Notable Examples in Isabelle/Pure

December 17, 2025

1 A simple formulation of First-Order Logic

The subsequent theory development illustrates single-sorted intuitionistic first-order logic with equality, formulated within the Pure framework.

theory *First_Order_Logic*
  imports *Pure*
begin

1.1 Abstract syntax

typedecl *i*
typedecl *o*

judgment *Trueprop* :: $o \Rightarrow prop$ (‹_› *5*)

1.2 Propositional logic

axiomatization *false* :: o (‹⊥›)
  where *falseE* [*elim*]: $\bot \Longrightarrow A$

axiomatization *imp* :: $o \Rightarrow o \Rightarrow o$ (**infixr** ‹⟶› *25*)
  where *impI* [*intro*]: $(A \Longrightarrow B) \Longrightarrow A \longrightarrow B$
    and *mp* [*dest*]: $A \longrightarrow B \Longrightarrow A \Longrightarrow B$

axiomatization *conj* :: $o \Rightarrow o \Rightarrow o$ (**infixr** ‹∧› *35*)
  where *conjI* [*intro*]: $A \Longrightarrow B \Longrightarrow A \land B$
    and *conjD1*: $A \land B \Longrightarrow A$
    and *conjD2*: $A \land B \Longrightarrow B$

theorem *conjE* [*elim*]:
  assumes $A \land B$
  obtains A **and** B
⟨*proof*⟩

axiomatization *disj* :: $o \Rightarrow o \Rightarrow o$ (**infixr** ‹∨› *30*)
  where *disjE* [*elim*]: $A \vee B \Longrightarrow (A \Longrightarrow C) \Longrightarrow (B \Longrightarrow C) \Longrightarrow C$
    and *disjI1* [*intro*]: $A \Longrightarrow A \vee B$
    and *disjI2* [*intro*]: $B \Longrightarrow A \vee B$

definition *true* :: o (‹⊤›)
  where $\top \equiv \bot \longrightarrow \bot$

theorem *trueI* [*intro*]: $\top$
  ⟨*proof*⟩

definition *not* :: $o \Rightarrow o$ (‹¬ _› [*40*] *40*)
  where $\neg\ A \equiv A \longrightarrow \bot$

theorem *notI* [*intro*]: $(A \Longrightarrow \bot) \Longrightarrow \neg\ A$
  ⟨*proof*⟩

theorem *notE* [*elim*]: $\neg\ A \Longrightarrow A \Longrightarrow B$
  ⟨*proof*⟩

definition *iff* :: $o \Rightarrow o \Rightarrow o$ (**infixr** ‹⟷› *25*)
  where $A \longleftrightarrow B \equiv (A \longrightarrow B) \wedge (B \longrightarrow A)$

theorem *iffI* [*intro*]:
  assumes $A \Longrightarrow B$
    and $B \Longrightarrow A$
  shows $A \longleftrightarrow B$
  ⟨*proof*⟩

theorem *iff1* [*elim*]:
  assumes $A \longleftrightarrow B$ **and** A
  shows B
⟨*proof*⟩

theorem *iff2* [*elim*]:
  assumes $A \longleftrightarrow B$ **and** B
  shows A
⟨*proof*⟩

1.3 Equality

axiomatization *equal* :: $i \Rightarrow i \Rightarrow o$ (**infixl** ‹=› *50*)
  where *refl* [*intro*]: $x = x$
    and *subst*: $x = y \Longrightarrow P\ x \Longrightarrow P\ y$

theorem *trans* [*trans*]: $x = y \Longrightarrow y = z \Longrightarrow x = z$
⟨*proof*⟩

theorem *sym* [*sym*]: $x = y \Longrightarrow y = x$
⟨*proof*⟩

1.4 Quantifiers

axiomatization *All* :: $(i \Rightarrow o) \Rightarrow o$ (**binder** ‹∀› *10*)
where *allI* [*intro*]: $(\bigwedge x.\ P\ x) \Longrightarrow \forall x.\ P\ x$
and *allD* [*dest*]: $\forall x.\ P\ x \Longrightarrow P\ a$

axiomatization *Ex* :: $(i \Rightarrow o) \Rightarrow o$ (**binder** ‹∃› *10*)
where *exI* [*intro*]: $P\ a \Longrightarrow \exists x.\ P\ x$
and *exE* [*elim*]: $\exists x.\ P\ x \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow C) \Longrightarrow C$

lemma $(\exists x.\ P\ (f\ x)) \longrightarrow (\exists y.\ P\ y)$
⟨*proof*⟩

lemma $(\exists x.\ \forall y.\ R\ x\ y) \longrightarrow (\forall y.\ \exists x.\ R\ x\ y)$
⟨*proof*⟩

end

2 Foundations of HOL

theory *Higher_Order_Logic*
imports *Pure*
begin

The following theory development illustrates the foundations of Higher-Order Logic. The "HOL" logic that is given here resembles [2] and its predecessor [1], but the order of axiomatizations and defined connectives has be adapted to modern presentations of λ-calculus and Constructive Type Theory. Thus it fits nicely to the underlying Natural Deduction framework of Isabelle/Pure and Isabelle/Isar.

