

Miscellaneous FOL Examples

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1 Natural numbers

```
theory Natural-Numbers
imports FOL
begin
```

Theory of the natural numbers: Peano's axioms, primitive recursion. (Modernized version of Larry Paulson's theory "Nat".)

```
typedecl nat
instance nat :: <term> <proof>

axiomatization
  Zero :: <nat>    (<0>) and
  Suc  :: <nat => nat> and
  rec  :: <[nat, 'a, [nat, 'a] => 'a] => 'a>
where
  induct [case-names 0 Suc, induct type: nat]:
    <P(0) ==> (!x. P(x) ==> P(Suc(x))) ==> P(n)> and
  Suc-inject: <Suc(m) = Suc(n) ==> m = n> and
  Suc-neq-0: <Suc(m) = 0 ==> R> and
  rec-0: <rec(0, a, f) = a> and
```

```

    rec-Suc:  $\langle \text{rec}(\text{Suc}(m), a, f) = f(m, \text{rec}(m, a, f)) \rangle$ 

lemma Suc-n-not-n:  $\langle \text{Suc}(k) \neq k \rangle$ 
 $\langle \text{proof} \rangle$ 

definition add ::  $\langle \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat} \rangle$     (infixl  $\langle + \rangle$  60)
  where  $\langle m + n = \text{rec}(m, n, \lambda x y. \text{Suc}(y)) \rangle$ 

lemma add-0 [simp]:  $\langle 0 + n = n \rangle$ 
 $\langle \text{proof} \rangle$ 

lemma add-Suc [simp]:  $\langle \text{Suc}(m) + n = \text{Suc}(m + n) \rangle$ 
 $\langle \text{proof} \rangle$ 

lemma add-assoc:  $\langle (k + m) + n = k + (m + n) \rangle$ 
 $\langle \text{proof} \rangle$ 

lemma add-0-right:  $\langle m + 0 = m \rangle$ 
 $\langle \text{proof} \rangle$ 

lemma add-Suc-right:  $\langle m + \text{Suc}(n) = \text{Suc}(m + n) \rangle$ 
 $\langle \text{proof} \rangle$ 

lemma
  assumes  $\langle !!n. f(\text{Suc}(n)) = \text{Suc}(f(n)) \rangle$ 
  shows  $\langle f(i + j) = i + f(j) \rangle$ 
 $\langle \text{proof} \rangle$ 

end

```

2 Examples for the manual “Introduction to Isabelle”

```

theory Intro
imports FOL
begin

```

2.0.1 Some simple backward proofs

```

lemma mythm:  $\langle P \vee P \longrightarrow P \rangle$ 
 $\langle \text{proof} \rangle$ 

lemma  $\langle (P \wedge Q) \vee R \longrightarrow (P \vee R) \rangle$ 
 $\langle \text{proof} \rangle$ 

```

Correct version, delaying use of *spec* until last.

```

lemma  $\langle (\forall x y. P(x,y)) \longrightarrow (\forall z w. P(w,z)) \rangle$ 

```

$\langle proof \rangle$

2.0.2 Demonstration of *fast*

lemma $\langle (\exists y. \forall x. J(y, x) \longleftrightarrow \neg J(x, x)) \longrightarrow \neg (\forall x. \exists y. \forall z. J(z, y) \longleftrightarrow \neg J(z, x)) \rangle$
 $\langle proof \rangle$

lemma $\langle \forall x. P(x, f(x)) \longleftrightarrow (\exists y. (\forall z. P(z, y) \longrightarrow P(z, f(x))) \wedge P(x, y)) \rangle$
 $\langle proof \rangle$

2.0.3 Derivation of conjunction elimination rule

lemma
 assumes *major*: $\langle P \wedge Q \rangle$
 and *minor*: $\langle \llbracket P; Q \rrbracket \Longrightarrow R \rangle$
 shows $\langle R \rangle$
 $\langle proof \rangle$

2.1 Derived rules involving definitions

Derivation of negation introduction

lemma
 assumes $\langle P \Longrightarrow False \rangle$
 shows $\langle \neg P \rangle$
 $\langle proof \rangle$

lemma
 assumes *major*: $\langle \neg P \rangle$
 and *minor*: $\langle P \rangle$
 shows $\langle R \rangle$
 $\langle proof \rangle$

Alternative proof of the result above

lemma
 assumes *major*: $\langle \neg P \rangle$
 and *minor*: $\langle P \rangle$
 shows $\langle R \rangle$
 $\langle proof \rangle$

end

3 Theory of the natural numbers: Peano's axioms, primitive recursion

theory *Nat*
 imports *FOL*
begin

typedec1 *nat*
instance *nat* :: $\langle \text{term} \rangle \langle \text{proof} \rangle$

axiomatization
Zero :: $\langle \text{nat} \rangle \langle 0 \rangle$ **and**
Suc :: $\langle \text{nat} \Rightarrow \text{nat} \rangle$ **and**
rec :: $\langle [\text{nat}, 'a, [\text{nat}, 'a] \Rightarrow 'a] \Rightarrow 'a \rangle$
where
induct: $\langle \llbracket P(0); \bigwedge x. P(x) \implies P(\text{Suc}(x)) \rrbracket \implies P(n) \rangle$ **and**
Suc-inject: $\langle \text{Suc}(m) = \text{Suc}(n) \implies m = n \rangle$ **and**
Suc-neq-0: $\langle \text{Suc}(m) = 0 \implies \text{False} \rangle$ **and**
rec-0: $\langle \text{rec}(0, a, f) = a \rangle$ **and**
rec-Suc: $\langle \text{rec}(\text{Suc}(m), a, f) = f(m, \text{rec}(m, a, f)) \rangle$

definition *add* :: $\langle [\text{nat}, \text{nat}] \Rightarrow \text{nat} \rangle$ (**infixl** $\langle + \rangle$ 60)
where $\langle m + n \equiv \text{rec}(m, n, \lambda x y. \text{Suc}(y)) \rangle$

3.1 Proofs about the natural numbers

lemma *Suc-n-not-n*: $\langle \text{Suc}(k) \neq k \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (k+m)+n = k+(m+n) \rangle$
 $\langle \text{proof} \rangle$

lemma *add-0* [*simp*]: $\langle 0+n = n \rangle$
 $\langle \text{proof} \rangle$

lemma *add-Suc* [*simp*]: $\langle \text{Suc}(m)+n = \text{Suc}(m+n) \rangle$
 $\langle \text{proof} \rangle$

lemma *add-assoc*: $\langle (k+m)+n = k+(m+n) \rangle$
 $\langle \text{proof} \rangle$

lemma *add-0-right*: $\langle m+0 = m \rangle$
 $\langle \text{proof} \rangle$

lemma *add-Suc-right*: $\langle m+\text{Suc}(n) = \text{Suc}(m+n) \rangle$
 $\langle \text{proof} \rangle$

lemma
assumes *prem*: $\langle \bigwedge n. f(\text{Suc}(n)) = \text{Suc}(f(n)) \rangle$
shows $\langle f(i+j) = i+f(j) \rangle$
 $\langle \text{proof} \rangle$

end

4 Theory of the natural numbers: Peano's axioms, primitive recursion

```
theory Nat-Class
  imports FOL
begin
```

This is an abstract version of `Nat.thy`. Instead of axiomatizing a single type `nat`, it defines the class of all these types (up to isomorphism).

Note: The `rec` operator has been made *monomorphic*, because class axioms cannot contain more than one type variable.

```
class nat =
  fixes Zero :: 'a <math>\langle 0 \rangle</math>
    and Suc :: 'a  $\Rightarrow$  'a
    and rec :: 'a  $\Rightarrow$  'a  $\Rightarrow$  ('a  $\Rightarrow$  'a  $\Rightarrow$  'a)  $\Rightarrow$  'a
  assumes induct:  $\langle P(0) \Longrightarrow (\bigwedge x. P(x) \Longrightarrow P(\text{Suc}(x))) \Longrightarrow P(n) \rangle$ 
    and Suc-inject:  $\langle \text{Suc}(m) = \text{Suc}(n) \Longrightarrow m = n \rangle$ 
    and Suc-neq-Zero:  $\langle \text{Suc}(m) = 0 \Longrightarrow \text{False} \rangle$ 
    and rec-Zero:  $\langle \text{rec}(0, a, f) = a \rangle$ 
    and rec-Suc:  $\langle \text{rec}(\text{Suc}(m), a, f) = f(m, \text{rec}(m, a, f)) \rangle$ 
begin
```

```
definition add :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a (infixl <math>\langle + \rangle</math> 60)
  where  $\langle m + n = \text{rec}(m, n, \lambda x y. \text{Suc}(y)) \rangle$ 
```

```
lemma Suc-n-not-n:  $\langle \text{Suc}(k) \neq (k::'a) \rangle$ 
  <math>\langle \text{proof} \rangle</math>
```

```
lemma  $\langle (k + m) + n = k + (m + n) \rangle$ 
  <math>\langle \text{proof} \rangle</math>
```

```
lemma add-Zero [simp]:  $\langle 0 + n = n \rangle$ 
  <math>\langle \text{proof} \rangle</math>
```

```
lemma add-Suc [simp]:  $\langle \text{Suc}(m) + n = \text{Suc}(m + n) \rangle$ 
  <math>\langle \text{proof} \rangle</math>
```

```
lemma add-assoc:  $\langle (k + m) + n = k + (m + n) \rangle$ 
  <math>\langle \text{proof} \rangle</math>
```

```
lemma add-Zero-right:  $\langle m + 0 = m \rangle$ 
  <math>\langle \text{proof} \rangle</math>
```

```
lemma add-Suc-right:  $\langle m + \text{Suc}(n) = \text{Suc}(m + n) \rangle$ 
  <math>\langle \text{proof} \rangle</math>
```

```
lemma
  assumes prem:  $\langle \bigwedge n. f(\text{Suc}(n)) = \text{Suc}(f(n)) \rangle$ 
```

```

    shows  $\langle f(i + j) = i + f(j) \rangle$ 
     $\langle proof \rangle$ 

```

```

end

```

```

end

```

5 Intuitionistic FOL: Examples from The Foundation of a Generic Theorem Prover

```

theory Foundation
imports IFOL
begin

```

```

lemma  $\langle A \wedge B \longrightarrow (C \longrightarrow A \wedge C) \rangle$ 
 $\langle proof \rangle$ 

```

A form of conj-elimination

```

lemma
  assumes  $\langle A \wedge B \rangle$ 
  and  $\langle A \Longrightarrow B \Longrightarrow C \rangle$ 
  shows  $\langle C \rangle$ 
 $\langle proof \rangle$ 

```

```

lemma
  assumes  $\langle \bigwedge A. \neg \neg A \Longrightarrow A \rangle$ 
  shows  $\langle B \vee \neg B \rangle$ 
 $\langle proof \rangle$ 

