*$(*s* − *i*) ∗ (*c*^*s*) $ *n*) {*0*..*s*}) {*0*..*n*}
  ⟨*proof*⟩

lemma *fps-compose-mult-distrib*:
  assumes *c0*: *c* $ *0* = (*0*::′*a*::*idom*)
  shows (*a* ∗ *b*) *oo c* = (*a oo c*) ∗ (*b oo c*)
⟨*proof*⟩

lemma *fps-compose-prod-distrib*:
  assumes *c0*: *c*$*0* = (*0*::′*a*::*idom*)
  shows *prod a S oo c* = *prod* (λ*k*. *a k oo c*) *S*
⟨*proof*⟩

lemma *fps-compose-divide*:
  assumes [*simp*]: *g dvd f h* $ *0* = *0*
  shows *fps-compose f h* = *fps-compose* (*f* / *g* :: ′*a* :: *field fps*) *h* ∗ *fps-compose g h*
⟨*proof*⟩

lemma *fps-compose-divide-distrib*:
  assumes *g dvd f h* $ *0* = *0 fps-compose g h* ≠ *0*
  shows *fps-compose* (*f* / *g* :: ′*a* :: *field fps*) *h* = *fps-compose f h* / *fps-compose g h*
  ⟨*proof*⟩

lemma *fps-compose-power*:
  assumes *c0*: *c*$*0* = (*0*::′*a*::*idom*)
  shows (*a oo c*)^*n* = *a*^*n oo c*
⟨*proof*⟩

lemma *fps-compose-uminus*: − (*a*::′*a*::*ring-1 fps*) *oo c* = − (*a oo c*)
  ⟨*proof*⟩

lemma *fps-compose-sub-distrib*: (*a* − *b*) *oo* (*c*::′*a*::*ring-1 fps*) = (*a oo c*) − (*b oo c*)
  ⟨*proof*⟩

lemma *fps-X-fps-compose*: *fps-X oo a* = *Abs-fps* (λ*n*. *if n* = *0 then* (*0*::′*a*::*comm-ring-1*) *else a*$*n*)

⟨*proof*⟩

lemma *fps-compose-eq-0-iff*:
fixes *F G* :: *'a* :: *idom fps*
assumes *fps-nth G 0* = *0*
shows *fps-compose F G* = *0* ⟷ *F* = *0* ∨ (*G* = *0* ∧ *fps-nth F 0* = *0*)
⟨*proof*⟩

lemma *subdegree-fps-compose* [*simp*]:
fixes *F G* :: *'a* :: *idom fps*
assumes [*simp*]: *fps-nth G 0* = *0*
shows *subdegree* (*fps-compose F G*) = *subdegree F* ∗ *subdegree G*
⟨*proof*⟩

lemma *fps-inverse-compose*:
assumes *b0*: (*b*$*0* :: *'a*::*field*) = *0*
and *a0*: *a*$*0* ≠ *0*
shows *inverse a oo b* = *inverse* (*a oo b*)
⟨*proof*⟩

lemma *fps-divide-compose*:
assumes *c0*: (*c*$*0* :: *'a*::*field*) = *0*
and *b0*: *b*$*0* ≠ *0*
shows (*a*/*b*) *oo c* = (*a oo c*) / (*b oo c*)
⟨*proof*⟩

lemma *gp*:
assumes *a0*: *a*$*0* = (*0*::*'a*::*field*)
shows (*Abs-fps* (λ*n*. *1*)) *oo a* = *1*/(*1* − *a*)
(**is** *?one oo a* = -)
⟨*proof*⟩

lemma *fps-compose-radical*:
assumes *b0*: *b*$*0* = (*0*::*'a*::*field-char-0*)
and *ra0*: *r* (*Suc k*) (*a*$*0*) ^ *Suc k* = *a*$*0*
and *a0*: *a*$*0* ≠ *0*
shows *fps-radical r* (*Suc k*) *a oo b* = *fps-radical r* (*Suc k*) (*a oo b*)
⟨*proof*⟩

lemma *fps-const-mult-apply-left*: *fps-const c* ∗ (*a oo b*) = (*fps-const c* ∗ *a*) *oo b*
⟨*proof*⟩

lemma *fps-const-mult-apply-right*:
(*a oo b*) ∗ *fps-const* (*c*::*'a*::*comm-semiring-1*) = (*fps-const c* ∗ *a*) *oo b*
⟨*proof*⟩

lemma *fps-compose-assoc*:
assumes *c0*: *c*$*0* = (*0*::*'a*::*idom*)
and *b0*: *b*$*0* = *0*

shows *a oo (b oo c) = a oo b oo c* (**is** *?l = ?r*)
⟨*proof*⟩

lemma *fps-X-power-compose*:
assumes *a0*: *a$0=0*
shows *fps-X^k oo a = (a::'a::idom fps)^k*
(**is** *?l = ?r*)
⟨*proof*⟩

lemma *fps-inv-right*:
assumes *a0*: *a$0 = 0*
and *a1*: *a$1 ≠ 0*
shows *a oo fps-inv a = fps-X*
⟨*proof*⟩

lemma *fps-inv-deriv*:
assumes *a0*: *a$0 = (0::'a::field)*
and *a1*: *a$1 ≠ 0*
shows *fps-deriv (fps-inv a) = inverse (fps-deriv a oo fps-inv a)*
⟨*proof*⟩

lemma *fps-inv-idempotent*:
assumes *a0*: *a$0 = 0*
and *a1*: *a$1 ≠ 0*
shows *fps-inv (fps-inv a) = a*
⟨*proof*⟩

lemma *fps-ginv-ginv*:
assumes *a0*: *a$0 = 0*
and *a1*: *a$1 ≠ 0*
and *c0*: *c$0 = 0*
and *c1*: *c$1 ≠ 0*
shows *fps-ginv b (fps-ginv c a) = b oo a oo fps-inv c*
⟨*proof*⟩

lemma *fps-ginv-deriv*:
assumes *a0*:*a$0 = (0::'a::field)*
and *a1*: *a$1 ≠ 0*
shows *fps-deriv (fps-ginv b a) = (fps-deriv b / fps-deriv a) oo fps-ginv fps-X a*
⟨*proof*⟩

lemma *fps-compose-linear*:
*fps-compose (f :: 'a :: comm-ring-1 fps) (fps-const c * fps-X) = Abs-fps (λn. c^n * f $ n)*
⟨*proof*⟩

lemma *fps-compose-uminus'*:
*fps-compose f (−fps-X :: 'a :: comm-ring-1 fps) = Abs-fps (λn. (−1)^n * f $ n)*

⟨*proof*⟩
lemma *fps-nth-compose-linear* [*simp*]:
fixes *f* :: *'a* :: *comm-ring-1 fps*
shows *fps-nth* (*fps-compose f* (*fps-const c* ∗ *fps-X*)) *n* = *c* ^ *n* ∗ *fps-nth f n*
⟨*proof*⟩

5.19 Elementary series

5.19.1 Exponential series

definition *fps-exp x* = *Abs-fps* (λ*n*. *x*^*n* / *of-nat* (*fact n*))

lemma *fps-exp-deriv*[*simp*]: *fps-deriv* (*fps-exp a*) = *fps-const* (*a*::*'a*::*field-char-0*) ∗ *fps-exp a*
(**is** *?l* = *?r*)
⟨*proof*⟩

lemma *fps-exp-unique-ODE*:
fps-deriv a = *fps-const c* ∗ *a* ⟷ *a* = *fps-const* (*a*$*0*) ∗ *fps-exp* (*c*::*'a*::*field-char-0*)
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *fps-exp-add-mult*: *fps-exp* (*a* + *b*) = *fps-exp* (*a*::*'a*::*field-char-0*) ∗ *fps-exp b* (**is** *?l* = *?r*)
⟨*proof*⟩

lemma *fps-exp-nth*[*simp*]: *fps-exp a* $ *n* = *a*^*n* / *of-nat* (*fact n*)
⟨*proof*⟩

lemma *fps-exp-0*[*simp*]: *fps-exp* (*0*::*'a*::*field*) = *1*
⟨*proof*⟩

lemma *fps-exp-neg*: *fps-exp* (− *a*) = *inverse* (*fps-exp* (*a*::*'a*::*field-char-0*))
⟨*proof*⟩

lemma *fps-exp-nth-deriv*[*simp*]:
fps-nth-deriv n (*fps-exp* (*a*::*'a*::*field-char-0*)) = (*fps-const a*)^*n* ∗ (*fps-exp a*)
⟨*proof*⟩

lemma *fps-X-compose-fps-exp*[*simp*]: *fps-X oo fps-exp* (*a*::*'a*::*field*) = *fps-exp a* − *1*
⟨*proof*⟩

lemma *fps-inv-fps-exp-compose*:
assumes *a*: *a* ≠ *0*
shows *fps-inv* (*fps-exp a* − *1*) *oo* (*fps-exp a* − *1*) = *fps-X*
and (*fps-exp a* − *1*) *oo fps-inv* (*fps-exp a* − *1*) = *fps-X*
⟨*proof*⟩

lemma *fps-exp-power-mult*: (*fps-exp* (*c*::*'a*::*field-char-0*))^*n* = *fps-exp* (*of-nat n* ∗

c)
⟨*proof*⟩

lemma *radical-fps-exp*:
assumes *r*: *r* (*Suc k*) *1* = *1*
shows *fps-radical r* (*Suc k*) (*fps-exp* (*c*::'*a*::*field-char-0*)) = *fps-exp* (*c* / *of-nat* (*Suc k*))
⟨*proof*⟩

lemma *fps-exp-compose-linear* [*simp*]:
fps-exp (*d*::'*a*::*field-char-0*) *oo* (*fps-const c* ∗ *fps-X*) = *fps-exp* (*c* ∗ *d*)
⟨*proof*⟩

lemma *fps-fps-exp-compose-minus* [*simp*]:
fps-compose (*fps-exp c*) (−*fps-X*) = *fps-exp* (−*c* :: '*a* :: *field-char-0*)
⟨*proof*⟩

lemma *fps-exp-eq-iff* [*simp*]: *fps-exp c* = *fps-exp d* ⟷ *c* = (*d* :: '*a* :: *field-char-0*)
⟨*proof*⟩

lemma *fps-exp-eq-fps-const-iff* [*simp*]:
fps-exp (*c* :: '*a* :: *field-char-0*) = *fps-const c'* ⟷ *c* = *0* ∧ *c'* = *1*
⟨*proof*⟩

lemma *fps-exp-neq-0* [*simp*]: ¬*fps-exp* (*c* :: '*a* :: *field-char-0*) = *0*
⟨*proof*⟩

lemma *fps-exp-eq-1-iff* [*simp*]: *fps-exp* (*c* :: '*a* :: *field-char-0*) = *1* ⟷ *c* = *0*
⟨*proof*⟩

lemma *fps-exp-neq-numeral-iff* [*simp*]:
fps-exp (*c* :: '*a* :: *field-char-0*) = *numeral n* ⟷ *c* = *0* ∧ *n* = *Num.One*
⟨*proof*⟩

5.19.2 Logarithmic series

lemma *Abs-fps-if-0*:
Abs-fps (λ*n*. *if n* = *0 then* (*v*::'*a*::*ring-1*) *else f n*) =
fps-const v + *fps-X* ∗ *Abs-fps* (λ*n*. *f* (*Suc n*))
⟨*proof*⟩

definition *fps-ln* :: '*a*::*field-char-0* ⇒ '*a fps*
where *fps-ln c* = *fps-const* (*1*/*c*) ∗ *Abs-fps* (λ*n*. *if n* = *0 then 0 else* (− *1*) ^ (*n* − *1*) / *of-nat n*)

lemma *fps-ln-deriv*: *fps-deriv* (*fps-ln c*) = *fps-const* (*1*/*c*) ∗ *inverse* (*1* + *fps-X*)
⟨*proof*⟩

lemma *fps-ln-nth*: *fps-ln c* $ *n* = (*if n* = *0 then 0 else 1*/*c* ∗ ((− *1*) ^ (*n* − *1*) /

of-nat n))
⟨*proof*⟩

lemma *fps-ln-0* [*simp*]: *fps-ln c $ 0 = 0* ⟨*proof*⟩

lemma *fps-ln-fps-exp-inv*:
fixes *a* :: *'a*::*field-char-0*
assumes *a*: *a ≠ 0*
shows *fps-ln a = fps-inv* (*fps-exp a − 1*) (**is** *?l = ?r*)
⟨*proof*⟩

lemma *fps-ln-mult-add*:
assumes *c0*: *c≠0*
and *d0*: *d≠0*
shows *fps-ln c + fps-ln d = fps-const* (*c+d*) ∗ *fps-ln* (*c∗d*)
(**is** *?r = ?l*)
⟨*proof*⟩

lemma *fps-X-dvd-fps-ln* [*simp*]: *fps-X dvd fps-ln c*
⟨*proof*⟩

5.19.3 Binomial series

definition *fps-binomial a = Abs-fps* (*λn. a gchoose n*)

lemma *fps-binomial-nth*[*simp*]: *fps-binomial a $ n = a gchoose n*
⟨*proof*⟩

lemma *fps-binomial-ODE-unique*:
fixes *c* :: *'a*::*field-char-0*
shows *fps-deriv a* = (*fps-const c ∗ a*) / (*1 + fps-X*) ⟷ *a = fps-const* (*a$0*) ∗ *fps-binomial c*
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *fps-binomial-ODE-unique'*:
(*fps-deriv a = fps-const c ∗ a* / (*1 + fps-X*) ∧ *a $ 0 = 1*) ⟷ (*a = fps-binomial c*)
⟨*proof*⟩

lemma *fps-binomial-deriv*: *fps-deriv* (*fps-binomial c*) = *fps-const c ∗ fps-binomial c* / (*1 + fps-X*)
⟨*proof*⟩

lemma *fps-binomial-add-mult*: *fps-binomial* (*c+d*) = *fps-binomial c ∗ fps-binomial d* (**is** *?l = ?r*)
⟨*proof*⟩

lemma *fps-binomial-minus-one*: *fps-binomial* (*− 1*) = *inverse* (*1 + fps-X*)

(**is** *?l = inverse ?r*)
⟨*proof*⟩

lemma *fps-binomial-of-nat*: *fps-binomial* (*of-nat n*) = (*1 + fps-X* :: *'a* :: *field-char-0 fps*) ^ *n*
⟨*proof*⟩

lemma *fps-binomial-0* [*simp*]: *fps-binomial 0 = 1*
⟨*proof*⟩

lemma *fps-binomial-power*: *fps-binomial a* ^ *n = fps-binomial* (*of-nat n* ∗ *a*)
⟨*proof*⟩

lemma *fps-binomial-1*: *fps-binomial 1 = 1 + fps-X*
⟨*proof*⟩

lemma *fps-binomial-minus-of-nat*:
fps-binomial (− *of-nat n*) = *inverse* ((*1 + fps-X* :: *'a* :: *field-char-0 fps*) ^ *n*)
⟨*proof*⟩

lemma *one-minus-const-fps-X-power*:
c ≠ *0* ⟹ (*1 − fps-const c* ∗ *fps-X*) ^ *n* =
fps-compose (*fps-binomial* (*of-nat n*)) (−*fps-const c* ∗ *fps-X*)
⟨*proof*⟩

lemma *one-minus-fps-X-const-neg-power*:
inverse ((*1 − fps-const c* ∗ *fps-X*) ^ *n*) =
fps-compose (*fps-binomial* (−*of-nat n*)) (−*fps-const c* ∗ *fps-X*)
⟨*proof*⟩

lemma *fps-X-plus-const-power*:
c ≠ *0* ⟹ (*fps-X + fps-const c*) ^ *n* =
fps-const (*c*^*n*) ∗ *fps-compose* (*fps-binomial* (*of-nat n*)) (*fps-const* (*inverse c*) ∗ *fps-X*)
⟨*proof*⟩

lemma *fps-X-plus-const-neg-power*:
c ≠ *0* ⟹ *inverse* ((*fps-X + fps-const c*) ^ *n*) =
fps-const (*inverse c*^*n*) ∗ *fps-compose* (*fps-binomial* (−*of-nat n*)) (*fps-const* (*inverse c*) ∗ *fps-X*)
⟨*proof*⟩

lemma *one-minus-const-fps-X-neg-power′*:
fixes *c* :: *'a* :: *field-char-0*
assumes *n > 0*
shows *inverse* ((*1 − fps-const c* ∗ *fps-X*) ^ *n*) = *Abs-fps* (λ*k*. *of-nat* ((*n + k − 1*) *choose k*) ∗ *c*^*k*)
⟨*proof*⟩

Vandermonde's Identity as a consequence.

lemma *gbinomial-Vandermonde*:
sum (λ*k*. (*a gchoose k*) ∗ (*b gchoose* (*n* − *k*))) {*0*..*n*} = (*a* + *b*) *gchoose n*
⟨*proof*⟩

lemma *binomial-Vandermonde*:
sum (λ*k*. (*a choose k*) ∗ (*b choose* (*n* − *k*))) {*0*..*n*} = (*a* + *b*) *choose n*
⟨*proof*⟩

lemma *binomial-Vandermonde-same*: *sum* (λ*k*. (*n choose k*)^2) {*0*..*n*} = (*2* ∗ *n*) *choose n*
⟨*proof*⟩

lemma *Vandermonde-pochhammer-lemma*:
fixes *a* :: 'a::*field-char-0*
assumes *b*: ⋀*j*. *j*<*n* ⟹ *b* ≠ *of-nat j*
shows *sum* (λ*k*. (*pochhammer* (− *a*) *k* ∗ *pochhammer* (− (*of-nat n*)) *k*) /
(*of-nat* (*fact k*) ∗ *pochhammer* (*b* − *of-nat n* + *1*) *k*)) {*0*..*n*} =
pochhammer (− (*a* + *b*)) *n* / *pochhammer* (− *b*) *n*
(**is** *?l* = *?r*)
⟨*proof*⟩

lemma *Vandermonde-pochhammer*:
fixes *a* :: 'a::*field-char-0*
assumes *c*: ∀ *i* ∈ {*0*..< *n*}. *c* ≠ − *of-nat i*
shows *sum* (λ*k*. (*pochhammer a k* ∗ *pochhammer* (− (*of-nat n*)) *k*) /
(*of-nat* (*fact k*) ∗ *pochhammer c k*)) {*0*..*n*} = *pochhammer* (*c* − *a*) *n* / *pochhammer c n*
⟨*proof*⟩

5.19.4 Trigonometric functions

definition *fps-sin* (*c*::'a::*field-char-0*) =
Abs-fps (λ*n*. *if even n then 0 else* (− *1*) ^((*n* − *1*) *div 2*) ∗ *c*^*n* /(*of-nat* (*fact n*)))

definition *fps-cos* (*c*::'a::*field-char-0*) =
Abs-fps (λ*n*. *if even n then* (− *1*) ^(*n div 2*) ∗ *c*^*n* / (*of-nat* (*fact n*)) *else 0*)

lemma *fps-sin-0* [*simp*]: *fps-sin 0* = *0*
⟨*proof*⟩

lemma *fps-cos-0* [*simp*]: *fps-cos 0* = *1*
⟨*proof*⟩

lemma *fps-sin-deriv*:
fps-deriv (*fps-sin c*) = *fps-const c* ∗ *fps-cos c*
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *fps-cos-deriv*: *fps-deriv* (*fps-cos c*) = *fps-const* (− *c*)∗ (*fps-sin c*)
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *fps-sin-cos-sum-of-squares*: (*fps-cos c*)^2 + (*fps-sin c*)^2 = *1*
(**is** *?lhs* = -)
⟨*proof*⟩

lemma *fps-sin-nth-0* [*simp*]: *fps-sin c* $ *0* = *0*
⟨*proof*⟩

lemma *fps-sin-nth-1* [*simp*]: *fps-sin c* $ *Suc 0* = *c*
⟨*proof*⟩

lemma *fps-sin-nth-add-2*:
fps-sin c $ (*n* + *2*) = − (*c* ∗ *c* ∗ *fps-sin c* $ *n* / (*of-nat* (*n* + *1*) ∗ *of-nat* (*n* + *2*)))
⟨*proof*⟩

lemma *fps-cos-nth-0* [*simp*]: *fps-cos c* $ *0* = *1*
⟨*proof*⟩

lemma *fps-cos-nth-1* [*simp*]: *fps-cos c* $ *Suc 0* = *0*
⟨*proof*⟩

lemma *fps-cos-nth-add-2*:
fps-cos c $ (*n* + *2*) = − (*c* ∗ *c* ∗ *fps-cos c* $ *n* / (*of-nat* (*n* + *1*) ∗ *of-nat* (*n* + *2*)))
⟨*proof*⟩

lemma *nat-add-1-add-1*: (*n*::*nat*) + *1* + *1* = *n* + *2*
⟨*proof*⟩

lemma *eq-fps-sin*:
assumes *a0*: *a* $ *0* = *0*
and *a1*: *a* $ *1* = *c*
and *a2*: *fps-deriv* (*fps-deriv a*) = − (*fps-const c* ∗ *fps-const c* ∗ *a*)
shows *fps-sin c* = *a*
⟨*proof*⟩

lemma *eq-fps-cos*:
assumes *a0*: *a* $ *0* = *1*
and *a1*: *a* $ *1* = *0*
and *a2*: *fps-deriv* (*fps-deriv a*) = − (*fps-const c* ∗ *fps-const c* ∗ *a*)
shows *fps-cos c* = *a*
⟨*proof*⟩

lemma *fps-sin-add*: *fps-sin* $(a + b)$ = *fps-sin* a $*$ *fps-cos* b + *fps-cos* a $*$ *fps-sin* b
⟨*proof*⟩

lemma *fps-cos-add*: *fps-cos* $(a + b)$ = *fps-cos* a $*$ *fps-cos* b − *fps-sin* a $*$ *fps-sin* b
⟨*proof*⟩

lemma *fps-sin-even*: *fps-sin* $(-c)$ = − *fps-sin* c
⟨*proof*⟩

lemma *fps-cos-odd*: *fps-cos* $(-c)$ = *fps-cos* c
⟨*proof*⟩

definition *fps-tan* c = *fps-sin* c / *fps-cos* c

lemma *fps-tan-0* [*simp*]: *fps-tan* 0 = 0
⟨*proof*⟩

lemma *fps-tan-deriv*: *fps-deriv* (*fps-tan* c) = *fps-const* c / (*fps-cos* c)^2
⟨*proof*⟩

Connection to *fps-exp* over the complex numbers — Euler and de Moivre.

lemma *fps-exp-ii-sin-cos*: *fps-exp* (i $*$ c) = *fps-cos* c + *fps-const* i $*$ *fps-sin* c
(**is** *?l* = *?r*)
⟨*proof*⟩

lemma *fps-exp-minus-ii-sin-cos*: *fps-exp* $(-$ (i $*$ $c))$ = *fps-cos* c − *fps-const* i $*$ *fps-sin* c
⟨*proof*⟩

lemma *fps-cos-fps-exp-ii*: *fps-cos* c = (*fps-exp* (i $*$ c) + *fps-exp* $(-$ i $*$ $c))$ / *fps-const* 2
⟨*proof*⟩

lemma *fps-sin-fps-exp-ii*: *fps-sin* c = (*fps-exp* (i $*$ c) − *fps-exp* $(-$ i $*$ $c))$ / *fps-const* (2$*$i)
⟨*proof*⟩

lemma *fps-tan-fps-exp-ii*:
fps-tan c = (*fps-exp* (i $*$ c) − *fps-exp* $(-$ i $*$ $c))$ /
(*fps-const* i $*$ (*fps-exp* (i $*$ c) + *fps-exp* $(-$ i $*$ $c)))$
⟨*proof*⟩

lemma *fps-demoivre*:
(*fps-cos* a + *fps-const* i $*$ *fps-sin* $a)\hat{}n$ =
fps-cos (*of-nat* n $*$ a) + *fps-const* i $*$ *fps-sin* (*of-nat* n $*$ a)
⟨*proof*⟩

5.20 Hypergeometric series

definition *fps-hypergeo as bs* (*c*::*'a*::*field-char-0*) =
  Abs-fps (λ*n*. (*foldl* (λ*r a*. *r*∗ *pochhammer a n*) *1 as* ∗ *c*^*n*) /
    (*foldl* (λ*r b*. *r* ∗ *pochhammer b n*) *1 bs* ∗ *of-nat* (*fact n*)))

lemma *fps-hypergeo-nth*[*simp*]: *fps-hypergeo as bs c* $ *n* =
  (*foldl* (λ*r a*. *r*∗ *pochhammer a n*) *1 as* ∗ *c*^*n*) /
    (*foldl* (λ*r b*. *r* ∗ *pochhammer b n*) *1 bs* ∗ *of-nat* (*fact n*))
  ⟨*proof*⟩

lemma *foldl-mult-start*:
  fixes *v* :: *'a*::*comm-ring-1*
  shows *foldl* (λ*r x*. *r* ∗ *f x*) *v as* ∗ *x* = *foldl* (λ*r x*. *r* ∗ *f x*) (*v* ∗ *x*) *as*
  ⟨*proof*⟩

lemma *foldr-mult-foldl*:
  fixes *v* :: *'a*::*comm-ring-1*
  shows *foldr* (λ*x r*. *r* ∗ *f x*) *as v* = *foldl* (λ*r x*. *r* ∗ *f x*) *v as*
  ⟨*proof*⟩

lemma *fps-hypergeo-nth-alt*:
  fps-hypergeo as bs c $ *n* = *foldr* (λ*a r*. *r* ∗ *pochhammer a n*) *as* (*c* ^ *n*) /
    foldr (λ*b r*. *r* ∗ *pochhammer b n*) *bs* (*of-nat* (*fact n*))
  ⟨*proof*⟩

lemma *fps-hypergeo-fps-exp*[*simp*]: *fps-hypergeo* [] [] *c* = *fps-exp c*
  ⟨*proof*⟩

lemma *fps-hypergeo-1-0*[*simp*]: *fps-hypergeo* [*1*] [] *c* = *1*/(*1* − *fps-const c* ∗ *fps-X*)
⟨*proof*⟩

lemma *fps-hypergeo-B*[*simp*]: *fps-hypergeo* [−*a*] [] (− *1*) = *fps-binomial a*
  ⟨*proof*⟩

lemma *fps-hypergeo-0*[*simp*]: *fps-hypergeo as bs c* $ *0* = *1*
⟨*proof*⟩

lemma *foldl-prod-prod*:
  foldl (λ(*r*::*'b*::*comm-ring-1*) (*x*::*'a*::*comm-ring-1*). *r* ∗ *f x*) *v as* ∗ *foldl* (λ*r x*. *r* ∗
g x) *w as* =
    foldl (λ*r x*. *r* ∗ *f x* ∗ *g x*) (*v* ∗ *w*) *as*
  ⟨*proof*⟩

lemma *fps-hypergeo-rec*:
  fps-hypergeo as bs c $ *Suc n* = ((*foldl* (λ*r a*. *r*∗ (*a* + *of-nat n*)) *c as*) /
    (*foldl* (λ*r b*. *r* ∗ (*b* + *of-nat n*)) (*of-nat* (*Suc n*)) *bs*)) ∗ *fps-hypergeo as bs c* $
n
  ⟨*proof*⟩

lemma *fps-XD-nth*[*simp*]: *fps-XD a $ n = of-nat n * a$n*
⟨*proof*⟩

lemma *fps-XD-0th*[*simp*]: *fps-XD a $ 0 = 0*
⟨*proof*⟩
lemma *fps-XD-Suc*[*simp*]: *fps-XD a $ Suc n = of-nat (Suc n) * a $ Suc n*
⟨*proof*⟩

definition *fps-XDp c a = fps-XD a + fps-const c * a*

lemma *fps-XDp-nth*[*simp*]: *fps-XDp c a $ n = (c + of-nat n) * a$n*
⟨*proof*⟩

lemma *fps-XDp-commute*: *fps-XDp b ∘ fps-XDp (c::'a::comm-ring-1) = fps-XDp c ∘ fps-XDp b*
⟨*proof*⟩

lemma *fps-XDp0* [*simp*]: *fps-XDp 0 = fps-XD*
⟨*proof*⟩

lemma *fps-XDp-fps-integral* [*simp*]:
fixes *a :: 'a::{division-ring,ring-char-0} fps*
shows *fps-XDp 0 (fps-integral a c) = fps-X * a*
⟨*proof*⟩

lemma *fps-hypergeo-minus-nat*:
fps-hypergeo [− *of-nat n*] [− *of-nat* (*n* + *m*)] (*c::'a::field-char-0*) $ *k* =
(*if k ≤ n then*
pochhammer (− *of-nat n*) *k* * *c* ^ *k* / (*pochhammer* (− *of-nat* (*n* + *m*)) *k* * *of-nat* (*fact k*))
else 0)
fps-hypergeo [− *of-nat m*] [− *of-nat* (*m* + *n*)] (*c::'a::field-char-0*) $ *k* =
(*if k ≤ m then*
pochhammer (− *of-nat m*) *k* * *c* ^ *k* / (*pochhammer* (− *of-nat* (*m* + *n*)) *k* * *of-nat* (*fact k*))
else 0)
⟨*proof*⟩

lemma *pochhammer-rec-if*: *pochhammer a n = (if n = 0 then 1 else a * pochhammer* (*a* + *1*) (*n* − *1*))
⟨*proof*⟩

lemma *fps-XDp-foldr-nth* [*simp*]: *foldr* (λ*c r*. *fps-XDp c ∘ r*) *cs* (λ*c*. *fps-XDp c a*) *c0 $ n* =
foldr (λ*c r*. (*c* + *of-nat n*) * *r*) *cs* (*c0* + *of-nat n*) * *a$n*
⟨*proof*⟩

lemma *genric-fps-XDp-foldr-nth*:

assumes *f*: $\forall n\ c\ a.\ f\ c\ a\ \$\ n = (of\text{-}nat\ n + k\ c) * a\$n$
shows $foldr\ (\lambda c\ r.\ f\ c \circ r)\ cs\ (\lambda c.\ g\ c\ a)\ c0\ \$\ n =$
$foldr\ (\lambda c\ r.\ (k\ c + of\text{-}nat\ n) * r)\ cs\ (g\ c0\ a\ \$\ n)$
⟨*proof*⟩

lemma *dist-less-imp-nth-equal*:
assumes $dist\ f\ g < inverse\ (2\ \widehat{}\ i)$
and $j \leq i$
shows $f\ \$\ j = g\ \$\ j$
⟨*proof*⟩

lemma *nth-equal-imp-dist-less*:
assumes $\bigwedge j.\ j \leq i \Longrightarrow f\ \$\ j = g\ \$\ j$
shows $dist\ f\ g < inverse\ (2\ \widehat{}\ i)$
⟨*proof*⟩

lemma *dist-less-eq-nth-equal*: $dist\ f\ g < inverse\ (2\ \widehat{}\ i) \longleftrightarrow (\forall j \leq i.\ f\ \$\ j = g\ \$\ j)$
⟨*proof*⟩

instance *fps* :: (*comm-ring-1*) *complete-space*
⟨*proof*⟩

bundle *fps-syntax*
begin
notation *fps-nth* (**infixl** ‹$› 75)
end

unbundle *no fps-syntax*

end

6 Converting polynomials to formal power series

theory *Polynomial-FPS*
imports *Polynomial Formal-Power-Series*
begin

context
includes *fps-syntax*
begin

definition *fps-of-poly* **where**
$fps\text{-}of\text{-}poly\ p = Abs\text{-}fps\ (coeff\ p)$

lemma *fps-of-poly-eq-iff*: $fps\text{-}of\text{-}poly\ p = fps\text{-}of\text{-}poly\ q \longleftrightarrow p = q$
⟨*proof*⟩

lemma *fps-of-poly-nth* [*simp*]: *fps-of-poly p $ n = coeff p n*
⟨*proof*⟩

lemma *fps-of-poly-const*: *fps-of-poly* [:*c*:] = *fps-const c*
⟨*proof*⟩

lemma *fps-of-poly-0* [*simp*]: *fps-of-poly 0 = 0*
⟨*proof*⟩

lemma *fps-of-poly-1* [*simp*]: *fps-of-poly 1 = 1*
⟨*proof*⟩

lemma *fps-of-poly-1'* [*simp*]: *fps-of-poly* [:*1*:] = *1*
⟨*proof*⟩

lemma *fps-of-poly-numeral* [*simp*]: *fps-of-poly* (*numeral n*) = *numeral n*
⟨*proof*⟩

lemma *fps-of-poly-numeral'* [*simp*]: *fps-of-poly* [:*numeral n*:] = *numeral n*
⟨*proof*⟩

lemma *fps-of-poly-fps-X* [*simp*]: *fps-of-poly* [:*0*, *1*:] = *fps-X*
⟨*proof*⟩

lemma *fps-of-poly-add*: *fps-of-poly* (*p* + *q*) = *fps-of-poly p* + *fps-of-poly q*
⟨*proof*⟩

lemma *fps-of-poly-diff*: *fps-of-poly* (*p* − *q*) = *fps-of-poly p* − *fps-of-poly q*
⟨*proof*⟩

lemma *fps-of-poly-uminus*: *fps-of-poly* (−*p*) = −*fps-of-poly p*
⟨*proof*⟩

lemma *fps-of-poly-mult*: *fps-of-poly* (*p* ∗ *q*) = *fps-of-poly p* ∗ *fps-of-poly q*
⟨*proof*⟩

lemma *fps-of-poly-smult*:
fps-of-poly (*smult c p*) = *fps-const c* ∗ *fps-of-poly p*
⟨*proof*⟩

lemma *fps-of-poly-sum*: *fps-of-poly* (*sum f A*) = *sum* (λ*x*. *fps-of-poly* (*f x*)) *A*
⟨*proof*⟩

lemma *fps-of-poly-sum-list*: *fps-of-poly* (*sum-list xs*) = *sum-list* (*map fps-of-poly xs*)
⟨*proof*⟩

lemma *fps-of-poly-prod*: *fps-of-poly* (*prod f A*) = *prod* (λ*x*. *fps-of-poly* (*f x*)) *A*
⟨*proof*⟩

lemma *fps-of-poly-prod-list*: *fps-of-poly* (*prod-list xs*) = *prod-list* (*map fps-of-poly xs*)
⟨*proof*⟩

lemma *fps-of-poly-pCons*:
fps-of-poly (*pCons* (*c* :: *'a* :: *semiring-1*) *p*) = *fps-const c* + *fps-of-poly p* ∗ *fps-X*
⟨*proof*⟩

lemma *fps-of-poly-pderiv*: *fps-of-poly* (*pderiv p*) = *fps-deriv* (*fps-of-poly p*)
⟨*proof*⟩

lemma *fps-of-poly-power*: *fps-of-poly* (*p* ^ *n*) = *fps-of-poly p* ^ *n*
⟨*proof*⟩

lemma *fps-of-poly-monom*: *fps-of-poly* (*monom* (*c* :: *'a* :: *comm-ring-1*) *n*) = *fps-const c* ∗ *fps-X* ^ *n*
⟨*proof*⟩

lemma *fps-of-poly-monom'*: *fps-of-poly* (*monom* (*1* :: *'a* :: *comm-ring-1*) *n*) = *fps-X* ^ *n*
⟨*proof*⟩

lemma *fps-of-poly-div*:
assumes (*q* :: *'a* :: *field poly*) *dvd p*
shows *fps-of-poly* (*p div q*) = *fps-of-poly p* / *fps-of-poly q*
⟨*proof*⟩

lemma *fps-of-poly-divide-numeral*:
fps-of-poly (*smult* (*inverse* (*numeral c* :: *'a* :: *field*)) *p*) = *fps-of-poly p* / *numeral c*
⟨*proof*⟩

lemma *subdegree-fps-of-poly*:
assumes *p* ≠ *0*
defines *n* ≡ *Polynomial.order 0 p*
shows *subdegree* (*fps-of-poly p*) = *n*
⟨*proof*⟩

lemma *fps-of-poly-dvd*:
assumes *p dvd q*
shows *fps-of-poly* (*p* :: *'a* :: *field poly*) *dvd fps-of-poly q*
⟨*proof*⟩

lemmas *fps-of-poly-simps* =
fps-of-poly-0 fps-of-poly-1 fps-of-poly-numeral fps-of-poly-const fps-of-poly-fps-X
fps-of-poly-add fps-of-poly-diff fps-of-poly-uminus fps-of-poly-mult fps-of-poly-smult

fps-of-poly-sum fps-of-poly-sum-list fps-of-poly-prod fps-of-poly-prod-list fps-of-poly-pCons fps-of-poly-pderiv fps-of-poly-power fps-of-poly-monom fps-of-poly-divide-numeral

lemma *fps-of-poly-pcompose*:
assumes *coeff q 0* = (*0* :: *'a* :: *idom*)
shows *fps-of-poly* (*pcompose p q*) = *fps-compose* (*fps-of-poly p*) (*fps-of-poly q*)
⟨*proof*⟩

lemmas *reify-fps-atom* =
fps-of-poly-0 fps-of-poly-1' fps-of-poly-numeral' fps-of-poly-const fps-of-poly-fps-X

The following simproc can reduce the equality of two polynomial FPSs two equality of the respective polynomials. A polynomial FPS is one that only has finitely many non-zero coefficients and can therefore be written as *fps-of-poly p* for some polynomial *p*.