3 HOL syntax within Pure

class *type*
default_sort *type*

typedecl *o*
instance *o* :: *type* ⟨*proof*⟩
instance *fun* :: (*type*, *type*) *type* ⟨*proof*⟩

judgment *Trueprop* :: $o \Rightarrow prop$ (‹_› *5*)

4 Minimal logic (axiomatization)

axiomatization *imp* :: $o \Rightarrow o \Rightarrow o$ (**infixr** ‹⟶› *25*)
  where *impI* [*intro*]: $(A \Longrightarrow B) \Longrightarrow A \longrightarrow B$
    and *impE* [*dest*, *trans*]: $A \longrightarrow B \Longrightarrow A \Longrightarrow B$

axiomatization *All* :: $('a \Rightarrow o) \Rightarrow o$ (**binder** ‹∀› *10*)
  where *allI* [*intro*]: $(\bigwedge x.\ P\ x) \Longrightarrow \forall x.\ P\ x$
    and *allE* [*dest*]: $\forall x.\ P\ x \Longrightarrow P\ a$

lemma *atomize_imp* [*atomize*]: $(A \Longrightarrow B) \equiv$ *Trueprop* $(A \longrightarrow B)$
  ⟨*proof*⟩

lemma *atomize_all* [*atomize*]: $(\bigwedge x.\ P\ x) \equiv$ *Trueprop* $(\forall x.\ P\ x)$
  ⟨*proof*⟩

4.0.1 Derived connectives

definition *False* :: o
  where *False* $\equiv \forall A.\ A$

lemma *FalseE* [*elim*]:
  assumes *False*
  shows A
⟨*proof*⟩

definition *True* :: o
  where *True* $\equiv$ *False* $\longrightarrow$ *False*

lemma *TrueI* [*intro*]: *True*
  ⟨*proof*⟩

definition *not* :: $o \Rightarrow o$ (‹¬ _› [*40*] *40*)
  where *not* $\equiv \lambda A.\ A \longrightarrow$ *False*

lemma *notI* [*intro*]:
  assumes $A \Longrightarrow$ *False*
  shows $\neg\ A$
  ⟨*proof*⟩

lemma *notE* [*elim*]:
  assumes $\neg\ A$ **and** A
  shows B
⟨*proof*⟩

lemma *notE'*: $A \Longrightarrow \neg\ A \Longrightarrow B$
  ⟨*proof*⟩

lemmas *contradiction* = *notE notE′* — proof by contradiction in any order

definition *conj* :: $o \Rightarrow o \Rightarrow o$ (**infixr** ‹∧› *35*)
where $A \wedge B \equiv \forall C.\ (A \longrightarrow B \longrightarrow C) \longrightarrow C$

lemma *conjI* [*intro*]:
assumes A **and** B
shows $A \wedge B$
⟨*proof*⟩

lemma *conjE* [*elim*]:
assumes $A \wedge B$
obtains A **and** B
⟨*proof*⟩

definition *disj* :: $o \Rightarrow o \Rightarrow o$ (**infixr** ‹∨› *30*)
where $A \vee B \equiv \forall C.\ (A \longrightarrow C) \longrightarrow (B \longrightarrow C) \longrightarrow C$

lemma *disjI1* [*intro*]:
assumes A
shows $A \vee B$
⟨*proof*⟩

lemma *disjI2* [*intro*]:
assumes B
shows $A \vee B$
⟨*proof*⟩

lemma *disjE* [*elim*]:
assumes $A \vee B$
obtains $(a)\ A \mid (b)\ B$
⟨*proof*⟩

definition *Ex* :: $('a \Rightarrow o) \Rightarrow o$ (**binder** ‹∃› *10*)
where $\exists x.\ P\ x \equiv \forall C.\ (\forall x.\ P\ x \longrightarrow C) \longrightarrow C$

lemma *exI* [*intro*]: $P\ a \Longrightarrow \exists x.\ P\ x$
⟨*proof*⟩

lemma *exE* [*elim*]:
assumes $\exists x.\ P\ x$
obtains (*that*) x **where** $P\ x$
⟨*proof*⟩