```

```

lemma
  assumes  $\langle \bigwedge A. \neg \neg A \Longrightarrow A \rangle$ 
  shows  $\langle B \vee \neg B \rangle$ 
 $\langle proof \rangle$ 

```

```

lemma
  assumes  $\langle A \vee \neg A \rangle$ 
  and  $\langle \neg \neg A \rangle$ 
  shows  $\langle A \rangle$ 
 $\langle proof \rangle$ 

```

5.1 Examples with quantifiers

```

lemma
  assumes  $\langle \forall z. G(z) \rangle$ 
  shows  $\langle \forall z. G(z) \vee H(z) \rangle$ 
 $\langle proof \rangle$ 

```

lemma $\langle \forall x. \exists y. x = y \rangle$
 $\langle proof \rangle$

lemma $\langle \exists y. \forall x. x = y \rangle$
 $\langle proof \rangle$

Parallel lifting example.

lemma $\langle \exists u. \forall x. \exists v. \forall y. \exists w. P(u, x, v, y, w) \rangle$
 $\langle proof \rangle$

lemma
assumes $\langle (\exists z. F(z)) \wedge B \rangle$
shows $\langle \exists z. F(z) \wedge B \rangle$
 $\langle proof \rangle$

A bigger demonstration of quantifiers – not in the paper.

lemma $\langle (\exists y. \forall x. Q(x, y)) \longrightarrow (\forall x. \exists y. Q(x, y)) \rangle$
 $\langle proof \rangle$

end

6 First-Order Logic: PROLOG examples

theory *Prolog*
imports *FOL*
begin

typeddecl *'a list*
instance *list* :: $(\langle term \rangle) \langle term \rangle \langle proof \rangle$

axiomatization
 Nil :: $\langle 'a list \rangle$ **and**
 $Cons$:: $\langle ['a, 'a list] \Rightarrow 'a list \rangle$ (**infixr** $\langle : \rangle$ 60) **and**
 app :: $\langle ['a list, 'a list, 'a list] \Rightarrow o \rangle$ **and**
 rev :: $\langle ['a list, 'a list] \Rightarrow o \rangle$
where
 $appNil$: $\langle app(Nil, ys, ys) \rangle$ **and**
 $appCons$: $\langle app(xs, ys, zs) \Rightarrow app(x:xs, ys, x:zs) \rangle$ **and**
 $revNil$: $\langle rev(Nil, Nil) \rangle$ **and**
 $revCons$: $\langle [| rev(xs, ys); app(ys, x:Nil, zs) |] \Rightarrow rev(x:xs, zs) \rangle$

schematic-goal $\langle app(a:b:c:Nil, d:e:Nil, ?x) \rangle$
 $\langle proof \rangle$

schematic-goal $\langle app(?x, c:d:Nil, a:b:c:d:Nil) \rangle$
 $\langle proof \rangle$

schematic-goal $\langle app(?x, ?y, a:b:c:d:Nil) \rangle$

⟨proof⟩

lemmas *rules* = *appNil appCons revNil revCons*

schematic-goal ⟨*rev(a:b:c:d:Nil, ?x)*⟩
⟨proof⟩

schematic-goal ⟨*rev(a:b:c:d:e:f:g:h:i:j:k:l:m:n:Nil, ?w)*⟩
⟨proof⟩

schematic-goal ⟨*rev(?x, a:b:c:Nil)*⟩
⟨proof⟩

⟨ML⟩

schematic-goal ⟨*rev(?x, a:b:c:Nil)*⟩
⟨proof⟩

schematic-goal ⟨*rev(a: ?x:c: ?y:Nil, d: ?z:b: ?u)*⟩
⟨proof⟩

schematic-goal ⟨*rev(a:b:c:d:e:f:g:h:i:j:k:l:m:n:o:p:Nil, ?w)*⟩
⟨proof⟩

schematic-goal ⟨*a:b:c:d:e:f:g:h:i:j:k:l:m:n:o:p:Nil = ?x ∧ app(?x, ?x, ?y) ∧ rev(?y, ?w)*⟩
⟨proof⟩

end

7 Intuitionistic First-Order Logic

theory *Intuitionistic*
imports *IFOL*
begin

Metatheorem (for *propositional* formulae): P is classically provable iff $\neg\neg P$ is intuitionistically provable. Therefore $\neg P$ is classically provable iff it is intuitionistically provable.

Proof: Let Q be the conjunction of the propositions $A \vee \neg A$, one for each atom A in P . Now $\neg\neg Q$ is intuitionistically provable because $\neg\neg(A \vee \neg A)$ is and because double-negation distributes over conjunction. If P is provable classically, then clearly $Q \rightarrow P$ is provable intuitionistically, so $\neg\neg(Q \rightarrow P)$

is also provable intuitionistically. The latter is intuitionistically equivalent to $\neg\neg Q \rightarrow \neg\neg P$, hence to $\neg\neg P$, since $\neg\neg Q$ is intuitionistically provable. Finally, if P is a negation then $\neg\neg P$ is intuitionistically equivalent to P . [Andy Pitts]

lemma $\langle \neg\neg (P \wedge Q) \longleftrightarrow \neg\neg P \wedge \neg\neg Q \rangle$
 $\langle proof \rangle$

lemma $\langle \neg\neg ((\neg P \rightarrow Q) \rightarrow (\neg P \rightarrow \neg Q) \rightarrow P) \rangle$
 $\langle proof \rangle$

Double-negation does NOT distribute over disjunction.

lemma $\langle \neg\neg (P \rightarrow Q) \longleftrightarrow (\neg\neg P \rightarrow \neg\neg Q) \rangle$
 $\langle proof \rangle$

lemma $\langle \neg\neg\neg P \longleftrightarrow \neg P \rangle$
 $\langle proof \rangle$

lemma $\langle \neg\neg ((P \rightarrow Q \vee R) \rightarrow (P \rightarrow Q) \vee (P \rightarrow R)) \rangle$
 $\langle proof \rangle$

lemma $\langle (P \longleftrightarrow Q) \longleftrightarrow (Q \longleftrightarrow P) \rangle$
 $\langle proof \rangle$

lemma $\langle ((P \rightarrow (Q \vee (Q \rightarrow R))) \rightarrow R) \rightarrow R \rangle$
 $\langle proof \rangle$

lemma
 $\langle (((G \rightarrow A) \rightarrow J) \rightarrow D \rightarrow E) \rightarrow (((H \rightarrow B) \rightarrow I) \rightarrow C \rightarrow J)$
 $\rightarrow (A \rightarrow H) \rightarrow F \rightarrow G \rightarrow (((C \rightarrow B) \rightarrow I) \rightarrow D) \rightarrow (A \rightarrow C)$
 $\rightarrow (((F \rightarrow A) \rightarrow B) \rightarrow I) \rightarrow E \rangle$
 $\langle proof \rangle$

Admissibility of the excluded middle for negated formulae

lemma $\langle (P \vee \neg P \rightarrow \neg Q) \rightarrow \neg Q \rangle$
 $\langle proof \rangle$

The same in a more general form, no ex falso quodlibet

lemma $\langle (P \vee (P \rightarrow R) \rightarrow Q \rightarrow R) \rightarrow Q \rightarrow R \rangle$
 $\langle proof \rangle$

7.1 Lemmas for the propositional double-negation translation

lemma $\langle P \rightarrow \neg\neg P \rangle$
 $\langle proof \rangle$

lemma $\langle \neg\neg (\neg\neg P \rightarrow P) \rangle$
 $\langle proof \rangle$

lemma $\langle \neg \neg P \wedge \neg \neg (P \longrightarrow Q) \longrightarrow \neg \neg Q \rangle$
 $\langle proof \rangle$

The following are classically but not constructively valid. The attempt to prove them terminates quickly!

lemma $\langle ((P \longrightarrow Q) \longrightarrow P) \longrightarrow P \rangle$
 $\langle proof \rangle$

lemma $\langle (P \wedge Q \longrightarrow R) \longrightarrow (P \longrightarrow R) \vee (Q \longrightarrow R) \rangle$
 $\langle proof \rangle$

7.2 de Bruijn formulae

de Bruijn formula with three predicates

lemma
 $\langle ((P \longleftrightarrow Q) \longrightarrow P \wedge Q \wedge R) \wedge$
 $((Q \longleftrightarrow R) \longrightarrow P \wedge Q \wedge R) \wedge$
 $((R \longleftrightarrow P) \longrightarrow P \wedge Q \wedge R) \longrightarrow P \wedge Q \wedge R \rangle$
 $\langle proof \rangle$

de Bruijn formula with five predicates

lemma
 $\langle ((P \longleftrightarrow Q) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T) \wedge$
 $((Q \longleftrightarrow R) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T) \wedge$
 $((R \longleftrightarrow S) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T) \wedge$
 $((S \longleftrightarrow T) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T) \wedge$
 $((T \longleftrightarrow P) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T) \longrightarrow P \wedge Q \wedge R \wedge S \wedge T \rangle$
 $\langle proof \rangle$

Problems from of Sahlin, Franzen and Haridi, An Intuitionistic Predicate Logic Theorem Prover. J. Logic and Comp. 2 (5), October 1992, 619-656.

Problem 1.1

lemma
 $\langle (\forall x. \exists y. \forall z. p(x) \wedge q(y) \wedge r(z)) \longleftrightarrow$
 $(\forall z. \exists y. \forall x. p(x) \wedge q(y) \wedge r(z)) \rangle$
 $\langle proof \rangle$

Problem 3.1

lemma $\langle \neg (\exists x. \forall y. mem(y, x) \longleftrightarrow \neg mem(x, x)) \rangle$
 $\langle proof \rangle$

Problem 4.1: hopeless!

lemma
 $\langle (\forall x. p(x) \longrightarrow p(h(x)) \vee p(g(x))) \wedge (\exists x. p(x)) \wedge (\forall x. \neg p(h(x)))$
 $\longrightarrow (\exists x. p(g(g(g(g(x)))))) \rangle$
 $\langle proof \rangle$

7.3 Intuitionistic FOL: propositional problems based on Pelletier.

$\neg\neg 1$

lemma $\langle \neg \neg ((P \longrightarrow Q) \longleftrightarrow (\neg Q \longrightarrow \neg P)) \rangle$
 $\langle proof \rangle$

$\neg\neg 2$

lemma $\langle \neg \neg (\neg \neg P \longleftrightarrow P) \rangle$
 $\langle proof \rangle$

3

lemma $\langle \neg (P \longrightarrow Q) \longrightarrow (Q \longrightarrow P) \rangle$
 $\langle proof \rangle$

$\neg\neg 4$

lemma $\langle \neg \neg ((\neg P \longrightarrow Q) \longleftrightarrow (\neg Q \longrightarrow P)) \rangle$
 $\langle proof \rangle$