This may sound trivial, but it covers a number of annoying side conditions like *1* + *fps-X* ≠ *0* that would otherwise not be solved automatically.

⟨*ML*⟩

lemma *fps-of-poly-linear*: *fps-of-poly* [:*a*,*1* :: *'a* :: *field*:] = *fps-X* + *fps-const a*
⟨*proof*⟩

lemma *fps-of-poly-linear'*: *fps-of-poly* [:*1*,*a* :: *'a* :: *field*:] = *1* + *fps-const a* ∗ *fps-X*
⟨*proof*⟩

lemma *fps-of-poly-cutoff* [*simp*]:
fps-of-poly (*poly-cutoff n p*) = *fps-cutoff n* (*fps-of-poly p*)
⟨*proof*⟩

lemma *fps-of-poly-shift* [*simp*]: *fps-of-poly* (*poly-shift n p*) = *fps-shift n* (*fps-of-poly p*)
⟨*proof*⟩

definition *poly-subdegree* :: *'a*::*zero poly* ⇒ *nat* **where**
poly-subdegree p = *subdegree* (*fps-of-poly p*)

lemma *coeff-less-poly-subdegree*:
k < *poly-subdegree p* ⟹ *coeff p k* = *0*
⟨*proof*⟩

definition *prefix-length* :: (*'a* ⇒ *bool*) ⇒ *'a list* ⇒ *nat* **where**
prefix-length P xs = *length* (*takeWhile P xs*)

primrec *prefix-length-aux* :: (*'a* ⇒ *bool*) ⇒ *nat* ⇒ *'a list* ⇒ *nat* **where**
prefix-length-aux P acc [] = *acc*

| *prefix-length-aux P acc (x#xs) = (if P x then prefix-length-aux P (Suc acc) xs else acc)*

lemma *prefix-length-aux-correct*: *prefix-length-aux P acc xs = prefix-length P xs + acc*
⟨*proof*⟩

lemma *prefix-length-code* [*code*]: *prefix-length P xs = prefix-length-aux P 0 xs*
⟨*proof*⟩

lemma *prefix-length-le-length*: *prefix-length P xs ≤ length xs*
⟨*proof*⟩

lemma *prefix-length-less-length*: *(∃ x∈set xs. ¬P x) ⟹ prefix-length P xs < length xs*
⟨*proof*⟩

lemma *nth-prefix-length*:
(∃ x∈set xs. ¬P x) ⟹ ¬P (xs ! prefix-length P xs)
⟨*proof*⟩

lemma *nth-less-prefix-length*:
n < prefix-length P xs ⟹ P (xs ! n)
⟨*proof*⟩

lemma *poly-subdegree-code* [*code*]: *poly-subdegree p = prefix-length ((=) 0) (coeffs p)*
⟨*proof*⟩

end

Truncation of formal power series: all monomials cx^k with $k \geq n$ are removed; the remainder is a polynomial of degree at most $n - 1$.

lift-definition *truncate-fps* :: *nat ⇒ 'a fps ⇒ 'a :: zero poly* **is**
λn F k. if k ≥ n then 0 else fps-nth F k
⟨*proof*⟩

lemma *coeff-truncate-fps'* [*simp*]:
k ≥ n ⟹ coeff (truncate-fps n F) k = 0
k < n ⟹ coeff (truncate-fps n F) k = fps-nth F k
⟨*proof*⟩

lemma *coeff-truncate-fps*: *coeff (truncate-fps n F) k = (if k < n then fps-nth F k else 0)*
⟨*proof*⟩

lemma *truncate-0-fps* [*simp*]: *truncate-fps 0 F = 0*
⟨*proof*⟩

lemma *degree-truncate-fps*: $n > 0 \Longrightarrow$ *degree* (*truncate-fps n F*) $< n$
⟨*proof*⟩

lemma *truncate-fps-0* [*simp*]: *truncate-fps n 0* = *0*
⟨*proof*⟩

lemma *truncate-fps-add*: *truncate-fps n* (*f* + *g*) = *truncate-fps n f* + *truncate-fps n g*
⟨*proof*⟩

lemma *truncate-fps-diff*: *truncate-fps n* (*f* − *g*) = *truncate-fps n f* − *truncate-fps n g*
⟨*proof*⟩

lemma *truncate-fps-uminus*: *truncate-fps n* (−*f*) = −*truncate-fps n f*
⟨*proof*⟩

lemma *fps-of-poly-truncate* [*simp*]: *fps-of-poly* (*truncate-fps n f*) = *fps-cutoff n f*
⟨*proof*⟩

end

7 A formalization of formal Laurent series

theory *Formal-Laurent-Series*
imports
Polynomial-FPS
begin

7.1 The type of formal Laurent series

7.1.1 Type definition

typedef (**overloaded**) *'a fls* = {*f*::*int* ⇒ *'a*::*zero*. ∀$_\infty$ *n*::*nat*. *f* (− *int n*) = *0*}
morphisms *fls-nth Abs-fls*
⟨*proof*⟩

setup-lifting *type-definition-fls*

unbundle *fps-syntax*
notation *fls-nth* (**infixl** ‹$$› *75*)

lemmas *fls-eqI* = *iffD1*[*OF fls-nth-inject*, *OF iffD2*, *OF fun-eq-iff*, *OF allI*]

lemma *fls-eq-iff*: *f* = *g* ⟷ (∀ *n*. *f* $$ *n* = *g* $$ *n*)
⟨*proof*⟩

lemma *nth-Abs-fls* [*simp*]: ∀$_\infty$*n*. *f* (− *int n*) = *0* ⟹ *Abs-fls f* $$ *n* = *f n*

⟨*proof*⟩

lemmas *nth-Abs-fls-finite-nonzero-neg-nth* = *nth-Abs-fls*[*OF iffD2*, *OF eventually-cofinite*]
lemmas *nth-Abs-fls-ex-nat-lower-bound* = *nth-Abs-fls*[*OF iffD2*, *OF MOST-nat*]
lemmas *nth-Abs-fls-nat-lower-bound* = *nth-Abs-fls-ex-nat-lower-bound*[*OF exI*]

lemma *nth-Abs-fls-ex-lower-bound*:
  assumes ∃ *N*. ∀ *n*<*N*. *f n* = *0*
  shows *Abs-fls f* $$ *n* = *f n*
⟨*proof*⟩

lemmas *nth-Abs-fls-lower-bound* = *nth-Abs-fls-ex-lower-bound*[*OF exI*]

lemmas *MOST-fls-neg-nth-eq-0* [*simp*] = *CollectD*[*OF fls-nth*]
lemmas *fls-finite-nonzero-neg-nth* = *iffD1*[*OF eventually-cofinite MOST-fls-neg-nth-eq-0*]

lemma *fls-nth-vanishes-below-natE*:
  fixes *f* :: *'a*::*zero fls*
  obtains *N* :: *nat*
  where ∀ *n*>*N*. *f*$$(−*int n*) = *0*
  ⟨*proof*⟩

lemma *fls-nth-vanishes-belowE*:
  fixes *f* :: *'a*::*zero fls*
  obtains *N* :: *int*
  where ∀ *n*<*N*. *f*$$*n* = *0*
⟨*proof*⟩

7.1.2 Definition of basic Laurent series

instantiation *fls* :: (*zero*) *zero*
begin
  lift-definition *zero-fls* :: *'a fls* **is** λ-. *0* ⟨*proof*⟩
  instance ⟨*proof*⟩
end

lemma *fls-zero-nth* [*simp*]: *0* $$ *n* = *0*
⟨*proof*⟩

lemma *fls-zero-eqI*: (⋀*n*. *f*$$*n* = *0*) ⟹ *f* = *0*
  ⟨*proof*⟩

lemma *fls-nonzeroI*: *f*$$*n* ≠ *0* ⟹ *f* ≠ *0*
  ⟨*proof*⟩

lemma *fls-nonzero-nth*: *f* ≠ *0* ⟷ (∃ *n*. *f* $$ *n* ≠ *0*)
  ⟨*proof*⟩

lemma *fls-trivial-delta-eq-zero* [*simp*]: $b = 0 \Longrightarrow$ *Abs-fls* ($\lambda n.$ *if* $n{=}a$ *then* b *else* 0) $= 0$
⟨*proof*⟩

lemma *fls-delta-nth* [*simp*]:
Abs-fls ($\lambda n.$ *if* $n{=}a$ *then* b *else* 0) $\$\$\ n =$ (*if* $n{=}a$ *then* b *else* 0)
⟨*proof*⟩

instantiation *fls* :: ({*zero*,*one*}) *one*
begin
lift-definition *one-fls* :: *'a fls* **is** $\lambda k.$ *if* $k = 0$ *then* 1 *else* 0
⟨*proof*⟩
instance ⟨*proof*⟩
end

lemma *fls-one-nth* [*simp*]:
$1\ \$\$\ n =$ (*if* $n = 0$ *then* 1 *else* 0)
⟨*proof*⟩

instance *fls* :: (*zero-neq-one*) *zero-neq-one*
⟨*proof*⟩

definition *fls-const* :: *'a*::*zero* ⇒ *'a fls*
where *fls-const* $c \equiv$ *Abs-fls* ($\lambda n.$ *if* $n = 0$ *then* c *else* 0)

lemma *fls-const-nth* [*simp*]: *fls-const* $c\ \$\$\ n =$ (*if* $n = 0$ *then* c *else* 0)
⟨*proof*⟩

lemma *fls-const-0* [*simp*]: *fls-const* $0 = 0$
⟨*proof*⟩

lemma *fls-const-nonzero*: $c \neq 0 \Longrightarrow$ *fls-const* $c \neq 0$
⟨*proof*⟩

lemma *fls-const-eq-0-iff* [*simp*]: *fls-const* $c = 0 \longleftrightarrow c = 0$
⟨*proof*⟩

lemma *fls-const-1* [*simp*]: *fls-const* $1 = 1$
⟨*proof*⟩

lemma *fls-const-eq-1-iff* [*simp*]: *fls-const* $c = 1 \longleftrightarrow c = 1$
⟨*proof*⟩

lift-definition *fls-X* :: *'a*::{*zero*,*one*} *fls*
is $\lambda n.$ *if* $n = 1$ *then* 1 *else* 0
⟨*proof*⟩

lemma *fls-X-nth* [*simp*]:
fls-X $\$\$\ n =$ (*if* $n = 1$ *then* 1 *else* 0)

⟨*proof*⟩

lemma *fls-X-nonzero* [*simp*]: (*fls-X* :: *'a* :: *zero-neq-one fls*) ≠ *0*
⟨*proof*⟩

lift-definition *fls-X-inv* :: *'a*::{*zero*,*one*} *fls*
is λ*n*. *if n* = −*1 then 1 else 0*
⟨*proof*⟩

lemma *fls-X-inv-nth* [*simp*]:
fls-X-inv $$ *n* = (*if n* = −*1 then 1 else 0*)
⟨*proof*⟩

lemma *fls-X-inv-nonzero* [*simp*]: (*fls-X-inv* :: *'a* :: *zero-neq-one fls*) ≠ *0*
⟨*proof*⟩

7.2 Subdegrees

lemma *unique-fls-subdegree*:
assumes *f* ≠ *0*
shows ∃!*n*. *f*$$*n* ≠ *0* ∧ (∀ *m*. *f*$$*m* ≠ *0* ⟶ *n* ≤ *m*)
⟨*proof*⟩

definition *fls-subdegree* :: (*'a*::*zero*) *fls* ⇒ *int*
where *fls-subdegree f* ≡ (*if f* = *0 then 0 else LEAST n*::*int*. *f*$$*n* ≠ *0*)

lemma *fls-zero-subdegree* [*simp*]: *fls-subdegree 0* = *0*
⟨*proof*⟩

lemma *nth-fls-subdegree-nonzero* [*simp*]: *f* ≠ *0* ⟹ *f* $$ *fls-subdegree f* ≠ *0*
⟨*proof*⟩

lemma *nth-fls-subdegree-zero-iff*: (*f* $$ *fls-subdegree f* = *0*) ⟷ (*f* = *0*)
⟨*proof*⟩

lemma *fls-subdegree-leI*: *f* $$ *n* ≠ *0* ⟹ *fls-subdegree f* ≤ *n*
⟨*proof*⟩

lemma *fls-subdegree-leI'*: *f* $$ *n* ≠ *0* ⟹ *n* ≤ *m* ⟹ *fls-subdegree f* ≤ *m*
⟨*proof*⟩

lemma *fls-eq0-below-subdegree* [*simp*]: *n* < *fls-subdegree f* ⟹ *f* $$ *n* = *0*
⟨*proof*⟩

lemma *fls-subdegree-geI*: *f* ≠ *0* ⟹ (⋀*k*. *k* < *n* ⟹ *f* $$ *k* = *0*) ⟹ *n* ≤ *fls-subdegree f*
⟨*proof*⟩

lemma *fls-subdegree-ge0I*: (⋀*k*. *k* < *0* ⟹ *f* $$ *k* = *0*) ⟹ *0* ≤ *fls-subdegree f*

⟨*proof*⟩

lemma *fls-subdegree-greaterI*:
assumes $f \neq 0$ $\bigwedge k.\ k \leq n \Longrightarrow f\ \$\$\ k = 0$
shows $n < \textit{fls-subdegree}\ f$
⟨*proof*⟩

lemma *fls-subdegree-eqI*: $f\ \$\$\ n \neq 0 \Longrightarrow (\bigwedge k.\ k < n \Longrightarrow f\ \$\$\ k = 0) \Longrightarrow \textit{fls-subdegree}\ f = n$
⟨*proof*⟩

lemma *fls-delta-subdegree* [*simp*]:
$b \neq 0 \Longrightarrow \textit{fls-subdegree}\ (\textit{Abs-fls}\ (\lambda n.\ \textit{if}\ n{=}a\ \textit{then}\ b\ \textit{else}\ 0)) = a$
⟨*proof*⟩

lemma *fls-delta0-subdegree*: $\textit{fls-subdegree}\ (\textit{Abs-fls}\ (\lambda n.\ \textit{if}\ n{=}0\ \textit{then}\ a\ \textit{else}\ 0)) = 0$
⟨*proof*⟩

lemma *fls-one-subdegree* [*simp*]: $\textit{fls-subdegree}\ 1 = 0$
⟨*proof*⟩

lemma *fls-const-subdegree* [*simp*]: $\textit{fls-subdegree}\ (\textit{fls-const}\ c) = 0$
⟨*proof*⟩

lemma *fls-X-subdegree* [*simp*]: $\textit{fls-subdegree}\ (\textit{fls-X}::'a::\{\textit{zero-neq-one}\}\ \textit{fls}) = 1$
⟨*proof*⟩

lemma *fls-X-inv-subdegree* [*simp*]: $\textit{fls-subdegree}\ (\textit{fls-X-inv}::'a::\{\textit{zero-neq-one}\}\ \textit{fls}) = -1$
⟨*proof*⟩

lemma *fls-eq-above-subdegreeI*:
assumes $N \leq \textit{fls-subdegree}\ f$ $N \leq \textit{fls-subdegree}\ g$ $\forall k{\geq}N.\ f\ \$\$\ k = g\ \$\$\ k$
shows $f = g$
⟨*proof*⟩

7.3 Shifting

7.3.1 Shift definition

definition *fls-shift* :: $\textit{int} \Rightarrow ('a::\textit{zero})\ \textit{fls} \Rightarrow 'a\ \textit{fls}$
where $\textit{fls-shift}\ n\ f \equiv \textit{Abs-fls}\ (\lambda k.\ f\ \$\$\ (k{+}n))$
— Since the index set is unbounded in both directions, we can shift in either direction.

lemma *fls-shift-nth* [*simp*]: $\textit{fls-shift}\ m\ f\ \$\$\ n = f\ \$\$\ (n{+}m)$
⟨*proof*⟩

lemma *fls-shift-eq-iff*: $(\textit{fls-shift}\ m\ f = \textit{fls-shift}\ m\ g) \longleftrightarrow (f = g)$
⟨*proof*⟩

lemma *fls-shift-0* [*simp*]: *fls-shift 0 f = f*
⟨*proof*⟩

lemma *fls-shift-subdegree* [*simp*]:
f ≠ 0 ⟹ fls-subdegree (fls-shift n f) = fls-subdegree f − n
⟨*proof*⟩

lemma *fls-shift-fls-shift* [*simp*]: *fls-shift m (fls-shift k f) = fls-shift (k+m) f*
⟨*proof*⟩

lemma *fls-shift-fls-shift-reorder*:
fls-shift m (fls-shift k f) = fls-shift k (fls-shift m f)
⟨*proof*⟩

lemma *fls-shift-zero* [*simp*]: *fls-shift m 0 = 0*
⟨*proof*⟩

lemma *fls-shift-eq0-iff*: *fls-shift m f = 0 ⟷ f = 0*
⟨*proof*⟩

lemma *fls-shift-eq-1-iff*: *fls-shift n f = 1 ⟷ f = fls-shift (−n) 1*
⟨*proof*⟩

lemma *fls-shift-nonneg-subdegree*: *m ≤ fls-subdegree f ⟹ fls-subdegree (fls-shift m f) ≥ 0*
⟨*proof*⟩

lemma *fls-shift-delta*:
fls-shift m (Abs-fls (λn. if n=a then b else 0)) = Abs-fls (λn. if n=a−m then b else 0)
⟨*proof*⟩

lemma *fls-shift-const*:
fls-shift m (fls-const c) = Abs-fls (λn. if n=−m then c else 0)
⟨*proof*⟩

lemma *fls-shift-const-nth*:
fls-shift m (fls-const c) $$ n = (if n=−m then c else 0)
⟨*proof*⟩

lemma *fls-X-conv-shift-1*: *fls-X = fls-shift (−1) 1*
⟨*proof*⟩

lemma *fls-X-shift-to-one* [*simp*]: *fls-shift 1 fls-X = 1*
⟨*proof*⟩

lemma *fls-X-inv-conv-shift-1*: *fls-X-inv = fls-shift 1 1*
⟨*proof*⟩

lemma *fls-X-inv-shift-to-one* [*simp*]: *fls-shift* (−*1*) *fls-X-inv* = *1*
⟨*proof*⟩

lemma *fls-X-fls-X-inv-conv*:
fls-X = *fls-shift* (−*2*) *fls-X-inv fls-X-inv* = *fls-shift 2 fls-X*
⟨*proof*⟩

7.3.2 Base factor

Similarly to the *unit-factor* for formal power series, we can decompose a formal Laurent series as a power of the implied variable times a series of subdegree 0. (See lemma *fls-base-factor-X-power-decompose*.) But we will call this something other *unit-factor* because it will not satisfy assumption *is-unit-unit-factor* of *semidom-divide-unit-factor*.

definition *fls-base-factor* :: ('*a*::*zero*) *fls* ⇒ '*a fls*
where *fls-base-factor-def*[*simp*]: *fls-base-factor f* = *fls-shift* (*fls-subdegree f*) *f*

lemma *fls-base-factor-nth*: *fls-base-factor f* $$ *n* = *f* $$ (*n* + *fls-subdegree f*)
⟨*proof*⟩

lemma *fls-base-factor-nonzero* [*simp*]: *f* ≠ *0* ⟹ *fls-base-factor f* ≠ *0*
⟨*proof*⟩

lemma *fls-base-factor-subdegree* [*simp*]: *fls-subdegree* (*fls-base-factor f*) = *0*
⟨*proof*⟩

lemma *fls-base-factor-base* [*simp*]:
fls-base-factor f $$ *fls-subdegree* (*fls-base-factor f*) = *f* $$ *fls-subdegree f*
⟨*proof*⟩

lemma *fls-conv-base-factor-shift-subdegree*:
f = *fls-shift* (−*fls-subdegree f*) (*fls-base-factor f*)
⟨*proof*⟩

lemma *fls-base-factor-idem*:
fls-base-factor (*fls-base-factor* (*f*::'*a*::*zero fls*)) = *fls-base-factor f*
⟨*proof*⟩

lemma *fls-base-factor-zero*: *fls-base-factor* (*0*::'*a*::*zero fls*) = *0*
⟨*proof*⟩

lemma *fls-base-factor-zero-iff*: *fls-base-factor* (*f*::'*a*::*zero fls*) = *0* ⟷ *f* = *0*
⟨*proof*⟩

lemma *fls-base-factor-nth-0*: *f* ≠ *0* ⟹ *fls-base-factor f* $$ *0* ≠ *0*
⟨*proof*⟩

lemma *fls-base-factor-one*: *fls-base-factor* (*1*::*'a*::{*zero*,*one*} *fls*) = *1*
⟨*proof*⟩

lemma *fls-base-factor-const*: *fls-base-factor* (*fls-const c*) = *fls-const c*
⟨*proof*⟩

lemma *fls-base-factor-delta*:
fls-base-factor (*Abs-fls* (λ*n*. *if n*=*a then c else 0*)) = *fls-const c*
⟨*proof*⟩

lemma *fls-base-factor-X*: *fls-base-factor* (*fls-X*::*'a*::{*zero-neq-one*} *fls*) = *1*
⟨*proof*⟩

lemma *fls-base-factor-X-inv*: *fls-base-factor* (*fls-X-inv*::*'a*::{*zero-neq-one*} *fls*) = *1*
⟨*proof*⟩

lemma *fls-base-factor-shift* [*simp*]: *fls-base-factor* (*fls-shift n f*) = *fls-base-factor f*
⟨*proof*⟩

7.4 Conversion between formal power and Laurent series

7.4.1 Converting Laurent to power series

We can truncate a Laurent series at index 0 to create a power series, called the regular part.

lift-definition *fls-regpart* :: (*'a*::*zero*) *fls* ⇒ *'a fps*
is λ*f*. *Abs-fps* (λ*n*. *f* (*int n*))
⟨*proof*⟩

lemma *fls-regpart-nth* [*simp*]: *fls-regpart f* $ *n* = *f* $$ (*int n*)
⟨*proof*⟩

lemma *fls-regpart-zero* [*simp*]: *fls-regpart 0* = *0*
⟨*proof*⟩

lemma *fls-regpart-one* [*simp*]: *fls-regpart 1* = *1*
⟨*proof*⟩

lemma *fls-regpart-Abs-fls*:
∀ ∞*n*. *F* (− *int n*) = *0* ⟹ *fls-regpart* (*Abs-fls F*) = *Abs-fps* (λ*n*. *F* (*int n*))
⟨*proof*⟩

lemma *fls-regpart-delta*:
fls-regpart (*Abs-fls* (λ*n*. *if n*=*a then b else 0*)) =
(*if a* < *0 then 0 else Abs-fps* (λ*n*. *if n*=*nat a then b else 0*))
⟨*proof*⟩

lemma *fls-regpart-const* [*simp*]: *fls-regpart* (*fls-const c*) = *fps-const c*
⟨*proof*⟩

lemma *fls-regpart-fls-X* [*simp*]: *fls-regpart fls-X* = *fps-X*
⟨*proof*⟩

lemma *fls-regpart-fls-X-inv* [*simp*]: *fls-regpart fls-X-inv* = *0*
⟨*proof*⟩

lemma *fls-regpart-eq0-imp-nonpos-subdegree*:
assumes *fls-regpart f* = *0*
shows *fls-subdegree f* ≤ *0*
⟨*proof*⟩

lemma *fls-subdegree-lt-fls-regpart-subdegree*:
fls-subdegree f ≤ *int* (*subdegree* (*fls-regpart f*))
⟨*proof*⟩

lemma *fls-regpart-subdegree-conv*:
assumes *fls-subdegree f* ≥ *0*
shows *subdegree* (*fls-regpart f*) = *nat* (*fls-subdegree f*)
— This is the best we can do since if the subdegree is negative, we might still have the bad luck that the term at index 0 is equal to 0.
⟨*proof*⟩

lemma *fls-eq-conv-fps-eqI*:
assumes *0* ≤ *fls-subdegree f 0* ≤ *fls-subdegree g fls-regpart f* = *fls-regpart g*
shows *f* = *g*
⟨*proof*⟩

lemma *fls-regpart-shift-conv-fps-shift*:
m ≥ *0* ⟹ *fls-regpart* (*fls-shift m f*) = *fps-shift* (*nat m*) (*fls-regpart f*)
⟨*proof*⟩

lemma *fps-shift-fls-regpart-conv-fls-shift*:
fps-shift m (*fls-regpart f*) = *fls-regpart* (*fls-shift m f*)
⟨*proof*⟩

lemma *fps-unit-factor-fls-regpart*:
fls-subdegree f ≥ *0* ⟹ *unit-factor* (*fls-regpart f*) = *fls-regpart* (*fls-base-factor f*)
⟨*proof*⟩

The terms below the zeroth form a polynomial in the inverse of the implied variable, called the principle part.

lift-definition *fls-prpart* :: ('*a*::*zero*) *fls* ⇒ '*a poly*
is λ*f*. *Abs-poly* (λ*n*. *if n* = *0 then 0 else f* (− *int n*))
⟨*proof*⟩

lemma *fls-prpart-coeff* [*simp*]: *coeff* (*fls-prpart f*) *n* = (*if n* = *0 then 0 else f* $$ (− *int n*))
⟨*proof*⟩

lemma *fls-prpart-eq0-iff*: (*fls-prpart f = 0*) ⟷ (*fls-subdegree f ≥ 0*)
⟨*proof*⟩

lemma *fls-prpart0* [*simp*]: *fls-prpart 0 = 0*
⟨*proof*⟩

lemma *fls-prpart-one* [*simp*]: *fls-prpart 1 = 0*
⟨*proof*⟩

lemma *fls-prpart-delta*:
fls-prpart (*Abs-fls* (λ*n*. *if n=a then b else 0*)) =
(*if a<0 then Poly* (*replicate* (*nat* (−*a*)) *0* @ [*b*]) *else 0*)
⟨*proof*⟩

lemma *fls-prpart-const* [*simp*]: *fls-prpart* (*fls-const c*) = *0*
⟨*proof*⟩

lemma *fls-prpart-X* [*simp*]: *fls-prpart fls-X = 0*
⟨*proof*⟩

lemma *fls-prpart-X-inv*: *fls-prpart fls-X-inv* = [:*0*,*1*:]
⟨*proof*⟩

lemma *degree-fls-prpart* [*simp*]:
degree (*fls-prpart f*) = *nat* (−*fls-subdegree f*)
⟨*proof*⟩

lemma *fls-prpart-shift*:
assumes *m ≤ 0*
shows *fls-prpart* (*fls-shift m f*) = *pCons 0* (*poly-shift* (*Suc* (*nat* (−*m*)))
(*fls-prpart f*))
⟨*proof*⟩

lemma *fls-prpart-base-factor*: *fls-prpart* (*fls-base-factor f*) = *0*
⟨*proof*⟩

The essential data of a formal Laurant series resides from the subdegree up.

abbreviation *fls-base-factor-to-fps* :: ('*a*::*zero*) *fls* ⇒ '*a fps*
where *fls-base-factor-to-fps f* ≡ *fls-regpart* (*fls-base-factor f*)

lemma *fls-base-factor-to-fps-conv-fps-shift*:
assumes *fls-subdegree f ≥ 0*
shows *fls-base-factor-to-fps f* = *fps-shift* (*nat* (*fls-subdegree f*)) (*fls-regpart f*)
⟨*proof*⟩

lemma *fls-base-factor-to-fps-nth*:
fls-base-factor-to-fps f $ *n* = *f* $$ (*fls-subdegree f* + *int n*)
⟨*proof*⟩

lemma *fls-base-factor-to-fps-base*: $f \neq 0 \Longrightarrow$ *fls-base-factor-to-fps f $ 0 ≠ 0*
⟨*proof*⟩

lemma *fls-base-factor-to-fps-nonzero*: $f \neq 0 \Longrightarrow$ *fls-base-factor-to-fps f ≠ 0*
⟨*proof*⟩

lemma *fls-base-factor-to-fps-subdegree* [*simp*]: *subdegree* (*fls-base-factor-to-fps f*) = *0*
⟨*proof*⟩

lemma *fls-base-factor-to-fps-trivial*:
fls-subdegree f = 0 $\Longrightarrow$ *fls-base-factor-to-fps f = fls-regpart f*
⟨*proof*⟩

lemma *fls-base-factor-to-fps-zero*: *fls-base-factor-to-fps 0 = 0*
⟨*proof*⟩

lemma *fls-base-factor-to-fps-one*: *fls-base-factor-to-fps 1 = 1*
⟨*proof*⟩

lemma *fls-base-factor-to-fps-delta*:
fls-base-factor-to-fps (*Abs-fls* (λ*n. if n=a then c else 0*)) = *fps-const c*
⟨*proof*⟩

lemma *fls-base-factor-to-fps-const*:
fls-base-factor-to-fps (*fls-const c*) = *fps-const c*
⟨*proof*⟩

lemma *fls-base-factor-to-fps-X*:
fls-base-factor-to-fps (*fls-X*::*'a*::{*zero-neq-one*} *fls*) = *1*
⟨*proof*⟩

lemma *fls-base-factor-to-fps-X-inv*:
fls-base-factor-to-fps (*fls-X-inv*::*'a*::{*zero-neq-one*} *fls*) = *1*
⟨*proof*⟩

lemma *fls-base-factor-to-fps-shift*:
fls-base-factor-to-fps (*fls-shift m f*) = *fls-base-factor-to-fps f*
⟨*proof*⟩

lemma *fls-base-factor-to-fps-base-factor*:
fls-base-factor-to-fps (*fls-base-factor f*) = *fls-base-factor-to-fps f*
⟨*proof*⟩

lemma *fps-unit-factor-fls-base-factor*:
unit-factor (*fls-base-factor-to-fps f*) = *fls-base-factor-to-fps f*
⟨*proof*⟩

7.4.2 Converting power to Laurent series

We can extend a power series by 0s below to create a Laurent series.