4.0.2 Extensional equality

axiomatization *equal* :: $'a \Rightarrow 'a \Rightarrow o$ (**infixl** ‹=› *50*)
where *refl* [*intro*]: $x = x$
and *subst*: $x = y \Longrightarrow P\ x \Longrightarrow P\ y$

abbreviation *not_equal* :: $'a \Rightarrow 'a \Rightarrow o$ (**infixl** ‹≠› *50*)
where $x \neq y \equiv \neg\ (x = y)$

abbreviation *iff* :: $o \Rightarrow o \Rightarrow o$ (**infixr** ‹⟷› *25*)
where $A \longleftrightarrow B \equiv A = B$

axiomatization
where *ext* [*intro*]: $(\bigwedge x.\ f\ x = g\ x) \Longrightarrow f = g$
and *iff* [*intro*]: $(A \Longrightarrow B) \Longrightarrow (B \Longrightarrow A) \Longrightarrow A \longleftrightarrow B$
for $f\ g :: 'a \Rightarrow 'b$

lemma *sym* [*sym*]: $y = x$ **if** $x = y$
⟨*proof*⟩

lemma [*trans*]: $x = y \Longrightarrow P\ y \Longrightarrow P\ x$
⟨*proof*⟩

lemma [*trans*]: $P\ x \Longrightarrow x = y \Longrightarrow P\ y$
⟨*proof*⟩

lemma *arg_cong*: $f\ x = f\ y$ **if** $x = y$
⟨*proof*⟩

lemma *fun_cong*: $f\ x = g\ x$ **if** $f = g$
⟨*proof*⟩

lemma *trans* [*trans*]: $x = y \Longrightarrow y = z \Longrightarrow x = z$
⟨*proof*⟩

lemma *iff1* [*elim*]: $A \longleftrightarrow B \Longrightarrow A \Longrightarrow B$
⟨*proof*⟩

lemma *iff2* [*elim*]: $A \longleftrightarrow B \Longrightarrow B \Longrightarrow A$
⟨*proof*⟩

4.1 Cantor's Theorem

Cantor's Theorem states that there is no surjection from a set to its powerset. The subsequent formulation uses elementary λ-calculus and predicate logic, with standard introduction and elimination rules.

lemma *iff_contradiction*:
assumes $*$: $\neg\ A \longleftrightarrow A$
shows C

$\langle proof \rangle$

theorem *Cantor*: $\neg (\exists f :: 'a \Rightarrow 'a \Rightarrow o. \forall A. \exists x. A = f\ x)$
$\langle proof \rangle$

4.2 Characterization of Classical Logic

The subsequent rules of classical reasoning are all equivalent.

locale *classical* =
assumes *classical*: $(\neg A \Longrightarrow A) \Longrightarrow A$
— predicate definition and hypothetical context
begin

lemma *classical_contradiction*:
assumes $\neg A \Longrightarrow False$
shows A
$\langle proof \rangle$

lemma *double_negation*:
assumes $\neg \neg A$
shows A
$\langle proof \rangle$

lemma *tertium_non_datur*: $A \vee \neg A$
$\langle proof \rangle$

lemma *classical_cases*:
obtains $A \mid \neg A$
$\langle proof \rangle$

end

lemma *classical_if_cases*: *classical*
if *cases*: $\bigwedge A\ C. (A \Longrightarrow C) \Longrightarrow (\neg A \Longrightarrow C) \Longrightarrow C$
$\langle proof \rangle$

5 Peirce's Law

Peirce's Law is another characterization of classical reasoning. Its statement only requires implication.

theorem (**in** *classical*) *Peirce's_Law*: $((A \longrightarrow B) \longrightarrow A) \longrightarrow A$
$\langle proof \rangle$

6 Hilbert's choice operator (axiomatization)

axiomatization *Eps* :: $('a \Rightarrow o) \Rightarrow 'a$
where *someI*: $P\ x \Longrightarrow P\ (Eps\ P)$

syntax *_Eps* :: *pttrn* ⇒ *o* ⇒ *'a* (‹(‹*indent=3 notation=*‹*binder SOME*››*SOME _./ _*)› [*0*, *10*] *10*)
syntax_consts *_Eps* ⇌ *Eps*
translations *SOME x. P* ⇌ *CONST Eps* (λ*x. P*)

It follows a derivation of the classical law of tertium-non-datur by means of Hilbert's choice operator (due to Berghofer, Beeson, Harrison, based on a proof by Diaconescu).

theorem *Diaconescu*: *A* ∨ ¬ *A*
⟨*proof*⟩

This means, the hypothetical predicate *classical* always holds unconditionally (with all consequences).

interpretation *classical*
⟨*proof*⟩

thm *classical*
classical_contradiction
double_negation
tertium_non_datur
classical_cases
Peirce's_Law

end

References

[1] A. Church. A formulation of the simple theory of types. *Journal of Symbolic Logic*, 5:56–68, 1940.

[2] M. J. C. Gordon. HOL: A machine oriented formulation of higher order logic. Technical Report 68, University of Cambridge Computer Laboratory, 1985.