$\neg\neg 5$

lemma $\langle \neg \neg ((P \vee Q \longrightarrow P \vee R) \longrightarrow P \vee (Q \longrightarrow R)) \rangle$
 $\langle proof \rangle$

$\neg\neg 6$

lemma $\langle \neg \neg (P \vee \neg P) \rangle$
 $\langle proof \rangle$

$\neg\neg 7$

lemma $\langle \neg \neg (P \vee \neg \neg \neg P) \rangle$
 $\langle proof \rangle$

$\neg\neg 8$. Peirce's law

lemma $\langle \neg \neg (((P \longrightarrow Q) \longrightarrow P) \longrightarrow P) \rangle$
 $\langle proof \rangle$

9

lemma $\langle ((P \vee Q) \wedge (\neg P \vee Q) \wedge (P \vee \neg Q)) \longrightarrow \neg (\neg P \vee \neg Q) \rangle$
 $\langle proof \rangle$

10

lemma $\langle (Q \longrightarrow R) \longrightarrow (R \longrightarrow P \wedge Q) \longrightarrow (P \longrightarrow (Q \vee R)) \longrightarrow (P \longleftrightarrow Q) \rangle$
 $\langle proof \rangle$

7.4 11. Proved in each direction (incorrectly, says Pelletier!!)

lemma $\langle P \longleftrightarrow P \rangle$
 $\langle proof \rangle$

$\neg\neg$ 12. Dijkstra's law

lemma $\langle \neg\neg ((P \longleftrightarrow Q) \longleftrightarrow R) \longleftrightarrow (P \longleftrightarrow (Q \longleftrightarrow R)) \rangle$
 $\langle proof \rangle$

lemma $\langle ((P \longleftrightarrow Q) \longleftrightarrow R) \longrightarrow \neg\neg (P \longleftrightarrow (Q \longleftrightarrow R)) \rangle$
 $\langle proof \rangle$

13. Distributive law

lemma $\langle P \vee (Q \wedge R) \longleftrightarrow (P \vee Q) \wedge (P \vee R) \rangle$
 $\langle proof \rangle$

$\neg\neg$ 14

lemma $\langle \neg\neg ((P \longleftrightarrow Q) \longleftrightarrow ((Q \vee \neg P) \wedge (\neg Q \vee P))) \rangle$
 $\langle proof \rangle$

$\neg\neg$ 15

lemma $\langle \neg\neg ((P \longrightarrow Q) \longleftrightarrow (\neg P \vee Q)) \rangle$
 $\langle proof \rangle$

$\neg\neg$ 16

lemma $\langle \neg\neg ((P \longrightarrow Q) \vee (Q \longrightarrow P)) \rangle$
 $\langle proof \rangle$

$\neg\neg$ 17

lemma $\langle \neg\neg (((P \wedge (Q \longrightarrow R)) \longrightarrow S) \longleftrightarrow ((\neg P \vee Q \vee S) \wedge (\neg P \vee \neg R \vee S))) \rangle$
 $\langle proof \rangle$

Dijkstra's "Golden Rule"

lemma $\langle (P \wedge Q) \longleftrightarrow P \longleftrightarrow Q \longleftrightarrow (P \vee Q) \rangle$
 $\langle proof \rangle$

8 Examples with quantifiers

8.1 The converse is classical in the following implications ...

lemma $\langle (\exists x. P(x) \longrightarrow Q) \longrightarrow (\forall x. P(x)) \longrightarrow Q \rangle$
 $\langle proof \rangle$

lemma $\langle ((\forall x. P(x)) \longrightarrow Q) \longrightarrow \neg (\forall x. P(x) \wedge \neg Q) \rangle$
 $\langle proof \rangle$

lemma $\langle ((\forall x. \neg P(x)) \longrightarrow Q) \longrightarrow \neg (\forall x. \neg (P(x) \vee Q)) \rangle$

$\langle proof \rangle$

lemma $\langle (\forall x. P(x)) \vee Q \longrightarrow (\forall x. P(x) \vee Q) \rangle$
 $\langle proof \rangle$

lemma $\langle (\exists x. P \longrightarrow Q(x)) \longrightarrow (P \longrightarrow (\exists x. Q(x))) \rangle$
 $\langle proof \rangle$

8.2 The following are not constructively valid!

The attempt to prove them terminates quickly!

lemma $\langle ((\forall x. P(x)) \longrightarrow Q) \longrightarrow (\exists x. P(x) \longrightarrow Q) \rangle$
 $\langle proof \rangle$

lemma $\langle (P \longrightarrow (\exists x. Q(x))) \longrightarrow (\exists x. P \longrightarrow Q(x)) \rangle$
 $\langle proof \rangle$

lemma $\langle (\forall x. P(x) \vee Q) \longrightarrow ((\forall x. P(x)) \vee Q) \rangle$
 $\langle proof \rangle$

lemma $\langle (\forall x. \neg \neg P(x)) \longrightarrow \neg \neg (\forall x. P(x)) \rangle$
 $\langle proof \rangle$

Classically but not intuitionistically valid. Proved by a bug in 1986!

lemma $\langle \exists x. Q(x) \longrightarrow (\forall x. Q(x)) \rangle$
 $\langle proof \rangle$

8.3 Hard examples with quantifiers

The ones that have not been proved are not known to be valid! Some will require quantifier duplication – not currently available.

$\neg\neg 18$

lemma $\langle \neg \neg (\exists y. \forall x. P(y) \longrightarrow P(x)) \rangle$
 $\langle proof \rangle$

$\neg\neg 19$

lemma $\langle \neg \neg (\exists x. \forall y z. (P(y) \longrightarrow Q(z)) \longrightarrow (P(x) \longrightarrow Q(x))) \rangle$
 $\langle proof \rangle$

20

lemma
 $\langle (\forall x y. \exists z. \forall w. (P(x) \wedge Q(y) \longrightarrow R(z) \wedge S(w)))$
 $\longrightarrow (\exists x y. P(x) \wedge Q(y)) \longrightarrow (\exists z. R(z)) \rangle$
 $\langle proof \rangle$

21

lemma $\langle (\exists x. P \longrightarrow Q(x)) \wedge (\exists x. Q(x) \longrightarrow P) \longrightarrow \neg \neg (\exists x. P \longleftrightarrow Q(x)) \rangle$
 $\langle proof \rangle$

22

lemma $\langle (\forall x. P \longleftrightarrow Q(x)) \longrightarrow (P \longleftrightarrow (\forall x. Q(x))) \rangle$
 $\langle proof \rangle$

$\neg\neg$ 23

lemma $\langle \neg \neg ((\forall x. P \vee Q(x)) \longleftrightarrow (P \vee (\forall x. Q(x)))) \rangle$
 $\langle proof \rangle$

24

lemma
 $\langle \neg (\exists x. S(x) \wedge Q(x)) \wedge (\forall x. P(x) \longrightarrow Q(x) \vee R(x)) \wedge$
 $(\neg (\exists x. P(x)) \longrightarrow (\exists x. Q(x))) \wedge (\forall x. Q(x) \vee R(x) \longrightarrow S(x))$
 $\longrightarrow \neg \neg (\exists x. P(x) \wedge R(x)) \rangle$

Not clear why *fast-tac*, *best-tac*, *ASTAR* and *ITER-DEEPEN* all take forever.

$\langle proof \rangle$

25

lemma
 $\langle (\exists x. P(x)) \wedge$
 $(\forall x. L(x) \longrightarrow \neg (M(x) \wedge R(x))) \wedge$
 $(\forall x. P(x) \longrightarrow (M(x) \wedge L(x))) \wedge$
 $((\forall x. P(x) \longrightarrow Q(x)) \vee (\exists x. P(x) \wedge R(x)))$
 $\longrightarrow (\exists x. Q(x) \wedge P(x)) \rangle$
 $\langle proof \rangle$

$\neg\neg$ 26

lemma
 $\langle (\neg \neg (\exists x. p(x)) \longleftrightarrow \neg \neg (\exists x. q(x))) \wedge$
 $(\forall x. \forall y. p(x) \wedge q(y) \longrightarrow (r(x) \longleftrightarrow s(y)))$
 $\longrightarrow ((\forall x. p(x) \longrightarrow r(x)) \longleftrightarrow (\forall x. q(x) \longrightarrow s(x))) \rangle$
 $\langle proof \rangle$

27

lemma
 $\langle (\exists x. P(x) \wedge \neg Q(x)) \wedge$
 $(\forall x. P(x) \longrightarrow R(x)) \wedge$
 $(\forall x. M(x) \wedge L(x) \longrightarrow P(x)) \wedge$
 $((\exists x. R(x) \wedge \neg Q(x)) \longrightarrow (\forall x. L(x) \longrightarrow \neg R(x)))$
 $\longrightarrow (\forall x. M(x) \longrightarrow \neg L(x)) \rangle$
 $\langle proof \rangle$

$\neg\neg$ 28. AMENDED

lemma

$$\begin{aligned}
&\langle (\forall x. P(x) \longrightarrow (\forall x. Q(x))) \wedge \\
&\quad (\neg \neg (\forall x. Q(x) \vee R(x)) \longrightarrow (\exists x. Q(x) \wedge S(x))) \wedge \\
&\quad (\neg \neg (\exists x. S(x)) \longrightarrow (\forall x. L(x) \longrightarrow M(x))) \\
&\quad \longrightarrow (\forall x. P(x) \wedge L(x) \longrightarrow M(x)) \rangle \\
&\langle proof \rangle
\end{aligned}$$

29. Essentially the same as Principia Mathematica *11.71

lemma

$$\begin{aligned}
&\langle (\exists x. P(x)) \wedge (\exists y. Q(y)) \\
&\quad \longrightarrow ((\forall x. P(x) \longrightarrow R(x)) \wedge (\forall y. Q(y) \longrightarrow S(y)) \longleftrightarrow \\
&\quad (\forall x y. P(x) \wedge Q(y) \longrightarrow R(x) \wedge S(y))) \rangle \\
&\langle proof \rangle
\end{aligned}$$

$\neg\neg$ 30

lemma

$$\begin{aligned}
&\langle (\forall x. (P(x) \vee Q(x)) \longrightarrow \neg R(x)) \wedge \\
&\quad (\forall x. (Q(x) \longrightarrow \neg S(x)) \longrightarrow P(x) \wedge R(x)) \\
&\quad \longrightarrow (\forall x. \neg \neg S(x)) \rangle \\
&\langle proof \rangle
\end{aligned}$$

31

lemma

$$\begin{aligned}
&\langle \neg (\exists x. P(x) \wedge (Q(x) \vee R(x))) \wedge \\
&\quad (\exists x. L(x) \wedge P(x)) \wedge \\
&\quad (\forall x. \neg R(x) \longrightarrow M(x)) \\
&\quad \longrightarrow (\exists x. L(x) \wedge M(x)) \rangle \\
&\langle proof \rangle
\end{aligned}$$