definition *fps-to-fls* :: ('*a*::*zero*) *fps* ⇒ '*a fls*
  where *fps-to-fls f* ≡ *Abs-fls* (λ*k*::*int*. *if k*<*0 then 0 else f* $ (*nat k*))

lemma *fps-to-fls-nth* [*simp*]:
  (*fps-to-fls f*) $$ *n* = (*if n* < *0 then 0 else f*$(*nat n*))
  ⟨*proof*⟩

lemma *fps-to-fls-eq-imp-fps-eq*:
  assumes *fps-to-fls f* = *fps-to-fls g*
  shows *f* = *g*
⟨*proof*⟩

lemma *fps-to-fls-eq-iff* [*simp*]: *fps-to-fls f* = *fps-to-fls g* ⟷ *f* = *g*
  ⟨*proof*⟩

lemma *fps-zero-to-fls* [*simp*]: *fps-to-fls 0* = *0*
  ⟨*proof*⟩

lemma *fps-to-fls-nonzeroI*: *f* ≠ *0* ⟹ *fps-to-fls f* ≠ *0*
  ⟨*proof*⟩

lemma *fps-one-to-fls* [*simp*]: *fps-to-fls 1* = *1*
  ⟨*proof*⟩

lemma *fps-to-fls-Abs-fps*:
  fps-to-fls (*Abs-fps F*) = *Abs-fls* (λ*n*. *if n*<*0 then 0 else F* (*nat n*))
  ⟨*proof*⟩

lemma *fps-delta-to-fls*:
  fps-to-fls (*Abs-fps* (λ*n*. *if n*=*a then b else 0*)) = *Abs-fls* (λ*n*. *if n*=*int a then b else 0*)
  ⟨*proof*⟩

lemma *fps-const-to-fls* [*simp*]: *fps-to-fls* (*fps-const c*) = *fls-const c*
  ⟨*proof*⟩

lemma *fps-X-to-fls* [*simp*]: *fps-to-fls fps-X* = *fls-X*
  ⟨*proof*⟩

lemma *fps-to-fls-eq-0-iff* [*simp*]: (*fps-to-fls f* = *0*) ⟷ (*f*=*0*)
  ⟨*proof*⟩

lemma *fps-to-fls-eq-1-iff* [*simp*]: *fps-to-fls f* = *1* ⟷ *f* = *1*
  ⟨*proof*⟩

lemma *fls-subdegree-fls-to-fps-gt0*: *fls-subdegree* (*fps-to-fls f*) ≥ *0*

⟨*proof*⟩

lemma *fls-subdegree-fls-to-fps*: *fls-subdegree* (*fps-to-fls f*) = *int* (*subdegree f*)
⟨*proof*⟩

lemma *fps-shift-to-fls* [*simp*]:
n ≤ *subdegree f* ⟹ *fps-to-fls* (*fps-shift n f*) = *fls-shift* (*int n*) (*fps-to-fls f*)
⟨*proof*⟩

lemma *fls-base-factor-fps-to-fls*: *fls-base-factor* (*fps-to-fls f*) = *fps-to-fls* (*unit-factor f*)
⟨*proof*⟩

lemma *fls-regpart-to-fls-trivial* [*simp*]:
fls-subdegree f ≥ *0* ⟹ *fps-to-fls* (*fls-regpart f*) = *f*
⟨*proof*⟩

lemma *fls-regpart-fps-trivial* [*simp*]: *fls-regpart* (*fps-to-fls f*) = *f*
⟨*proof*⟩

lemma *fps-to-fls-base-factor-to-fps*:
fps-to-fls (*fls-base-factor-to-fps f*) = *fls-base-factor f*
⟨*proof*⟩

lemma *fls-conv-base-factor-to-fps-shift-subdegree*:
f = *fls-shift* (−*fls-subdegree f*) (*fps-to-fls* (*fls-base-factor-to-fps f*))
⟨*proof*⟩

lemma *fls-base-factor-to-fps-to-fls*:
fls-base-factor-to-fps (*fps-to-fls f*) = *unit-factor f*
⟨*proof*⟩

lemma *fls-as-fps*:
fixes *f* :: *'a* :: *zero fls* **and** *n* :: *int*
assumes *n*: *n* ≥ −*fls-subdegree f*
obtains *f'* **where** *f* = *fls-shift n* (*fps-to-fls f'*)
⟨*proof*⟩

lemma *fls-as-fps'*:
fixes *f* :: *'a* :: *zero fls* **and** *n* :: *int*
assumes *n*: *n* ≥ −*fls-subdegree f*
shows ∃*f'*. *f* = *fls-shift n* (*fps-to-fls f'*)
⟨*proof*⟩

abbreviation
fls-regpart-as-fls f ≡ *fps-to-fls* (*fls-regpart f*)
abbreviation
fls-prpart-as-fls f ≡
fls-shift (−*fls-subdegree f*) (*fps-to-fls* (*fps-of-poly* (*reflect-poly* (*fls-prpart f*))))

lemma *fls-regpart-as-fls-nth*:
fls-regpart-as-fls f $$ n = (if n < 0 then 0 else f $$ n)
⟨*proof*⟩

lemma *fls-regpart-idem*:
fls-regpart (fls-regpart-as-fls f) = fls-regpart f
⟨*proof*⟩

lemma *fls-prpart-as-fls-nth*:
fls-prpart-as-fls f $$ n = (if n < 0 then f $$ n else 0)
⟨*proof*⟩

lemma *fls-prpart-idem* [*simp*]: *fls-prpart (fls-prpart-as-fls f) = fls-prpart f*
⟨*proof*⟩

lemma *fls-regpart-prpart*: *fls-regpart (fls-prpart-as-fls f) = 0*
⟨*proof*⟩

lemma *fls-prpart-regpart*: *fls-prpart (fls-regpart-as-fls f) = 0*
⟨*proof*⟩

7.5 Algebraic structures

7.5.1 Addition

instantiation *fls* :: (*monoid-add*) *plus*
begin
lift-definition *plus-fls* :: *'a fls ⇒ 'a fls ⇒ 'a fls* **is** *λf g n. f n + g n*
⟨*proof*⟩
instance ⟨*proof*⟩
end

lemma *fls-plus-nth* [*simp*]: *(f + g) $$ n = f $$ n + g $$ n*
⟨*proof*⟩

lemma *fls-plus-const*: *fls-const x + fls-const y = fls-const (x+y)*
⟨*proof*⟩

lemma *fls-plus-subdegree*:
f + g ≠ 0 ⟹ fls-subdegree (f + g) ≥ min (fls-subdegree f) (fls-subdegree g)
⟨*proof*⟩

lemma *fls-shift-plus* [*simp*]:
fls-shift m (f + g) = (fls-shift m f) + (fls-shift m g)
⟨*proof*⟩

lemma *fls-regpart-plus* [*simp*]: *fls-regpart (f + g) = fls-regpart f + fls-regpart g*
⟨*proof*⟩

lemma *fls-prpart-plus* [*simp*] : *fls-prpart* (*f* + *g*) = *fls-prpart f* + *fls-prpart g*
⟨*proof*⟩

lemma *fls-decompose-reg-pr-parts*:
fixes *f* :: *'a* :: *monoid-add fls*
defines *R* ≡ *fls-regpart-as-fls f*
and *P* ≡ *fls-prpart-as-fls f*
shows *f* = *P* + *R*
and *f* = *R* + *P*
⟨*proof*⟩

lemma *fps-to-fls-plus* [*simp*]: *fps-to-fls* (*f* + *g*) = *fps-to-fls f* + *fps-to-fls g*
⟨*proof*⟩

instance *fls* :: (*monoid-add*) *monoid-add*
⟨*proof*⟩

instance *fls* :: (*comm-monoid-add*) *comm-monoid-add*
⟨*proof*⟩

lemma *fls-nth-sum*: *fls-nth* (∑ *x*∈*A*. *f x*) *n* = (∑ *x*∈*A*. *fls-nth* (*f x*) *n*)
⟨*proof*⟩

7.5.2 Subtraction and negatives

instantiation *fls* :: (*group-add*) *minus*
begin
lift-definition *minus-fls* :: *'a fls* ⇒ *'a fls* ⇒ *'a fls* **is** λ*f g n*. *f n* − *g n*
⟨*proof*⟩
instance ⟨*proof*⟩
end

lemma *fls-minus-nth* [*simp*]: (*f* − *g*) $$ *n* = *f* $$ *n* − *g* $$ *n*
⟨*proof*⟩

lemma *fls-minus-const*: *fls-const x* − *fls-const y* = *fls-const* (*x*−*y*)
⟨*proof*⟩

lemma *fls-subdegree-minus*:
f − *g* ≠ *0* ⟹ *fls-subdegree* (*f* − *g*) ≥ *min* (*fls-subdegree f*) (*fls-subdegree g*)
⟨*proof*⟩

lemma *fls-shift-minus* [*simp*]: *fls-shift m* (*f* − *g*) = (*fls-shift m f*) − (*fls-shift m g*)
⟨*proof*⟩

lemma *fls-regpart-minus* [*simp*]: *fls-regpart* (*f* − *g*) = *fls-regpart f* − *fls-regpart g*
⟨*proof*⟩

lemma *fls-prpart-minus* [*simp*] : *fls-prpart* (*f* − *g*) = *fls-prpart f* − *fls-prpart g*
⟨*proof*⟩

lemma *fps-to-fls-minus* [*simp*]: *fps-to-fls* (*f* − *g*) = *fps-to-fls f* − *fps-to-fls g*
⟨*proof*⟩

instantiation *fls* :: (*group-add*) *uminus*
begin
lift-definition *uminus-fls* :: *'a fls* ⇒ *'a fls* **is** λ*f n*. − *f n*
⟨*proof*⟩
instance ⟨*proof*⟩
end

lemma *fls-uminus-nth* [*simp*]: (−*f*) $$ *n* = − (*f* $$ *n*)
⟨*proof*⟩

lemma *fls-const-uminus*[*simp*]: *fls-const* (−*x*) = −*fls-const x*
⟨*proof*⟩

lemma *fls-shift-uminus* [*simp*]: *fls-shift m* (− *f*) = − (*fls-shift m f*)
⟨*proof*⟩

lemma *fls-regpart-uminus* [*simp*]: *fls-regpart* (− *f*) = − *fls-regpart f*
⟨*proof*⟩

lemma *fls-prpart-uminus* [*simp*] : *fls-prpart* (− *f*) = − *fls-prpart f*
⟨*proof*⟩

lemma *fps-to-fls-uminus* [*simp*]: *fps-to-fls* (− *f*) = − *fps-to-fls f*
⟨*proof*⟩

instance *fls* :: (*group-add*) *group-add*
⟨*proof*⟩

instance *fls* :: (*ab-group-add*) *ab-group-add*
⟨*proof*⟩

lemma *fls-uminus-subdegree* [*simp*]: *fls-subdegree* (−*f*) = *fls-subdegree f*
⟨*proof*⟩

lemma *fls-subdegree-minus-sym*: *fls-subdegree* (*g* − *f*) = *fls-subdegree* (*f* − *g*)
⟨*proof*⟩

lemma *fls-regpart-sub-prpart*: *fls-regpart* (*f* − *fls-prpart-as-fls f*) = *fls-regpart f*
⟨*proof*⟩

lemma *fls-prpart-sub-regpart*: *fls-prpart* (*f* − *fls-regpart-as-fls f*) = *fls-prpart f*
⟨*proof*⟩

7.5.3 Multiplication

instantiation *fls* :: ({*comm-monoid-add*, *times*}) *times*
begin
definition *fls-times-def*:
(∗) = (λ*f g*.
fls-shift
(− (*fls-subdegree f* + *fls-subdegree g*))
(*fps-to-fls* (*fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*))
)
instance ⟨*proof*⟩
end

lemma *fls-times-nth-eq0*: *n* < *fls-subdegree f* + *fls-subdegree g* ⟹ (*f* ∗ *g*) $$ *n* = *0*
⟨*proof*⟩

lemma *fls-times-nth*:
fixes *f df g dg*
defines *df* ≡ *fls-subdegree f* **and** *dg* ≡ *fls-subdegree g*
shows (*f* ∗ *g*) $$ *n* = (∑ *i*=*df* + *dg*..*n*. *f* $$ (*i* − *dg*) ∗ *g* $$ (*dg* + *n* − *i*))
and (*f* ∗ *g*) $$ *n* = (∑ *i*=*df*..*n* − *dg*. *f* $$ *i* ∗ *g* $$ (*n* − *i*))
and (*f* ∗ *g*) $$ *n* = (∑ *i*=*dg*..*n* − *df*. *f* $$ (*df* + *i* − *dg*) ∗ *g* $$ (*dg* + *n* − *df* − *i*))
and (*f* ∗ *g*) $$ *n* = (∑ *i*=*0*..*n* − (*df* + *dg*). *f* $$ (*df* + *i*) ∗ *g* $$ (*n* − *df* − *i*))
⟨*proof*⟩

lemma *fls-times-base* [*simp*]:
(*f* ∗ *g*) $$ (*fls-subdegree f* + *fls-subdegree g*) =
(*f* $$ *fls-subdegree f*) ∗ (*g* $$ *fls-subdegree g*)
⟨*proof*⟩

instance *fls* :: ({*comm-monoid-add*, *mult-zero*}) *mult-zero*
⟨*proof*⟩

lemma *fls-mult-one*:
fixes *f* :: 'a::{*comm-monoid-add*, *mult-zero*, *monoid-mult*} *fls*
shows *1* ∗ *f* = *f*
and *f* ∗ *1* = *f*
⟨*proof*⟩

lemma *fls-mult-const-nth* [*simp*]:
fixes *f* :: 'a::{*comm-monoid-add*, *mult-zero*} *fls*
shows (*fls-const x* ∗ *f*) $$ *n* = *x* ∗ *f*$$*n*
and (*f* ∗ *fls-const x*) $$ *n* = *f*$$*n* ∗ *x*
⟨*proof*⟩

lemma *fls-const-mult-const*[*simp*]:
fixes *x y* :: 'a::{*comm-monoid-add*, *mult-zero*}

shows *fls-const x * fls-const y = fls-const (x*y)*
⟨*proof*⟩

lemma *fls-subdegree-add-eq1*:
assumes *f ≠ 0 fls-subdegree f < fls-subdegree g*
shows *fls-subdegree (f + g) = fls-subdegree f*
⟨*proof*⟩

lemma *fls-subdegree-add-eq2*:
assumes *g ≠ 0 fls-subdegree g < fls-subdegree f*
shows *fls-subdegree (f + g) = fls-subdegree g*
⟨*proof*⟩

lemma *fls-subdegree-diff-eq1*:
assumes *f ≠ 0 fls-subdegree f < fls-subdegree g*
shows *fls-subdegree (f − g) = fls-subdegree f*
⟨*proof*⟩

lemma *fls-subdegree-diff-eq2*:
assumes *g ≠ 0 fls-subdegree g < fls-subdegree f*
shows *fls-subdegree (f − g) = fls-subdegree g*
⟨*proof*⟩

lemma *nat-minus-fls-subdegree-plus-const-eq*:
nat (−fls-subdegree (F + fls-const c)) = nat (−fls-subdegree F)
⟨*proof*⟩

lemma *fls-mult-subdegree-ge*:
fixes *f g :: 'a::{comm-monoid-add,mult-zero} fls*
assumes *f*g ≠ 0*
shows *fls-subdegree (f*g) ≥ fls-subdegree f + fls-subdegree g*
⟨*proof*⟩

lemma *fls-mult-subdegree-ge-0*:
fixes *f g :: 'a::{comm-monoid-add,mult-zero} fls*
assumes *fls-subdegree f ≥ 0 fls-subdegree g ≥ 0*
shows *fls-subdegree (f*g) ≥ 0*
⟨*proof*⟩

lemma *fls-mult-nonzero-base-subdegree-eq*:
fixes *f g :: 'a::{comm-monoid-add,mult-zero} fls*
assumes *f $$ (fls-subdegree f) * g $$ (fls-subdegree g) ≠ 0*
shows *fls-subdegree (f*g) = fls-subdegree f + fls-subdegree g*
⟨*proof*⟩

lemma *fls-subdegree-mult* [*simp*]:
fixes *f g :: 'a::semiring-no-zero-divisors fls*
assumes *f ≠ 0 g ≠ 0*
shows *fls-subdegree (f * g) = fls-subdegree f + fls-subdegree g*

⟨*proof*⟩

lemma *fls-shifted-times-simps*:
fixes *f g* :: *'a*::{*comm-monoid-add*, *mult-zero*} *fls*
shows *f* ∗ (*fls-shift n g*) = *fls-shift n* (*f*∗*g*) (*fls-shift n f*) ∗ *g* = *fls-shift n* (*f*∗*g*)
⟨*proof*⟩

lemma *fls-shifted-times-transfer*:
fixes *f g* :: *'a*::{*comm-monoid-add*, *mult-zero*} *fls*
shows *fls-shift n f* ∗ *g* = *f* ∗ *fls-shift n g*
⟨*proof*⟩

lemma *fls-times-both-shifted-simp*:
fixes *f g* :: *'a*::{*comm-monoid-add*, *mult-zero*} *fls*
shows (*fls-shift m f*) ∗ (*fls-shift n g*) = *fls-shift* (*m*+*n*) (*f*∗*g*)
⟨*proof*⟩

lemma *fls-base-factor-mult-base-factor*:
fixes *f g* :: *'a*::{*comm-monoid-add*, *mult-zero*} *fls*
shows *fls-base-factor* (*f* ∗ *fls-base-factor g*) = *fls-base-factor* (*f* ∗ *g*)
and *fls-base-factor* (*fls-base-factor f* ∗ *g*) = *fls-base-factor* (*f* ∗ *g*)
⟨*proof*⟩

lemma *fls-base-factor-mult-both-base-factor*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fls*
shows *fls-base-factor* (*fls-base-factor f* ∗ *fls-base-factor g*) = *fls-base-factor* (*f* ∗ *g*)
⟨*proof*⟩

lemma *fls-base-factor-mult*:
fixes *f g* :: *'a*::*semiring-no-zero-divisors fls*
shows *fls-base-factor* (*f* ∗ *g*) = *fls-base-factor f* ∗ *fls-base-factor g*
⟨*proof*⟩

lemma *fls-times-conv-base-factor-times*:
fixes *f g* :: *'a*::{*comm-monoid-add*, *mult-zero*} *fls*
shows
f ∗ *g* =
fls-shift (−(*fls-subdegree f* + *fls-subdegree g*)) (*fls-base-factor f* ∗ *fls-base-factor g*)
⟨*proof*⟩

lemma *fls-times-base-factor-conv-shifted-times*:
— Convenience form of lemma *fls-times-both-shifted-simp*.
fixes *f g* :: *'a*::{*comm-monoid-add*, *mult-zero*} *fls*
shows
fls-base-factor f ∗ *fls-base-factor g* = *fls-shift* (*fls-subdegree f* + *fls-subdegree g*) (*f* ∗ *g*)
⟨*proof*⟩

lemma *fls-times-conv-regpart*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fls*
assumes *fls-subdegree f* ≥ *0 fls-subdegree g* ≥ *0*
shows *fls-regpart* (*f* ∗ *g*) = *fls-regpart f* ∗ *fls-regpart g*
⟨*proof*⟩

lemma *fls-base-factor-to-fps-mult-conv-unit-factor*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fls*
shows
fls-base-factor-to-fps (*f* ∗ *g*) =
unit-factor (*fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*)
⟨*proof*⟩

lemma *fls-base-factor-to-fps-mult'*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fls*
assumes (*f* $$ *fls-subdegree f*) ∗ (*g* $$ *fls-subdegree g*) ≠ *0*
shows *fls-base-factor-to-fps* (*f* ∗ *g*) = *fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*
⟨*proof*⟩

lemma *fls-base-factor-to-fps-mult*:
fixes *f g* :: *'a*::*semiring-no-zero-divisors fls*
shows *fls-base-factor-to-fps* (*f* ∗ *g*) = *fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*
⟨*proof*⟩

lemma *fls-times-conv-fps-times*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fls*
assumes *fls-subdegree f* ≥ *0 fls-subdegree g* ≥ *0*
shows *f* ∗ *g* = *fps-to-fls* (*fls-regpart f* ∗ *fls-regpart g*)
⟨*proof*⟩

lemma *fps-times-conv-fls-times*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fps*
shows *f* ∗ *g* = *fls-regpart* (*fps-to-fls f* ∗ *fps-to-fls g*)
⟨*proof*⟩

lemma *fls-times-fps-to-fls*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fps*
shows *fps-to-fls* (*f* ∗ *g*) = *fps-to-fls f* ∗ *fps-to-fls g*
⟨*proof*⟩

lemma *fls-X-times-conv-shift*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fls*
shows *fls-X* ∗ *f* = *fls-shift* (−*1*) *f f* ∗ *fls-X* = *fls-shift* (−*1*) *f*
⟨*proof*⟩

lemmas *fls-X-times-comm* = *trans-sym*[*OF fls-X-times-conv-shift*]

lemma *fls-subdegree-mult-fls-X*:
fixes $f :: 'a::\{comm\text{-}monoid\text{-}add, mult\text{-}zero, monoid\text{-}mult\}$ *fls*
assumes $f \neq 0$
shows *fls-subdegree* (*fls-X* $*$ f) = *fls-subdegree* $f + 1$
and *fls-subdegree* (f $*$ *fls-X*) = *fls-subdegree* $f + 1$
⟨*proof*⟩

lemma *fls-mult-fls-X-nonzero*:
fixes $f :: 'a::\{comm\text{-}monoid\text{-}add, mult\text{-}zero, monoid\text{-}mult\}$ *fls*
assumes $f \neq 0$
shows *fls-X* $*$ $f \neq 0$
and f $*$ *fls-X* $\neq 0$
⟨*proof*⟩

lemma *fls-base-factor-mult-fls-X*:
fixes $f :: 'a::\{comm\text{-}monoid\text{-}add, monoid\text{-}mult, mult\text{-}zero\}$ *fls*
shows *fls-base-factor* (*fls-X* $*$ f) = *fls-base-factor* f
and *fls-base-factor* (f $*$ *fls-X*) = *fls-base-factor* f
⟨*proof*⟩

lemma *fls-X-inv-times-conv-shift*:
fixes $f :: 'a::\{comm\text{-}monoid\text{-}add, mult\text{-}zero, monoid\text{-}mult\}$ *fls*
shows *fls-X-inv* $*$ f = *fls-shift* 1 f f $*$ *fls-X-inv* = *fls-shift* 1 f
⟨*proof*⟩

lemmas *fls-X-inv-times-comm* = *trans-sym*[*OF fls-X-inv-times-conv-shift*]

lemma *fls-subdegree-mult-fls-X-inv*:
fixes $f :: 'a::\{comm\text{-}monoid\text{-}add, mult\text{-}zero, monoid\text{-}mult\}$ *fls*
assumes $f \neq 0$
shows *fls-subdegree* (*fls-X-inv* $*$ f) = *fls-subdegree* $f - 1$
and *fls-subdegree* (f $*$ *fls-X-inv*) = *fls-subdegree* $f - 1$
⟨*proof*⟩

lemma *fls-mult-fls-X-inv-nonzero*:
fixes $f :: 'a::\{comm\text{-}monoid\text{-}add, mult\text{-}zero, monoid\text{-}mult\}$ *fls*
assumes $f \neq 0$
shows *fls-X-inv* $*$ $f \neq 0$
and f $*$ *fls-X-inv* $\neq 0$
⟨*proof*⟩

lemma *fls-base-factor-mult-fls-X-inv*:
fixes $f :: 'a::\{comm\text{-}monoid\text{-}add, monoid\text{-}mult, mult\text{-}zero\}$ *fls*
shows *fls-base-factor* (*fls-X-inv* $*$ f) = *fls-base-factor* f
and *fls-base-factor* (f $*$ *fls-X-inv*) = *fls-base-factor* f
⟨*proof*⟩

lemma *fls-mult-assoc-subdegree-ge-0*:

fixes $f\ g\ h :: 'a$::*semiring-0 fls*
assumes *fls-subdegree* $f \geq 0$ *fls-subdegree* $g \geq 0$ *fls-subdegree* $h \geq 0$
shows $f * g * h = f * (g * h)$
⟨*proof*⟩

lemma *fls-mult-assoc-base-factor*:
fixes $a\ b\ c :: 'a$::*semiring-0 fls*
shows
fls-base-factor a $*$ *fls-base-factor b* $*$ *fls-base-factor c* =
fls-base-factor a $*$ (*fls-base-factor b* $*$ *fls-base-factor c*)
⟨*proof*⟩

lemma *fls-mult-distrib-subdegree-ge-0*:
fixes $f\ g\ h :: 'a$::*semiring-0 fls*
assumes *fls-subdegree* $f \geq 0$ *fls-subdegree* $g \geq 0$ *fls-subdegree* $h \geq 0$
shows $(f + g) * h = f * h + g * h$
and $h * (f + g) = h * f + h * g$
⟨*proof*⟩

lemma *fls-mult-distrib-base-factor*:
fixes $a\ b\ c :: 'a$::*semiring-0 fls*
shows
fls-base-factor a $*$ (*fls-base-factor b* $+$ *fls-base-factor c*) =
fls-base-factor a $*$ *fls-base-factor b* $+$ *fls-base-factor a* $*$ *fls-base-factor c*
⟨*proof*⟩

instance *fls* :: (*semiring-0*) *semiring-0*
⟨*proof*⟩

lemma *fls-mult-commute-subdegree-ge-0*:
fixes $f\ g :: 'a$::*comm-semiring-0 fls*
assumes *fls-subdegree* $f \geq 0$ *fls-subdegree* $g \geq 0$
shows $f * g = g * f$
⟨*proof*⟩

lemma *fls-mult-commute-base-factor*:
fixes $a\ b\ c :: 'a$::*comm-semiring-0 fls*
shows *fls-base-factor a* $*$ *fls-base-factor b* = *fls-base-factor b* $*$ *fls-base-factor a*
⟨*proof*⟩

instance *fls* :: (*comm-semiring-0*) *comm-semiring-0*
⟨*proof*⟩

instance *fls* :: (*semiring-1*) *semiring-1*
⟨*proof*⟩

lemma *fls-of-nat*: (*of-nat n* :: $'a$::*semiring-1 fls*) = *fls-const* (*of-nat n*)
⟨*proof*⟩

lemma *fls-of-nat-nth*: *of-nat n $$ k = (if k=0 then of-nat n else 0)*
⟨*proof*⟩

lemma *fls-mult-of-nat-nth* [*simp*]:
shows *(of-nat k * f) $$ n = of-nat k * f$$n*
and *(f * of-nat k) $$ n = f$$n * of-nat k*
⟨*proof*⟩

lemma *fls-subdegree-of-nat* [*simp*]: *fls-subdegree (of-nat n) = 0*
⟨*proof*⟩

lemma *fls-shift-of-nat-nth*:
fls-shift k (of-nat a) $$ n = (if n=−k then of-nat a else 0)
⟨*proof*⟩

lemma *fls-base-factor-of-nat* [*simp*]:
fls-base-factor (of-nat n :: 'a::semiring-1 fls) = (of-nat n :: 'a fls)
⟨*proof*⟩

lemma *fls-regpart-of-nat* [*simp*]: *fls-regpart (of-nat n) = (of-nat n :: 'a::semiring-1 fps)*
⟨*proof*⟩

lemma *fls-prpart-of-nat* [*simp*]: *fls-prpart (of-nat n) = 0*
⟨*proof*⟩

lemma *fls-base-factor-to-fps-of-nat*:
fls-base-factor-to-fps (of-nat n) = (of-nat n :: 'a::semiring-1 fps)
⟨*proof*⟩

lemma *fps-to-fls-of-nat*:
fps-to-fls (of-nat n) = (of-nat n :: 'a::semiring-1 fls)
⟨*proof*⟩

lemma *fps-to-fls-numeral* [*simp*]: *fps-to-fls (numeral n) = numeral n*
⟨*proof*⟩

lemma *fls-const-power*: *fls-const (a ^ b) = fls-const a ^ b*
⟨*proof*⟩

lemma *fls-const-numeral* [*simp*]: *fls-const (numeral n) = numeral n*
⟨*proof*⟩

lemma *fls-mult-of-numeral-nth* [*simp*]:
shows *(numeral k * f) $$ n = numeral k * f $$ n*
and *(f * numeral k) $$ n = f $$ n * numeral k*
⟨*proof*⟩

lemma *fls-nth-numeral'* [*simp*]:

numeral n $$ 0 = numeral n k ≠ 0 ⟹ numeral n $$ k = 0
⟨*proof*⟩

instance *fls* :: (*comm-semiring-1*) *comm-semiring-1*
⟨*proof*⟩

instance *fls* :: (*ring*) *ring* ⟨*proof*⟩

instance *fls* :: (*comm-ring*) *comm-ring* ⟨*proof*⟩

instance *fls* :: (*ring-1*) *ring-1* ⟨*proof*⟩

lemma *fls-of-int-nonneg*: (*of-int* (*int n*) :: *'a*::*ring-1 fls*) = *fls-const* (*of-int* (*int n*))
⟨*proof*⟩

lemma *fls-of-int*: (*of-int i* :: *'a*::*ring-1 fls*) = *fls-const* (*of-int i*)
⟨*proof*⟩

lemma *fls-of-int-nth*: *of-int n $$ k* = (*if k=0 then of-int n else 0*)
⟨*proof*⟩

lemma *fls-mult-of-int-nth* [*simp*]:
shows (*of-int k* ∗ *f*) $$ *n* = *of-int k* ∗ *f*$$*n*
and (*f* ∗ *of-int k*) $$ *n* = *f*$$*n* ∗ *of-int k*
⟨*proof*⟩

lemma *fls-subdegree-of-int* [*simp*]: *fls-subdegree* (*of-int i*) = *0*
⟨*proof*⟩

lemma *fls-shift-of-int-nth*:
fls-shift k (*of-int i*) $$ *n* = (*if n=−k then of-int i else 0*)
⟨*proof*⟩

lemma *fls-base-factor-of-int* [*simp*]:
fls-base-factor (*of-int i* :: *'a*::*ring-1 fls*) = (*of-int i* :: *'a fls*)
⟨*proof*⟩

lemma *fls-regpart-of-int* [*simp*]:
fls-regpart (*of-int i*) = (*of-int i* :: *'a*::*ring-1 fps*)
⟨*proof*⟩

lemma *fls-prpart-of-int* [*simp*]: *fls-prpart* (*of-int n*) = *0*
⟨*proof*⟩

lemma *fls-base-factor-to-fps-of-int*:
fls-base-factor-to-fps (*of-int i*) = (*of-int i* :: *'a*::*ring-1 fps*)
⟨*proof*⟩

lemma *fps-to-fls-of-int*:
fps-to-fls (*of-int i*) = (*of-int i* :: *'a*::*ring-1 fls*)
⟨*proof*⟩

instance *fls* :: (*comm-ring-1*) *comm-ring-1* ⟨*proof*⟩

instance *fls* :: (*semiring-no-zero-divisors*) *semiring-no-zero-divisors*
⟨*proof*⟩

instance *fls* :: (*semiring-1-no-zero-divisors*) *semiring-1-no-zero-divisors* ⟨*proof*⟩

instance *fls* :: (*ring-no-zero-divisors*) *ring-no-zero-divisors* ⟨*proof*⟩

instance *fls* :: (*ring-1-no-zero-divisors*) *ring-1-no-zero-divisors* ⟨*proof*⟩

instance *fls* :: (*idom*) *idom* ⟨*proof*⟩

lemma *semiring-char-fls* [*simp*]: *CHAR*(*'a* :: *comm-semiring-1 fls*) = *CHAR*(*'a*)
⟨*proof*⟩

instance *fls* :: ({*semiring-prime-char*,*comm-semiring-1*}) *semiring-prime-char*
⟨*proof*⟩
instance *fls* :: ({*comm-semiring-prime-char*,*comm-semiring-1*}) *comm-semiring-prime-char*
⟨*proof*⟩
instance *fls* :: ({*comm-ring-prime-char*,*comm-semiring-1*}) *comm-ring-prime-char*
⟨*proof*⟩
instance *fls* :: ({*idom-prime-char*,*comm-semiring-1*}) *idom-prime-char*
⟨*proof*⟩

lemma *fls-subdegree-numeral* [*simp*]: *fls-subdegree* (*numeral n*) = *0*
⟨*proof*⟩

lemma *fls-regpart-numeral* [*simp*]: *fls-regpart* (*numeral n*) = *numeral n*
⟨*proof*⟩

7.5.4 Powers

lemma *fls-subdegree-prod*:
fixes *F* :: *'a* ⇒ *'b* :: *field-char-0 fls*
assumes ⋀*x*. *x* ∈ *I* ⟹ *F x* ≠ *0*
shows *fls-subdegree* (∏ *x*∈*I*. *F x*) = (∑ *x*∈*I*. *fls-subdegree* (*F x*))
⟨*proof*⟩

lemma *fls-subdegree-prod'*:
fixes *F* :: *'a* ⇒ *'b* :: *field-char-0 fls*
assumes ⋀*x*. *x* ∈ *I* ⟹ *fls-subdegree* (*F x*) ≠ *0*
shows *fls-subdegree* (∏ *x*∈*I*. *F x*) = (∑ *x*∈*I*. *fls-subdegree* (*F x*))
⟨*proof*⟩

lemma *fls-pow-subdegree-ge*:
*f^n ≠ 0 ⟹ fls-subdegree (f^n) ≥ n * fls-subdegree f*
⟨*proof*⟩

lemma *fls-pow-nth-below-subdegree*:
*k < n * fls-subdegree f ⟹ (f^n) $$ k = 0*
⟨*proof*⟩

lemma *fls-pow-base* [*simp*]:
*(f ^ n) $$ (n * fls-subdegree f) = (f $$ fls-subdegree f) ^ n*
⟨*proof*⟩

lemma *fls-pow-subdegree-eqI*:
*(f $$ fls-subdegree f) ^ n ≠ 0 ⟹ fls-subdegree (f^n) = n * fls-subdegree f*
⟨*proof*⟩

lemma *fls-unit-base-subdegree-power*:
*x * f $$ fls-subdegree f = 1 ⟹ fls-subdegree (f ^ n) = n * fls-subdegree f*
*f $$ fls-subdegree f * y = 1 ⟹ fls-subdegree (f ^ n) = n * fls-subdegree f*
⟨*proof*⟩

lemma *fls-base-dvd1-subdegree-power*:
*f $$ fls-subdegree f dvd 1 ⟹ fls-subdegree (f ^ n) = n * fls-subdegree f*
⟨*proof*⟩

lemma *fls-pow-subdegree-ge0*:
assumes *fls-subdegree f ≥ 0*
shows *fls-subdegree (f^n) ≥ 0*
⟨*proof*⟩

lemma *fls-subdegree-pow*:
fixes *f* :: *'a*::*semiring-1-no-zero-divisors fls*
shows *fls-subdegree (f ^ n) = n * fls-subdegree f*
⟨*proof*⟩

lemma *fls-shifted-pow*:
*(fls-shift m f) ^ n = fls-shift (n*m) (f ^ n)*
⟨*proof*⟩

lemma *fls-pow-conv-fps-pow*:
assumes *fls-subdegree f ≥ 0*
shows *f ^ n = fps-to-fls ((fls-regpart f) ^ n)*
⟨*proof*⟩

lemma *fps-to-fls-power*: *fps-to-fls (f ^ n) = fps-to-fls f ^ n*
⟨*proof*⟩

lemma *fls-pow-conv-regpart*:

fls-subdegree f $\geq$ *0* $\Longrightarrow$ *fls-regpart* (*f* $\hat{}$ *n*) = (*fls-regpart f*) $\hat{}$ *n*
⟨*proof*⟩

These two lemmas show that shifting 1 is equivalent to powers of the implied variable.

lemma *fls-X-power-conv-shift-1*: *fls-X* $\hat{}$ *n* = *fls-shift* (−*n*) *1*
⟨*proof*⟩

lemma *fls-X-inv-power-conv-shift-1*: *fls-X-inv* $\hat{}$ *n* = *fls-shift n 1*
⟨*proof*⟩

abbreviation *fls-X-intpow* ≡ (λ*i*. *fls-shift* (−*i*) *1*)
— Unifies *fls-X* and *fls-X-inv* so that *fls-X-intpow* returns the equivalent of the implied variable raised to the supplied integer argument of *fls-X-intpow*, whether positive or negative.

lemma *fls-X-intpow-nonzero*[*simp*]: (*fls-X-intpow i* :: *'a*::*zero-neq-one fls*) ≠ *0*
⟨*proof*⟩

lemma *fls-X-intpow-power*: (*fls-X-intpow i*) $\hat{}$ *n* = *fls-X-intpow* (*n* ∗ *i*)
⟨*proof*⟩

lemma *fls-X-power-nth* [*simp*]: *fls-X* $\hat{}$ *n* $$ *k* = (*if k*=*n then 1 else 0*)
⟨*proof*⟩

lemma *fls-X-inv-power-nth* [*simp*]: *fls-X-inv* $\hat{}$ *n* $$ *k* = (*if k*=−*n then 1 else 0*)
⟨*proof*⟩

lemma *fls-X-pow-nonzero*[*simp*]: (*fls-X* $\hat{}$ *n* :: *'a* :: *semiring-1 fls*) ≠ *0*
⟨*proof*⟩

lemma *fls-X-inv-pow-nonzero*[*simp*]: (*fls-X-inv* $\hat{}$ *n* :: *'a* :: *semiring-1 fls*) ≠ *0*
⟨*proof*⟩

lemma *fls-subdegree-fls-X-pow* [*simp*]: *fls-subdegree* (*fls-X* $\hat{}$ *n*) = *n*
⟨*proof*⟩

lemma *fls-subdegree-fls-X-inv-pow* [*simp*]: *fls-subdegree* (*fls-X-inv* $\hat{}$ *n*) = −*n*
⟨*proof*⟩

lemma *fls-subdegree-fls-X-intpow* [*simp*]:
fls-subdegree ((*fls-X-intpow i*) :: *'a*::*zero-neq-one fls*) = *i*
⟨*proof*⟩

lemma *fls-X-pow-conv-fps-X-pow*: *fls-regpart* (*fls-X* $\hat{}$ *n*) = *fps-X* $\hat{}$ *n*
⟨*proof*⟩

lemma *fls-X-inv-pow-regpart*: *n* > *0* $\Longrightarrow$ *fls-regpart* (*fls-X-inv* $\hat{}$ *n*) = *0*
⟨*proof*⟩

lemma *fls-X-intpow-regpart*:
fls-regpart (*fls-X-intpow i*) = (*if i≥0 then fps-X ^ nat i else 0*)
⟨*proof*⟩

lemma *fls-X-power-times-conv-shift*:
*fls-X ^ n * f = fls-shift* (−*int n*) *f f * fls-X ^ n = fls-shift* (−*int n*) *f*
⟨*proof*⟩

lemma *fls-X-inv-power-times-conv-shift*:
*fls-X-inv ^ n * f = fls-shift* (*int n*) *f f * fls-X-inv ^ n = fls-shift* (*int n*) *f*
⟨*proof*⟩

lemma *fls-X-intpow-times-conv-shift*:
fixes *f* :: *'a*::*semiring-1 fls*
shows *fls-X-intpow i * f = fls-shift* (−*i*) *f f * fls-X-intpow i = fls-shift* (−*i*) *f*
⟨*proof*⟩

lemmas *fls-X-power-times-comm* = *trans-sym*[*OF fls-X-power-times-conv-shift*]
lemmas *fls-X-inv-power-times-comm* = *trans-sym*[*OF fls-X-inv-power-times-conv-shift*]

lemma *fls-X-intpow-times-comm*:
fixes *f* :: *'a*::*semiring-1 fls*
shows *fls-X-intpow i * f = f * fls-X-intpow i*
⟨*proof*⟩

lemma *fls-X-intpow-times-fls-X-intpow*:
(*fls-X-intpow i* :: *'a*::*semiring-1 fls*) * *fls-X-intpow j = fls-X-intpow* (*i*+*j*)
⟨*proof*⟩

lemma *fls-X-intpow-diff-conv-times*:
fls-X-intpow (*i*−*j*) = (*fls-X-intpow i* :: *'a*::*semiring-1 fls*) * *fls-X-intpow* (−*j*)
⟨*proof*⟩

lemma *fls-mult-fls-X-power-nonzero*:
assumes *f* ≠ *0*
shows *fls-X ^ n * f* ≠ *0 f * fls-X ^ n* ≠ *0*
⟨*proof*⟩

lemma *fls-mult-fls-X-inv-power-nonzero*:
assumes *f* ≠ *0*
shows *fls-X-inv ^ n * f* ≠ *0 f * fls-X-inv ^ n* ≠ *0*
⟨*proof*⟩

lemma *fls-mult-fls-X-intpow-nonzero*:
fixes *f* :: *'a*::*semiring-1 fls*
assumes *f* ≠ *0*
shows *fls-X-intpow i * f* ≠ *0 f * fls-X-intpow i* ≠ *0*
⟨*proof*⟩

lemma *fls-subdegree-mult-fls-X-power*:
assumes $f \neq 0$
shows *fls-subdegree* (*fls-X* $\hat{}$ $n * f$) = *fls-subdegree* $f + n$
and *fls-subdegree* (f * *fls-X* $\hat{}$ n) = *fls-subdegree* $f + n$
⟨*proof*⟩

lemma *fls-subdegree-mult-fls-X-inv-power*:
assumes $f \neq 0$
shows *fls-subdegree* (*fls-X-inv* $\hat{}$ $n * f$) = *fls-subdegree* $f - n$
and *fls-subdegree* (f * *fls-X-inv* $\hat{}$ n) = *fls-subdegree* $f - n$
⟨*proof*⟩

lemma *fls-subdegree-mult-fls-X-intpow*:
fixes f :: $'a$::*semiring-1 fls*
assumes $f \neq 0$
shows *fls-subdegree* (*fls-X-intpow* $i * f$) = *fls-subdegree* $f + i$
and *fls-subdegree* (f * *fls-X-intpow* i) = *fls-subdegree* $f + i$
⟨*proof*⟩

lemma *fls-X-shift*:
fls-shift ($-$*int* n) *fls-X* = *fls-X* $\hat{}$ *Suc* n
fls-shift (*int* (*Suc* n)) *fls-X* = *fls-X-inv* $\hat{}$ n
⟨*proof*⟩

lemma *fls-X-inv-shift*:
fls-shift (*int* n) *fls-X-inv* = *fls-X-inv* $\hat{}$ *Suc* n
fls-shift ($-$ *int* (*Suc* n)) *fls-X-inv* = *fls-X* $\hat{}$ n
⟨*proof*⟩

lemma *fls-X-power-base-factor*: *fls-base-factor* (*fls-X* $\hat{}$ n) = *1*
⟨*proof*⟩

lemma *fls-X-inv-power-base-factor*: *fls-base-factor* (*fls-X-inv* $\hat{}$ n) = *1*
⟨*proof*⟩

lemma *fls-X-intpow-base-factor*: *fls-base-factor* (*fls-X-intpow* i) = *1*
⟨*proof*⟩

lemma *fls-base-factor-mult-fls-X-power*:
shows *fls-base-factor* (*fls-X* $\hat{}$ $n * f$) = *fls-base-factor* f
and *fls-base-factor* (f * *fls-X* $\hat{}$ n) = *fls-base-factor* f
⟨*proof*⟩

lemma *fls-base-factor-mult-fls-X-inv-power*:
shows *fls-base-factor* (*fls-X-inv* $\hat{}$ $n * f$) = *fls-base-factor* f
and *fls-base-factor* (f * *fls-X-inv* $\hat{}$ n) = *fls-base-factor* f
⟨*proof*⟩

lemma *fls-base-factor-mult-fls-X-intpow*:
fixes f :: *'a*::*semiring-1 fls*
shows *fls-base-factor* (*fls-X-intpow* $i * f$) = *fls-base-factor* f
and *fls-base-factor* (f * *fls-X-intpow* i) = *fls-base-factor* f
⟨*proof*⟩

lemma *fls-X-power-base-factor-to-fps*: *fls-base-factor-to-fps* (*fls-X* ^ n) = *1*
⟨*proof*⟩

lemma *fls-X-inv-power-base-factor-to-fps*: *fls-base-factor-to-fps* (*fls-X-inv* ^ n) = *1*
⟨*proof*⟩

lemma *fls-X-intpow-base-factor-to-fps*: *fls-base-factor-to-fps* (*fls-X-intpow* i) = *1*
⟨*proof*⟩

lemma *fls-base-factor-X-power-decompose*:
fixes f :: *'a*::*semiring-1 fls*
shows f = *fls-base-factor* f * *fls-X-intpow* (*fls-subdegree* f)
and f = *fls-X-intpow* (*fls-subdegree* f) * *fls-base-factor* f
⟨*proof*⟩

lemma *fls-normalized-product-of-inverses*:
assumes $f * g = 1$
shows *fls-base-factor* f * *fls-base-factor* g =
fls-X ^ (*nat* (−(*fls-subdegree* f+*fls-subdegree* g)))
and *fls-base-factor* f * *fls-base-factor* g =
fls-X-intpow (−(*fls-subdegree* f+*fls-subdegree* g))
⟨*proof*⟩

lemma *fls-fps-normalized-product-of-inverses*:
assumes $f * g = 1$
shows *fls-base-factor-to-fps* f * *fls-base-factor-to-fps* g =
fps-X ^ (*nat* (−(*fls-subdegree* f+*fls-subdegree* g)))
⟨*proof*⟩

7.5.5 Inverses

abbreviation *fls-left-inverse* ::
'a::{*comm-monoid-add*,*uminus*,*times*} *fls* ⇒ *'a* ⇒ *'a fls*
where
fls-left-inverse f x ≡
fls-shift (*fls-subdegree* f) (*fps-to-fls* (*fps-left-inverse* (*fls-base-factor-to-fps* f) x))

abbreviation *fls-right-inverse* ::
'a::{*comm-monoid-add*,*uminus*,*times*} *fls* ⇒ *'a* ⇒ *'a fls*
where
fls-right-inverse f y ≡
fls-shift (*fls-subdegree* f) (*fps-to-fls* (*fps-right-inverse* (*fls-base-factor-to-fps* f)

y))

instantiation *fls* :: ({*comm-monoid-add*,*uminus*,*times*,*inverse*}) *inverse*
begin
definition *fls-divide-def*:
f div g =
fls-shift (*fls-subdegree g* − *fls-subdegree f*) (
fps-to-fls ((*fls-base-factor-to-fps f*) *div* (*fls-base-factor-to-fps g*))
)

definition *fls-inverse-def*:
inverse f = *fls-shift* (*fls-subdegree f*) (*fps-to-fls* (*inverse* (*fls-base-factor-to-fps f*)))
instance ⟨*proof*⟩
end

lemma *fls-inverse-def'*:
inverse f = *fls-right-inverse f* (*inverse* (*f* $$ *fls-subdegree f*))
⟨*proof*⟩

lemma *fls-lr-inverse-base*:
fls-left-inverse f x $$ (−*fls-subdegree f*) = *x*
fls-right-inverse f y $$ (−*fls-subdegree f*) = *y*
⟨*proof*⟩

lemma *fls-inverse-base*:
f ≠ *0* ⟹ *inverse f* $$ (−*fls-subdegree f*) = *inverse* (*f* $$ *fls-subdegree f*)
⟨*proof*⟩

lemma *fls-lr-inverse-starting0*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*} *fls*
and *g* :: *'b*::{*ab-group-add*,*mult-zero*} *fls*
shows *fls-left-inverse f 0* = *0*
and *fls-right-inverse g 0* = *0*
⟨*proof*⟩

lemma *fls-lr-inverse-eq0-imp-starting0*:
fls-left-inverse f x = *0* ⟹ *x* = *0*
fls-right-inverse f x = *0* ⟹ *x* = *0*
⟨*proof*⟩

lemma *fls-lr-inverse-eq-0-iff*:
fixes *x* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*}
and *y* :: *'b*::{*ab-group-add*,*mult-zero*}
shows *fls-left-inverse f x* = *0* ⟷ *x* = *0*
and *fls-right-inverse g y* = *0* ⟷ *y* = *0*
⟨*proof*⟩

lemma *fls-inverse-eq-0-iff'*:

fixes $f :: 'a::\{ab\text{-}group\text{-}add,inverse,mult\text{-}zero\}$ *fls*
shows *inverse* $f = 0 \longleftrightarrow (inverse\ (f\ \$\$\ fls\text{-}subdegree\ f) = 0)$
⟨*proof*⟩

lemma *fls-inverse-eq-0-iff*[*simp*]:
inverse $f = (0:: ('a::division\text{-}ring)\ fls) \longleftrightarrow f\ \$\$\ fls\text{-}subdegree\ f = 0$
⟨*proof*⟩

lemmas *fls-inverse-eq-0'* = *iffD2*[*OF fls-inverse-eq-0-iff'*]
lemmas *fls-inverse-eq-0* = *iffD2*[*OF fls-inverse-eq-0-iff*]

lemma *fls-lr-inverse-const*:
fixes $a :: 'a::\{ab\text{-}group\text{-}add,mult\text{-}zero\}$
and $b :: 'b::\{comm\text{-}monoid\text{-}add,mult\text{-}zero,uminus\}$
shows *fls-left-inverse* (*fls-const a*) *x* = *fls-const x*
and *fls-right-inverse* (*fls-const b*) *y* = *fls-const y*
⟨*proof*⟩

lemma *fls-inverse-const*:
fixes $a :: 'a::\{comm\text{-}monoid\text{-}add,inverse,mult\text{-}zero,uminus\}$
shows *inverse* (*fls-const a*) = *fls-const* (*inverse a*)
⟨*proof*⟩

lemma *fls-lr-inverse-of-nat*:
fixes $x :: 'a::\{ring\text{-}1,mult\text{-}zero\}$
and $y :: 'b::\{semiring\text{-}1,uminus\}$
shows *fls-left-inverse* (*of-nat n*) *x* = *fls-const x*
and *fls-right-inverse* (*of-nat n*) *y* = *fls-const y*
⟨*proof*⟩

lemma *fls-inverse-of-nat*:
inverse (*of-nat n* :: *'a* :: {*semiring-1*,*inverse*,*uminus*} *fls*) = *fls-const* (*inverse* (*of-nat n*))
⟨*proof*⟩

lemma *fls-lr-inverse-of-int*:
fixes $x :: 'a::\{ring\text{-}1,mult\text{-}zero\}$
shows *fls-left-inverse* (*of-int n*) *x* = *fls-const x*
and *fls-right-inverse* (*of-int n*) *x* = *fls-const x*
⟨*proof*⟩

lemma *fls-inverse-of-int*:
inverse (*of-int n* :: *'a* :: {*ring-1*,*inverse*,*uminus*} *fls*) = *fls-const* (*inverse* (*of-int n*))
⟨*proof*⟩

lemma *fls-lr-inverse-zero*:
fixes $x :: 'a::\{ab\text{-}group\text{-}add,mult\text{-}zero\}$
and $y :: 'b::\{comm\text{-}monoid\text{-}add,mult\text{-}zero,uminus\}$

shows *fls-left-inverse 0 x = fls-const x*
and *fls-right-inverse 0 y = fls-const y*
⟨*proof*⟩

lemma *fls-inverse-zero-conv-fls-const*:
inverse (0::'a::{comm-monoid-add,mult-zero,uminus,inverse} fls) = fls-const (inverse 0)
⟨*proof*⟩

lemma *fls-inverse-zero'*:
assumes *inverse (0::'a::{comm-monoid-add,inverse,mult-zero,uminus}) = 0*
shows *inverse (0::'a fls) = 0*
⟨*proof*⟩

lemma *fls-inverse-zero* [*simp*]: *inverse (0::'a::division-ring fls) = 0*
⟨*proof*⟩

lemma *fls-inverse-base2*:
fixes *f :: 'a::{comm-monoid-add,mult-zero,uminus,inverse} fls*
shows *inverse f $$ (−fls-subdegree f) = inverse (f $$ fls-subdegree f)*
⟨*proof*⟩

lemma *fls-lr-inverse-one*:
fixes *x :: 'a::{ab-group-add,mult-zero,one}*
and *y :: 'b::{comm-monoid-add,mult-zero,uminus,one}*
shows *fls-left-inverse 1 x = fls-const x*
and *fls-right-inverse 1 y = fls-const y*
⟨*proof*⟩

lemma *fls-lr-inverse-one-one*:
fls-left-inverse 1 1 =
(1::'a::{ab-group-add,mult-zero,one} fls)
fls-right-inverse 1 1 =
(1::'b::{comm-monoid-add,mult-zero,uminus,one} fls)
⟨*proof*⟩

lemma *fls-inverse-one*:
assumes *inverse (1::'a::{comm-monoid-add,inverse,mult-zero,uminus,one}) = 1*
shows *inverse (1::'a fls) = 1*
⟨*proof*⟩

lemma *fls-left-inverse-delta*:
fixes *b :: 'a::{ab-group-add,mult-zero}*
assumes *b ≠ 0*
shows *fls-left-inverse (Abs-fls (λn. if n=a then b else 0)) x =*
Abs-fls (λn. if n=−a then x else 0)
⟨*proof*⟩

lemma *fls-right-inverse-delta*:

fixes $b :: 'a::\{comm\text{-}monoid\text{-}add,mult\text{-}zero,uminus\}$
assumes $b \neq 0$
shows *fls-right-inverse* (*Abs-fls* ($\lambda n.$ *if* $n{=}a$ *then* b *else* 0)) $x =$
Abs-fls ($\lambda n.$ *if* $n{=}{-}a$ *then* x *else* 0)
⟨*proof*⟩

lemma *fls-inverse-delta-nonzero*:
fixes $b :: 'a::\{comm\text{-}monoid\text{-}add,inverse,mult\text{-}zero,uminus\}$
assumes $b \neq 0$
shows *inverse* (*Abs-fls* ($\lambda n.$ *if* $n{=}a$ *then* b *else* 0)) =
Abs-fls ($\lambda n.$ *if* $n{=}{-}a$ *then inverse* b *else* 0)
⟨*proof*⟩

lemma *fls-inverse-delta*:
fixes $b :: 'a::division\text{-}ring$
shows *inverse* (*Abs-fls* ($\lambda n.$ *if* $n{=}a$ *then* b *else* 0)) =
Abs-fls ($\lambda n.$ *if* $n{=}{-}a$ *then inverse* b *else* 0)
⟨*proof*⟩

lemma *fls-lr-inverse-X*:
fixes $x :: 'a::\{ab\text{-}group\text{-}add,mult\text{-}zero,zero\text{-}neq\text{-}one\}$
and $y :: 'b::\{comm\text{-}monoid\text{-}add,uminus,mult\text{-}zero,zero\text{-}neq\text{-}one\}$
shows *fls-left-inverse fls-X x* = *fls-shift 1* (*fls-const x*)
and *fls-right-inverse fls-X y* = *fls-shift 1* (*fls-const y*)
⟨*proof*⟩

lemma *fls-lr-inverse-X'*:
fixes $x :: 'a::\{ab\text{-}group\text{-}add,mult\text{-}zero,zero\text{-}neq\text{-}one,monoid\text{-}mult\}$
and $y :: 'b::\{comm\text{-}monoid\text{-}add,uminus,mult\text{-}zero,zero\text{-}neq\text{-}one,monoid\text{-}mult\}$
shows *fls-left-inverse fls-X x* = *fls-const x* ∗ *fls-X-inv*
and *fls-right-inverse fls-X y* = *fls-const y* ∗ *fls-X-inv*
⟨*proof*⟩

lemma *fls-inverse-X'*:
assumes *inverse* $1 = (1::'a::\{comm\text{-}monoid\text{-}add,inverse,mult\text{-}zero,uminus,zero\text{-}neq\text{-}one\})$
shows *inverse* (*fls-X*::*'a fls*) = *fls-X-inv*
⟨*proof*⟩

lemma *fls-inverse-X*: *inverse* (*fls-X*::*'a*::*division-ring fls*) = *fls-X-inv*
⟨*proof*⟩

lemma *fls-lr-inverse-X-inv*:
fixes $x :: 'a::\{ab\text{-}group\text{-}add,mult\text{-}zero,zero\text{-}neq\text{-}one\}$
and $y :: 'b::\{comm\text{-}monoid\text{-}add,uminus,mult\text{-}zero,zero\text{-}neq\text{-}one\}$
shows *fls-left-inverse fls-X-inv x* = *fls-shift* (-1) (*fls-const x*)
and *fls-right-inverse fls-X-inv y* = *fls-shift* (-1) (*fls-const y*)
⟨*proof*⟩

lemma *fls-lr-inverse-X-inv'*:

fixes $x :: 'a::\{ab\text{-}group\text{-}add,mult\text{-}zero,zero\text{-}neq\text{-}one,monoid\text{-}mult\}$
and $y :: 'b::\{comm\text{-}monoid\text{-}add,uminus,mult\text{-}zero,zero\text{-}neq\text{-}one,monoid\text{-}mult\}$
shows *fls-left-inverse fls-X-inv x = fls-const x * fls-X*
and *fls-right-inverse fls-X-inv y = fls-const y * fls-X*
⟨*proof*⟩

lemma *fls-inverse-X-inv'*:
assumes *inverse 1 = (1::'a::{comm-monoid-add,inverse,mult-zero,uminus,zero-neq-one})*
shows *inverse (fls-X-inv::'a fls) = fls-X*
⟨*proof*⟩

lemma *fls-inverse-X-inv*: *inverse (fls-X-inv::'a::division-ring fls) = fls-X*
⟨*proof*⟩

lemma *fls-lr-inverse-subdegree*:
assumes *x ≠ 0*
shows *fls-subdegree (fls-left-inverse f x) = − fls-subdegree f*
and *fls-subdegree (fls-right-inverse f x) = − fls-subdegree f*
⟨*proof*⟩

lemma *fls-inverse-subdegree'*:
inverse (f $$ fls-subdegree f) ≠ 0 ⟹ fls-subdegree (inverse f) = − fls-subdegree f
⟨*proof*⟩

lemma *fls-inverse-subdegree* [*simp*]:
fixes *f :: 'a::division-ring fls*
shows *fls-subdegree (inverse f) = − fls-subdegree f*
⟨*proof*⟩

lemma *fls-inverse-subdegree-base-nonzero*:
assumes *f ≠ 0 inverse (f $$ fls-subdegree f) ≠ 0*
shows *inverse f $$ (fls-subdegree (inverse f)) = inverse (f $$ fls-subdegree f)*
⟨*proof*⟩

lemma *fls-inverse-subdegree-base*:
fixes *f :: 'a::{ab-group-add,inverse,mult-zero} fls*
shows *inverse f $$ (fls-subdegree (inverse f)) = inverse (f $$ fls-subdegree f)*
⟨*proof*⟩

lemma *fls-lr-inverse-subdegree-0*:
assumes *fls-subdegree f = 0*
shows *fls-subdegree (fls-left-inverse f x) ≥ 0*
and *fls-subdegree (fls-right-inverse f x) ≥ 0*
⟨*proof*⟩

lemma *fls-inverse-subdegree-0*:
fls-subdegree f = 0 ⟹ fls-subdegree (inverse f) ≥ 0
⟨*proof*⟩

lemma *fls-lr-inverse-shift-nonzero*:
fixes $f :: 'a::\{comm\text{-}monoid\text{-}add,mult\text{-}zero,uminus\}$ *fls*
assumes $f \neq 0$
shows *fls-left-inverse* (*fls-shift* m f) x = *fls-shift* $(-m)$ (*fls-left-inverse* f x)
and *fls-right-inverse* (*fls-shift* m f) x = *fls-shift* $(-m)$ (*fls-right-inverse* f x)
⟨*proof*⟩

lemma *fls-inverse-shift-nonzero*:
fixes $f :: 'a::\{comm\text{-}monoid\text{-}add,inverse,mult\text{-}zero,uminus\}$ *fls*
assumes $f \neq 0$
shows *inverse* (*fls-shift* m f) = *fls-shift* $(-m)$ (*inverse* f)
⟨*proof*⟩

lemma *fls-inverse-shift*:
fixes $f :: 'a::division\text{-}ring$ *fls*
shows *inverse* (*fls-shift* m f) = *fls-shift* $(-m)$ (*inverse* f)
⟨*proof*⟩

lemma *fls-left-inverse-base-factor*:
fixes $x :: 'a::\{ab\text{-}group\text{-}add,mult\text{-}zero\}$
assumes $x \neq 0$
shows *fls-left-inverse* (*fls-base-factor* f) x = *fls-base-factor* (*fls-left-inverse* f x)
⟨*proof*⟩

lemma *fls-right-inverse-base-factor*:
fixes $y :: 'a::\{comm\text{-}monoid\text{-}add,mult\text{-}zero,uminus\}$
assumes $y \neq 0$
shows *fls-right-inverse* (*fls-base-factor* f) y = *fls-base-factor* (*fls-right-inverse* f y)
⟨*proof*⟩

lemma *fls-inverse-base-factor'*:
fixes $f :: 'a::\{comm\text{-}monoid\text{-}add,inverse,mult\text{-}zero,uminus\}$ *fls*
assumes *inverse* (f $\$\$$ *fls-subdegree* f) $\neq 0$
shows *inverse* (*fls-base-factor* f) = *fls-base-factor* (*inverse* f)
⟨*proof*⟩

lemma *fls-inverse-base-factor*:
fixes $f :: 'a::\{ab\text{-}group\text{-}add,inverse,mult\text{-}zero\}$ *fls*
shows *inverse* (*fls-base-factor* f) = *fls-base-factor* (*inverse* f)
⟨*proof*⟩

lemma *fls-lr-inverse-regpart*:
assumes *fls-subdegree* $f = 0$
shows *fls-regpart* (*fls-left-inverse* f x) = *fps-left-inverse* (*fls-regpart* f) x
and *fls-regpart* (*fls-right-inverse* f y) = *fps-right-inverse* (*fls-regpart* f) y
⟨*proof*⟩

lemma *fls-inverse-regpart*:
assumes *fls-subdegree f = 0*
shows *fls-regpart (inverse f) = inverse (fls-regpart f)*
⟨*proof*⟩

lemma *fls-base-factor-to-fps-left-inverse*:
fixes *x :: 'a::{ab-group-add,mult-zero}*
shows *fls-base-factor-to-fps (fls-left-inverse f x) =*
fps-left-inverse (fls-base-factor-to-fps f) x
⟨*proof*⟩

lemma *fls-base-factor-to-fps-right-inverse-nonzero*:
fixes *y :: 'a::{comm-monoid-add,mult-zero,uminus}*
assumes *y ≠ 0*
shows *fls-base-factor-to-fps (fls-right-inverse f y) =*
fps-right-inverse (fls-base-factor-to-fps f) y
⟨*proof*⟩

lemma *fls-base-factor-to-fps-right-inverse*:
fixes *y :: 'a::{ab-group-add,mult-zero}*
shows *fls-base-factor-to-fps (fls-right-inverse f y) =*
fps-right-inverse (fls-base-factor-to-fps f) y
⟨*proof*⟩

lemma *fls-base-factor-to-fps-inverse-nonzero*:
fixes *f :: 'a::{comm-monoid-add,inverse,mult-zero,uminus} fls*
assumes *inverse (f $$ fls-subdegree f) ≠ 0*
shows *fls-base-factor-to-fps (inverse f) = inverse (fls-base-factor-to-fps f)*
⟨*proof*⟩

lemma *fls-base-factor-to-fps-inverse*:
fixes *f :: 'a::{ab-group-add,inverse,mult-zero} fls*
shows *fls-base-factor-to-fps (inverse f) = inverse (fls-base-factor-to-fps f)*
⟨*proof*⟩

lemma *fls-lr-inverse-fps-to-fls*:
assumes *subdegree f = 0*
shows *fls-left-inverse (fps-to-fls f) x = fps-to-fls (fps-left-inverse f x)*
and *fls-right-inverse (fps-to-fls f) x = fps-to-fls (fps-right-inverse f x)*
⟨*proof*⟩

lemma *fls-inverse-fps-to-fls*:
subdegree f = 0 ⟹ inverse (fps-to-fls f) = fps-to-fls (inverse f)
⟨*proof*⟩

lemma *fls-lr-inverse-X-power*:
fixes *x :: 'a::ring-1*
and *y :: 'b::{semiring-1,uminus}*
shows *fls-left-inverse (fls-X ^ n) x = fls-shift n (fls-const x)*

and *fls-right-inverse* (*fls-X* ^ *n*) *y* = *fls-shift n* (*fls-const y*)
⟨*proof*⟩

lemma *fls-lr-inverse-X-power'*:
fixes *x* :: *'a*::*ring-1*
and *y* :: *'b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*fls-X* ^ *n*) *x* = *fls-const x* ∗ *fls-X-inv* ^ *n*
and *fls-right-inverse* (*fls-X* ^ *n*) *y* = *fls-const y* ∗ *fls-X-inv* ^ *n*
⟨*proof*⟩

lemma *fls-inverse-X-power'*:
assumes *inverse 1* = (*1*::*'a*::{*semiring-1*,*uminus*,*inverse*})
shows *inverse* ((*fls-X* ^ *n*)::*'a fls*) = *fls-X-inv* ^ *n*
⟨*proof*⟩

lemma *fls-inverse-X-power*:
inverse ((*fls-X*::*'a*::*division-ring fls*) ^ *n*) = *fls-X-inv* ^ *n*
⟨*proof*⟩

lemma *fls-lr-inverse-X-inv-power*:
fixes *x* :: *'a*::*ring-1*
and *y* :: *'b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*fls-X-inv* ^ *n*) *x* = *fls-shift* (−*n*) (*fls-const x*)
and *fls-right-inverse* (*fls-X-inv* ^ *n*) *y* = *fls-shift* (−*n*) (*fls-const y*)
⟨*proof*⟩

lemma *fls-lr-inverse-X-inv-power'*:
fixes *x* :: *'a*::*ring-1*
and *y* :: *'b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*fls-X-inv* ^ *n*) *x* = *fls-const x* ∗ *fls-X* ^ *n*
and *fls-right-inverse* (*fls-X-inv* ^ *n*) *y* = *fls-const y* ∗ *fls-X* ^ *n*
⟨*proof*⟩

lemma *fls-inverse-X-inv-power'*:
assumes *inverse 1* = (*1*::*'a*::{*semiring-1*,*uminus*,*inverse*})
shows *inverse* ((*fls-X-inv* ^ *n*)::*'a fls*) = *fls-X* ^ *n*
⟨*proof*⟩

lemma *fls-inverse-X-inv-power*:
inverse ((*fls-X-inv*::*'a*::*division-ring fls*) ^ *n*) = *fls-X* ^ *n*
⟨*proof*⟩

lemma *fls-lr-inverse-X-intpow*:
fixes *x* :: *'a*::*ring-1*
and *y* :: *'b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*fls-X-intpow i*) *x* = *fls-shift i* (*fls-const x*)
and *fls-right-inverse* (*fls-X-intpow i*) *y* = *fls-shift i* (*fls-const y*)
⟨*proof*⟩

lemma *fls-lr-inverse-X-intpow'*:
fixes $x :: 'a::ring\text{-}1$
and $y :: 'b::\{semiring\text{-}1, uminus\}$
shows *fls-left-inverse* (*fls-X-intpow* i) x = *fls-const* x $*$ *fls-X-intpow* $(-i)$
and *fls-right-inverse* (*fls-X-intpow* i) y = *fls-const* y $*$ *fls-X-intpow* $(-i)$
⟨*proof*⟩

lemma *fls-inverse-X-intpow'*:
assumes *inverse* $1 = (1::'a::\{semiring\text{-}1, uminus, inverse\})$
shows *inverse* (*fls-X-intpow* $i :: 'a$ *fls*) = *fls-X-intpow* $(-i)$
⟨*proof*⟩

lemma *fls-inverse-X-intpow*:
inverse (*fls-X-intpow* $i :: 'a::division\text{-}ring$ *fls*) = *fls-X-intpow* $(-i)$
⟨*proof*⟩

lemma *fls-left-inverse*:
fixes $f :: 'a::ring\text{-}1$ *fls*
assumes $x * f$ $$ *fls-subdegree* $f = 1$
shows *fls-left-inverse* f $x * f = 1$
⟨*proof*⟩

lemma *fls-right-inverse*:
fixes $f :: 'a::ring\text{-}1$ *fls*
assumes f $$ *fls-subdegree* $f * y = 1$
shows $f *$ *fls-right-inverse* f $y = 1$
⟨*proof*⟩
lemma *fls-left-inverse-eq-fls-right-inverse*:
fixes $f :: 'a::ring\text{-}1$ *fls*
assumes $x * f$ $$ *fls-subdegree* $f = 1$ f $$ *fls-subdegree* $f * y = 1$
— These assumptions imply x equals y, but no need to assume that.
shows *fls-left-inverse* f x = *fls-right-inverse* f y
⟨*proof*⟩

lemma *fls-left-inverse-eq-inverse*:
fixes $f :: 'a::division\text{-}ring$ *fls*
shows *fls-left-inverse* f (*inverse* (f $$ *fls-subdegree* f)) = *inverse* f
⟨*proof*⟩

lemma *fls-right-inverse-eq-inverse*:
fixes $f :: 'a::division\text{-}ring$ *fls*
shows *fls-right-inverse* f (*inverse* (f $$ *fls-subdegree* f)) = *inverse* f
⟨*proof*⟩

lemma *fls-left-inverse-eq-fls-right-inverse-comm*:
fixes $f :: 'a::comm\text{-}ring\text{-}1$ *fls*
assumes $x * f$ $$ *fls-subdegree* $f = 1$
shows *fls-left-inverse* f x = *fls-right-inverse* f x
⟨*proof*⟩

lemma *fls-left-inverse'*:
fixes $f :: 'a$::*ring-1 fls*
assumes $x * f$ $$ *fls-subdegree* $f = 1$ f $$ *fls-subdegree* $f * y = 1$
— These assumptions imply x equals y, but no need to assume that.
shows *fls-right-inverse* $f\ y * f = 1$
⟨*proof*⟩

lemma *fls-right-inverse'*:
fixes $f :: 'a$::*ring-1 fls*
assumes $x * f$ $$ *fls-subdegree* $f = 1$ f $$ *fls-subdegree* $f * y = 1$
— These assumptions imply x equals y, but no need to assume that.
shows $f *$ *fls-left-inverse* $f\ x = 1$
⟨*proof*⟩

lemma *fls-mult-left-inverse-base-factor*:
fixes $f :: 'a$::*ring-1 fls*
assumes $x * (f$ $$ *fls-subdegree* $f) = 1$
shows *fls-left-inverse* (*fls-base-factor* f) $x * f =$ *fls-X-intpow* (*fls-subdegree* f)
⟨*proof*⟩

lemma *fls-mult-right-inverse-base-factor*:
fixes $f :: 'a$::*ring-1 fls*
assumes $(f$ $$ *fls-subdegree* $f) * y = 1$
shows $f *$ *fls-right-inverse* (*fls-base-factor* f) $y =$ *fls-X-intpow* (*fls-subdegree* f)
⟨*proof*⟩