32

lemma

$$\begin{aligned}
&\langle (\forall x. P(x) \wedge (Q(x) \vee R(x)) \longrightarrow S(x)) \wedge \\
&\quad (\forall x. S(x) \wedge R(x) \longrightarrow L(x)) \wedge \\
&\quad (\forall x. M(x) \longrightarrow R(x)) \\
&\quad \longrightarrow (\forall x. P(x) \wedge M(x) \longrightarrow L(x)) \rangle \\
&\langle proof \rangle
\end{aligned}$$

$\neg\neg$ 33

lemma

$$\begin{aligned}
&\langle (\forall x. \neg \neg (P(a) \wedge (P(x) \longrightarrow P(b)) \longrightarrow P(c))) \longleftrightarrow \\
&\quad (\forall x. \neg \neg ((\neg P(a) \vee P(x) \vee P(c)) \wedge (\neg P(a) \vee \neg P(b) \vee P(c)))) \rangle \\
&\langle proof \rangle
\end{aligned}$$

36

lemma

$$\begin{aligned}
&\langle (\forall x. \exists y. J(x, y)) \wedge \\
&\quad (\forall x. \exists y. G(x, y)) \wedge \\
&\quad (\forall x y. J(x, y) \vee G(x, y) \longrightarrow (\forall z. J(y, z) \vee G(y, z) \longrightarrow H(x, z))) \\
&\quad \longrightarrow (\forall x. \exists y. H(x, y)) \rangle
\end{aligned}$$

$\langle \text{proof} \rangle$

37

lemma

$\langle (\forall z. \exists w. \forall x. \exists y. \\ \neg \neg (P(x,z) \longrightarrow P(y,w)) \wedge P(y,z) \wedge (P(y,w) \longrightarrow (\exists u. Q(u,w)))) \wedge \\ (\forall x z. \neg P(x,z) \longrightarrow (\exists y. Q(y,z))) \wedge \\ (\neg \neg (\exists x y. Q(x,y)) \longrightarrow (\forall x. R(x,x))) \\ \longrightarrow \neg \neg (\forall x. \exists y. R(x,y)) \rangle$
 $\langle \text{proof} \rangle$

39

lemma $\langle \neg (\exists x. \forall y. F(y,x) \longleftrightarrow \neg F(y,y)) \rangle$
 $\langle \text{proof} \rangle$

40. AMENDED

lemma

$\langle (\exists y. \forall x. F(x,y) \longleftrightarrow F(x,x)) \longrightarrow \\ \neg (\forall x. \exists y. \forall z. F(z,y) \longleftrightarrow \neg F(z,x)) \rangle$
 $\langle \text{proof} \rangle$

44

lemma

$\langle (\forall x. f(x) \longrightarrow \\ (\exists y. g(y) \wedge h(x,y) \wedge (\exists y. g(y) \wedge \neg h(x,y)))) \wedge \\ (\exists x. j(x) \wedge (\forall y. g(y) \longrightarrow h(x,y))) \\ \longrightarrow (\exists x. j(x) \wedge \neg f(x)) \rangle$
 $\langle \text{proof} \rangle$

48

lemma $\langle (a = b \vee c = d) \wedge (a = c \vee b = d) \longrightarrow a = d \vee b = c \rangle$
 $\langle \text{proof} \rangle$

51

lemma

$\langle (\exists z w. \forall x y. P(x,y) \longleftrightarrow (x = z \wedge y = w)) \longrightarrow \\ (\exists z. \forall x y. \exists w. (\forall y. P(x,y) \longleftrightarrow y = w) \longleftrightarrow x = z) \rangle$
 $\langle \text{proof} \rangle$

52

Almost the same as 51.

lemma

$\langle (\exists z w. \forall x y. P(x,y) \longleftrightarrow (x = z \wedge y = w)) \longrightarrow \\ (\exists w. \forall y. \exists z. (\forall x. P(x,y) \longleftrightarrow x = z) \longleftrightarrow y = w) \rangle$
 $\langle \text{proof} \rangle$

56

lemma $\langle (\forall x. (\exists y. P(y) \wedge x = f(y)) \longrightarrow P(x)) \longleftrightarrow (\forall x. P(x) \longrightarrow P(f(x))) \rangle$
 $\langle proof \rangle$

57

lemma
 $\langle P(f(a,b), f(b,c)) \wedge P(f(b,c), f(a,c)) \wedge$
 $(\forall x y z. P(x,y) \wedge P(y,z) \longrightarrow P(x,z)) \longrightarrow P(f(a,b), f(a,c)) \rangle$
 $\langle proof \rangle$

60

lemma $\langle \forall x. P(x, f(x)) \longleftrightarrow (\exists y. (\forall z. P(z, y) \longrightarrow P(z, f(x))) \wedge P(x, y)) \rangle$
 $\langle proof \rangle$

end

9 First-Order Logic: propositional examples (intuitionistic version)

theory *Propositional-Int*
imports *IFOL*
begin

commutative laws of \wedge and \vee

lemma $\langle P \wedge Q \longrightarrow Q \wedge P \rangle$
 $\langle proof \rangle$

lemma $\langle P \vee Q \longrightarrow Q \vee P \rangle$
 $\langle proof \rangle$

associative laws of \wedge and \vee

lemma $\langle (P \wedge Q) \wedge R \longrightarrow P \wedge (Q \wedge R) \rangle$
 $\langle proof \rangle$

lemma $\langle (P \vee Q) \vee R \longrightarrow P \vee (Q \vee R) \rangle$
 $\langle proof \rangle$

distributive laws of \wedge and \vee

lemma $\langle (P \wedge Q) \vee R \longrightarrow (P \vee R) \wedge (Q \vee R) \rangle$
 $\langle proof \rangle$

lemma $\langle (P \vee R) \wedge (Q \vee R) \longrightarrow (P \wedge Q) \vee R \rangle$
 $\langle proof \rangle$

lemma $\langle (P \vee Q) \wedge R \longrightarrow (P \wedge R) \vee (Q \wedge R) \rangle$
 $\langle proof \rangle$

lemma $\langle (P \wedge R) \vee (Q \wedge R) \longrightarrow (P \vee Q) \wedge R \rangle$

$\langle \text{proof} \rangle$

Laws involving implication

lemma $\langle (P \rightarrow R) \wedge (Q \rightarrow R) \leftrightarrow (P \vee Q \rightarrow R) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (P \wedge Q \rightarrow R) \leftrightarrow (P \rightarrow (Q \rightarrow R)) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle ((P \rightarrow R) \rightarrow R) \rightarrow ((Q \rightarrow R) \rightarrow R) \rightarrow (P \wedge Q \rightarrow R) \rightarrow R \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle \neg (P \rightarrow R) \rightarrow \neg (Q \rightarrow R) \rightarrow \neg (P \wedge Q \rightarrow R) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (P \rightarrow Q \wedge R) \leftrightarrow (P \rightarrow Q) \wedge (P \rightarrow R) \rangle$
 $\langle \text{proof} \rangle$

Propositions-as-types

lemma $\langle P \rightarrow (Q \rightarrow P) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (P \rightarrow Q \rightarrow R) \rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow R) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (P \rightarrow Q) \vee (P \rightarrow R) \rightarrow (P \rightarrow Q \vee R) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P) \rangle$
 $\langle \text{proof} \rangle$

Schwichtenberg's examples (via T. Nipkow)

lemma *stab-imp*: $\langle (((Q \rightarrow R) \rightarrow R) \rightarrow Q) \rightarrow (((P \rightarrow Q) \rightarrow R) \rightarrow R) \rightarrow P \rightarrow Q \rangle$
 $\langle \text{proof} \rangle$

lemma *stab-to-peirce*:
 $\langle (((P \rightarrow R) \rightarrow R) \rightarrow P) \rightarrow (((Q \rightarrow R) \rightarrow R) \rightarrow Q) \rightarrow ((P \rightarrow Q) \rightarrow P) \rightarrow P \rangle$
 $\langle \text{proof} \rangle$

lemma *peirce-imp1*:
 $\langle (((Q \rightarrow R) \rightarrow Q) \rightarrow Q) \rightarrow (((P \rightarrow Q) \rightarrow R) \rightarrow P \rightarrow Q) \rightarrow P \rightarrow Q \rangle$
 $\langle \text{proof} \rangle$

lemma *peirce-imp2*: $\langle (((P \rightarrow R) \rightarrow P) \rightarrow P) \rightarrow ((P \rightarrow Q \rightarrow R) \rightarrow P) \rightarrow P \rangle$
 $\langle \text{proof} \rangle$

lemma *mints*: $\langle (((P \rightarrow Q) \rightarrow P) \rightarrow P) \rightarrow Q \rightarrow Q \rangle$

$\langle \text{proof} \rangle$

lemma *mits-solovev*: $\langle (P \longrightarrow (Q \longrightarrow R) \longrightarrow Q) \longrightarrow ((P \longrightarrow Q) \longrightarrow R) \longrightarrow R \rangle$
 $\langle \text{proof} \rangle$

lemma *tatsuta*:

$\langle (((P7 \longrightarrow P1) \longrightarrow P10) \longrightarrow P4 \longrightarrow P5)$
 $\longrightarrow (((P8 \longrightarrow P2) \longrightarrow P9) \longrightarrow P3 \longrightarrow P10)$
 $\longrightarrow (P1 \longrightarrow P8) \longrightarrow P6 \longrightarrow P7$
 $\longrightarrow (((P3 \longrightarrow P2) \longrightarrow P9) \longrightarrow P4)$
 $\longrightarrow (P1 \longrightarrow P3) \longrightarrow (((P6 \longrightarrow P1) \longrightarrow P2) \longrightarrow P9) \longrightarrow P5 \rangle$
 $\langle \text{proof} \rangle$

lemma *tatsuta1*:

$\langle (((P8 \longrightarrow P2) \longrightarrow P9) \longrightarrow P3 \longrightarrow P10)$
 $\longrightarrow (((P3 \longrightarrow P2) \longrightarrow P9) \longrightarrow P4)$
 $\longrightarrow (((P6 \longrightarrow P1) \longrightarrow P2) \longrightarrow P9)$
 $\longrightarrow (((P7 \longrightarrow P1) \longrightarrow P10) \longrightarrow P4 \longrightarrow P5)$
 $\longrightarrow (P1 \longrightarrow P3) \longrightarrow (P1 \longrightarrow P8) \longrightarrow P6 \longrightarrow P7 \longrightarrow P5 \rangle$
 $\langle \text{proof} \rangle$

end

10 First-Order Logic: quantifier examples (intuitionistic version)

theory *Quantifiers-Int*

imports *IFOL*

begin

lemma $\langle (\forall x y. P(x,y)) \longrightarrow (\forall y x. P(x,y)) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (\exists x y. P(x,y)) \longrightarrow (\exists y x. P(x,y)) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (\forall x. P(x)) \vee (\forall x. Q(x)) \longrightarrow (\forall x. P(x) \vee Q(x)) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (\forall x. P \longrightarrow Q(x)) \longleftrightarrow (P \longrightarrow (\forall x. Q(x))) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (\forall x. P(x) \longrightarrow Q) \longleftrightarrow ((\exists x. P(x)) \longrightarrow Q) \rangle$
 $\langle \text{proof} \rangle$