lemma *fls-mult-inverse-base-factor*:
fixes $f :: 'a$::*division-ring fls*
assumes $f \neq 0$
shows $f *$ *inverse* (*fls-base-factor* f) = *fls-X-intpow* (*fls-subdegree* f)
⟨*proof*⟩

lemma *fls-left-inverse-idempotent-ring1*:
fixes $f :: 'a$::*ring-1 fls*
assumes $x * f$ $$ *fls-subdegree* $f = 1$ $y * x = 1$
— These assumptions imply y equals f $$ *fls-subdegree* f, but no need to assume that.
shows *fls-left-inverse* (*fls-left-inverse* $f\ x$) $y = f$
⟨*proof*⟩

lemma *fls-left-inverse-idempotent-comm-ring1*:
fixes $f :: 'a$::*comm-ring-1 fls*
assumes $x * f$ $$ *fls-subdegree* $f = 1$
shows *fls-left-inverse* (*fls-left-inverse* $f\ x$) $(f$ $$ *fls-subdegree* $f) = f$
⟨*proof*⟩

lemma *fls-right-inverse-idempotent-ring1*:
fixes $f :: 'a$::*ring-1 fls*

assumes f $$ *fls-subdegree* $f * x = 1$ $x * y = 1$
— These assumptions imply y equals f $$ *fls-subdegree* f, but no need to assume that.
shows *fls-right-inverse* (*fls-right-inverse* f x) $y = f$
⟨*proof*⟩

lemma *fls-right-inverse-idempotent-comm-ring1*:
fixes $f :: 'a$::*comm-ring-1 fls*
assumes f $$ *fls-subdegree* $f * x = 1$
shows *fls-right-inverse* (*fls-right-inverse* f x) (f $$ *fls-subdegree* f) $= f$
⟨*proof*⟩

lemma *fls-lr-inverse-unique-ring1*:
fixes f $g :: 'a ::$ *ring-1 fls*
assumes *fg*: $f * g = 1$ g $$ *fls-subdegree* $g * f$ $$ *fls-subdegree* $f = 1$
shows *fls-left-inverse* g (f $$ *fls-subdegree* f) $= f$
and *fls-right-inverse* f (g $$ *fls-subdegree* g) $= g$
⟨*proof*⟩

lemma *fls-lr-inverse-unique-divring*:
fixes f $g :: 'a$ ::*division-ring fls*
assumes *fg*: $f * g = 1$
shows *fls-left-inverse* g (f $$ *fls-subdegree* f) $= f$
and *fls-right-inverse* f (g $$ *fls-subdegree* g) $= g$
⟨*proof*⟩

lemma *fls-lr-inverse-minus*:
fixes $f :: 'a$::*ring-1 fls*
shows *fls-left-inverse* $(-f)$ $(-x) = -$ *fls-left-inverse* f x
and *fls-right-inverse* $(-f)$ $(-x) = -$ *fls-right-inverse* f x
⟨*proof*⟩

lemma *fls-inverse-minus* [*simp*]: *inverse* $(-f) = -$*inverse* ($f :: 'a ::$ *division-ring fls*)
⟨*proof*⟩

lemma *fls-lr-inverse-mult-ring1*:
fixes f $g :: 'a$::*ring-1 fls*
assumes *x*: $x * f$ $$ *fls-subdegree* $f = 1$ f $$ *fls-subdegree* $f * x = 1$
and *y*: $y * g$ $$ *fls-subdegree* $g = 1$ g $$ *fls-subdegree* $g * y = 1$
shows *fls-left-inverse* $(f * g)$ $(y*x) =$ *fls-left-inverse* g $y *$ *fls-left-inverse* f x
and *fls-right-inverse* $(f * g)$ $(y*x) =$ *fls-right-inverse* g $y *$ *fls-right-inverse* f x
⟨*proof*⟩

lemma *fls-lr-inverse-power-ring1*:
fixes $f :: 'a$::*ring-1 fls*
assumes *x*: $x * f$ $$ *fls-subdegree* $f = 1$ f $$ *fls-subdegree* $f * x = 1$
shows *fls-left-inverse* $(f \hat{} n)$ $(x \hat{} n) =$ (*fls-left-inverse* f x) $\hat{} n$

fls-right-inverse $(f \hat{} n)$ $(x \hat{} n) = ($*fls-right-inverse* $f\ x) \hat{} n$
⟨*proof*⟩

lemma *fls-divide-convert-times-inverse*:
fixes $f\ g$ :: $'a$::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*} *fls*
shows $f\ /\ g = f * inverse\ g$
⟨*proof*⟩

instance *fls* :: (*division-ring*) *division-ring*
⟨*proof*⟩

lemma *fls-lr-inverse-mult-divring*:
fixes $f\ g$ :: $'a$::*division-ring fls*
and *df dg* :: *int*
defines $df \equiv$ *fls-subdegree f*
and $dg \equiv$ *fls-subdegree g*
shows *fls-left-inverse* $(f*g)$ $($*inverse* $((f*g)\$\$(df+dg))) =$
fls-left-inverse g $($*inverse* $(g\$\$dg)) *$ *fls-left-inverse f* $($*inverse* $(f\$\$df))$
and *fls-right-inverse* $(f*g)$ $($*inverse* $((f*g)\$\$(df+dg))) =$
fls-right-inverse g $($*inverse* $(g\$\$dg)) *$ *fls-right-inverse f* $($*inverse* $(f\$\$df))$
⟨*proof*⟩

lemma *fls-lr-inverse-power-divring*:
fls-left-inverse $(f \hat{} n)$ $((inverse\ (f\ \$\$\ fls\text{-}subdegree\ f)) \hat{} n) =$
$(fls\text{-}left\text{-}inverse\ f\ (inverse\ (f\ \$\$\ fls\text{-}subdegree\ f))) \hat{} n$ (**is** *?P*)
and *fls-right-inverse* $(f \hat{} n)$ $((inverse\ (f\ \$\$\ fls\text{-}subdegree\ f)) \hat{} n) =$
$(fls\text{-}right\text{-}inverse\ f\ (inverse\ (f\ \$\$\ fls\text{-}subdegree\ f))) \hat{} n$ (**is** *?Q*)
for f :: $'a$::*division-ring fls*
⟨*proof*⟩

lemma *one-plus-fls-X-powi-eq*:
$(1 + fls\text{-}X)$ *powi* $n =$ *fps-to-fls* $($*fps-binomial* $($*of-int* n :: $'a$ :: *field-char-0*$))$
⟨*proof*⟩

instance *fls* :: (*field*) *field*
⟨*proof*⟩

instance *fls* :: ({*field-prime-char*,*comm-semiring-1*}) *field-prime-char*
⟨*proof*⟩

instance *fls* :: (*semiring-char-0*) *semiring-char-0*
⟨*proof*⟩

instance *fls* :: (*field-char-0*) *field-char-0* ⟨*proof*⟩

lemma *fls-subdegree-power-int* [*simp*]:
fixes F :: $'a$ :: *field fls*
shows *fls-subdegree* $(F$ *powi* $n) = n *$ *fls-subdegree F*

⟨*proof*⟩

7.5.6 Division

lemma *fls-divide-nth-below*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*uminus*,*times*,*inverse*} *fls*
shows *n* < *fls-subdegree f* − *fls-subdegree g* ⟹ (*f div g*) $$ *n* = *0*
⟨*proof*⟩

lemma *fls-divide-nth-base*:
fixes *f g* :: *'a*::*division-ring fls*
shows
(*f div g*) $$ (*fls-subdegree f* − *fls-subdegree g*) =
f $$ *fls-subdegree f* / *g* $$ *fls-subdegree g*
⟨*proof*⟩

lemma *fls-div-zero* [*simp*]:
0 div (*g* :: *'a* :: {*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*} *fls*) = *0*
⟨*proof*⟩

lemma *fls-div-by-zero*:
fixes *g* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*} *fls*
assumes *inverse* (*0*::*'a*) = *0*
shows *g div 0* = *0*
⟨*proof*⟩

lemma *fls-divide-times*:
fixes *f g* :: *'a*::{*semiring-0*,*inverse*,*uminus*} *fls*
shows (*f* ∗ *g*) / *h* = *f* ∗ (*g* / *h*)
⟨*proof*⟩

lemma *fls-divide-times2*:
fixes *f g* :: *'a*::{*comm-semiring-0*,*inverse*,*uminus*} *fls*
shows (*f* ∗ *g*) / *h* = (*f* / *h*) ∗ *g*
⟨*proof*⟩

lemma *fls-divide-subdegree-ge*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*uminus*,*times*,*inverse*} *fls*
assumes *f* / *g* ≠ *0*
shows *fls-subdegree* (*f* / *g*) ≥ *fls-subdegree f* − *fls-subdegree g*
⟨*proof*⟩

lemma *fls-divide-subdegree*:
fixes *f g* :: *'a*::*division-ring fls*
assumes *f* ≠ *0 g* ≠ *0*
shows *fls-subdegree* (*f* / *g*) = *fls-subdegree f* − *fls-subdegree g*
⟨*proof*⟩

lemma *fls-divide-shift-numer-nonzero*:

fixes $f\ g :: 'a :: \{comm\text{-}monoid\text{-}add, inverse, times, uminus\}\ fls$
assumes $f \neq 0$
shows $fls\text{-}shift\ m\ f\ /\ g = fls\text{-}shift\ m\ (f/g)$
⟨*proof*⟩

lemma *fls-divide-shift-numer*:
fixes $f\ g :: 'a :: \{comm\text{-}monoid\text{-}add, inverse, mult\text{-}zero, uminus\}\ fls$
shows $fls\text{-}shift\ m\ f\ /\ g = fls\text{-}shift\ m\ (f/g)$
⟨*proof*⟩

lemma *fls-divide-shift-denom-nonzero*:
fixes $f\ g :: 'a :: \{comm\text{-}monoid\text{-}add, inverse, times, uminus\}\ fls$
assumes $g \neq 0$
shows $f\ /\ fls\text{-}shift\ m\ g = fls\text{-}shift\ (-m)\ (f/g)$
⟨*proof*⟩

lemma *fls-divide-shift-denom*:
fixes $f\ g :: 'a :: division\text{-}ring\ fls$
shows $f\ /\ fls\text{-}shift\ m\ g = fls\text{-}shift\ (-m)\ (f/g)$
⟨*proof*⟩

lemma *fls-divide-shift-both-nonzero*:
fixes $f\ g :: 'a :: \{comm\text{-}monoid\text{-}add, inverse, times, uminus\}\ fls$
assumes $f \neq 0\ g \neq 0$
shows $fls\text{-}shift\ n\ f\ /\ fls\text{-}shift\ m\ g = fls\text{-}shift\ (n-m)\ (f/g)$
⟨*proof*⟩

lemma *fls-divide-shift-both* [*simp*]:
fixes $f\ g :: 'a :: division\text{-}ring\ fls$
shows $fls\text{-}shift\ n\ f\ /\ fls\text{-}shift\ m\ g = fls\text{-}shift\ (n-m)\ (f/g)$
⟨*proof*⟩

lemma *fls-divide-base-factor-numer*:
$fls\text{-}base\text{-}factor\ f\ /\ g = fls\text{-}shift\ (fls\text{-}subdegree\ f)\ (f/g)$
⟨*proof*⟩

lemma *fls-divide-base-factor-denom*:
$f\ /\ fls\text{-}base\text{-}factor\ g = fls\text{-}shift\ (-fls\text{-}subdegree\ g)\ (f/g)$
⟨*proof*⟩

lemma *fls-divide-base-factor'*:
$fls\text{-}base\text{-}factor\ f\ /\ fls\text{-}base\text{-}factor\ g = fls\text{-}shift\ (fls\text{-}subdegree\ f - fls\text{-}subdegree\ g)$
(f/g)
⟨*proof*⟩

lemma *fls-divide-base-factor*:
fixes $f\ g :: 'a :: division\text{-}ring\ fls$
shows $fls\text{-}base\text{-}factor\ f\ /\ fls\text{-}base\text{-}factor\ g = fls\text{-}base\text{-}factor\ (f/g)$
⟨*proof*⟩

lemma *fls-divide-regpart*:
fixes *f g* :: *'a*::{*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*} *fls*
assumes *fls-subdegree f* ≥ *0 fls-subdegree g* ≥ *0*
shows *fls-regpart* (*f* / *g*) = *fls-regpart f* / *fls-regpart g*
⟨*proof*⟩

lemma *fls-divide-fls-base-factor-to-fps'*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*uminus*,*inverse*,*mult-zero*} *fls*
shows
fls-base-factor-to-fps f / *fls-base-factor-to-fps g* =
fls-regpart (*fls-shift* (*fls-subdegree f* − *fls-subdegree g*) (*f* / *g*))
⟨*proof*⟩

lemma *fls-divide-fls-base-factor-to-fps*:
fixes *f g* :: *'a*::*division-ring fls*
shows *fls-base-factor-to-fps f* / *fls-base-factor-to-fps g* = *fls-base-factor-to-fps* (*f* / *g*)
⟨*proof*⟩

lemma *fls-divide-fps-to-fls*:
fixes *f g* :: *'a*::{*inverse*,*ab-group-add*,*mult-zero*} *fps*
assumes *subdegree f* ≥ *subdegree g*
shows *fps-to-fls f* / *fps-to-fls g* = *fps-to-fls* (*f*/*g*)
⟨*proof*⟩

lemma *fls-divide-1'*:
fixes *f* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*zero-neq-one*,*monoid-mult*} *fls*
assumes *inverse* (*1*::*'a*) = *1*
shows *f* / *1* = *f*
⟨*proof*⟩

lemma *fls-divide-1* [*simp*]: *a* / *1* = (*a*::*'a*::*division-ring fls*)
⟨*proof*⟩

lemma *fls-const-divide-const*:
fixes *x y* :: *'a*::*division-ring*
shows *fls-const x* / *fls-const y* = *fls-const* (*x*/*y*)
⟨*proof*⟩

lemma *fls-divide-X'*:
fixes *f* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*zero-neq-one*,*monoid-mult*} *fls*
assumes *inverse* (*1*::*'a*) = *1*
shows *f* / *fls-X* = *fls-shift 1 f*
⟨*proof*⟩

lemma *fls-divide-X* [*simp*]:

fixes $f :: 'a::division\text{-}ring\ fls$
shows $f\ /\ fls\text{-}X = fls\text{-}shift\ 1\ f$
⟨*proof*⟩

lemma *fls-divide-X-power'*:
fixes $f :: 'a::\{semiring\text{-}1,inverse,uminus\}\ fls$
assumes $inverse\ (1::'a) = 1$
shows $f\ /\ (fls\text{-}X\ \hat{}\ n) = fls\text{-}shift\ n\ f$
⟨*proof*⟩

lemma *fls-divide-X-power* [*simp*]:
fixes $f :: 'a::division\text{-}ring\ fls$
shows $f\ /\ (fls\text{-}X\ \hat{}\ n) = fls\text{-}shift\ n\ f$
⟨*proof*⟩

lemma *fls-divide-X-inv'*:
fixes $f :: 'a::\{comm\text{-}monoid\text{-}add,inverse,mult\text{-}zero,uminus,zero\text{-}neq\text{-}one,monoid\text{-}mult\}$ *fls*
assumes $inverse\ (1::'a) = 1$
shows $f\ /\ fls\text{-}X\text{-}inv = fls\text{-}shift\ (-1)\ f$
⟨*proof*⟩

lemma *fls-divide-X-inv* [*simp*]:
fixes $f :: 'a::division\text{-}ring\ fls$
shows $f\ /\ fls\text{-}X\text{-}inv = fls\text{-}shift\ (-1)\ f$
⟨*proof*⟩

lemma *fls-divide-X-inv-power'*:
fixes $f :: 'a::\{semiring\text{-}1,inverse,uminus\}\ fls$
assumes $inverse\ (1::'a) = 1$
shows $f\ /\ (fls\text{-}X\text{-}inv\ \hat{}\ n) = fls\text{-}shift\ (-int\ n)\ f$
⟨*proof*⟩

lemma *fls-divide-X-inv-power* [*simp*]:
fixes $f :: 'a::division\text{-}ring\ fls$
shows $f\ /\ (fls\text{-}X\text{-}inv\ \hat{}\ n) = fls\text{-}shift\ (-int\ n)\ f$
⟨*proof*⟩

lemma *fls-divide-X-intpow'*:
fixes $f :: 'a::\{semiring\text{-}1,inverse,uminus\}\ fls$
assumes $inverse\ (1::'a) = 1$
shows $f\ /\ (fls\text{-}X\text{-}intpow\ i) = fls\text{-}shift\ i\ f$
⟨*proof*⟩

lemma *fls-divide-X-intpow-conv-times'*:
fixes $f :: 'a::\{semiring\text{-}1,inverse,uminus\}\ fls$
assumes $inverse\ (1::'a) = 1$
shows $f\ /\ (fls\text{-}X\text{-}intpow\ i) = f * fls\text{-}X\text{-}intpow\ (-i)$
⟨*proof*⟩

lemma *fls-divide-X-intpow*:
fixes f :: *'a*::*division-ring fls*
shows f / (*fls-X-intpow* i) = *fls-shift* i f
⟨*proof*⟩

lemma *fls-divide-X-intpow-conv-times*:
fixes f :: *'a*::*division-ring fls*
shows f / (*fls-X-intpow* i) = f ∗ *fls-X-intpow* $(-i)$
⟨*proof*⟩

lemma *fls-X-intpow-div-fls-X-intpow-semiring1*:
assumes *inverse* (*1*::*'a*::{*semiring-1*,*inverse*,*uminus*}) = *1*
shows (*fls-X-intpow* i :: *'a fls*) / *fls-X-intpow* j = *fls-X-intpow* $(i-j)$
⟨*proof*⟩

lemma *fls-X-intpow-div-fls-X-intpow*:
(*fls-X-intpow* i :: *'a*::*division-ring fls*) / *fls-X-intpow* j = *fls-X-intpow* $(i-j)$
⟨*proof*⟩

lemma *fls-divide-add*:
fixes f g h :: *'a*::{*semiring-0*,*inverse*,*uminus*} *fls*
shows $(f + g)$ / h = f / h + g / h
⟨*proof*⟩

lemma *fls-divide-diff*:
fixes f g h :: *'a*::{*ring*,*inverse*} *fls*
shows $(f - g)$ / h = f / h − g / h
⟨*proof*⟩

lemma *fls-divide-uminus*:
fixes f g h :: *'a*::{*ring*,*inverse*} *fls*
shows $(-f)$ / g = − $(f$ / $g)$
⟨*proof*⟩

lemma *fls-divide-uminus'*:
fixes f g h :: *'a*::*division-ring fls*
shows f / $(-g)$ = − $(f$ / $g)$
⟨*proof*⟩

7.5.7 Units

lemma *fls-is-left-unit-iff-base-is-left-unit*:
fixes f :: *'a* :: *ring-1-no-zero-divisors fls*
shows $(\exists g.\ 1 = f * g) \longleftrightarrow (\exists k.\ 1 = f$ $$ *fls-subdegree* $f * k)$
⟨*proof*⟩

lemma *fls-is-right-unit-iff-base-is-right-unit*:
fixes f :: *'a* :: *ring-1-no-zero-divisors fls*

shows $(\exists g.\ 1 = g * f) \longleftrightarrow (\exists k.\ 1 = k * f\ \$\$\ \mathit{fls\text{-}subdegree}\ f)$
⟨*proof*⟩

7.6 Composition

definition *fls-compose-fps* :: *'a* :: *field fls* ⇒ *'a fps* ⇒ *'a fls* **where**
fls-compose-fps F G =
fps-to-fls (*fps-compose* (*fls-base-factor-to-fps F*) *G*) ∗ *fps-to-fls G powi fls-subdegree F*

lemma *fps-compose-of-nat* [*simp*]: *fps-compose* (*of-nat n* :: *'a* :: *comm-ring-1 fps*) *H* = *of-nat n*
and *fps-compose-of-int* [*simp*]: *fps-compose* (*of-int i*) *H* = *of-int i*
⟨*proof*⟩

lemmas [*simp*] = *fps-to-fls-of-nat fps-to-fls-of-int*

lemma *fls-compose-fps-0* [*simp*]: *fls-compose-fps 0 H* = *0*
and *fls-compose-fps-1* [*simp*]: *fls-compose-fps 1 H* = *1*
and *fls-compose-fps-const* [*simp*]: *fls-compose-fps* (*fls-const c*) *H* = *fls-const c*
and *fls-compose-fps-of-nat* [*simp*]: *fls-compose-fps* (*of-nat n*) *H* = *of-nat n*
and *fls-compose-fps-of-int* [*simp*]: *fls-compose-fps* (*of-int i*) *H* = *of-int i*
and *fls-compose-fps-X* [*simp*]: *fls-compose-fps fls-X F* = *fps-to-fls F*
⟨*proof*⟩

lemma *fls-compose-fps-0-right*:
fls-compose-fps F 0 = (*if 0* ≤ *fls-subdegree F then fls-const* (*F* $$ *0*) *else 0*)
⟨*proof*⟩

lemma *fls-compose-fps-shift*:
assumes *H* ≠ *0*
shows *fls-compose-fps* (*fls-shift n F*) *H* = *fls-compose-fps F H* ∗ *fps-to-fls H powi* (−*n*)
⟨*proof*⟩

lemma *fls-compose-fps-to-fls* [*simp*]:
assumes [*simp*]: *G* ≠ *0 fps-nth G 0* = *0*
shows *fls-compose-fps* (*fps-to-fls F*) *G* = *fps-to-fls* (*fps-compose F G*)
⟨*proof*⟩

lemma *fls-compose-fps-mult*:
assumes [*simp*]: *H* ≠ *0 fps-nth H 0* = *0*
shows *fls-compose-fps* (*F* ∗ *G*) *H* = *fls-compose-fps F H* ∗ *fls-compose-fps G H*
⟨*proof*⟩

lemma *fls-compose-fps-power*:
assumes [*simp*]: *G* ≠ *0 fps-nth G 0* = *0*
shows *fls-compose-fps* (*F* ^ *n*) *G* = *fls-compose-fps F G* ^ *n*

⟨*proof*⟩

lemma *fls-compose-fps-add*:
assumes [*simp*]: $H \neq 0$ *fps-nth* $H\ 0 = 0$
shows *fls-compose-fps* $(F + G)\ H =$ *fls-compose-fps* $F\ H +$ *fls-compose-fps* $G\ H$
⟨*proof*⟩

lemma *fls-compose-fps-uminus* [*simp*]: *fls-compose-fps* $(-F)\ H = -$*fls-compose-fps* $F\ H$
⟨*proof*⟩

lemma *fls-compose-fps-diff*:
assumes [*simp*]: $H \neq 0$ *fps-nth* $H\ 0 = 0$
shows *fls-compose-fps* $(F - G)\ H =$ *fls-compose-fps* $F\ H -$ *fls-compose-fps* $G\ H$
⟨*proof*⟩

lemma *fls-compose-fps-eq-0-iff*:
assumes $H \neq 0$ *fps-nth* $H\ 0 = 0$
shows *fls-compose-fps* $F\ H = 0 \longleftrightarrow F = 0$
⟨*proof*⟩

lemma *fls-compose-fps-inverse*:
assumes [*simp*]: $H \neq 0$ *fps-nth* $H\ 0 = 0$
shows *fls-compose-fps* (*inverse* F) $H =$ *inverse* (*fls-compose-fps* $F\ H$)
⟨*proof*⟩

lemma *fls-compose-fps-divide*:
assumes [*simp*]: $H \neq 0$ *fps-nth* $H\ 0 = 0$
shows *fls-compose-fps* $(F\ /\ G)\ H =$ *fls-compose-fps* $F\ H\ /$ *fls-compose-fps* $G\ H$
⟨*proof*⟩

lemma *fls-compose-fps-powi*:
assumes [*simp*]: $H \neq 0$ *fps-nth* $H\ 0 = 0$
shows *fls-compose-fps* $(F$ *powi* $n)\ H =$ *fls-compose-fps* $F\ H$ *powi* n
⟨*proof*⟩

lemma *fls-compose-fps-assoc*:
assumes [*simp*]: $G \neq 0$ *fps-nth* $G\ 0 = 0\ H \neq 0$ *fps-nth* $H\ 0 = 0$
shows *fls-compose-fps* (*fls-compose-fps* $F\ G$) $H =$ *fls-compose-fps* F (*fps-compose* $G\ H$)
⟨*proof*⟩

lemma *subdegree-pos-iff*: *subdegree* $F > 0 \longleftrightarrow F \neq 0 \wedge$ *fps-nth* $F\ 0 = 0$
⟨*proof*⟩

lemma *fls-X-power-int* [*simp*]: *fls-X* *powi* $n =$ (*fls-X-intpow* $n ::$ *'a* $::$ *division-ring*

fls)
⟨*proof*⟩

lemma *fls-const-power-int*: *fls-const* (*c powi n*) = *fls-const* (*c* :: *'a* :: *division-ring*) *powi n*
⟨*proof*⟩

lemma *fls-nth-fls-compose-fps-linear*:
fixes *c* :: *'a* :: *field*
assumes [*simp*]: *c* ≠ *0*
shows *fls-compose-fps F* (*fps-const c* ∗ *fps-X*) $$ *n* = *F* $$ *n* ∗ *c powi n*
⟨*proof*⟩

lemma *fls-const-transfer* [*transfer-rule*]:
rel-fun (=) (*pcr-fls* (=))
(λ*c n*. *if n* = *0 then c else 0*) *fls-const*
⟨*proof*⟩

lemma *fls-shift-transfer* [*transfer-rule*]:
rel-fun (=) (*rel-fun* (*pcr-fls* (=)) (*pcr-fls* (=)))
(λ*n f k*. *f* (*k*+*n*)) *fls-shift*
⟨*proof*⟩

lift-definition *fls-compose-power* :: *'a* :: *zero fls* ⇒ *nat* ⇒ *'a fls* **is**
λ*f d n*. *if d* > *0* ∧ *int d dvd n then f* (*n div int d*) *else 0*
⟨*proof*⟩

lemma *fls-nth-compose-power*:
assumes *d* > *0*
shows *fls-compose-power f d* $$ *n* = (*if int d dvd n then f* $$ (*n div int d*) *else 0*)
⟨*proof*⟩

lemma *fls-compose-power-0-left* [*simp*]: *fls-compose-power 0 d* = *0*
⟨*proof*⟩

lemma *fls-compose-power-1-left* [*simp*]: *d* > *0* ⟹ *fls-compose-power 1 d* = *1*
⟨*proof*⟩

lemma *fls-compose-power-const-left* [*simp*]:
d > *0* ⟹ *fls-compose-power* (*fls-const c*) *d* = *fls-const c*
⟨*proof*⟩

lemma *fls-compose-power-shift* [*simp*]:
d > *0* ⟹ *fls-compose-power* (*fls-shift n f*) *d* = *fls-shift* (*d* ∗ *n*) (*fls-compose-power f d*)
⟨*proof*⟩

lemma *fls-compose-power-X-intpow* [*simp*]:
$d > 0 \Longrightarrow$ *fls-compose-power* (*fls-X-intpow* n) $d =$ *fls-X-intpow* (*int* $d * n$)
⟨*proof*⟩

lemma *fls-compose-power-X* [*simp*]:
$d > 0 \Longrightarrow$ *fls-compose-power fls-X* $d =$ *fls-X-intpow* (*int* d)
⟨*proof*⟩

lemma *fls-compose-power-X-inv* [*simp*]:
$d > 0 \Longrightarrow$ *fls-compose-power fls-X-inv* $d =$ *fls-X-intpow* $(-$*int* $d)$
⟨*proof*⟩

lemma *fls-compose-power-0-right* [*simp*]: *fls-compose-power* $f\ 0 = 0$
⟨*proof*⟩

lemma *fls-compose-power-add* [*simp*]:
fls-compose-power $(f + g)\ d =$ *fls-compose-power* $f\ d +$ *fls-compose-power* $g\ d$
⟨*proof*⟩

lemma *fls-compose-power-diff* [*simp*]:
fls-compose-power $(f - g)\ d =$ *fls-compose-power* $f\ d -$ *fls-compose-power* $g\ d$
⟨*proof*⟩

lemma *fls-compose-power-uminus* [*simp*]:
fls-compose-power $(-f)\ d = -$*fls-compose-power* $f\ d$
⟨*proof*⟩

lemma *fps-nth-compose-X-power*:
fps-nth (*f oo* (*fps-X* $\hat{}$ d)) $n =$ (*if* d *dvd* n *then fps-nth* f $(n$ *div* $d)$ *else* 0)
⟨*proof*⟩

lemma *fls-compose-power-fps-to-fls*:
assumes $d > 0$
shows *fls-compose-power* (*fps-to-fls* f) $d =$ *fps-to-fls* (*fps-compose* f (*fps-X* $\hat{}$ d))
⟨*proof*⟩

lemma *fls-compose-power-mult* [*simp*]:
fls-compose-power $(f * g :: 'a ::$ *idom fls*$)\ d =$ *fls-compose-power* $f\ d *$ *fls-compose-power* $g\ d$
⟨*proof*⟩

lemma *fls-compose-power-power* [*simp*]:
assumes $d > 0 \vee n > 0$
shows *fls-compose-power* $(f \hat{} n :: 'a ::$ *idom fls*$)\ d =$ *fls-compose-power* $f\ d \hat{} n$
⟨*proof*⟩

lemma *fls-nth-compose-power'* [*simp*]:
$d = 0 \vee \neg d$ *dvd* $n \Longrightarrow$ *fls-compose-power* $f\ d$ \$\$ *int* $n = 0$

d dvd n $\Longrightarrow$ *d > 0* $\Longrightarrow$ *fls-compose-power f d $$ int n = f $$ int (n div d)*
⟨*proof*⟩

lemma *subdegree-fls-compose-fps* [*simp*]:
fixes *G* :: *'a* :: *field fps*
assumes [*simp*]: *fps-nth G 0 = 0*
shows *fls-subdegree (fls-compose-fps F G) = fls-subdegree F * subdegree G*
⟨*proof*⟩

7.7 Formal differentiation and integration

7.7.1 Derivative

definition *fls-deriv f = Abs-fls* (λ*n. of-int (n+1) * f$$(n+1)*)

lemma *fls-deriv-nth*[*simp*]: *fls-deriv f $$ n = of-int (n+1) * f$$(n+1)*
⟨*proof*⟩

lemma *fls-deriv-residue*: *fls-deriv f $$ −1 = 0*
⟨*proof*⟩

lemma *fls-deriv-const*[*simp*]: *fls-deriv (fls-const x) = 0*
⟨*proof*⟩

lemma *fls-deriv-of-nat*[*simp*]: *fls-deriv (of-nat n) = 0*
⟨*proof*⟩

lemma *fls-deriv-of-int*[*simp*]: *fls-deriv (of-int i) = 0*
⟨*proof*⟩

lemma *fls-deriv-zero*[*simp*]: *fls-deriv 0 = 0*
⟨*proof*⟩

lemma *fls-deriv-one*[*simp*]: *fls-deriv 1 = 0*
⟨*proof*⟩

lemma *fls-deriv-numeral* [*simp*]: *fls-deriv (numeral n) = 0*
⟨*proof*⟩

lemma *fls-deriv-subdegree′*:
assumes *of-int (fls-subdegree f) * f $$ fls-subdegree f ≠ 0*
shows *fls-subdegree (fls-deriv f) = fls-subdegree f − 1*
⟨*proof*⟩

lemma *fls-deriv-subdegree0*:
assumes *fls-subdegree f = 0*
shows *fls-subdegree (fls-deriv f) ≥ 0*
⟨*proof*⟩

lemma *fls-subdegree-deriv′*:

fixes $f :: 'a::ring\text{-}1\text{-}no\text{-}zero\text{-}divisors\ fls$
assumes $(of\text{-}int\ (fls\text{-}subdegree\ f) :: 'a) \neq 0$
shows $fls\text{-}subdegree\ (fls\text{-}deriv\ f) = fls\text{-}subdegree\ f - 1$
⟨*proof*⟩

lemma *fls-subdegree-deriv*:
fixes $f :: 'a::\{ring\text{-}1\text{-}no\text{-}zero\text{-}divisors, ring\text{-}char\text{-}0\}\ fls$
assumes $fls\text{-}subdegree\ f \neq 0$
shows $fls\text{-}subdegree\ (fls\text{-}deriv\ f) = fls\text{-}subdegree\ f - 1$
⟨*proof*⟩

lemma *fps-deriv-fls-regpart*: $fps\text{-}deriv\ (fls\text{-}regpart\ F) = fls\text{-}regpart\ (fls\text{-}deriv\ F)$
⟨*proof*⟩

Shifting is like multiplying by a power of the implied variable, and so satisfies a product-like rule.

lemma *fls-deriv-shift*:
$fls\text{-}deriv\ (fls\text{-}shift\ n\ f) = of\text{-}int\ (-n) * fls\text{-}shift\ (n+1)\ f + fls\text{-}shift\ n\ (fls\text{-}deriv\ f)$
⟨*proof*⟩

lemma *fls-deriv-X* [*simp*]: $fls\text{-}deriv\ fls\text{-}X = 1$
⟨*proof*⟩

lemma *fls-deriv-X-inv* [*simp*]: $fls\text{-}deriv\ fls\text{-}X\text{-}inv = -\ (fls\text{-}X\text{-}inv^2)$
⟨*proof*⟩

lemma *fls-deriv-delta*:
$fls\text{-}deriv\ (Abs\text{-}fls\ (\lambda n.\ if\ n{=}m\ then\ c\ else\ 0)) =$
$Abs\text{-}fls\ (\lambda n.\ if\ n{=}m{-}1\ then\ of\text{-}int\ m * c\ else\ 0)$
⟨*proof*⟩

lemma *fls-deriv-base-factor*:
$fls\text{-}deriv\ (fls\text{-}base\text{-}factor\ f) =$
$of\text{-}int\ (-fls\text{-}subdegree\ f) * fls\text{-}shift\ (fls\text{-}subdegree\ f + 1)\ f +$
$fls\text{-}shift\ (fls\text{-}subdegree\ f)\ (fls\text{-}deriv\ f)$
⟨*proof*⟩

lemma *fls-regpart-deriv*: $fls\text{-}regpart\ (fls\text{-}deriv\ f) = fps\text{-}deriv\ (fls\text{-}regpart\ f)$
⟨*proof*⟩

lemma *fls-prpart-deriv*:
fixes $f :: 'a :: \{comm\text{-}ring\text{-}1, ring\text{-}no\text{-}zero\text{-}divisors\}\ fls$
— Commutivity and no zero divisors are required by the definition of *pderiv*.
shows $fls\text{-}prpart\ (fls\text{-}deriv\ f) = -\ pCons\ 0\ (pCons\ 0\ (pderiv\ (fls\text{-}prpart\ f)))$
⟨*proof*⟩

lemma *pderiv-fls-prpart*:
$pderiv\ (fls\text{-}prpart\ f) = -\ poly\text{-}shift\ 2\ (fls\text{-}prpart\ (fls\text{-}deriv\ f))$
⟨*proof*⟩

lemma *fls-deriv-fps-to-fls*: *fls-deriv* (*fps-to-fls f*) = *fps-to-fls* (*fps-deriv f*)
⟨*proof*⟩

7.7.2 Algebraic rules of the derivative

lemma *fls-deriv-add* [*simp*]: *fls-deriv* (*f*+*g*) = *fls-deriv f* + *fls-deriv g*
⟨*proof*⟩

lemma *fls-deriv-sub* [*simp*]: *fls-deriv* (*f*−*g*) = *fls-deriv f* − *fls-deriv g*
⟨*proof*⟩

lemma *fls-deriv-neg* [*simp*]: *fls-deriv* (−*f*) = − *fls-deriv f*
⟨*proof*⟩

lemma *fls-deriv-mult* [*simp*]:
fls-deriv (*f*∗*g*) = *f* ∗ *fls-deriv g* + *fls-deriv f* ∗ *g*
⟨*proof*⟩

lemma *fls-deriv-mult-const-left*:
fls-deriv (*fls-const c* ∗ *f*) = *fls-const c* ∗ *fls-deriv f*
⟨*proof*⟩

lemma *fls-deriv-linear*:
fls-deriv (*fls-const a* ∗ *f* + *fls-const b* ∗ *g*) =
fls-const a ∗ *fls-deriv f* + *fls-const b* ∗ *fls-deriv g*
⟨*proof*⟩

lemma *fls-deriv-mult-const-right*:
fls-deriv (*f* ∗ *fls-const c*) = *fls-deriv f* ∗ *fls-const c*
⟨*proof*⟩

lemma *fls-deriv-linear2*:
fls-deriv (*f* ∗ *fls-const a* + *g* ∗ *fls-const b*) =
fls-deriv f ∗ *fls-const a* + *fls-deriv g* ∗ *fls-const b*
⟨*proof*⟩

lemma *fls-deriv-sum*:
fls-deriv (*sum f S*) = *sum* (λ*i*. *fls-deriv* (*f i*)) *S*
⟨*proof*⟩

lemma *fls-deriv-power*:
fixes *f* :: *'a*::*comm-ring-1 fls*
shows *fls-deriv* (*f*^*n*) = *of-nat n* ∗ *f*^(*n*−*1*) ∗ *fls-deriv f*
⟨*proof*⟩

lemma *fls-deriv-X-power*:
fls-deriv (*fls-X* ^ *n*) = *of-nat n* ∗ *fls-X* ^ (*n*−*1*)
⟨*proof*⟩

lemma *fls-deriv-X-inv-power*:
fls-deriv (*fls-X-inv* ^ *n*) = − *of-nat n* ∗ *fls-X-inv* ^ (*Suc n*)
⟨*proof*⟩

lemma *fls-deriv-X-intpow*:
fls-deriv (*fls-X-intpow i*) = *of-int i* ∗ *fls-X-intpow* (*i*−*1*)
⟨*proof*⟩

lemma *fls-deriv-lr-inverse*:
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1 f* $$ *fls-subdegree f* ∗ *y* = *1*
— These assumptions imply x equals y, but no need to assume that.
shows *fls-deriv* (*fls-left-inverse f x*) =
− *fls-left-inverse f x* ∗ *fls-deriv f* ∗ *fls-left-inverse f x*
and *fls-deriv* (*fls-right-inverse f y*) =
− *fls-right-inverse f y* ∗ *fls-deriv f* ∗ *fls-right-inverse f y*
⟨*proof*⟩

lemma *fls-deriv-lr-inverse-comm*:
fixes *x y* :: *'a*::*comm-ring-1*
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1*
shows *fls-deriv* (*fls-left-inverse f x*) = − *fls-deriv f* ∗ (*fls-left-inverse f x*)^2
and *fls-deriv* (*fls-right-inverse f x*) = − *fls-deriv f* ∗ (*fls-right-inverse f x*)^2
⟨*proof*⟩

lemma *fls-inverse-deriv-divring*:
fixes *a* :: *'a*::*division-ring fls*
shows *fls-deriv* (*inverse a*) = − *inverse a* ∗ *fls-deriv a* ∗ *inverse a*
⟨*proof*⟩

lemma *fls-inverse-deriv*:
fixes *a* :: *'a*::*field fls*
shows *fls-deriv* (*inverse a*) = − *fls-deriv a* ∗ (*inverse a*)^2
⟨*proof*⟩

lemma *fls-inverse-deriv'*:
fixes *a* :: *'a*::*field fls*
shows *fls-deriv* (*inverse a*) = − *fls-deriv a* / *a*^{2}
⟨*proof*⟩

7.7.3 Equality of derivatives

lemma *fls-deriv-eq-0-iff*:
fls-deriv f = *0* ⟷ *f* = *fls-const* (*f*$$*0* :: *'a*::{*ring-1-no-zero-divisors*,*ring-char-0*})
⟨*proof*⟩

lemma *fls-deriv-eq-iff*:
fixes *f g* :: *'a*::{*ring-1-no-zero-divisors*,*ring-char-0*} *fls*
shows *fls-deriv f* = *fls-deriv g* ⟷ (*f* = *fls-const*(*f*$$*0* − *g*$$*0*) + *g*)

⟨*proof*⟩

lemma *fls-deriv-eq-iff-ex*:
fixes *f g* :: *'a*::{*ring-1-no-zero-divisors*,*ring-char-0*} *fls*
shows (*fls-deriv f* = *fls-deriv g*) ⟷ (∃ *c*. *f* = *fls-const c* + *g*)
⟨*proof*⟩

7.7.4 Residues

definition *fls-residue-def*[*simp*]: *fls-residue f* ≡ *f* $$ −*1*

lemma *fls-residue-deriv*: *fls-residue* (*fls-deriv f*) = *0*
⟨*proof*⟩

lemma *fls-residue-add*: *fls-residue* (*f*+*g*) = *fls-residue f* + *fls-residue g*
⟨*proof*⟩

lemma *fls-residue-times-deriv*:
fls-residue (*fls-deriv f* ∗ *g*) = − *fls-residue* (*f* ∗ *fls-deriv g*)
⟨*proof*⟩

lemma *fls-residue-power-series*: *fls-subdegree f* ≥ *0* ⟹ *fls-residue f* = *0*
⟨*proof*⟩

lemma *fls-residue-fls-X-intpow*:
fls-residue (*fls-X-intpow i*) = (*if i*=−*1 then 1 else 0*)
⟨*proof*⟩

lemma *fls-residue-shift-nth*:
fixes *f* :: *'a*::*semiring-1 fls*
shows *f*$$*n* = *fls-residue* (*fls-X-intpow* (−*n*−*1*) ∗ *f*)
⟨*proof*⟩

lemma *fls-residue-fls-const-times*:
fixes *f* :: *'a*::{*comm-monoid-add*, *mult-zero*} *fls*
shows *fls-residue* (*fls-const c* ∗ *f*) = *c* ∗ *fls-residue f*
and *fls-residue* (*f* ∗ *fls-const c*) = *fls-residue f* ∗ *c*
⟨*proof*⟩

lemma *fls-residue-of-int-times*:
fixes *f* :: *'a*::*ring-1 fls*
shows *fls-residue* (*of-int i* ∗ *f*) = *of-int i* ∗ *fls-residue f*
and *fls-residue* (*f* ∗ *of-int i*) = *fls-residue f* ∗ *of-int i*
⟨*proof*⟩

lemma *fls-residue-deriv-times-lr-inverse-eq-subdegree*:
fixes *f g* :: *'a*::*ring-1 fls*
assumes *y* ∗ (*f* $$ *fls-subdegree f*) = *1* (*f* $$ *fls-subdegree f*) ∗ *y* = *1*
shows *fls-residue* (*fls-deriv f* ∗ *fls-right-inverse f y*) = *of-int* (*fls-subdegree f*)

and *fls-residue (fls-deriv f * fls-left-inverse f y) = of-int (fls-subdegree f)*
and *fls-residue (fls-left-inverse f y * fls-deriv f) = of-int (fls-subdegree f)*
and *fls-residue (fls-right-inverse f y * fls-deriv f) = of-int (fls-subdegree f)*
⟨*proof*⟩

lemma *fls-residue-deriv-times-inverse-eq-subdegree*:
fixes *f g* :: *'a*::*division-ring fls*
shows *fls-residue (fls-deriv f * inverse f) = of-int (fls-subdegree f)*
and *fls-residue (inverse f * fls-deriv f) = of-int (fls-subdegree f)*
⟨*proof*⟩

7.7.5 Integral definition and basic properties

definition *fls-integral* :: *'a*::{*ring-1*,*inverse*} *fls* ⇒ *'a fls*
where *fls-integral a = Abs-fls (λn. if n=0 then 0 else inverse (of-int n) * a$$(n − 1))*

lemma *fls-integral-nth* [*simp*]:
*fls-integral a $$ n = (if n=0 then 0 else inverse (of-int n) * a$$(n−1))*
⟨*proof*⟩

lemma *fls-integral-conv-fps-zeroth-integral*:
assumes *fls-subdegree a ≥ 0*
shows *fls-integral a = fps-to-fls (fps-integral0 (fls-regpart a))*
⟨*proof*⟩

lemma *fls-integral-zero* [*simp*]: *fls-integral 0 = 0*
⟨*proof*⟩

lemma *fls-integral-const'*:
fixes *x* :: *'a*::{*ring-1*,*inverse*}
assumes *inverse (1*::*'a) = 1*
shows *fls-integral (fls-const x) = fls-const x * fls-X*
⟨*proof*⟩

lemma *fls-integral-const*:
fixes *x* :: *'a*::*division-ring*
shows *fls-integral (fls-const x) = fls-const x * fls-X*
⟨*proof*⟩

lemma *fls-integral-of-nat'*:
assumes *inverse (1*::*'a*::{*ring-1*,*inverse*}) *= 1*
shows *fls-integral (of-nat n* :: *'a fls) = of-nat n * fls-X*
⟨*proof*⟩

lemma *fls-integral-of-nat*:
fls-integral (of-nat n :: *'a*::*division-ring fls) = of-nat n * fls-X*
⟨*proof*⟩

lemma *fls-integral-of-int'*:
assumes *inverse* (*1*::*'a*::{*ring-1*,*inverse*}) = *1*
shows *fls-integral* (*of-int i* :: *'a fls*) = *of-int i* ∗ *fls-X*
⟨*proof*⟩

lemma *fls-integral-of-int*:
fls-integral (*of-int i* :: *'a*::*division-ring fls*) = *of-int i* ∗ *fls-X*
⟨*proof*⟩

lemma *fls-integral-one'*:
assumes *inverse* (*1*::*'a*::{*ring-1*,*inverse*}) = *1*
shows *fls-integral* (*1*::*'a fls*) = *fls-X*
⟨*proof*⟩

lemma *fls-integral-one*: *fls-integral* (*1*::*'a*::*division-ring fls*) = *fls-X*
⟨*proof*⟩

lemma *fls-subdegree-integral-ge*:
fls-integral f ≠ *0* ⟹ *fls-subdegree* (*fls-integral f*) ≥ *fls-subdegree f* + *1*
⟨*proof*⟩

lemma *fls-subdegree-integral*:
fixes *f* :: *'a*::{*division-ring*,*ring-char-0*} *fls*
assumes *f* ≠ *0 fls-subdegree f* ≠ −*1*
shows *fls-subdegree* (*fls-integral f*) = *fls-subdegree f* + *1*
⟨*proof*⟩

lemma *fls-integral-X* [*simp*]:
fls-integral (*fls-X*::*'a*::{*ring-1*,*inverse*} *fls*) =
fls-const (*inverse* (*of-int 2*)) ∗ *fls-X*^{2}
⟨*proof*⟩

lemma *fls-integral-X-power*:
fls-integral (*fls-X* ^ *n* ::*'a* :: {*ring-1*,*inverse*} *fls*) =
fls-const (*inverse* (*of-nat* (*Suc n*))) ∗ *fls-X* ^ *Suc n*
⟨*proof*⟩

lemma *fls-integral-X-power-char0*:
fls-integral (*fls-X* ^ *n* :: *'a* :: {*ring-char-0*,*inverse*} *fls*) =
inverse (*of-nat* (*Suc n*)) ∗ *fls-X* ^ *Suc n*
⟨*proof*⟩

lemma *fls-integral-X-inv* [*simp*]: *fls-integral* (*fls-X-inv*::*'a*::{*ring-1*,*inverse*} *fls*) =
0
⟨*proof*⟩

lemma *fls-integral-X-inv-power*:
assumes *n* ≥ *2*
shows

fls-integral (*fls-X-inv* ^ *n* :: *'a* :: {*ring-1*,*inverse*} *fls*) =
fls-const (*inverse* (*of-int* (*1* − *int n*))) ∗ *fls-X-inv* ^ (*n*−*1*)
⟨*proof*⟩

lemma *fls-integral-X-inv-power-char0*:
assumes *n* ≥ *2*
shows
fls-integral (*fls-X-inv* ^ *n* :: *'a* :: {*ring-char-0*,*inverse*} *fls*) =
inverse (*of-int* (*1* − *int n*)) ∗ *fls-X-inv* ^ (*n*−*1*)
⟨*proof*⟩

lemma *fls-integral-X-inv-power'*:
assumes *n* ≥ *1*
shows
fls-integral (*fls-X-inv* ^ *n* :: *'a* :: *division-ring fls*) =
− *fls-const* (*inverse* (*of-nat* (*n*−*1*))) ∗ *fls-X-inv* ^ (*n*−*1*)
⟨*proof*⟩

lemma *fls-integral-X-inv-power-char0'*:
assumes *n* ≥ *1*
shows
fls-integral (*fls-X-inv* ^ *n* :: *'a* :: {*division-ring*,*ring-char-0*} *fls*) =
− *inverse* (*of-nat* (*n*−*1*)) ∗ *fls-X-inv* ^ (*n*−*1*)
⟨*proof*⟩

lemma *fls-integral-delta*:
assumes *m* ≠ −*1*
shows
fls-integral (*Abs-fls* (λ*n*. *if n*=*m then c else 0*)) =
Abs-fls (λ*n*. *if n*=*m*+*1 then inverse* (*of-int* (*m*+*1*)) ∗ *c else 0*)
⟨*proof*⟩

lemma *fls-regpart-integral*:
fls-regpart (*fls-integral f*) = *fps-integral0* (*fls-regpart f*)
⟨*proof*⟩

lemma *fls-integral-fps-to-fls*:
fls-integral (*fps-to-fls f*) = *fps-to-fls* (*fps-integral0 f*)
⟨*proof*⟩

7.7.6 Algebraic rules of the integral

lemma *fls-integral-add* [*simp*]: *fls-integral* (*f*+*g*) = *fls-integral f* + *fls-integral g*
⟨*proof*⟩

lemma *fls-integral-sub* [*simp*]: *fls-integral* (*f*−*g*) = *fls-integral f* − *fls-integral g*
⟨*proof*⟩

lemma *fls-integral-neg* [*simp*]: *fls-integral* (−*f*) = − *fls-integral f*

⟨*proof*⟩

lemma *fls-integral-mult-const-left*:
fls-integral (*fls-const c* ∗ *f*) = *fls-const c* ∗ *fls-integral* (*f* :: *'a*::*division-ring fls*)
⟨*proof*⟩

lemma *fls-integral-mult-const-left-comm*:
fixes *f* :: *'a*::{*comm-ring-1*,*inverse*} *fls*
shows *fls-integral* (*fls-const c* ∗ *f*) = *fls-const c* ∗ *fls-integral f*
⟨*proof*⟩

lemma *fls-integral-linear*:
fixes *f g* :: *'a*::*division-ring fls*
shows
fls-integral (*fls-const a* ∗ *f* + *fls-const b* ∗ *g*) =
fls-const a ∗ *fls-integral f* + *fls-const b* ∗ *fls-integral g*
⟨*proof*⟩

lemma *fls-integral-linear-comm*:
fixes *f g* :: *'a*::{*comm-ring-1*,*inverse*} *fls*
shows
fls-integral (*fls-const a* ∗ *f* + *fls-const b* ∗ *g*) =
fls-const a ∗ *fls-integral f* + *fls-const b* ∗ *fls-integral g*
⟨*proof*⟩

lemma *fls-integral-mult-const-right*:
fls-integral (*f* ∗ *fls-const c*) = *fls-integral f* ∗ *fls-const c*
⟨*proof*⟩

lemma *fls-integral-linear2*:
fls-integral (*f* ∗ *fls-const a* + *g* ∗ *fls-const b*) =
fls-integral f ∗ *fls-const a* + *fls-integral g* ∗ *fls-const b*
⟨*proof*⟩

lemma *fls-integral-sum*:
fls-integral (*sum f S*) = *sum* (λ*i*. *fls-integral* (*f i*)) *S*
⟨*proof*⟩

7.7.7 Derivatives of integrals and vice versa

lemma *fls-integral-fls-deriv*:
fixes *a* :: *'a*::{*division-ring*,*ring-char-0*} *fls*
shows *fls-integral* (*fls-deriv a*) + *fls-const* (*a*$$*0*) = *a*
⟨*proof*⟩

lemma *fls-deriv-fls-integral*:
fixes *a* :: *'a*::{*division-ring*,*ring-char-0*} *fls*
assumes *fls-residue a* = *0*
shows *fls-deriv* (*fls-integral a*) = *a*

⟨*proof*⟩

Series with zero residue are precisely the derivatives.

lemma *fls-residue-nonzero-ex-antiderivative*:
fixes $f :: 'a::\{division\text{-}ring, ring\text{-}char\text{-}0\}$ *fls*
assumes *fls-residue* $f = 0$
shows $\exists F.$ *fls-deriv* $F = f$
⟨*proof*⟩

lemma *fls-ex-antiderivative-residue-nonzero*:
assumes $\exists F.$ *fls-deriv* $F = f$
shows *fls-residue* $f = 0$
⟨*proof*⟩

lemma *fls-residue-nonzero-ex-anitderivative-iff*:
fixes $f :: 'a::\{division\text{-}ring, ring\text{-}char\text{-}0\}$ *fls*
shows *fls-residue* $f = 0 \longleftrightarrow (\exists F.$ *fls-deriv* $F = f)$
⟨*proof*⟩

7.8 Topology

instantiation *fls* :: (*group-add*) *metric-space*
begin

definition
dist-fls-def:
dist (*a* :: '*a fls*) *b* =
(*if a* = *b*
then 0
else if fls-subdegree (*a*−*b*) ≥ *0*
then inverse (*2* ^ *nat* (*fls-subdegree* (*a*−*b*)))
else 2 ^ *nat* (−*fls-subdegree* (*a*−*b*))
)

lemma *dist-fls-ge0*: *dist* (*a* :: '*a fls*) *b* ≥ *0*
⟨*proof*⟩

definition *uniformity-fls-def* [*code del*]:
(*uniformity* :: ('*a fls* × '*a fls*) *filter*) = (*INF e* ∈ {*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

definition *open-fls-def'* [*code del*]:
open (*U* :: '*a fls set*) ⟷ (∀ *x*∈*U*. *eventually* (λ(*x'*, *y*). *x'* = *x* ⟶ *y* ∈ *U*) *uniformity*)

lemma *dist-fls-sym*: *dist* (*a* :: '*a fls*) *b* = *dist b a*
⟨*proof*⟩

context

begin

private lemma *instance-helper*:
fixes *a b c* :: *'a fls*
assumes *neq*: *a≠b a≠c*
and *dist-ineq*: *dist a b > dist a c*
shows *fls-subdegree* (*a − b*) < *fls-subdegree* (*a − c*)
⟨*proof*⟩

instance
⟨*proof*⟩

end
end

declare *uniformity-Abort*[**where** *'a='a* :: *group-add fls*, *code*]

lemma *open-fls-def*:
open (*S* :: *'a*::*group-add fls set*) = (∀ *a* ∈ *S*. ∃ *r*. *r* >*0* ∧ {*y*. *dist y a* < *r*} ⊆ *S*)
⟨*proof*⟩

7.9 Notation

bundle *fps-syntax*
begin
notation *fls-nth* (**infixl** ‹$$› *75*)
end

unbundle *no fps-syntax*

end

8 The fraction field of any integral domain

theory *Fraction-Field*
imports *Main*
begin

8.1 General fractions construction

8.1.1 Construction of the type of fractions

context *idom* **begin**

definition *fractrel* :: *'a* × *'a* ⇒ *'a* ∗ *'a* ⇒ *bool* **where**
fractrel = (λ*x y*. *snd x* ≠ *0* ∧ *snd y* ≠ *0* ∧ *fst x* ∗ *snd y* = *fst y* ∗ *snd x*)

lemma *fractrel-iff* [*simp*]:
fractrel x y ⟷ *snd x* ≠ *0* ∧ *snd y* ≠ *0* ∧ *fst x* ∗ *snd y* = *fst y* ∗ *snd x*

⟨*proof*⟩

lemma *symp-fractrel*: *symp fractrel*
⟨*proof*⟩

lemma *transp-fractrel*: *transp fractrel*
⟨*proof*⟩

lemma *part-equivp-fractrel*: *part-equivp fractrel*
⟨*proof*⟩

end

quotient-type (**overloaded**) *'a fract* = *'a* :: *idom* × *'a* / *partial*: *fractrel*
⟨*proof*⟩

8.1.2 Representation and basic operations

lift-definition *Fract* :: *'a* :: *idom* ⇒ *'a* ⇒ *'a fract*
is λ*a b*. *if b* = *0 then* (*0*, *1*) *else* (*a*, *b*)
⟨*proof*⟩

lemma *Fract-cases* [*cases type*: *fract*]:
obtains (*Fract*) *a b* **where** *q* = *Fract a b b* ≠ *0*
⟨*proof*⟩

lemma *Fract-induct* [*case-names Fract*, *induct type*: *fract*]:
(⋀*a b*. *b* ≠ *0* ⟹ *P* (*Fract a b*)) ⟹ *P q*
⟨*proof*⟩

lemma *eq-fract*:
shows ⋀*a b c d*. *b* ≠ *0* ⟹ *d* ≠ *0* ⟹ *Fract a b* = *Fract c d* ⟷ *a* ∗ *d* = *c* ∗ *b*
and ⋀*a*. *Fract a 0* = *Fract 0 1*
and ⋀*a c*. *Fract 0 a* = *Fract 0 c*
⟨*proof*⟩

instantiation *fract* :: (*idom*) *comm-ring-1*
begin

lift-definition *zero-fract* :: *'a fract* **is** (*0*, *1*) ⟨*proof*⟩

lemma *Zero-fract-def*: *0* = *Fract 0 1*
⟨*proof*⟩

lift-definition *one-fract* :: *'a fract* **is** (*1*, *1*) ⟨*proof*⟩

lemma *One-fract-def*: *1* = *Fract 1 1*
⟨*proof*⟩

lift-definition *plus-fract* :: *'a fract* ⇒ *'a fract* ⇒ *'a fract*
  is λ*q r*. (*fst q* ∗ *snd r* + *fst r* ∗ *snd q*, *snd q* ∗ *snd r*)
⟨*proof*⟩

lemma *add-fract* [*simp*]:
  ⟦ *b* ≠ *0*; *d* ≠ *0* ⟧ ⟹ *Fract a b* + *Fract c d* = *Fract* (*a* ∗ *d* + *c* ∗ *b*) (*b* ∗ *d*)
⟨*proof*⟩

lift-definition *uminus-fract* :: *'a fract* ⇒ *'a fract*
  is λ*x*. (− *fst x*, *snd x*)
⟨*proof*⟩

lemma *minus-fract* [*simp*]:
  fixes *a b* :: *'a*::*idom*
  shows − *Fract a b* = *Fract* (− *a*) *b*
⟨*proof*⟩

lemma *minus-fract-cancel* [*simp*]: *Fract* (− *a*) (− *b*) = *Fract a b*
  ⟨*proof*⟩

definition *diff-fract-def*: *q* − *r* = *q* + − (*r*::*'a fract*)

lemma *diff-fract* [*simp*]:
  ⟦ *b* ≠ *0*; *d* ≠ *0* ⟧ ⟹ *Fract a b* − *Fract c d* = *Fract* (*a* ∗ *d* − *c* ∗ *b*) (*b* ∗ *d*)
  ⟨*proof*⟩

lift-definition *times-fract* :: *'a fract* ⇒ *'a fract* ⇒ *'a fract*
  is λ*q r*. (*fst q* ∗ *fst r*, *snd q* ∗ *snd r*)
⟨*proof*⟩

lemma *mult-fract* [*simp*]: *Fract* (*a*::*'a*::*idom*) *b* ∗ *Fract c d* = *Fract* (*a* ∗ *c*) (*b* ∗ *d*)
⟨*proof*⟩

lemma *mult-fract-cancel*:
  c ≠ *0* ⟹ *Fract* (*c* ∗ *a*) (*c* ∗ *b*) = *Fract a b*
⟨*proof*⟩

instance
⟨*proof*⟩

end

lemma *of-nat-fract*: *of-nat k* = *Fract* (*of-nat k*) *1*
  ⟨*proof*⟩

lemma *Fract-of-nat-eq*: *Fract* (*of-nat k*) *1* = *of-nat k*
  ⟨*proof*⟩

lemma *fract-collapse*:

Fract 0 k = 0
Fract 1 1 = 1
Fract k 0 = 0
⟨*proof*⟩

lemma *fract-expand*:
0 = Fract 0 1
1 = Fract 1 1
⟨*proof*⟩

lemma *Fract-cases-nonzero*:
obtains (*Fract*) *a b* **where** *q = Fract a b* **and** *b ≠ 0* **and** *a ≠ 0*
| (*0*) *q = 0*
⟨*proof*⟩

8.1.3 The field of rational numbers

context *idom*
begin

subclass *ring-no-zero-divisors* ⟨*proof*⟩

end

instantiation *fract* :: (*idom*) *field*
begin

lift-definition *inverse-fract* :: *'a fract ⇒ 'a fract*
is *λx. if fst x = 0 then (0, 1) else (snd x, fst x)*
⟨*proof*⟩

lemma *inverse-fract* [*simp*]: *inverse* (*Fract a b*) = *Fract* (*b*::*'a*::*idom*) *a*
⟨*proof*⟩

definition *divide-fract-def*: *q div r = q * inverse* (*r*:: *'a fract*)

lemma *divide-fract* [*simp*]: *Fract a b div Fract c d = Fract* (*a * d*) (*b * c*)
⟨*proof*⟩

instance
⟨*proof*⟩

end

8.1.4 The ordered field of fractions over an ordered idom

instantiation *fract* :: (*linordered-idom*) *linorder*
begin

lemma *less-eq-fract-respect*:

fixes $a\ b\ a'\ b'\ c\ d\ c'\ d' :: 'a$
assumes *neq*: $b \neq 0\ \ b' \neq 0\ \ d \neq 0\ \ d' \neq 0$
assumes *eq1*: $a * b' = a' * b$
assumes *eq2*: $c * d' = c' * d$
shows $((a * d) * (b * d) \leq (c * b) * (b * d)) \longleftrightarrow ((a' * d') * (b' * d') \leq (c' * b') * (b' * d'))$
⟨*proof*⟩

lift-definition *less-eq-fract* :: $'a\ fract \Rightarrow 'a\ fract \Rightarrow bool$
is $\lambda q\ r.\ (fst\ q * snd\ r) * (snd\ q * snd\ r) \leq (fst\ r * snd\ q) * (snd\ q * snd\ r)$
⟨*proof*⟩

definition *less-fract-def*: $z < (w::'a\ fract) \longleftrightarrow z \leq w \wedge \neg\ w \leq z$

lemma *le-fract* [*simp*]:
$[\![\ b \neq 0;\ d \neq 0\ ]\!] \Longrightarrow Fract\ a\ b \leq Fract\ c\ d \longleftrightarrow (a * d) * (b * d) \leq (c * b) * (b * d)$
⟨*proof*⟩

lemma *less-fract* [*simp*]:
$[\![\ b \neq 0;\ d \neq 0\ ]\!] \Longrightarrow Fract\ a\ b < Fract\ c\ d \longleftrightarrow (a * d) * (b * d) < (c * b) * (b * d)$
⟨*proof*⟩

instance
⟨*proof*⟩

end

instantiation *fract* :: (*linordered-idom*) *linordered-field*
begin

definition *abs-fract-def2*:
$|q| = (if\ q < 0\ then\ -q\ else\ (q::'a\ fract))$

definition *sgn-fract-def*:
$sgn\ (q::'a\ fract) = (if\ q = 0\ then\ 0\ else\ if\ 0 < q\ then\ 1\ else\ -\ 1)$

theorem *abs-fract* [*simp*]: $|Fract\ a\ b| = Fract\ |a|\ |b|$
⟨*proof*⟩

instance ⟨*proof*⟩

end

instantiation *fract* :: (*linordered-idom*) *distrib-lattice*
begin

definition *inf-fract-def*:

(*inf* :: *'a fract* ⇒ *'a fract* ⇒ *'a fract*) = *min*

definition *sup-fract-def*:
(*sup* :: *'a fract* ⇒ *'a fract* ⇒ *'a fract*) = *max*

instance
⟨*proof*⟩

end

lemma *fract-induct-pos* [*case-names Fract*]:
fixes *P* :: *'a*::*linordered-idom fract* ⇒ *bool*
assumes *step*: ⋀*a b*. *0* < *b* ⟹ *P* (*Fract a b*)
shows *P q*
⟨*proof*⟩

lemma *zero-less-Fract-iff*: *0* < *b* ⟹ *0* < *Fract a b* ⟷ *0* < *a*
⟨*proof*⟩

lemma *Fract-less-zero-iff*: *0* < *b* ⟹ *Fract a b* < *0* ⟷ *a* < *0*
⟨*proof*⟩

lemma *zero-le-Fract-iff*: *0* < *b* ⟹ *0* ≤ *Fract a b* ⟷ *0* ≤ *a*
⟨*proof*⟩

lemma *Fract-le-zero-iff*: *0* < *b* ⟹ *Fract a b* ≤ *0* ⟷ *a* ≤ *0*
⟨*proof*⟩

lemma *one-less-Fract-iff*: *0* < *b* ⟹ *1* < *Fract a b* ⟷ *b* < *a*
⟨*proof*⟩

lemma *Fract-less-one-iff*: *0* < *b* ⟹ *Fract a b* < *1* ⟷ *a* < *b*
⟨*proof*⟩

lemma *one-le-Fract-iff*: *0* < *b* ⟹ *1* ≤ *Fract a b* ⟷ *b* ≤ *a*
⟨*proof*⟩

lemma *Fract-le-one-iff*: *0* < *b* ⟹ *Fract a b* ≤ *1* ⟷ *a* ≤ *b*
⟨*proof*⟩

end

9 Fundamental Theorem of Algebra

theory *Fundamental-Theorem-Algebra*
imports *Polynomial Complex-Main*
begin

9.1 More lemmas about module of complex numbers

The triangle inequality for cmod

lemma *complex-mod-triangle-sub*: *cmod w* ≤ *cmod* (*w* + *z*) + *norm z*
⟨*proof*⟩

9.2 Basic lemmas about polynomials

lemma *poly-bound-exists*:
fixes *p* :: *'a*::{*comm-semiring-0*,*real-normed-div-algebra*} *poly*
shows ∃ *m*. *m* > *0* ∧ (∀ *z*. *norm z* ≤ *r* ⟶ *norm* (*poly p z*) ≤ *m*)
⟨*proof*⟩

Offsetting the variable in a polynomial gives another of same degree

definition *offset-poly* :: *'a*::*comm-semiring-0 poly* ⇒ *'a* ⇒ *'a poly*
where *offset-poly p h* = *fold-coeffs* (λ*a q*. *smult h q* + *pCons a q*) *p 0*

lemma *offset-poly-0*: *offset-poly 0 h* = *0*
⟨*proof*⟩

lemma *offset-poly-pCons*:
offset-poly (*pCons a p*) *h* =
smult h (*offset-poly p h*) + *pCons a* (*offset-poly p h*)
⟨*proof*⟩

lemma *offset-poly-single* [*simp*]: *offset-poly* [:*a*:] *h* = [:*a*:]
⟨*proof*⟩

lemma *poly-offset-poly*: *poly* (*offset-poly p h*) *x* = *poly p* (*h* + *x*)
⟨*proof*⟩

lemma *offset-poly-eq-0-lemma*: *smult c p* + *pCons a p* = *0* ⟹ *p* = *0*
⟨*proof*⟩

lemma *offset-poly-eq-0-iff* [*simp*]: *offset-poly p h* = *0* ⟷ *p* = *0*
⟨*proof*⟩

lemma *degree-offset-poly* [*simp*]: *degree* (*offset-poly p h*) = *degree p*
⟨*proof*⟩

definition *psize p* = (*if p* = *0 then 0 else Suc* (*degree p*))

lemma *psize-eq-0-iff* [*simp*]: *psize p* = *0* ⟷ *p* = *0*
⟨*proof*⟩

lemma *poly-offset*:
fixes *p* :: *'a*::*comm-ring-1 poly*
shows ∃ *q*. *psize q* = *psize p* ∧ (∀ *x*. *poly q x* = *poly p* (*a* + *x*))
⟨*proof*⟩

An alternative useful formulation of completeness of the reals

lemma *real-sup-exists*:
assumes *ex*: $\exists\, x.\ P\ x$
and *bz*: $\exists\, z.\ \forall\, x.\ P\ x \longrightarrow x < z$
shows $\exists\, s{::}real.\ \forall\, y.\ (\exists\, x.\ P\ x \wedge y < x) \longleftrightarrow y < s$
⟨*proof*⟩

9.3 Fundamental theorem of algebra

lemma *unimodular-reduce-norm*:
assumes *md*: $cmod\ z = 1$
shows $cmod\ (z + 1) < 1 \vee cmod\ (z - 1) < 1 \vee cmod\ (z + \mathrm{i}) < 1 \vee cmod\ (z - \mathrm{i}) < 1$
⟨*proof*⟩

Hence we can always reduce modulus of $1 + b\ z\hat{}n$ if nonzero

lemma *reduce-poly-simple*:
assumes *b*: $b \neq 0$
and *n*: $n \neq 0$
shows $\exists\, z.\ cmod\ (1 + b * z\hat{}n) < 1$
⟨*proof*⟩