Some harder ones

lemma $\langle (\exists x. P(x) \vee Q(x)) \longleftrightarrow (\exists x. P(x)) \vee (\exists x. Q(x)) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (\exists x. P(x) \wedge Q(x)) \longrightarrow (\exists x. P(x)) \wedge (\exists x. Q(x)) \rangle$
 $\langle proof \rangle$

Basic test of quantifier reasoning

lemma $\langle (\exists y. \forall x. Q(x,y)) \longrightarrow (\forall x. \exists y. Q(x,y)) \rangle$
 $\langle proof \rangle$

lemma $\langle (\forall x. Q(x)) \longrightarrow (\exists x. Q(x)) \rangle$
 $\langle proof \rangle$

The following should fail, as they are false!

lemma $\langle (\forall x. \exists y. Q(x,y)) \longrightarrow (\exists y. \forall x. Q(x,y)) \rangle$
 $\langle proof \rangle$

lemma $\langle (\exists x. Q(x)) \longrightarrow (\forall x. Q(x)) \rangle$
 $\langle proof \rangle$

schematic-goal $\langle P(?a) \longrightarrow (\forall x. P(x)) \rangle$
 $\langle proof \rangle$

schematic-goal $\langle (P(?a) \longrightarrow (\forall x. Q(x))) \longrightarrow (\forall x. P(x) \longrightarrow Q(x)) \rangle$
 $\langle proof \rangle$

Back to things that are provable ...

lemma $\langle (\forall x. P(x) \longrightarrow Q(x)) \wedge (\exists x. P(x)) \longrightarrow (\exists x. Q(x)) \rangle$
 $\langle proof \rangle$

lemma $\langle (P \longrightarrow (\exists x. Q(x))) \wedge P \longrightarrow (\exists x. Q(x)) \rangle$
 $\langle proof \rangle$

schematic-goal $\langle (\forall x. P(x) \longrightarrow Q(f(x))) \wedge (\forall x. Q(x) \longrightarrow R(g(x))) \wedge P(d) \longrightarrow R(?a) \rangle$
 $\langle proof \rangle$

lemma $\langle (\forall x. Q(x)) \longrightarrow (\exists x. Q(x)) \rangle$
 $\langle proof \rangle$

Some slow ones

lemma $\langle (\forall x y. P(x) \longrightarrow Q(y)) \longleftrightarrow ((\exists x. P(x)) \longrightarrow (\forall y. Q(y))) \rangle$
 $\langle proof \rangle$

lemma $\langle (\exists x y. P(x) \wedge Q(x,y)) \longleftrightarrow (\exists x. P(x) \wedge (\exists y. Q(x,y))) \rangle$
 $\langle proof \rangle$

lemma $\langle (\exists y. \forall x. P(x) \longrightarrow Q(x,y)) \longrightarrow (\forall x. P(x) \longrightarrow (\exists y. Q(x,y))) \rangle$
 $\langle proof \rangle$

end

11 Classical Predicate Calculus Problems

theory *Classical*
imports *FOL*
begin

lemma $\langle (P \longrightarrow Q \vee R) \longrightarrow (P \longrightarrow Q) \vee (P \longrightarrow R) \rangle$
\langle proof \rangle

11.0.1 If and only if

lemma $\langle (P \longleftrightarrow Q) \longleftrightarrow (Q \longleftrightarrow P) \rangle$
\langle proof \rangle

lemma $\langle \neg (P \longleftrightarrow \neg P) \rangle$
\langle proof \rangle

11.1 Pelletier's examples

Sample problems from

- F. J. Pelletier, Seventy-Five Problems for Testing Automatic Theorem Provers, J. Automated Reasoning 2 (1986), 191-216. Errata, JAR 4 (1988), 236-236.

The hardest problems – judging by experience with several theorem provers, including matrix ones – are 34 and 43.

1

lemma $\langle (P \longrightarrow Q) \longleftrightarrow (\neg Q \longrightarrow \neg P) \rangle$
\langle proof \rangle

2

lemma $\langle \neg \neg P \longleftrightarrow P \rangle$
\langle proof \rangle

3

lemma $\langle \neg (P \longrightarrow Q) \longrightarrow (Q \longrightarrow P) \rangle$
\langle proof \rangle

4

lemma $\langle (\neg P \longrightarrow Q) \longleftrightarrow (\neg Q \longrightarrow P) \rangle$
\langle proof \rangle

5

lemma $\langle ((P \vee Q) \longrightarrow (P \vee R)) \longrightarrow (P \vee (Q \longrightarrow R)) \rangle$
\langle proof \rangle

6

lemma $\langle P \vee \neg P \rangle$
 $\langle proof \rangle$

7

lemma $\langle P \vee \neg \neg \neg P \rangle$
 $\langle proof \rangle$

8. Peirce's law

lemma $\langle (P \longrightarrow Q) \longrightarrow P \longrightarrow P \rangle$
 $\langle proof \rangle$

9

lemma $\langle ((P \vee Q) \wedge (\neg P \vee Q) \wedge (P \vee \neg Q)) \longrightarrow \neg (\neg P \vee \neg Q) \rangle$
 $\langle proof \rangle$

10

lemma $\langle (Q \longrightarrow R) \wedge (R \longrightarrow P \wedge Q) \wedge (P \longrightarrow Q \vee R) \longrightarrow (P \longleftrightarrow Q) \rangle$
 $\langle proof \rangle$

11. Proved in each direction (incorrectly, says Pelletier!!)

lemma $\langle P \longleftrightarrow P \rangle$
 $\langle proof \rangle$

12. "Dijkstra's law"

lemma $\langle ((P \longleftrightarrow Q) \longleftrightarrow R) \longleftrightarrow (P \longleftrightarrow (Q \longleftrightarrow R)) \rangle$
 $\langle proof \rangle$

13. Distributive law

lemma $\langle P \vee (Q \wedge R) \longleftrightarrow (P \vee Q) \wedge (P \vee R) \rangle$
 $\langle proof \rangle$

14

lemma $\langle (P \longleftrightarrow Q) \longleftrightarrow ((Q \vee \neg P) \wedge (\neg Q \vee P)) \rangle$
 $\langle proof \rangle$

15

lemma $\langle (P \longrightarrow Q) \longleftrightarrow (\neg P \vee Q) \rangle$
 $\langle proof \rangle$

16

lemma $\langle (P \longrightarrow Q) \vee (Q \longrightarrow P) \rangle$
 $\langle proof \rangle$

17

lemma $\langle ((P \wedge (Q \longrightarrow R)) \longrightarrow S) \longleftrightarrow ((\neg P \vee Q \vee S) \wedge (\neg P \vee \neg R \vee S)) \rangle$
 $\langle proof \rangle$

11.2 Classical Logic: examples with quantifiers

lemma $\langle (\forall x. P(x) \wedge Q(x)) \longleftrightarrow (\forall x. P(x)) \wedge (\forall x. Q(x)) \rangle$
 $\langle proof \rangle$

lemma $\langle (\exists x. P \longrightarrow Q(x)) \longleftrightarrow (P \longrightarrow (\exists x. Q(x))) \rangle$
 $\langle proof \rangle$

lemma $\langle (\exists x. P(x) \longrightarrow Q) \longleftrightarrow (\forall x. P(x)) \longrightarrow Q \rangle$
 $\langle proof \rangle$

lemma $\langle (\forall x. P(x)) \vee Q \longleftrightarrow (\forall x. P(x) \vee Q) \rangle$
 $\langle proof \rangle$

Discussed in Avron, Gentzen-Type Systems, Resolution and Tableaux, JAR 10 (265-281), 1993. Proof is trivial!

lemma $\langle \neg ((\exists x. \neg P(x)) \wedge ((\exists x. P(x)) \vee (\exists x. P(x) \wedge Q(x))) \wedge \neg (\exists x. P(x))) \rangle$
 $\langle proof \rangle$

11.3 Problems requiring quantifier duplication

Theorem B of Peter Andrews, Theorem Proving via General Matings, JACM 28 (1981).

lemma $\langle (\exists x. \forall y. P(x) \longleftrightarrow P(y)) \longrightarrow ((\exists x. P(x)) \longleftrightarrow (\forall y. P(y))) \rangle$
 $\langle proof \rangle$

Needs multiple instantiation of ALL.

lemma $\langle (\forall x. P(x) \longrightarrow P(f(x))) \wedge P(d) \longrightarrow P(f(f(f(d)))) \rangle$
 $\langle proof \rangle$

Needs double instantiation of the quantifier

lemma $\langle \exists x. P(x) \longrightarrow P(a) \wedge P(b) \rangle$
 $\langle proof \rangle$

lemma $\langle \exists z. P(z) \longrightarrow (\forall x. P(x)) \rangle$
 $\langle proof \rangle$

lemma $\langle \exists x. (\exists y. P(y)) \longrightarrow P(x) \rangle$
 $\langle proof \rangle$

V. Lifschitz, What Is the Inverse Method?, JAR 5 (1989), 1–23. NOT PROVED.

lemma
 $\langle \exists x x'. \forall y. \exists z z'.$
 $\quad (\neg P(y, y) \vee P(x, x) \vee \neg S(z, x)) \wedge$
 $\quad (S(x, y) \vee \neg S(y, z) \vee Q(z', z')) \wedge$
 $\quad (Q(x', y) \vee \neg Q(y, z') \vee S(x', x')) \rangle$
 $\langle proof \rangle$

11.4 Hard examples with quantifiers

18

lemma $\langle \exists y. \forall x. P(y) \longrightarrow P(x) \rangle$
 $\langle proof \rangle$

19

lemma $\langle \exists x. \forall y z. (P(y) \longrightarrow Q(z)) \longrightarrow (P(x) \longrightarrow Q(x)) \rangle$
 $\langle proof \rangle$

20

lemma $\langle (\forall x y. \exists z. \forall w. (P(x) \wedge Q(y) \longrightarrow R(z) \wedge S(w)))$
 $\longrightarrow (\exists x y. P(x) \wedge Q(y)) \longrightarrow (\exists z. R(z)) \rangle$
 $\langle proof \rangle$

21

lemma $\langle (\exists x. P \longrightarrow Q(x)) \wedge (\exists x. Q(x) \longrightarrow P) \longrightarrow (\exists x. P \longleftrightarrow Q(x)) \rangle$
 $\langle proof \rangle$

22

lemma $\langle (\forall x. P \longleftrightarrow Q(x)) \longrightarrow (P \longleftrightarrow (\forall x. Q(x))) \rangle$
 $\langle proof \rangle$