Bolzano-Weierstrass type property for closed disc in complex plane.

lemma *metric-bound-lemma*: $cmod\ (x - y) \leq |Re\ x - Re\ y| + |Im\ x - Im\ y|$
⟨*proof*⟩

lemma *Bolzano-Weierstrass-complex-disc*:
assumes *r*: $\forall\, n.\ cmod\ (s\ n) \leq r$
shows $\exists\, f\ z.\ strict\text{-}mono\ (f :: nat \Rightarrow nat) \wedge (\forall\, e > 0.\ \exists\, N.\ \forall\, n \geq N.\ cmod\ (s\ (f\ n) - z) < e)$
⟨*proof*⟩

Polynomial is continuous.

lemma *poly-cont*:
fixes *p* :: *'a*::{*comm-semiring-0*,*real-normed-div-algebra*} *poly*
assumes *ep*: $e > 0$
shows $\exists\, d > 0.\ \forall\, w.\ 0 < norm\ (w - z) \wedge norm\ (w - z) < d \longrightarrow norm\ (poly\ p\ w - poly\ p\ z) < e$
⟨*proof*⟩

Hence a polynomial attains minimum on a closed disc in the complex plane.

lemma *poly-minimum-modulus-disc*: $\exists\, z.\ \forall\, w.\ cmod\ w \leq r \longrightarrow cmod\ (poly\ p\ z) \leq cmod\ (poly\ p\ w)$
⟨*proof*⟩

Nonzero polynomial in z goes to infinity as z does.

lemma *poly-infinity*:
fixes *p*:: *'a*::{*comm-semiring-0*,*real-normed-div-algebra*} *poly*

assumes *ex*: $p \neq 0$
shows $\exists\, r.\ \forall\, z.\ r \leq \mathit{norm}\ z \longrightarrow d \leq \mathit{norm}\ (\mathit{poly}\ (\mathit{pCons}\ a\ p)\ z)$
⟨*proof*⟩

Hence polynomial's modulus attains its minimum somewhere.

lemma *poly-minimum-modulus*: $\exists\, z.\forall\, w.\ \mathit{cmod}\ (\mathit{poly}\ p\ z) \leq \mathit{cmod}\ (\mathit{poly}\ p\ w)$
⟨*proof*⟩

Constant function (non-syntactic characterization).

definition $\mathit{constant}\ f \longleftrightarrow (\forall\, x\ y.\ f\ x = f\ y)$

lemma *nonconstant-length*: $\neg\ \mathit{constant}\ (\mathit{poly}\ p) \Longrightarrow \mathit{psize}\ p \geq 2$
⟨*proof*⟩

lemma *poly-replicate-append*: $\mathit{poly}\ (\mathit{monom}\ 1\ n * p)\ (x::'a::\mathit{comm\text{-}ring\text{-}1}) = x\hat{}n * \mathit{poly}\ p\ x$
⟨*proof*⟩

Decomposition of polynomial, skipping zero coefficients after the first.

lemma *poly-decompose-lemma*:
assumes *nz*: $\neg\ (\forall\, z.\ z \neq 0 \longrightarrow \mathit{poly}\ p\ z = (0::'a::\mathit{idom}))$
shows $\exists\, k\ a\ q.\ a \neq 0 \wedge \mathit{Suc}\ (\mathit{psize}\ q + k) = \mathit{psize}\ p \wedge (\forall\, z.\ \mathit{poly}\ p\ z = z\hat{}k * \mathit{poly}\ (\mathit{pCons}\ a\ q)\ z)$
⟨*proof*⟩

lemma *poly-decompose*:
fixes $p :: 'a::\mathit{idom}\ \mathit{poly}$
assumes *nc*: $\neg\ \mathit{constant}\ (\mathit{poly}\ p)$
shows $\exists\, k\ a\ q.\ a \neq 0 \wedge k \neq 0 \wedge$
$\mathit{psize}\ q + k + 1 = \mathit{psize}\ p \wedge$
$(\forall\, z.\ \mathit{poly}\ p\ z = \mathit{poly}\ p\ 0 + z\hat{}k * \mathit{poly}\ (\mathit{pCons}\ a\ q)\ z)$
⟨*proof*⟩

Fundamental theorem of algebra

theorem *fundamental-theorem-of-algebra*:
assumes *nc*: $\neg\ \mathit{constant}\ (\mathit{poly}\ p)$
shows $\exists\, z::\mathit{complex}.\ \mathit{poly}\ p\ z = 0$
⟨*proof*⟩

Alternative version with a syntactic notion of constant polynomial.

lemma *fundamental-theorem-of-algebra-alt*:
assumes *nc*: $\neg\ (\exists\, a\ l.\ a \neq 0 \wedge l = 0 \wedge p = \mathit{pCons}\ a\ l)$
shows $\exists\, z.\ \mathit{poly}\ p\ z = (0::\mathit{complex})$
⟨*proof*⟩

9.4 Nullstellensatz, degrees and divisibility of polynomials

lemma *nullstellensatz-lemma*:

fixes p :: *complex poly*
assumes $\forall x.$ *poly* p $x = 0 \longrightarrow$ *poly* q $x = 0$
and *degree* $p = n$
and $n \neq 0$
shows p *dvd* $(q \mathbin{\hat{}} n)$
⟨*proof*⟩

lemma *nullstellensatz-univariate*:
$(\forall x.$ *poly* p $x =$ $(0$::*complex*$) \longrightarrow$ *poly* q $x = 0) \longleftrightarrow$
p *dvd* $(q \mathbin{\hat{}} ($*degree* $p)) \vee (p = 0 \wedge q = 0)$
⟨*proof*⟩

Useful lemma

lemma *constant-degree*:
fixes p :: *'a*::{*idom*,*ring-char-0*} *poly*
shows *constant* (*poly* p) $\longleftrightarrow$ *degree* $p = 0$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *complex-poly-decompose*:
smult (*lead-coeff* p) $(\prod z|$*poly* p $z = 0.$ $[:-z,$ $1:] \mathbin{\hat{}}$ *order* z $p) = (p$ :: *complex poly*$)$
⟨*proof*⟩

instance *complex* :: *alg-closed-field*
⟨*proof*⟩

lemma *size-proots-complex*: *size* (*proots* (p :: *complex poly*)) = *degree* p
⟨*proof*⟩

lemma *complex-poly-decompose-multiset*:
smult (*lead-coeff* p) $(\prod x \in \#$*proots* $p.$ $[:-x,$ $1:]) = (p$ :: *complex poly*$)$
⟨*proof*⟩

lemma *complex-poly-decompose'*:
obtains *root* **where** *smult* (*lead-coeff* p) $(\prod i<$*degree* $p.$ $[:-$*root* $i,$ $1:]) = (p$ :: *complex poly*$)$
⟨*proof*⟩

lemma *complex-poly-decompose-rsquarefree*:
assumes *rsquarefree* p
shows *smult* (*lead-coeff* p) $(\prod z|$*poly* p $z = 0.$ $[:-z,$ $1:]) = (p$ :: *complex poly*$)$
⟨*proof*⟩

Arithmetic operations on multivariate polynomials.

lemma *mpoly-base-conv*:
fixes x :: *'a*::*comm-ring-1*
shows $0 =$ *poly* 0 x $c =$ *poly* $[:c:]$ x $x =$ *poly* $[:0,1:]$ x
⟨*proof*⟩

lemma *mpoly-norm-conv*:

fixes $x :: 'a::comm\text{-}ring\text{-}1$
shows $poly\ [:0:]\ x = poly\ 0\ x\ poly\ [:poly\ 0\ y:]\ x = poly\ 0\ x$
⟨*proof*⟩

lemma *mpoly-sub-conv*:
fixes $x :: 'a::comm\text{-}ring\text{-}1$
shows $poly\ p\ x - poly\ q\ x = poly\ p\ x + -1 * poly\ q\ x$
⟨*proof*⟩

lemma *poly-pad-rule*: $poly\ p\ x = 0 \Longrightarrow poly\ (pCons\ 0\ p)\ x = 0$
⟨*proof*⟩

lemma *poly-cancel-eq-conv*:
fixes $x :: 'a::field$
shows $x = 0 \Longrightarrow a \neq 0 \Longrightarrow y = 0 \longleftrightarrow a * y - b * x = 0$
⟨*proof*⟩

lemma *poly-divides-pad-rule*:
fixes $p::\ ('a::comm\text{-}ring\text{-}1)\ poly$
assumes *pq*: $p\ dvd\ q$
shows $p\ dvd\ (pCons\ 0\ q)$
⟨*proof*⟩

lemma *poly-divides-conv0*:
fixes $p::\ 'a::field\ poly$
assumes *lgpq*: $degree\ q < degree\ p$ **and** *lq*: $p \neq 0$
shows $p\ dvd\ q \longleftrightarrow q = 0$
⟨*proof*⟩

lemma *poly-divides-conv1*:
fixes $p :: 'a::field\ poly$
assumes *a0*: $a \neq 0$
and *pp'*: $p\ dvd\ p'$
and *qrp'*: $smult\ a\ q - p' = r$
shows $p\ dvd\ q \longleftrightarrow p\ dvd\ r$
⟨*proof*⟩

lemma *basic-cqe-conv1*:
$(\exists x.\ poly\ p\ x = 0 \wedge poly\ 0\ x \neq 0) \longleftrightarrow False$
$(\exists x.\ poly\ 0\ x \neq 0) \longleftrightarrow False$
$(\exists x.\ poly\ [:c:]\ x \neq 0) \longleftrightarrow c \neq 0$
$(\exists x.\ poly\ 0\ x = 0) \longleftrightarrow True$
$(\exists x.\ poly\ [:c:]\ x = 0) \longleftrightarrow c = 0$
⟨*proof*⟩

lemma *basic-cqe-conv2*:
assumes *l*: $p \neq 0$
shows $\exists x.\ poly\ (pCons\ a\ (pCons\ b\ p))\ x = (0::complex)$
⟨*proof*⟩

lemma *basic-cqe-conv-2b*: $(\exists x.\ poly\ p\ x \neq (0::complex)) \longleftrightarrow p \neq 0$
⟨*proof*⟩

lemma *basic-cqe-conv3*:
fixes $p\ q$ :: *complex poly*
assumes l: $p \neq 0$
shows $(\exists x.\ poly\ (pCons\ a\ p)\ x = 0 \wedge poly\ q\ x \neq 0) \longleftrightarrow \neg\ (pCons\ a\ p)\ dvd\ (q\ \hat{}\ psize\ p)$
⟨*proof*⟩

lemma *basic-cqe-conv4*:
fixes $p\ q$ :: *complex poly*
assumes h: $\bigwedge x.\ poly\ (q\ \hat{}\ n)\ x = poly\ r\ x$
shows $p\ dvd\ (q\ \hat{}\ n) \longleftrightarrow p\ dvd\ r$
⟨*proof*⟩

lemma *poly-const-conv*:
fixes x :: *'a::comm-ring-1*
shows $poly\ [:c:]\ x = y \longleftrightarrow c = y$
⟨*proof*⟩

end

theory *Group-Closure*
imports
Main
begin

context *ab-group-add*
begin

inductive-set *group-closure* :: *'a set* ⇒ *'a set* **for** S
where *base*: $s \in insert\ 0\ S \Longrightarrow s \in group\text{-}closure\ S$
| *diff*: $s \in group\text{-}closure\ S \Longrightarrow t \in group\text{-}closure\ S \Longrightarrow s - t \in group\text{-}closure\ S$

lemma *zero-in-group-closure* [*simp*]:
$0 \in group\text{-}closure\ S$
⟨*proof*⟩

lemma *group-closure-minus-iff* [*simp*]:
$-\ s \in group\text{-}closure\ S \longleftrightarrow s \in group\text{-}closure\ S$
⟨*proof*⟩

lemma *group-closure-add*:
$s + t \in group\text{-}closure\ S$ **if** $s \in group\text{-}closure\ S$ **and** $t \in group\text{-}closure\ S$
⟨*proof*⟩

lemma *group-closure-empty* [*simp*]:
group-closure {} = {*0*}
⟨*proof*⟩

lemma *group-closure-insert-zero* [*simp*]:
group-closure (*insert 0 S*) = *group-closure S*
⟨*proof*⟩

end

context *comm-ring-1*
begin

lemma *group-closure-scalar-mult-left*:
of-nat n ∗ *s* ∈ *group-closure S* **if** *s* ∈ *group-closure S*
⟨*proof*⟩

lemma *group-closure-scalar-mult-right*:
s ∗ *of-nat n* ∈ *group-closure S* **if** *s* ∈ *group-closure S*
⟨*proof*⟩

end

lemma *group-closure-abs-iff* [*simp*]:
|*s*| ∈ *group-closure S* ⟷ *s* ∈ *group-closure S* **for** *s* :: *int*
⟨*proof*⟩

lemma *group-closure-mult-left*:
s ∗ *t* ∈ *group-closure S* **if** *s* ∈ *group-closure S* **for** *s t* :: *int*
⟨*proof*⟩

lemma *group-closure-mult-right*:
s ∗ *t* ∈ *group-closure S* **if** *t* ∈ *group-closure S* **for** *s t* :: *int*
⟨*proof*⟩

context *idom*
begin

lemma *group-closure-mult-all-eq*:
group-closure (*times k* ‘ *S*) = *times k* ‘ *group-closure S*
⟨*proof*⟩

end

lemma *Gcd-group-closure-eq-Gcd*:
Gcd (*group-closure S*) = *Gcd S* **for** *S* :: *int set*
⟨*proof*⟩

lemma *group-closure-sum*:

fixes S :: *int set*
assumes X: *finite* X $X \neq \{\}$ $X \subseteq S$
shows $(\sum x{\in}X.\ a\ x * x) \in$ *group-closure* S
⟨*proof*⟩

lemma *Gcd-group-closure-in-group-closure*:
Gcd (*group-closure* S) $\in$ *group-closure* S **for** S :: *int set*
⟨*proof*⟩

lemma *Gcd-in-group-closure*:
Gcd $S \in$ *group-closure* S **for** S :: *int set*
⟨*proof*⟩

lemma *group-closure-eq*:
group-closure S = *range* (*times* (*Gcd* S)) **for** S :: *int set*
⟨*proof*⟩

end

theory *Normalized-Fraction*
imports
Main
Euclidean-Algorithm
Fraction-Field
begin

lemma *unit-factor-1-imp-normalized*: *unit-factor* x = *1* $\Longrightarrow$ *normalize* x = x
⟨*proof*⟩

definition *quot-to-fract* :: $'a \times 'a \Rightarrow 'a$ :: *idom fract* **where**
quot-to-fract = (λ(*a*,*b*). *Fraction-Field.Fract a b*)

definition *normalize-quot* :: $'a$ :: {*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*}
$\times$ $'a \Rightarrow 'a \times 'a$ **where**
normalize-quot =
(λ(*a*,*b*). *if b = 0 then* (*0*,*1*) *else let d = gcd a b * unit-factor b in* (*a div d*, *b div d*))

lemma *normalize-quot-zero* [*simp*]:
normalize-quot (*a*, *0*) = (*0*, *1*)
⟨*proof*⟩

lemma *normalize-quot-proj*:
fst (*normalize-quot* (*a*, *b*)) = *a div* (*gcd a b * unit-factor b*)
snd (*normalize-quot* (*a*, *b*)) = *normalize b div gcd a b* **if** $b \neq 0$
⟨*proof*⟩

definition *normalized-fracts* :: ($'a$ :: {*ring-gcd*,*idom-divide*} $\times$ $'a$) *set* **where**

normalized-fracts = {(*a*,*b*). *coprime a b* ∧ *unit-factor b* = *1*}

lemma *not-normalized-fracts-0-denom* [*simp*]: (*a*, *0*) ∉ *normalized-fracts*
⟨*proof*⟩

lemma *unit-factor-snd-normalize-quot* [*simp*]:
unit-factor (*snd* (*normalize-quot x*)) = *1*
⟨*proof*⟩

lemma *snd-normalize-quot-nonzero* [*simp*]: *snd* (*normalize-quot x*) ≠ *0*
⟨*proof*⟩

lemma *normalize-quot-aux*:
fixes *a b*
assumes *b* ≠ *0*
defines *d* ≡ *gcd a b* ∗ *unit-factor b*
shows *a* = *fst* (*normalize-quot* (*a*,*b*)) ∗ *d b* = *snd* (*normalize-quot* (*a*,*b*)) ∗ *d*
d dvd a d dvd b d ≠ *0*
⟨*proof*⟩

lemma *normalize-quotE*:
assumes *b* ≠ *0*
obtains *d* **where** *a* = *fst* (*normalize-quot* (*a*,*b*)) ∗ *d b* = *snd* (*normalize-quot* (*a*,*b*)) ∗ *d*
d dvd a d dvd b d ≠ *0*
⟨*proof*⟩

lemma *normalize-quotE′*:
assumes *snd x* ≠ *0*
obtains *d* **where** *fst x* = *fst* (*normalize-quot x*) ∗ *d snd x* = *snd* (*normalize-quot x*) ∗ *d*
d dvd fst x d dvd snd x d ≠ *0*
⟨*proof*⟩

lemma *coprime-normalize-quot*:
coprime (*fst* (*normalize-quot x*)) (*snd* (*normalize-quot x*))
⟨*proof*⟩

lemma *normalize-quot-in-normalized-fracts* [*simp*]: *normalize-quot x* ∈ *normalized-fracts*
⟨*proof*⟩

lemma *normalize-quot-eq-iff*:
assumes *b* ≠ *0 d* ≠ *0*
shows *normalize-quot* (*a*,*b*) = *normalize-quot* (*c*,*d*) ⟷ *a* ∗ *d* = *b* ∗ *c*
⟨*proof*⟩

lemma *normalize-quot-eq-iff′*:
assumes *snd x* ≠ *0 snd y* ≠ *0*

shows *normalize-quot x = normalize-quot y ⟷ fst x * snd y = snd x * fst y*
⟨*proof*⟩

lemma *normalize-quot-id*: *x ∈ normalized-fracts ⟹ normalize-quot x = x*
⟨*proof*⟩

lemma *normalize-quot-idem* [*simp*]: *normalize-quot (normalize-quot x) = normalize-quot x*
⟨*proof*⟩

lemma *fractrel-iff-normalize-quot-eq*:
fractrel x y ⟷ normalize-quot x = normalize-quot y ∧ snd x ≠ 0 ∧ snd y ≠ 0
⟨*proof*⟩

lemma *fractrel-normalize-quot-left*:
assumes *snd x ≠ 0*
shows *fractrel (normalize-quot x) y ⟷ fractrel x y*
⟨*proof*⟩

lemma *fractrel-normalize-quot-right*:
assumes *snd x ≠ 0*
shows *fractrel y (normalize-quot x) ⟷ fractrel y x*
⟨*proof*⟩

lift-definition *quot-of-fract* ::
'a :: {*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*} *fract ⇒ 'a × 'a*
is *normalize-quot*
⟨*proof*⟩

lemma *quot-to-fract-quot-of-fract* [*simp*]: *quot-to-fract (quot-of-fract x) = x*
⟨*proof*⟩

lemma *quot-of-fract-quot-to-fract*: *quot-of-fract (quot-to-fract x) = normalize-quot x*
⟨*proof*⟩

lemma *quot-of-fract-quot-to-fract'*:
x ∈ normalized-fracts ⟹ quot-of-fract (quot-to-fract x) = x
⟨*proof*⟩

lemma *quot-of-fract-in-normalized-fracts* [*simp*]: *quot-of-fract x ∈ normalized-fracts*
⟨*proof*⟩

lemma *normalize-quotI*:
assumes *a * d = b * c b ≠ 0 (c, d) ∈ normalized-fracts*
shows *normalize-quot (a, b) = (c, d)*
⟨*proof*⟩

lemma *td-normalized-fract*:
type-definition quot-of-fract quot-to-fract normalized-fracts
⟨*proof*⟩

lemma *quot-of-fract-add-aux*:
assumes *snd x ≠ 0 snd y ≠ 0*
shows *(fst x * snd y + fst y * snd x) * (snd (normalize-quot x) * snd (normalize-quot y)) =*
*snd x * snd y * (fst (normalize-quot x) * snd (normalize-quot y) +*
*snd (normalize-quot x) * fst (normalize-quot y))*
⟨*proof*⟩

locale *fract-as-normalized-quot*
begin
setup-lifting *td-normalized-fract*
end

lemma *quot-of-fract-add*:
quot-of-fract (x + y) =
(let (a,b) = quot-of-fract x; (c,d) = quot-of-fract y
*in normalize-quot (a * d + b * c, b * d))*
⟨*proof*⟩

lemma *quot-of-fract-uminus*:
quot-of-fract (−x) = (let (a,b) = quot-of-fract x in (−a, b))
⟨*proof*⟩

lemma *quot-of-fract-diff*:
quot-of-fract (x − y) =
(let (a,b) = quot-of-fract x; (c,d) = quot-of-fract y
*in normalize-quot (a * d − b * c, b * d))* (**is** *- = ?rhs*)
⟨*proof*⟩

lemma *normalize-quot-mult-coprime*:
assumes *coprime a b coprime c d unit-factor b = 1 unit-factor d = 1*
defines *e ≡ fst (normalize-quot (a, d))* **and** *f ≡ snd (normalize-quot (a, d))*
and *g ≡ fst (normalize-quot (c, b))* **and** *h ≡ snd (normalize-quot (c, b))*
shows *normalize-quot (a * c, b * d) = (e * g, f * h)*
⟨*proof*⟩

lemma *normalize-quot-mult*:
assumes *snd x ≠ 0 snd y ≠ 0*
shows *normalize-quot (fst x * fst y, snd x * snd y) = normalize-quot*
*(fst (normalize-quot x) * fst (normalize-quot y),*
*snd (normalize-quot x) * snd (normalize-quot y))*
⟨*proof*⟩

lemma *quot-of-fract-mult*:
quot-of-fract $(x * y) =$
$(let\ (a,b) = quot\text{-}of\text{-}fract\ x;\ (c,d) = quot\text{-}of\text{-}fract\ y;$
$(e,f) = normalize\text{-}quot\ (a,d);\ (g,h) = normalize\text{-}quot\ (c,b)$
$in\ \ (e*g,\ f*h))$
⟨*proof*⟩

lemma *normalize-quot-0* [*simp*]:
$normalize\text{-}quot\ (0,\ x) = (0,\ 1)\ normalize\text{-}quot\ (x,\ 0) = (0,\ 1)$
⟨*proof*⟩

lemma *normalize-quot-eq-0-iff* [*simp*]: $fst\ (normalize\text{-}quot\ x) = 0 \longleftrightarrow fst\ x = 0 \vee snd\ x = 0$
⟨*proof*⟩

lemma *fst-quot-of-fract-0-imp*: $fst\ (quot\text{-}of\text{-}fract\ x) = 0 \Longrightarrow snd\ (quot\text{-}of\text{-}fract\ x) = 1$
⟨*proof*⟩

lemma *normalize-quot-swap*:
assumes $a \neq 0\ b \neq 0$
defines $a' \equiv fst\ (normalize\text{-}quot\ (a,\ b))$ **and** $b' \equiv snd\ (normalize\text{-}quot\ (a,\ b))$
shows $normalize\text{-}quot\ (b,\ a) = (b'\ div\ unit\text{-}factor\ a',\ a'\ div\ unit\text{-}factor\ a')$
⟨*proof*⟩

lemma *quot-of-fract-inverse*:
quot-of-fract $(inverse\ x) =$
$(let\ (a,b) = quot\text{-}of\text{-}fract\ x;\ d = unit\text{-}factor\ a$
$in\ \ if\ d = 0\ then\ (0,\ 1)\ else\ (b\ div\ d,\ a\ div\ d))$
⟨*proof*⟩

lemma *normalize-quot-div-unit-left*:
fixes $x\ y\ u$
assumes *is-unit* u
defines $x' \equiv fst\ (normalize\text{-}quot\ (x,\ y))$ **and** $y' \equiv snd\ (normalize\text{-}quot\ (x,\ y))$
shows $normalize\text{-}quot\ (x\ div\ u,\ y) = (x'\ div\ u,\ y')$
⟨*proof*⟩

lemma *normalize-quot-div-unit-right*:
fixes $x\ y\ u$
assumes *is-unit* u
defines $x' \equiv fst\ (normalize\text{-}quot\ (x,\ y))$ **and** $y' \equiv snd\ (normalize\text{-}quot\ (x,\ y))$
shows $normalize\text{-}quot\ (x,\ y\ div\ u) = (x' * u,\ y')$
⟨*proof*⟩

lemma *normalize-quot-normalize-left*:
fixes $x\ y\ u$
defines $x' \equiv fst\ (normalize\text{-}quot\ (x,\ y))$ **and** $y' \equiv snd\ (normalize\text{-}quot\ (x,\ y))$
shows $normalize\text{-}quot\ (normalize\ x,\ y) = (x'\ div\ unit\text{-}factor\ x,\ y')$

⟨*proof*⟩

lemma *normalize-quot-normalize-right*:
fixes *x y u*
defines $x' \equiv$ *fst* (*normalize-quot* (*x*, *y*)) **and** $y' \equiv$ *snd* (*normalize-quot* (*x*, *y*))
shows *normalize-quot* (*x*, *normalize y*) = (*x'* ∗ *unit-factor y*, *y'*)
⟨*proof*⟩

lemma *quot-of-fract-0* [*simp*]: *quot-of-fract 0* = (*0*, *1*)
⟨*proof*⟩

lemma *quot-of-fract-1* [*simp*]: *quot-of-fract 1* = (*1*, *1*)
⟨*proof*⟩

lemma *quot-of-fract-divide*:
quot-of-fract (*x* / *y*) = (*if y* = *0 then* (*0*, *1*) *else*
(*let* (*a*,*b*) = *quot-of-fract x*; (*c*,*d*) = *quot-of-fract y*;
(*e*,*f*) = *normalize-quot* (*a*,*c*); (*g*,*h*) = *normalize-quot* (*d*,*b*)
in (*e* ∗ *g*, *f* ∗ *h*))) (**is** - = *?rhs*)
⟨*proof*⟩

lemma *snd-quot-of-fract-nonzero* [*simp*]: *snd* (*quot-of-fract x*) ≠ *0*
⟨*proof*⟩

lemma *Fract-quot-of-fract* [*simp*]: *Fract* (*fst* (*quot-of-fract x*)) (*snd* (*quot-of-fract x*)) = *x*
⟨*proof*⟩

lemma *snd-quot-of-fract-Fract-whole*:
assumes *y dvd x*
shows *snd* (*quot-of-fract* (*Fract x y*)) = *1*
⟨*proof*⟩

lemma *fst-quot-of-fract-eq-0-iff* [*simp*]: *fst* (*quot-of-fract x*) = *0* ⟷ *x* = *0*
⟨*proof*⟩

lemma *coprime-quot-of-fract*:
coprime (*fst* (*quot-of-fract x*)) (*snd* (*quot-of-fract x*))
⟨*proof*⟩

lemma *unit-factor-snd-quot-of-fract*: *unit-factor* (*snd* (*quot-of-fract x*)) = *1*
⟨*proof*⟩

lemma *normalize-snd-quot-of-fract*: *normalize* (*snd* (*quot-of-fract x*)) = *snd* (*quot-of-fract x*)
⟨*proof*⟩

end

10 n-th powers and roots of naturals

theory *Nth-Powers*
imports *Primes*
begin

10.1 The set of n-th powers

definition *is-nth-power* :: *nat* ⇒ *'a* :: *monoid-mult* ⇒ *bool* **where**
is-nth-power n x ⟷ (∃ *y*. *x* = *y* ^ *n*)

lemma *is-nth-power-nth-power* [*simp*, *intro*]: *is-nth-power n* (*x* ^ *n*)
⟨*proof*⟩

lemma *is-nth-powerI* [*intro?*]: *x* = *y* ^ *n* ⟹ *is-nth-power n x*
⟨*proof*⟩

lemma *is-nth-powerE*: *is-nth-power n x* ⟹ (⋀*y*. *x* = *y* ^ *n* ⟹ *P*) ⟹ *P*
⟨*proof*⟩

abbreviation *is-square* **where** *is-square* ≡ *is-nth-power 2*

lemma *is-zeroth-power* [*simp*]: *is-nth-power 0 x* ⟷ *x* = *1*
⟨*proof*⟩

lemma *is-first-power* [*simp*]: *is-nth-power 1 x*
⟨*proof*⟩

lemma *is-first-power'* [*simp*]: *is-nth-power* (*Suc 0*) *x*
⟨*proof*⟩

lemma *is-nth-power-0* [*simp*]: *n* > *0* ⟹ *is-nth-power n* (*0* :: *'a* :: *semiring-1*)
⟨*proof*⟩

lemma *is-nth-power-0-iff* [*simp*]: *is-nth-power n* (*0* :: *'a* :: *semiring-1*) ⟷ *n* > *0*
⟨*proof*⟩

lemma *is-nth-power-1* [*simp*]: *is-nth-power n 1*
⟨*proof*⟩

lemma *is-nth-power-Suc-0* [*simp*]: *is-nth-power n* (*Suc 0*)
⟨*proof*⟩

lemma *is-nth-power-conv-multiplicity*:
fixes *x* :: *'a* :: {*factorial-semiring*, *normalization-semidom-multiplicative*}
assumes *n* > *0*
shows *is-nth-power n* (*normalize x*) ⟷ (∀ *p*. *prime p* ⟶ *n dvd multiplicity p x*)
⟨*proof*⟩

lemma *is-nth-power-conv-multiplicity-nat*:
assumes $n > 0$
shows *is-nth-power n* $(x :: nat) \longleftrightarrow (\forall p.\ prime\ p \longrightarrow n\ dvd\ multiplicity\ p\ x)$
⟨*proof*⟩

lemma *is-nth-power-mult*:
assumes *is-nth-power n a is-nth-power n b*
shows *is-nth-power n* $(a * b :: 'a :: comm\text{-}monoid\text{-}mult)$
⟨*proof*⟩

lemma *is-nth-power-mult-coprime-natD*:
fixes *a b* :: *nat*
assumes *coprime a b is-nth-power n* $(a * b)$ $a > 0$ $b > 0$
shows *is-nth-power n a is-nth-power n b*
⟨*proof*⟩

lemma *is-nth-power-mult-coprime-nat-iff*:
fixes *a b* :: *nat*
assumes *coprime a b*
shows *is-nth-power n* $(a * b) \longleftrightarrow$ *is-nth-power n a* $\wedge$*is-nth-power n b*
⟨*proof*⟩

lemma *is-nth-power-prime-power-nat-iff*:
fixes *p* :: *nat* **assumes** *prime p*
shows *is-nth-power n* $(p \mathbin{\hat{}} k) \longleftrightarrow n\ dvd\ k$
⟨*proof*⟩

lemma *is-nth-power-nth-power'*:
assumes *n dvd n'*
shows *is-nth-power n* $(m \mathbin{\hat{}} n')$
⟨*proof*⟩

definition *is-nth-power-nat* :: $nat \Rightarrow nat \Rightarrow bool$
where [*code-abbrev*]: *is-nth-power-nat = is-nth-power*

lemma *is-nth-power-nat-code* [*code*]:
is-nth-power-nat n m =
(*if n = 0 then m = 1*
else if m = 0 then n > 0
else if n = 1 then True
else $(\exists k \in \{1..m\}.\ k \mathbin{\hat{}} n = m)$)
⟨*proof*⟩

lemma *is-nth-power-mult-cancel-left*:
fixes *a b* :: $'a :: semiring\text{-}gcd$
assumes *is-nth-power n a a* $\neq 0$
shows *is-nth-power n* $(a * b) \longleftrightarrow$ *is-nth-power n b*
⟨*proof*⟩

lemma *is-nth-power-mult-cancel-right*:
fixes $a\ b :: 'a :: semiring\text{-}gcd$
assumes $is\text{-}nth\text{-}power\ n\ b\ b \neq 0$
shows $is\text{-}nth\text{-}power\ n\ (a * b) \longleftrightarrow is\text{-}nth\text{-}power\ n\ a$
⟨*proof*⟩

10.2 The n-root of a natural number

definition $nth\text{-}root\text{-}nat :: nat \Rightarrow nat \Rightarrow nat$ **where**
$nth\text{-}root\text{-}nat\ k\ n = (if\ k = 0\ then\ 0\ else\ Max\ \{m.\ m\ \hat{}\ k \leq n\})$

lemma *zeroth-root-nat* [*simp*]: $nth\text{-}root\text{-}nat\ 0\ n = 0$
⟨*proof*⟩

lemma *nth-root-nat-aux1*:
assumes $k > 0$
shows $\{m::nat.\ m\ \hat{}\ k \leq n\} \subseteq \{..n\}$
⟨*proof*⟩

lemma *nth-root-nat-aux2*:
assumes $k > 0$
shows $finite\ \{m::nat.\ m\ \hat{}\ k \leq n\}\ \{m::nat.\ m\ \hat{}\ k \leq n\} \neq \{\}$
⟨*proof*⟩

lemma
assumes $k > 0$
shows *nth-root-nat-power-le*: $nth\text{-}root\text{-}nat\ k\ n\ \hat{}\ k \leq n$
and *nth-root-nat-ge*: $x\ \hat{}\ k \leq n \Longrightarrow x \leq nth\text{-}root\text{-}nat\ k\ n$
⟨*proof*⟩

lemma *nth-root-nat-less*:
assumes $k > 0\ x\ \hat{}\ k > n$
shows $nth\text{-}root\text{-}nat\ k\ n < x$
⟨*proof*⟩

lemma *nth-root-nat-unique*:
assumes $m\ \hat{}\ k \leq n\ (m + 1)\ \hat{}\ k > n$
shows $nth\text{-}root\text{-}nat\ k\ n = m$
⟨*proof*⟩

lemma *nth-root-nat-0* [*simp*]: $nth\text{-}root\text{-}nat\ k\ 0 = 0$
⟨*proof*⟩

lemma *nth-root-nat-1* [*simp*]: $k > 0 \Longrightarrow nth\text{-}root\text{-}nat\ k\ 1 = 1$
⟨*proof*⟩

lemma *nth-root-nat-Suc-0* [*simp*]: $k > 0 \Longrightarrow nth\text{-}root\text{-}nat\ k\ (Suc\ 0) = Suc\ 0$
⟨*proof*⟩

lemma *first-root-nat* [*simp*]: *nth-root-nat 1 n = n*
⟨*proof*⟩

lemma *first-root-nat′* [*simp*]: *nth-root-nat (Suc 0) n = n*
⟨*proof*⟩

lemma *nth-root-nat-code-naive′*:
nth-root-nat k n = (if k = 0 then 0 else Max (Set.filter (λm. m ^ k ≤ n) {..n}))
⟨*proof*⟩

function *nth-root-nat-aux* :: *nat ⇒ nat ⇒ nat ⇒ nat ⇒ nat* **where**
nth-root-nat-aux m k acc n =
(let acc′ = (k + 1) ^ m
in if k ≥ n ∨ acc′ > n then k else nth-root-nat-aux m (k+1) acc′ n)
⟨*proof*⟩
termination ⟨*proof*⟩

lemma *nth-root-nat-aux-le*:
assumes *k ^ m ≤ n m > 0*
shows *nth-root-nat-aux m k (k ^ m) n ^ m ≤ n*
⟨*proof*⟩

lemma *nth-root-nat-aux-gt*:
assumes *m > 0*
shows *(nth-root-nat-aux m k (k ^ m) n + 1) ^ m > n*
⟨*proof*⟩

lemma *nth-root-nat-aux-correct*:
assumes *k ^ m ≤ n m > 0*
shows *nth-root-nat-aux m k (k ^ m) n = nth-root-nat m n*
⟨*proof*⟩

lemma *nth-root-nat-naive-code* [*code*]:
nth-root-nat m n = (if m = 0 ∨ n = 0 then 0 else if m = 1 ∨ n = 1 then n else
nth-root-nat-aux m 1 1 n)
⟨*proof*⟩

lemma *nth-root-nat-nth-power* [*simp*]: *k > 0 ⟹ nth-root-nat k (n ^ k) = n*
⟨*proof*⟩

lemma *nth-root-nat-nth-power′*:
assumes *k > 0 k dvd m*
shows *nth-root-nat k (n ^ m) = n ^ (m div k)*
⟨*proof*⟩

lemma *nth-root-nat-mono*:
assumes *m ≤ n*

shows *nth-root-nat k m* ≤ *nth-root-nat k n*
⟨*proof*⟩

end

11 Polynomials, fractions and rings

theory *Polynomial-Factorial*
imports
Complex-Main
Polynomial
Normalized-Fraction
begin

11.1 Lifting elements into the field of fractions

definition *to-fract* :: *'a* :: *idom* ⇒ *'a fract*
where *to-fract x* = *Fract x 1*
— FIXME: more idiomatic name, abbreviation

lemma *to-fract-0* [*simp*]: *to-fract 0* = *0*
⟨*proof*⟩

lemma *to-fract-1* [*simp*]: *to-fract 1* = *1*
⟨*proof*⟩

lemma *to-fract-add* [*simp*]: *to-fract* (*x* + *y*) = *to-fract x* + *to-fract y*
⟨*proof*⟩

lemma *to-fract-diff* [*simp*]: *to-fract* (*x* − *y*) = *to-fract x* − *to-fract y*
⟨*proof*⟩

lemma *to-fract-uminus* [*simp*]: *to-fract* (−*x*) = −*to-fract x*
⟨*proof*⟩

lemma *to-fract-mult* [*simp*]: *to-fract* (*x* ∗ *y*) = *to-fract x* ∗ *to-fract y*
⟨*proof*⟩

lemma *to-fract-eq-iff* [*simp*]: *to-fract x* = *to-fract y* ⟷ *x* = *y*
⟨*proof*⟩

lemma *to-fract-eq-0-iff* [*simp*]: *to-fract x* = *0* ⟷ *x* = *0*
⟨*proof*⟩

lemma *to-fract-quot-of-fract*:
assumes *snd* (*quot-of-fract x*) = *1*
shows *to-fract* (*fst* (*quot-of-fract x*)) = *x*
⟨*proof*⟩

lemma *Fract-conv-to-fract*: *Fract a b = to-fract a / to-fract b*
⟨*proof*⟩

lemma *quot-of-fract-to-fract* [*simp*]: *quot-of-fract (to-fract x) = (x, 1)*
⟨*proof*⟩

lemma *snd-quot-of-fract-to-fract* [*simp*]: *snd (quot-of-fract (to-fract x)) = 1*
⟨*proof*⟩

11.2 Lifting polynomial coefficients to the field of fractions

abbreviation (*input*) *fract-poly* :: ‹*'a*::*idom poly* ⇒ *'a fract poly*›
where *fract-poly* ≡ *map-poly to-fract*

abbreviation (*input*) *unfract-poly* :: ‹*'a*::{*ring-gcd*,*semiring-gcd-mult-normalize*,*idom-divide*}
fract poly ⇒ *'a poly*›
where *unfract-poly* ≡ *map-poly* (*fst* ∘ *quot-of-fract*)

lemma *fract-poly-smult* [*simp*]: *fract-poly* (*smult c p*) = *smult* (*to-fract c*) (*fract-poly*
p)
⟨*proof*⟩

lemma *fract-poly-0* [*simp*]: *fract-poly 0 = 0*
⟨*proof*⟩

lemma *fract-poly-1* [*simp*]: *fract-poly 1 = 1*
⟨*proof*⟩

lemma *fract-poly-add* [*simp*]:
fract-poly (*p* + *q*) = *fract-poly p* + *fract-poly q*
⟨*proof*⟩

lemma *fract-poly-diff* [*simp*]:
fract-poly (*p* − *q*) = *fract-poly p* − *fract-poly q*
⟨*proof*⟩

lemma *to-fract-sum* [*simp*]: *to-fract* (*sum f A*) = *sum* (λ*x*. *to-fract* (*f x*)) *A*
⟨*proof*⟩

lemma *fract-poly-mult* [*simp*]:
fract-poly (*p* ∗ *q*) = *fract-poly p* ∗ *fract-poly q*
⟨*proof*⟩

lemma *fract-poly-eq-iff* [*simp*]: *fract-poly p = fract-poly q* ⟷ *p = q*
⟨*proof*⟩

lemma *fract-poly-eq-0-iff* [*simp*]: *fract-poly p = 0* ⟷ *p = 0*
⟨*proof*⟩

lemma *fract-poly-dvd*: *p dvd q* $\Longrightarrow$ *fract-poly p dvd fract-poly q*
⟨*proof*⟩

lemma *prod-mset-fract-poly*:
(∏ *x*∈#*A*. *map-poly to-fract* (*f x*)) = *fract-poly* (*prod-mset* (*image-mset f A*))
⟨*proof*⟩

lemma *is-unit-fract-poly-iff*:
p dvd 1 $\longleftrightarrow$ *fract-poly p dvd 1* ∧ *content p* = *1*
⟨*proof*⟩

lemma *fract-poly-is-unit*: *p dvd 1* $\Longrightarrow$ *fract-poly p dvd 1*
⟨*proof*⟩

lemma *fract-poly-smult-eqE*:
fixes *c* :: *'a* :: {*idom-divide*,*ring-gcd*,*semiring-gcd-mult-normalize*} *fract*
assumes *fract-poly p* = *smult c* (*fract-poly q*)
obtains *a b*
where *c* = *to-fract b* / *to-fract a smult a p* = *smult b q coprime a b normalize*
a = *a*
⟨*proof*⟩

11.3 Fractional content

abbreviation (*input*) *Lcm-coeff-denoms*
:: *'a* :: {*semiring-Gcd*,*idom-divide*,*ring-gcd*,*semiring-gcd-mult-normalize*} *fract*
poly ⇒ *'a*
where *Lcm-coeff-denoms p* ≡ *Lcm* (*snd* ‘ *quot-of-fract* ‘ *set* (*coeffs p*))

definition *fract-content* ::
'a :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*}
fract poly ⇒ *'a fract* **where**
fract-content p =
(*let d* = *Lcm-coeff-denoms p in Fract* (*content* (*unfract-poly* (*smult* (*to-fract*
d) *p*))) *d*)

definition *primitive-part-fract* ::
'a :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*}
fract poly ⇒ *'a poly* **where**
primitive-part-fract p =
primitive-part (*unfract-poly* (*smult* (*to-fract* (*Lcm-coeff-denoms p*)) *p*))

lemma *primitive-part-fract-0* [*simp*]: *primitive-part-fract 0* = *0*
⟨*proof*⟩

lemma *fract-content-eq-0-iff* [*simp*]:
fract-content p = *0* $\longleftrightarrow$ *p* = *0*
⟨*proof*⟩

lemma *content-primitive-part-fract* [*simp*]:
  fixes *p* :: *'a* :: {*semiring-gcd-mult-normalize*,
    factorial-semiring, *ring-gcd*, *semiring-Gcd*,*idom-divide*} *fract poly*
  shows *p* ≠ *0* ⟹ *content* (*primitive-part-fract p*) = *1*
  ⟨*proof*⟩

lemma *content-times-primitive-part-fract*:
  smult (*fract-content p*) (*fract-poly* (*primitive-part-fract p*)) = *p*
⟨*proof*⟩

lemma *fract-content-fract-poly* [*simp*]: *fract-content* (*fract-poly p*) = *to-fract* (*content p*)
⟨*proof*⟩

lemma *content-decompose-fract*:
  fixes *p* :: *'a* :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,
    semiring-gcd-mult-normalize} *fract poly*
  obtains *c p'* **where** *p* = *smult c* (*map-poly to-fract p'*) *content p'* = *1*
⟨*proof*⟩

lemma *fract-poly-dvdD*:
  fixes *p* :: *'a* :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,
    semiring-gcd-mult-normalize} *poly*
  assumes *fract-poly p dvd fract-poly q content p* = *1*
  shows *p dvd q*
⟨*proof*⟩

11.4 Polynomials over a field are a Euclidean ring

context
begin

interpretation *field-poly*:
  normalization-euclidean-semiring-multiplicative **where** *zero* = *0* :: *'a* :: *field poly*
    and *one* = *1* **and** *plus* = *plus* **and** *minus* = *minus*
    and *times* = *times*
    and *normalize* = λ*p*. *smult* (*inverse* (*lead-coeff p*)) *p*
    and *unit-factor* = λ*p*. [:*lead-coeff p*:]
    and *euclidean-size* = λ*p*. *if p* = *0 then 0 else 2* ^ *degree p*
    and *divide* = *divide* **and** *modulo* = *modulo*
  rewrites *dvd.dvd* (*times* :: *'a poly* ⇒ -) = *Rings.dvd*
    and *comm-monoid-mult.prod-mset times 1* = *prod-mset*
    and *comm-semiring-1.irreducible times 1 0* = *irreducible*
    and *comm-semiring-1.prime-elem times 1 0* = *prime-elem*
⟨*proof*⟩

lemma *field-poly-irreducible-imp-prime*:
  prime-elem p **if** *irreducible p* **for** *p* :: *'a* :: *field poly*
  ⟨*proof*⟩

lemma *field-poly-prod-mset-prime-factorization*:
prod-mset (*field-poly.prime-factorization p*) = *smult* (*inverse* (*lead-coeff p*)) *p*
if $p \neq 0$ **for** $p :: 'a ::$ *field poly*
⟨*proof*⟩

lemma *field-poly-in-prime-factorization-imp-prime*:
prime-elem p **if** $p \in\#$ *field-poly.prime-factorization x*
for $p :: 'a ::$ *field poly*
⟨*proof*⟩

11.5 Primality and irreducibility in polynomial rings

lemma *nonconst-poly-irreducible-iff*:
fixes $p :: 'a ::$ {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*} *poly*
assumes *degree p* $\neq 0$
shows *irreducible p* $\longleftrightarrow$ *irreducible* (*fract-poly p*) $\wedge$ *content p* = 1
⟨*proof*⟩

lemma *irreducible-imp-prime-poly*:
fixes $p :: 'a ::$ {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*} *poly*
assumes *irreducible p*
shows *prime-elem p*
⟨*proof*⟩

lemma *degree-primitive-part-fract* [*simp*]:
degree (*primitive-part-fract p*) = *degree p*
⟨*proof*⟩

lemma *irreducible-primitive-part-fract*:
fixes $p :: 'a ::$ {*idom-divide*, *ring-gcd*, *factorial-semiring*, *semiring-Gcd*,*semiring-gcd-mult-normalize*} *fract poly*
assumes *irreducible p*
shows *irreducible* (*primitive-part-fract p*)
⟨*proof*⟩

lemma *prime-elem-primitive-part-fract*:
fixes $p :: 'a ::$ {*idom-divide*, *ring-gcd*, *factorial-semiring*, *semiring-Gcd*,*semiring-gcd-mult-normalize*} *fract poly*
shows *irreducible p* $\Longrightarrow$ *prime-elem* (*primitive-part-fract p*)
⟨*proof*⟩

lemma *irreducible-linear-field-poly*:
fixes *a b* :: 'a::*field*
assumes $b \neq 0$
shows *irreducible* [:*a*,*b*:]
⟨*proof*⟩

lemma *prime-elem-linear-field-poly*:
(*b* :: *'a* :: *field*) ≠ *0* ⟹ *prime-elem* [:*a*,*b*:]
⟨*proof*⟩

lemma *irreducible-linear-poly*:
fixes *a b* :: *'a*::{*idom-divide*,*ring-gcd*,*factorial-semiring*,*semiring-Gcd*,*semiring-gcd-mult-normalize*}
shows *b* ≠ *0* ⟹ *coprime a b* ⟹ *irreducible* [:*a*,*b*:]
⟨*proof*⟩

lemma *prime-elem-linear-poly*:
fixes *a b* :: *'a*::{*idom-divide*,*ring-gcd*,*factorial-semiring*,*semiring-Gcd*,*semiring-gcd-mult-normalize*}
shows *b* ≠ *0* ⟹ *coprime a b* ⟹ *prime-elem* [:*a*,*b*:]
⟨*proof*⟩

11.6 Prime factorisation of polynomials

lemma *poly-prime-factorization-exists-content-1*:
fixes *p* :: *'a* :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*}
poly
assumes *p* ≠ *0 content p* = *1*
shows ∃ *A*. (∀ *p*. *p* ∈# *A* ⟶ *prime-elem p*) ∧ *prod-mset A* = *normalize p*
⟨*proof*⟩

lemma *poly-prime-factorization-exists*:
fixes *p* :: *'a* :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*}
poly
assumes *p* ≠ *0*
shows ∃ *A*. (∀ *p*. *p* ∈# *A* ⟶ *prime-elem p*) ∧ *normalize* (*prod-mset A*) =
normalize p
⟨*proof*⟩

end

11.7 Typeclass instances

instance *poly* :: ({*factorial-ring-gcd*,*semiring-gcd-mult-normalize*}) *factorial-semiring*
⟨*proof*⟩

instantiation *poly* :: ({*factorial-ring-gcd*, *semiring-gcd-mult-normalize*}) *factorial-ring-gcd*
begin

definition *gcd-poly* :: *'a poly* ⇒ *'a poly* ⇒ *'a poly* **where**
[*code del*]: *gcd-poly* = *gcd-factorial*

definition *lcm-poly* :: *'a poly* ⇒ *'a poly* ⇒ *'a poly* **where**
[*code del*]: *lcm-poly* = *lcm-factorial*

definition *Gcd-poly* :: *'a poly set* ⇒ *'a poly* **where**
[*code del*]: *Gcd-poly* = *Gcd-factorial*

definition *Lcm-poly* :: *'a poly set* ⇒ *'a poly* **where**
[*code del*]: *Lcm-poly* = *Lcm-factorial*

instance ⟨*proof*⟩

end

instance *poly* :: ({*factorial-ring-gcd*, *semiring-gcd-mult-normalize*}) *semiring-gcd-mult-normalize*
⟨*proof*⟩

instance *poly* :: ({*field*,*factorial-ring-gcd*,*semiring-gcd-mult-normalize*})
normalization-euclidean-semiring ⟨*proof*⟩

instance *poly* :: ({*field*, *normalization-euclidean-semiring*, *factorial-ring-gcd*,
semiring-gcd-mult-normalize}) *euclidean-ring-gcd*
⟨*proof*⟩

instance *poly* :: ({*field*, *normalization-euclidean-semiring*, *factorial-ring-gcd*,
semiring-gcd-mult-normalize}) *factorial-semiring-multiplicative*
⟨*proof*⟩

11.8 Polynomial GCD

lemma *gcd-poly-decompose*:
fixes *p q* :: *'a* :: {*factorial-ring-gcd*,*semiring-gcd-mult-normalize*} *poly*
shows *gcd p q* =
smult (*gcd* (*content p*) (*content q*)) (*gcd* (*primitive-part p*) (*primitive-part q*))
⟨*proof*⟩

lemma *gcd-poly-pseudo-mod*:
fixes *p q* :: *'a* :: {*factorial-ring-gcd*,*semiring-gcd-mult-normalize*} *poly*
assumes *nz*: *q* ≠ *0* **and** *prim*: *content p* = *1 content q* = *1*
shows *gcd p q* = *gcd q* (*primitive-part* (*pseudo-mod p q*))
⟨*proof*⟩

lemma *degree-pseudo-mod-less*:
assumes *q* ≠ *0 pseudo-mod p q* ≠ *0*
shows *degree* (*pseudo-mod p q*) < *degree q*
⟨*proof*⟩

function *gcd-poly-code-aux* :: *'a* :: *factorial-ring-gcd poly* ⇒ *'a poly* ⇒ *'a poly*
where
gcd-poly-code-aux p q =
(*if q* = *0 then normalize p else gcd-poly-code-aux q* (*primitive-part* (*pseudo-mod p q*)))
⟨*proof*⟩

termination
⟨*proof*⟩

declare *gcd-poly-code-aux.simps* [*simp del*]

lemma *gcd-poly-code-aux-correct*:
assumes *content p = 1 q = 0 ∨ content q = 1*
shows *gcd-poly-code-aux p q = gcd p q*
⟨*proof*⟩

definition *gcd-poly-code*
:: *'a :: factorial-ring-gcd poly ⇒ 'a poly ⇒ 'a poly*
where *gcd-poly-code p q =*
(if p = 0 then normalize q else if q = 0 then normalize p else
smult (gcd (content p) (content q))
(gcd-poly-code-aux (primitive-part p) (primitive-part q)))

lemma *gcd-poly-code* [*code*]: *gcd p q = gcd-poly-code p q*
⟨*proof*⟩

lemma *lcm-poly-code* [*code*]:
fixes *p q :: 'a :: {factorial-ring-gcd,semiring-gcd-mult-normalize} poly*
shows *lcm p q = normalize (p * q div gcd p q)*
⟨*proof*⟩

lemmas *Gcd-poly-set-eq-fold* [*code*] =
Gcd-set-eq-fold [**where** *?'a = 'a :: {factorial-ring-gcd,semiring-gcd-mult-normalize} poly*]
lemmas *Lcm-poly-set-eq-fold* [*code*] =
Lcm-set-eq-fold [**where** *?'a = 'a :: {factorial-ring-gcd,semiring-gcd-mult-normalize} poly*]

end

12 Squarefreeness

theory *Squarefree*
imports *Primes*
begin

definition *squarefree :: 'a :: comm-monoid-mult ⇒ bool* **where**
squarefree n ⟷ (∀x. x ^ 2 dvd n ⟶ x dvd 1)

lemma *squarefreeI*: *(⋀x. x ^ 2 dvd n ⟹ x dvd 1) ⟹ squarefree n*
⟨*proof*⟩

lemma *squarefreeD*: *squarefree n ⟹ x ^ 2 dvd n ⟹ x dvd 1*

⟨*proof*⟩

lemma *not-squarefreeI*: $x \mathbin{\hat{}} 2$ *dvd* $n \Longrightarrow \neg x$ *dvd* $1 \Longrightarrow \neg$*squarefree* n
⟨*proof*⟩

lemma *not-squarefreeE* [*case-names square-dvd*]:
$\neg$*squarefree* $n \Longrightarrow (\bigwedge x.\ x \mathbin{\hat{}} 2$ *dvd* $n \Longrightarrow \neg x$ *dvd* $1 \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *not-squarefree-0* [*simp*]: $\neg$*squarefree* $(0 :: 'a ::$ *comm-semiring-1*$)$
⟨*proof*⟩

lemma *squarefree-factorial-semiring*:
assumes $n \neq 0$
shows *squarefree* $(n :: 'a ::$ *factorial-semiring*$) \longleftrightarrow (\forall p.$ *prime* $p \longrightarrow \neg p \mathbin{\hat{}} 2$ *dvd* $n)$
⟨*proof*⟩

lemma *squarefree-factorial-semiring'*:
assumes $n \neq 0$
shows *squarefree* $(n :: 'a ::$ *factorial-semiring*$) \longleftrightarrow$
$(\forall p \in$*prime-factors* $n.$ *multiplicity* $p\ n = 1)$
⟨*proof*⟩

lemma *squarefree-factorial-semiring''*:
assumes $n \neq 0$
shows *squarefree* $(n :: 'a ::$ *factorial-semiring*$) \longleftrightarrow$
$(\forall p.$ *prime* $p \longrightarrow$ *multiplicity* $p\ n \leq 1)$
⟨*proof*⟩

lemma *squarefree-unit* [*simp*]: *is-unit* $n \Longrightarrow$ *squarefree* n
⟨*proof*⟩

lemma *squarefree-1* [*simp*]: *squarefree* $(1 :: 'a ::$ *algebraic-semidom*$)$
⟨*proof*⟩

lemma *squarefree-minus* [*simp*]: *squarefree* $(-n :: 'a ::$ *comm-ring-1*$) \longleftrightarrow$ *squarefree* n
⟨*proof*⟩

lemma *squarefree-mono*: a *dvd* $b \Longrightarrow$ *squarefree* $b \Longrightarrow$ *squarefree* a
⟨*proof*⟩

lemma *squarefree-multD*:
assumes *squarefree* $(a * b)$
shows *squarefree* a *squarefree* b
⟨*proof*⟩

lemma *squarefree-prime-elem*:

assumes *prime-elem* (*p* :: *'a* :: *factorial-semiring*)
shows *squarefree p*
⟨*proof*⟩

lemma *squarefree-prime*:
assumes *prime* (*p* :: *'a* :: *factorial-semiring*)
shows *squarefree p*
⟨*proof*⟩

lemma *squarefree-mult-coprime*:
fixes *a b* :: *'a* :: *factorial-semiring-gcd*
assumes *coprime a b squarefree a squarefree b*
shows *squarefree* (*a* ∗ *b*)
⟨*proof*⟩

lemma *squarefree-prod-coprime*:
fixes *f* :: *'a* ⇒ *'b* :: *factorial-semiring-gcd*
assumes ⋀*a b*. *a* ∈ *A* ⟹ *b* ∈ *A* ⟹ *a* ≠ *b* ⟹ *coprime* (*f a*) (*f b*)
assumes ⋀*a*. *a* ∈ *A* ⟹ *squarefree* (*f a*)
shows *squarefree* (*prod f A*)
⟨*proof*⟩

lemma *squarefree-powerD*: *m* > *0* ⟹ *squarefree* (*n* ^ *m*) ⟹ *squarefree n*
⟨*proof*⟩

lemma *squarefree-power-iff*:
squarefree (*n* ^ *m*) ⟷ *m* = *0* ∨ *is-unit n* ∨ (*squarefree n* ∧ *m* = *1*)
⟨*proof*⟩

definition *squarefree-nat* :: *nat* ⇒ *bool* **where**
[*code-abbrev*]: *squarefree-nat* = *squarefree*

lemma *squarefree-nat-code-naive* [*code*]:
squarefree-nat n ⟷ *n* ≠ *0* ∧ (∀ *k*∈{*2*..*n*}. ¬*k* ^ *2 dvd n*)
⟨*proof*⟩

definition *square-part* :: *'a* :: *factorial-semiring* ⇒ *'a* **where**
square-part n = (*if n* = *0 then 0 else*
normalize (∏ *p*∈*prime-factors n*. *p* ^ (*multiplicity p n div 2*)))

lemma *square-part-nonzero*:
n ≠ *0* ⟹ *square-part n* = *normalize* (∏ *p*∈*prime-factors n*. *p* ^ (*multiplicity p n div 2*))
⟨*proof*⟩

lemma *square-part-0* [*simp*]: *square-part 0* = *0*
⟨*proof*⟩

lemma *square-part-unit* [*simp*]: *is-unit x* $\Longrightarrow$ *square-part x = 1*
⟨*proof*⟩

lemma *square-part-1* [*simp*]: *square-part 1 = 1*
⟨*proof*⟩

lemma *square-part-0-iff* [*simp*]: *square-part n = 0* $\longleftrightarrow$ *n = 0*
⟨*proof*⟩

lemma *normalize-uminus* [*simp*]:
normalize (−*x* :: *'a* :: {*normalization-semidom*, *comm-ring-1*}) = *normalize x*
⟨*proof*⟩

lemma *multiplicity-uminus-right* [*simp*]:
multiplicity (*x* :: *'a* :: {*factorial-semiring*, *comm-ring-1*}) (−*y*) = *multiplicity x y*
⟨*proof*⟩

lemma *multiplicity-uminus-left* [*simp*]:
multiplicity (−*x* :: *'a* :: {*factorial-semiring*, *comm-ring-1*}) *y* = *multiplicity x y*
⟨*proof*⟩

lemma *prime-factorization-uminus* [*simp*]:
prime-factorization (−*x* :: *'a* :: {*factorial-semiring*, *comm-ring-1*}) = *prime-factorization x*
⟨*proof*⟩

lemma *square-part-uminus* [*simp*]:
square-part (−*x* :: *'a* :: {*factorial-semiring*, *comm-ring-1*}) = *square-part x*
⟨*proof*⟩

lemma *prime-multiplicity-square-part*:
assumes *prime p*
shows *multiplicity p* (*square-part n*) = *multiplicity p n div 2*
⟨*proof*⟩

lemma *square-part-square-dvd* [*simp*, *intro*]: *square-part n* ^ *2 dvd n*
⟨*proof*⟩

lemma *prime-multiplicity-le-imp-dvd*:
assumes *x* ≠ *0 y* ≠ *0*
shows *x dvd y* $\longleftrightarrow$ (∀ *p*. *prime p* $\longrightarrow$ *multiplicity p x* ≤ *multiplicity p y*)
⟨*proof*⟩

lemma *dvd-square-part-iff*: *x dvd square-part n* $\longleftrightarrow$ *x* ^ *2 dvd n*
⟨*proof*⟩

definition *squarefree-part* :: *'a* :: *factorial-semiring* ⇒ *'a* **where**

squarefree-part n = (if n = 0 then 1 else n div square-part n ^ 2)

lemma *squarefree-part-0* [*simp*]: *squarefree-part 0 = 1*
⟨*proof*⟩

lemma *squarefree-part-unit* [*simp*]: *is-unit n ⟹ squarefree-part n = n*
⟨*proof*⟩

lemma *squarefree-part-1* [*simp*]: *squarefree-part 1 = 1*
⟨*proof*⟩

lemma *squarefree-decompose*: *n = squarefree-part n * square-part n ^ 2*
⟨*proof*⟩

lemma *squarefree-part-uminus* [*simp*]:
assumes *x ≠ 0*
shows *squarefree-part (−x :: 'a :: {factorial-semiring, comm-ring-1}) = −squarefree-part x*
⟨*proof*⟩

lemma *squarefree-part-nonzero* [*simp*]: *squarefree-part n ≠ 0*
⟨*proof*⟩

lemma *prime-multiplicity-squarefree-part*:
assumes *prime p*
shows *multiplicity p (squarefree-part n) = multiplicity p n mod 2*
⟨*proof*⟩

lemma *prime-multiplicity-squarefree-part-le-Suc-0* [*intro*]:
assumes *prime p*
shows *multiplicity p (squarefree-part n) ≤ Suc 0*
⟨*proof*⟩

lemma *squarefree-squarefree-part* [*simp*, *intro*]: *squarefree (squarefree-part n)*
⟨*proof*⟩

lemma *squarefree-decomposition-unique*:
assumes *square-part m = square-part n*
assumes *squarefree-part m = squarefree-part n*
shows *m = n*
⟨*proof*⟩

lemma *normalize-square-part* [*simp*]: *normalize (square-part x) = square-part x*
⟨*proof*⟩

lemma *square-part-even-power'*: *square-part (x ^ (2 * n)) = normalize (x ^ n)*
⟨*proof*⟩

lemma *square-part-even-power*: *even n ⟹ square-part (x ^ n) = normalize (x ^*

(*n div 2*))
⟨*proof*⟩

lemma *square-part-odd-power'*: *square-part* (*x* ^ (*Suc* (*2* * *n*))) = *normalize* (*x* ^ *n* * *square-part x*)
⟨*proof*⟩

lemma *square-part-odd-power*:
odd n ⟹ *square-part* (*x* ^ *n*) = *normalize* (*x* ^ (*n div 2*) * *square-part x*)
⟨*proof*⟩

end

13 Pieces of computational Algebra

theory *Computational-Algebra*
imports
Euclidean-Algorithm
Factorial-Ring
Formal-Laurent-Series
Fraction-Field
Fundamental-Theorem-Algebra
Group-Closure
Normalized-Fraction
Nth-Powers
Polynomial-FPS
Polynomial
Polynomial-Factorial
Primes
Squarefree
begin

end

theory *Field-as-Ring*
imports
Complex-Main
Euclidean-Algorithm
begin

context *field*
begin

subclass *idom-divide* ⟨*proof*⟩

definition *normalize-field* :: *'a* ⇒ *'a*
where [*simp*]: *normalize-field x* = (*if x* = *0 then 0 else 1*)
definition *unit-factor-field* :: *'a* ⇒ *'a*

where [*simp*]: *unit-factor-field x = x*
definition *euclidean-size-field* :: *'a ⇒ nat*
where [*simp*]: *euclidean-size-field x = (if x = 0 then 0 else 1)*
definition *mod-field* :: *'a ⇒ 'a ⇒ 'a*
where [*simp*]: *mod-field x y = (if y = 0 then x else 0)*

end

instantiation *real* ::
{*unique-euclidean-ring*, *normalization-euclidean-semiring*, *normalization-semidom-multiplicative*}
begin

definition [*simp*]: *normalize-real = (normalize-field* :: *real ⇒* -)
definition [*simp*]: *unit-factor-real = (unit-factor-field* :: *real ⇒* -)
definition [*simp*]: *modulo-real = (mod-field* :: *real ⇒* -)
definition [*simp*]: *euclidean-size-real = (euclidean-size-field* :: *real ⇒* -)
definition [*simp*]: *division-segment* (*x* :: *real*) = *1*

instance
⟨*proof*⟩

end

instantiation *real* :: *euclidean-ring-gcd*
begin

definition *gcd-real* :: *real ⇒ real ⇒ real* **where**
gcd-real = Euclidean-Algorithm.gcd
definition *lcm-real* :: *real ⇒ real ⇒ real* **where**
lcm-real = Euclidean-Algorithm.lcm
definition *Gcd-real* :: *real set ⇒ real* **where**
Gcd-real = Euclidean-Algorithm.Gcd
definition *Lcm-real* :: *real set ⇒ real* **where**
Lcm-real = Euclidean-Algorithm.Lcm

instance ⟨*proof*⟩

end

instance *real* :: *field-gcd* ⟨*proof*⟩

instantiation *rat* ::
{*unique-euclidean-ring*, *normalization-euclidean-semiring*, *normalization-semidom-multiplicative*}
begin

definition [*simp*]: *normalize-rat = (normalize-field* :: *rat ⇒* -)
definition [*simp*]: *unit-factor-rat = (unit-factor-field* :: *rat ⇒* -)
definition [*simp*]: *modulo-rat = (mod-field* :: *rat ⇒* -)

definition [*simp*]: *euclidean-size-rat* = (*euclidean-size-field* :: *rat* ⇒ -)
definition [*simp*]: *division-segment* (*x* :: *rat*) = *1*

instance
⟨*proof*⟩

end

instantiation *rat* :: *euclidean-ring-gcd*
begin

definition *gcd-rat* :: *rat* ⇒ *rat* ⇒ *rat* **where**
gcd-rat = *Euclidean-Algorithm.gcd*
definition *lcm-rat* :: *rat* ⇒ *rat* ⇒ *rat* **where**
lcm-rat = *Euclidean-Algorithm.lcm*
definition *Gcd-rat* :: *rat set* ⇒ *rat* **where**
Gcd-rat = *Euclidean-Algorithm.Gcd*
definition *Lcm-rat* :: *rat set* ⇒ *rat* **where**
Lcm-rat = *Euclidean-Algorithm.Lcm*

instance ⟨*proof*⟩

end

instance *rat* :: *field-gcd* ⟨*proof*⟩

instantiation *complex* ::
{*unique-euclidean-ring*, *normalization-euclidean-semiring*, *normalization-semidom-multiplicative*}
begin

definition [*simp*]: *normalize-complex* = (*normalize-field* :: *complex* ⇒ -)
definition [*simp*]: *unit-factor-complex* = (*unit-factor-field* :: *complex* ⇒ -)
definition [*simp*]: *modulo-complex* = (*mod-field* :: *complex* ⇒ -)
definition [*simp*]: *euclidean-size-complex* = (*euclidean-size-field* :: *complex* ⇒ -)
definition [*simp*]: *division-segment* (*x* :: *complex*) = *1*

instance
⟨*proof*⟩

end

instantiation *complex* :: *euclidean-ring-gcd*
begin

definition *gcd-complex* :: *complex* ⇒ *complex* ⇒ *complex* **where**
gcd-complex = *Euclidean-Algorithm.gcd*
definition *lcm-complex* :: *complex* ⇒ *complex* ⇒ *complex* **where**
lcm-complex = *Euclidean-Algorithm.lcm*

definition *Gcd-complex* :: *complex set* ⇒ *complex* **where**
Gcd-complex = *Euclidean-Algorithm.Gcd*
definition *Lcm-complex* :: *complex set* ⇒ *complex* **where**
Lcm-complex = *Euclidean-Algorithm.Lcm*

instance ⟨*proof*⟩

end

instance *complex* :: *field-gcd* ⟨*proof*⟩

end

14 Computation checks

theory *Computation-Checks*
imports *Primes Polynomial-Factorial HOL−Library.Discrete-Functions HOL−Library.Code-Target-Numeral*
begin

floor-sqrt 16476148165462159 = 128359449

prime 97

prime 97

prime 9973

prime 9973

Gcd {[:*1*, *2*, *3*:], [:*2*, *3*, *4*:]} = *1*

Lcm {[:*1*, *2*, *3*:], [:*2*, *3*, *4*:]} = [:[:*2*:], [:*7*:], [:*16*:], [:*17*:], [:*12*:]:]

end

References

[1] K. J. Nowak. Some elementary proofs of Puiseuxs theorems. *Univ. Iagel. Acta Math*, 38:279–282, 2000.