23

lemma $\langle (\forall x. P \vee Q(x)) \longleftrightarrow (P \vee (\forall x. Q(x))) \rangle$
 $\langle proof \rangle$

24

lemma
 $\langle \neg (\exists x. S(x) \wedge Q(x)) \wedge (\forall x. P(x) \longrightarrow Q(x) \vee R(x)) \wedge$
 $(\neg (\exists x. P(x)) \longrightarrow (\exists x. Q(x))) \wedge (\forall x. Q(x) \vee R(x) \longrightarrow S(x))$
 $\longrightarrow (\exists x. P(x) \wedge R(x)) \rangle$
 $\langle proof \rangle$

25

lemma
 $\langle (\exists x. P(x)) \wedge$
 $(\forall x. L(x) \longrightarrow \neg (M(x) \wedge R(x))) \wedge$
 $(\forall x. P(x) \longrightarrow (M(x) \wedge L(x))) \wedge$
 $((\forall x. P(x) \longrightarrow Q(x)) \vee (\exists x. P(x) \wedge R(x)))$
 $\longrightarrow (\exists x. Q(x) \wedge P(x)) \rangle$
 $\langle proof \rangle$

26

lemma
 $\langle ((\exists x. p(x)) \longleftrightarrow (\exists x. q(x))) \wedge$
 $(\forall x. \forall y. p(x) \wedge q(y) \longrightarrow (r(x) \longleftrightarrow s(y))) \rangle$

$$\longrightarrow ((\forall x. p(x) \longrightarrow r(x)) \longleftrightarrow (\forall x. q(x) \longrightarrow s(x))) \rangle$$

<proof>

27

lemma

$$\begin{aligned} &\langle (\exists x. P(x) \wedge \neg Q(x)) \wedge \\ &\quad (\forall x. P(x) \longrightarrow R(x)) \wedge \\ &\quad (\forall x. M(x) \wedge L(x) \longrightarrow P(x)) \wedge \\ &\quad ((\exists x. R(x) \wedge \neg Q(x)) \longrightarrow (\forall x. L(x) \longrightarrow \neg R(x))) \\ &\longrightarrow (\forall x. M(x) \longrightarrow \neg L(x)) \rangle \\ &\langle proof \rangle \end{aligned}$$

28. AMENDED

lemma

$$\begin{aligned} &\langle (\forall x. P(x) \longrightarrow (\forall x. Q(x))) \wedge \\ &\quad ((\forall x. Q(x) \vee R(x)) \longrightarrow (\exists x. Q(x) \wedge S(x))) \wedge \\ &\quad ((\exists x. S(x)) \longrightarrow (\forall x. L(x) \longrightarrow M(x))) \\ &\longrightarrow (\forall x. P(x) \wedge L(x) \longrightarrow M(x)) \rangle \\ &\langle proof \rangle \end{aligned}$$

29. Essentially the same as Principia Mathematica *11.71

lemma

$$\begin{aligned} &\langle (\exists x. P(x)) \wedge (\exists y. Q(y)) \\ &\longrightarrow ((\forall x. P(x) \longrightarrow R(x)) \wedge (\forall y. Q(y) \longrightarrow S(y)) \longleftrightarrow \\ &\quad (\forall x y. P(x) \wedge Q(y) \longrightarrow R(x) \wedge S(y))) \rangle \\ &\langle proof \rangle \end{aligned}$$

30

lemma

$$\begin{aligned} &\langle (\forall x. P(x) \vee Q(x) \longrightarrow \neg R(x)) \wedge \\ &\quad (\forall x. (Q(x) \longrightarrow \neg S(x)) \longrightarrow P(x) \wedge R(x)) \\ &\longrightarrow (\forall x. S(x)) \rangle \\ &\langle proof \rangle \end{aligned}$$

31

lemma

$$\begin{aligned} &\langle \neg (\exists x. P(x) \wedge (Q(x) \vee R(x))) \wedge \\ &\quad (\exists x. L(x) \wedge P(x)) \wedge \\ &\quad (\forall x. \neg R(x) \longrightarrow M(x)) \\ &\longrightarrow (\exists x. L(x) \wedge M(x)) \rangle \\ &\langle proof \rangle \end{aligned}$$

32

lemma

$$\begin{aligned} &\langle (\forall x. P(x) \wedge (Q(x) \vee R(x)) \longrightarrow S(x)) \wedge \\ &\quad (\forall x. S(x) \wedge R(x) \longrightarrow L(x)) \wedge \\ &\quad (\forall x. M(x) \longrightarrow R(x)) \\ &\longrightarrow (\forall x. P(x) \wedge M(x) \longrightarrow L(x)) \rangle \end{aligned}$$

$\langle proof \rangle$

33

lemma

$\langle (\forall x. P(a) \wedge (P(x) \longrightarrow P(b)) \longrightarrow P(c)) \longleftrightarrow$
 $(\forall x. (\neg P(a) \vee P(x) \vee P(c)) \wedge (\neg P(a) \vee \neg P(b) \vee P(c))) \rangle$
 $\langle proof \rangle$

34. AMENDED (TWICE!!). Andrews's challenge.

lemma

$\langle ((\exists x. \forall y. p(x) \longleftrightarrow p(y)) \longleftrightarrow ((\exists x. q(x)) \longleftrightarrow (\forall y. p(y)))) \longleftrightarrow$
 $((\exists x. \forall y. q(x) \longleftrightarrow q(y)) \longleftrightarrow ((\exists x. p(x)) \longleftrightarrow (\forall y. q(y)))) \rangle$
 $\langle proof \rangle$

35

lemma $\langle \exists x y. P(x,y) \longrightarrow (\forall u v. P(u,v)) \rangle$
 $\langle proof \rangle$

36

lemma

$\langle (\forall x. \exists y. J(x,y)) \wedge$
 $(\forall x. \exists y. G(x,y)) \wedge$
 $(\forall x y. J(x,y) \vee G(x,y) \longrightarrow (\forall z. J(y,z) \vee G(y,z) \longrightarrow H(x,z)))$
 $\longrightarrow (\forall x. \exists y. H(x,y)) \rangle$
 $\langle proof \rangle$

37

lemma

$\langle (\forall z. \exists w. \forall x. \exists y.$
 $(P(x,z) \longrightarrow P(y,w)) \wedge P(y,z) \wedge (P(y,w) \longrightarrow (\exists u. Q(u,w)))) \wedge$
 $(\forall x z. \neg P(x,z) \longrightarrow (\exists y. Q(y,z))) \wedge$
 $((\exists x y. Q(x,y)) \longrightarrow (\forall x. R(x,x)))$
 $\longrightarrow (\forall x. \exists y. R(x,y)) \rangle$
 $\langle proof \rangle$

38

lemma

$\langle (\forall x. p(a) \wedge (p(x) \longrightarrow (\exists y. p(y) \wedge r(x,y))) \longrightarrow$
 $(\exists z. \exists w. p(z) \wedge r(x,w) \wedge r(w,z))) \longleftrightarrow$
 $(\forall x. (\neg p(a) \vee p(x) \vee (\exists z. \exists w. p(z) \wedge r(x,w) \wedge r(w,z))) \wedge$
 $(\neg p(a) \vee \neg (\exists y. p(y) \wedge r(x,y)) \vee$
 $(\exists z. \exists w. p(z) \wedge r(x,w) \wedge r(w,z)))) \rangle$
 $\langle proof \rangle$

39

lemma $\langle \neg (\exists x. \forall y. F(y,x) \longleftrightarrow \neg F(y,y)) \rangle$
 $\langle proof \rangle$

40. AMENDED

lemma

$$\langle (\exists y. \forall x. F(x,y) \longleftrightarrow F(x,x)) \longrightarrow \\ \neg (\forall x. \exists y. \forall z. F(z,y) \longleftrightarrow \neg F(z,x)) \rangle \\ \langle proof \rangle$$

41

lemma

$$\langle (\forall z. \exists y. \forall x. f(x,y) \longleftrightarrow f(x,z) \wedge \neg f(x,x)) \\ \longrightarrow \neg (\exists z. \forall x. f(x,z)) \rangle \\ \langle proof \rangle$$

42

lemma $\langle \neg (\exists y. \forall x. p(x,y) \longleftrightarrow \neg (\exists z. p(x,z) \wedge p(z,x))) \rangle$
 $\langle proof \rangle$

43

lemma

$$\langle (\forall x. \forall y. q(x,y) \longleftrightarrow (\forall z. p(z,x) \longleftrightarrow p(z,y))) \\ \longrightarrow (\forall x. \forall y. q(x,y) \longleftrightarrow q(y,x)) \rangle \\ \langle proof \rangle$$

Other proofs: Can use *auto*, which cheats by using rewriting! *Deepen-tac* alone requires 253 secs. Or *by (mini-tac 1 THEN Deepen-tac 5 1)*.

44

lemma

$$\langle (\forall x. f(x) \longrightarrow (\exists y. g(y) \wedge h(x,y) \wedge (\exists y. g(y) \wedge \neg h(x,y)))) \wedge \\ (\exists x. j(x) \wedge (\forall y. g(y) \longrightarrow h(x,y))) \\ \longrightarrow (\exists x. j(x) \wedge \neg f(x)) \rangle \\ \langle proof \rangle$$

45

lemma

$$\langle (\forall x. f(x) \wedge (\forall y. g(y) \wedge h(x,y) \longrightarrow j(x,y)) \\ \longrightarrow (\forall y. g(y) \wedge h(x,y) \longrightarrow k(y))) \wedge \\ \neg (\exists y. l(y) \wedge k(y)) \wedge \\ (\exists x. f(x) \wedge (\forall y. h(x,y) \longrightarrow l(y)) \wedge (\forall y. g(y) \wedge h(x,y) \longrightarrow j(x,y))) \\ \longrightarrow (\exists x. f(x) \wedge \neg (\exists y. g(y) \wedge h(x,y))) \rangle \\ \langle proof \rangle$$

46

lemma

$$\langle (\forall x. f(x) \wedge (\forall y. f(y) \wedge h(y,x) \longrightarrow g(y)) \longrightarrow g(x)) \wedge \\ ((\exists x. f(x) \wedge \neg g(x)) \longrightarrow \\ (\exists x. f(x) \wedge \neg g(x) \wedge (\forall y. f(y) \wedge \neg g(y) \longrightarrow j(x,y)))) \wedge \\ (\forall x y. f(x) \wedge f(y) \wedge h(x,y) \longrightarrow \neg j(y,x)) \\ \longrightarrow (\forall x. f(x) \longrightarrow g(x)) \rangle \\ \langle proof \rangle$$

11.5 Problems (mainly) involving equality or functions

48

lemma $\langle (a = b \vee c = d) \wedge (a = c \vee b = d) \longrightarrow a = d \vee b = c \rangle$
 $\langle \text{proof} \rangle$

49. NOT PROVED AUTOMATICALLY. Hard because it involves substitution for Vars; the type constraint ensures that x,y,z have the same type as a,b,u.

lemma

$\langle (\exists x y :: 'a. \forall z. z = x \vee z = y) \wedge P(a) \wedge P(b) \wedge a \neq b \longrightarrow (\forall u :: 'a. P(u)) \rangle$
 $\langle \text{proof} \rangle$

50. (What has this to do with equality?)

lemma $\langle (\forall x. P(a,x) \vee (\forall y. P(x,y))) \longrightarrow (\exists x. \forall y. P(x,y)) \rangle$
 $\langle \text{proof} \rangle$

51

lemma

$\langle (\exists z w. \forall x y. P(x,y) \longleftrightarrow (x = z \wedge y = w)) \longrightarrow$
 $(\exists z. \forall x. \exists w. (\forall y. P(x,y) \longleftrightarrow y = w) \longleftrightarrow x = z) \rangle$
 $\langle \text{proof} \rangle$

52

Almost the same as 51.

lemma

$\langle (\exists z w. \forall x y. P(x,y) \longleftrightarrow (x = z \wedge y = w)) \longrightarrow$
 $(\exists w. \forall y. \exists z. (\forall x. P(x,y) \longleftrightarrow x = z) \longleftrightarrow y = w) \rangle$
 $\langle \text{proof} \rangle$

55

Non-equational version, from Manthey and Bry, CADE-9 (Springer, 1988).
fast DISCOVERS who killed Agatha.

schematic-goal

$\langle \text{lives}(\text{agatha}) \wedge \text{lives}(\text{butler}) \wedge \text{lives}(\text{charles}) \wedge$
 $(\text{killed}(\text{agatha}, \text{agatha}) \vee \text{killed}(\text{butler}, \text{agatha}) \vee \text{killed}(\text{charles}, \text{agatha})) \wedge$
 $(\forall x y. \text{killed}(x, y) \longrightarrow \text{hates}(x, y) \wedge \neg \text{richer}(x, y)) \wedge$
 $(\forall x. \text{hates}(\text{agatha}, x) \longrightarrow \neg \text{hates}(\text{charles}, x)) \wedge$
 $(\text{hates}(\text{agatha}, \text{agatha}) \wedge \text{hates}(\text{agatha}, \text{charles})) \wedge$
 $(\forall x. \text{lives}(x) \wedge \neg \text{richer}(x, \text{agatha}) \longrightarrow \text{hates}(\text{butler}, x)) \wedge$
 $(\forall x. \text{hates}(\text{agatha}, x) \longrightarrow \text{hates}(\text{butler}, x)) \wedge$
 $(\forall x. \neg \text{hates}(x, \text{agatha}) \vee \neg \text{hates}(x, \text{butler}) \vee \neg \text{hates}(x, \text{charles})) \longrightarrow$
 $\text{killed}(\text{?who}, \text{agatha}) \rangle$
 $\langle \text{proof} \rangle$

56

lemma $\langle (\forall x. (\exists y. P(y) \wedge x = f(y)) \longrightarrow P(x)) \longleftrightarrow (\forall x. P(x) \longrightarrow P(f(x))) \rangle$
 $\langle proof \rangle$

57

lemma
 $\langle P(f(a,b), f(b,c)) \wedge P(f(b,c), f(a,c)) \wedge$
 $(\forall x y z. P(x,y) \wedge P(y,z) \longrightarrow P(x,z)) \longrightarrow P(f(a,b), f(a,c)) \rangle$
 $\langle proof \rangle$

58 NOT PROVED AUTOMATICALLY

lemma $\langle (\forall x y. f(x) = g(y)) \longrightarrow (\forall x y. f(f(x)) = f(g(y))) \rangle$
 $\langle proof \rangle$

59

lemma $\langle (\forall x. P(x) \longleftrightarrow \neg P(f(x))) \longrightarrow (\exists x. P(x) \wedge \neg P(f(x))) \rangle$
 $\langle proof \rangle$

60

lemma $\langle \forall x. P(x, f(x)) \longleftrightarrow (\exists y. (\forall z. P(z, y) \longrightarrow P(z, f(x))) \wedge P(x, y)) \rangle$
 $\langle proof \rangle$

62 as corrected in JAR 18 (1997), page 135

lemma
 $\langle (\forall x. p(a) \wedge (p(x) \longrightarrow p(f(x))) \longrightarrow p(f(f(x)))) \longleftrightarrow$
 $(\forall x. (\neg p(a) \vee p(x) \vee p(f(f(x)))) \wedge$
 $(\neg p(a) \vee \neg p(f(x)) \vee p(f(f(x)))) \rangle$
 $\langle proof \rangle$

From Davis, Obvious Logical Inferences, IJCAI-81, 530-531 fast indeed copes!

lemma
 $\langle (\forall x. F(x) \wedge \neg G(x) \longrightarrow (\exists y. H(x,y) \wedge J(y))) \wedge$
 $(\exists x. K(x) \wedge F(x) \wedge (\forall y. H(x,y) \longrightarrow K(y))) \wedge$
 $(\forall x. K(x) \longrightarrow \neg G(x)) \longrightarrow (\exists x. K(x) \wedge J(x)) \rangle$
 $\langle proof \rangle$

From Rudnicki, Obvious Inferences, JAR 3 (1987), 383-393. It does seem obvious!

lemma
 $\langle (\forall x. F(x) \wedge \neg G(x) \longrightarrow (\exists y. H(x,y) \wedge J(y))) \wedge$
 $(\exists x. K(x) \wedge F(x) \wedge (\forall y. H(x,y) \longrightarrow K(y))) \wedge$
 $(\forall x. K(x) \longrightarrow \neg G(x)) \longrightarrow (\exists x. K(x) \longrightarrow \neg G(x)) \rangle$
 $\langle proof \rangle$

Halting problem: Formulation of Li Dafa (AAR Newsletter 27, Oct 1994.)
author U. Egly.

lemma
 $\langle ((\exists x. A(x) \wedge (\forall y. C(y) \longrightarrow (\forall z. D(x,y,z)))) \longrightarrow$

$$\begin{aligned}
& (\exists w. C(w) \wedge (\forall y. C(y) \longrightarrow (\forall z. D(w,y,z)))) \\
& \wedge \\
& (\forall w. C(w) \wedge (\forall u. C(u) \longrightarrow (\forall v. D(w,u,v))) \longrightarrow \\
& \quad (\forall y z. \\
& \quad \quad (C(y) \wedge P(y,z) \longrightarrow Q(w,y,z) \wedge OO(w,g)) \wedge \\
& \quad \quad (C(y) \wedge \neg P(y,z) \longrightarrow Q(w,y,z) \wedge OO(w,b)))) \\
& \wedge \\
& (\forall w. C(w) \wedge \\
& \quad (\forall y z. \\
& \quad \quad (C(y) \wedge P(y,z) \longrightarrow Q(w,y,z) \wedge OO(w,g)) \wedge \\
& \quad \quad (C(y) \wedge \neg P(y,z) \longrightarrow Q(w,y,z) \wedge OO(w,b))) \longrightarrow \\
& \quad (\exists v. C(v) \wedge \\
& \quad \quad (\forall y. ((C(y) \wedge Q(w,y,y)) \wedge OO(w,g) \longrightarrow \neg P(v,y)) \wedge \\
& \quad \quad ((C(y) \wedge Q(w,y,y)) \wedge OO(w,b) \longrightarrow P(v,y) \wedge OO(v,b)))))) \\
& \longrightarrow \neg (\exists x. A(x) \wedge (\forall y. C(y) \longrightarrow (\forall z. D(x,y,z)))) \rangle \\
& \langle proof \rangle
\end{aligned}$$

Halting problem II: credited to M. Bruschi by Li Dafa in JAR 18(1), p. 105.

lemma

$$\begin{aligned}
& \langle ((\exists x. A(x) \wedge (\forall y. C(y) \longrightarrow (\forall z. D(x,y,z)))) \longrightarrow \\
& \quad (\exists w. C(w) \wedge (\forall y. C(y) \longrightarrow (\forall z. D(w,y,z)))))) \\
& \wedge \\
& (\forall w. C(w) \wedge (\forall u. C(u) \longrightarrow (\forall v. D(w,u,v))) \longrightarrow \\
& \quad (\forall y z. \\
& \quad \quad (C(y) \wedge P(y,z) \longrightarrow Q(w,y,z) \wedge OO(w,g)) \wedge \\
& \quad \quad (C(y) \wedge \neg P(y,z) \longrightarrow Q(w,y,z) \wedge OO(w,b)))) \\
& \wedge \\
& ((\exists w. C(w) \wedge (\forall y. (C(y) \wedge P(y,y) \longrightarrow Q(w,y,y) \wedge OO(w,g)) \wedge \\
& \quad (C(y) \wedge \neg P(y,y) \longrightarrow Q(w,y,y) \wedge OO(w,b)))) \\
& \longrightarrow \\
& \quad (\exists v. C(v) \wedge (\forall y. (C(y) \wedge P(y,y) \longrightarrow P(v,y) \wedge OO(v,g)) \wedge \\
& \quad \quad (C(y) \wedge \neg P(y,y) \longrightarrow P(v,y) \wedge OO(v,b)))))) \\
& \longrightarrow \\
& ((\exists v. C(v) \wedge (\forall y. (C(y) \wedge P(y,y) \longrightarrow P(v,y) \wedge OO(v,g)) \wedge \\
& \quad (C(y) \wedge \neg P(y,y) \longrightarrow P(v,y) \wedge OO(v,b)))) \\
& \longrightarrow \\
& \quad (\exists u. C(u) \wedge (\forall y. (C(y) \wedge P(y,y) \longrightarrow \neg P(u,y)) \wedge \\
& \quad \quad (C(y) \wedge \neg P(y,y) \longrightarrow P(u,y) \wedge OO(u,b)))))) \\
& \longrightarrow \neg (\exists x. A(x) \wedge (\forall y. C(y) \longrightarrow (\forall z. D(x,y,z)))) \rangle \\
& \langle proof \rangle
\end{aligned}$$

Challenge found on info-hol.

lemma $\langle \forall x. \exists v w. \forall y z. P(x) \wedge Q(y) \longrightarrow (P(v) \vee R(w)) \wedge (R(z) \longrightarrow Q(v)) \rangle$
 $\langle proof \rangle$

Attributed to Lewis Carroll by S. G. Pulman. The first or last assumption can be deleted.

lemma

$$\langle (\forall x. \text{honest}(x) \wedge \text{industrious}(x) \longrightarrow \text{healthy}(x)) \wedge$$

$\neg (\exists x. \text{grocer}(x) \wedge \text{healthy}(x)) \wedge$
 $(\forall x. \text{industrious}(x) \wedge \text{grocer}(x) \longrightarrow \text{honest}(x)) \wedge$
 $(\forall x. \text{cyclist}(x) \longrightarrow \text{industrious}(x)) \wedge$
 $(\forall x. \neg \text{healthy}(x) \wedge \text{cyclist}(x) \longrightarrow \neg \text{honest}(x))$
 $\longrightarrow (\forall x. \text{grocer}(x) \longrightarrow \neg \text{cyclist}(x))\rangle$
 $\langle \text{proof} \rangle$

end

12 First-Order Logic: propositional examples (classical version)

theory *Propositional-Cla*
imports *FOL*
begin

commutative laws of \wedge and \vee

lemma $\langle P \wedge Q \longrightarrow Q \wedge P \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle P \vee Q \longrightarrow Q \vee P \rangle$
 $\langle \text{proof} \rangle$

associative laws of \wedge and \vee

lemma $\langle (P \wedge Q) \wedge R \longrightarrow P \wedge (Q \wedge R) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (P \vee Q) \vee R \longrightarrow P \vee (Q \vee R) \rangle$
 $\langle \text{proof} \rangle$

distributive laws of \wedge and \vee

lemma $\langle (P \wedge Q) \vee R \longrightarrow (P \vee R) \wedge (Q \vee R) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (P \vee R) \wedge (Q \vee R) \longrightarrow (P \wedge Q) \vee R \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (P \vee Q) \wedge R \longrightarrow (P \wedge R) \vee (Q \wedge R) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (P \wedge R) \vee (Q \wedge R) \longrightarrow (P \vee Q) \wedge R \rangle$
 $\langle \text{proof} \rangle$

Laws involving implication

lemma $\langle (P \longrightarrow R) \wedge (Q \longrightarrow R) \longleftrightarrow (P \vee Q \longrightarrow R) \rangle$

$\langle \text{proof} \rangle$

lemma $\langle (P \wedge Q \longrightarrow R) \longleftrightarrow (P \longrightarrow (Q \longrightarrow R)) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle ((P \longrightarrow R) \longrightarrow R) \longrightarrow ((Q \longrightarrow R) \longrightarrow R) \longrightarrow (P \wedge Q \longrightarrow R) \longrightarrow R \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle \neg (P \longrightarrow R) \longrightarrow \neg (Q \longrightarrow R) \longrightarrow \neg (P \wedge Q \longrightarrow R) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (P \longrightarrow Q \wedge R) \longleftrightarrow (P \longrightarrow Q) \wedge (P \longrightarrow R) \rangle$
 $\langle \text{proof} \rangle$

Propositions-as-types

lemma $\langle P \longrightarrow (Q \longrightarrow P) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (P \longrightarrow Q \longrightarrow R) \longrightarrow (P \longrightarrow Q) \longrightarrow (P \longrightarrow R) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (P \longrightarrow Q) \vee (P \longrightarrow R) \longrightarrow (P \longrightarrow Q \vee R) \rangle$
 $\langle \text{proof} \rangle$

lemma $\langle (P \longrightarrow Q) \longrightarrow (\neg Q \longrightarrow \neg P) \rangle$
 $\langle \text{proof} \rangle$

Schwichtenberg's examples (via T. Nipkow)

lemma *stab-imp*: $\langle (((Q \longrightarrow R) \longrightarrow R) \longrightarrow Q) \longrightarrow (((P \longrightarrow Q) \longrightarrow R) \longrightarrow R) \longrightarrow P \longrightarrow Q \rangle$
 $\langle \text{proof} \rangle$

lemma *stab-to-peirce*:
 $\langle (((P \longrightarrow R) \longrightarrow R) \longrightarrow P) \longrightarrow (((Q \longrightarrow R) \longrightarrow R) \longrightarrow Q) \longrightarrow ((P \longrightarrow Q) \longrightarrow P) \longrightarrow P \rangle$
 $\langle \text{proof} \rangle$

lemma *peirce-imp1*:
 $\langle (((Q \longrightarrow R) \longrightarrow Q) \longrightarrow Q) \longrightarrow (((P \longrightarrow Q) \longrightarrow R) \longrightarrow P \longrightarrow Q) \longrightarrow P \longrightarrow Q \rangle$
 $\langle \text{proof} \rangle$

lemma *peirce-imp2*: $\langle (((P \longrightarrow R) \longrightarrow P) \longrightarrow P) \longrightarrow ((P \longrightarrow Q \longrightarrow R) \longrightarrow P) \longrightarrow P \rangle$
 $\langle \text{proof} \rangle$

lemma *mits*: $\langle (((P \longrightarrow Q) \longrightarrow P) \longrightarrow P) \longrightarrow Q \longrightarrow Q \rangle$
 $\langle \text{proof} \rangle$

lemma *mits-solovev*: $\langle (P \longrightarrow (Q \longrightarrow R) \longrightarrow Q) \longrightarrow ((P \longrightarrow Q) \longrightarrow R) \longrightarrow R \rangle$
 $\langle \text{proof} \rangle$

lemma tatsuta:

$\langle (((P7 \rightarrow P1) \rightarrow P10) \rightarrow P4 \rightarrow P5)$
 $\rightarrow (((P8 \rightarrow P2) \rightarrow P9) \rightarrow P3 \rightarrow P10)$
 $\rightarrow (P1 \rightarrow P8) \rightarrow P6 \rightarrow P7$
 $\rightarrow (((P3 \rightarrow P2) \rightarrow P9) \rightarrow P4)$
 $\rightarrow (P1 \rightarrow P3) \rightarrow (((P6 \rightarrow P1) \rightarrow P2) \rightarrow P9) \rightarrow P5 \rangle$
 $\langle proof \rangle$

lemma tatsuta1:

$\langle (((P8 \rightarrow P2) \rightarrow P9) \rightarrow P3 \rightarrow P10)$
 $\rightarrow (((P3 \rightarrow P2) \rightarrow P9) \rightarrow P4)$
 $\rightarrow (((P6 \rightarrow P1) \rightarrow P2) \rightarrow P9)$
 $\rightarrow (((P7 \rightarrow P1) \rightarrow P10) \rightarrow P4 \rightarrow P5)$
 $\rightarrow (P1 \rightarrow P3) \rightarrow (P1 \rightarrow P8) \rightarrow P6 \rightarrow P7 \rightarrow P5 \rangle$
 $\langle proof \rangle$

end

13 First-Order Logic: quantifier examples (classical version)

theory *Quantifiers-Cla*

imports *FOL*

begin

lemma $\langle (\forall x y. P(x,y)) \rightarrow (\forall y x. P(x,y)) \rangle$
 $\langle proof \rangle$

lemma $\langle (\exists x y. P(x,y)) \rightarrow (\exists y x. P(x,y)) \rangle$
 $\langle proof \rangle$

Converse is false.

lemma $\langle (\forall x. P(x)) \vee (\forall x. Q(x)) \rightarrow (\forall x. P(x) \vee Q(x)) \rangle$
 $\langle proof \rangle$

lemma $\langle (\forall x. P \rightarrow Q(x)) \longleftrightarrow (P \rightarrow (\forall x. Q(x))) \rangle$
 $\langle proof \rangle$

lemma $\langle (\forall x. P(x) \rightarrow Q) \longleftrightarrow ((\exists x. P(x)) \rightarrow Q) \rangle$
 $\langle proof \rangle$

Some harder ones.

lemma $\langle (\exists x. P(x) \vee Q(x)) \longleftrightarrow (\exists x. P(x)) \vee (\exists x. Q(x)) \rangle$
 $\langle proof \rangle$

lemma $\langle (\exists x. P(x) \wedge Q(x)) \rightarrow (\exists x. P(x)) \wedge (\exists x. Q(x)) \rangle$
 $\langle proof \rangle$

Basic test of quantifier reasoning.

lemma $\langle (\exists y. \forall x. Q(x,y)) \longrightarrow (\forall x. \exists y. Q(x,y)) \rangle$
 $\langle proof \rangle$

lemma $\langle (\forall x. Q(x)) \longrightarrow (\exists x. Q(x)) \rangle$
 $\langle proof \rangle$

The following should fail, as they are false!

lemma $\langle (\forall x. \exists y. Q(x,y)) \longrightarrow (\exists y. \forall x. Q(x,y)) \rangle$
 $\langle proof \rangle$

lemma $\langle (\exists x. Q(x)) \longrightarrow (\forall x. Q(x)) \rangle$
 $\langle proof \rangle$

schematic-goal $\langle P(?a) \longrightarrow (\forall x. P(x)) \rangle$
 $\langle proof \rangle$

schematic-goal $\langle (P(?a) \longrightarrow (\forall x. Q(x))) \longrightarrow (\forall x. P(x) \longrightarrow Q(x)) \rangle$
 $\langle proof \rangle$

Back to things that are provable ...

lemma $\langle (\forall x. P(x) \longrightarrow Q(x)) \wedge (\exists x. P(x)) \longrightarrow (\exists x. Q(x)) \rangle$
 $\langle proof \rangle$

An example of why *exI* should be delayed as long as possible.

lemma $\langle (P \longrightarrow (\exists x. Q(x))) \wedge P \longrightarrow (\exists x. Q(x)) \rangle$
 $\langle proof \rangle$

schematic-goal $\langle (\forall x. P(x) \longrightarrow Q(f(x))) \wedge (\forall x. Q(x) \longrightarrow R(g(x))) \wedge P(d) \longrightarrow R(?a) \rangle$
 $\langle proof \rangle$

lemma $\langle (\forall x. Q(x)) \longrightarrow (\exists x. Q(x)) \rangle$
 $\langle proof \rangle$

Some slow ones

Principia Mathematica *11.53

lemma $\langle (\forall x y. P(x) \longrightarrow Q(y)) \longleftrightarrow ((\exists x. P(x)) \longrightarrow (\forall y. Q(y))) \rangle$
 $\langle proof \rangle$

lemma $\langle (\exists x y. P(x) \wedge Q(x,y)) \longleftrightarrow (\exists x. P(x) \wedge (\exists y. Q(x,y))) \rangle$
 $\langle proof \rangle$

lemma $\langle (\exists y. \forall x. P(x) \longrightarrow Q(x,y)) \longrightarrow (\forall x. P(x) \longrightarrow (\exists y. Q(x,y))) \rangle$
 $\langle proof \rangle$

end

theory *Miniscope*
imports *FOL*
begin

lemmas *ccontr* = *FalseE* [*THEN classical*]

13.1 Negation Normal Form

13.1.1 de Morgan laws

lemma *demorgans1*:

$\langle \neg (P \wedge Q) \longleftrightarrow \neg P \vee \neg Q \rangle$
 $\langle \neg (P \vee Q) \longleftrightarrow \neg P \wedge \neg Q \rangle$
 $\langle \neg \neg P \longleftrightarrow P \rangle$
 $\langle \text{proof} \rangle$

lemma *demorgans2*:

$\langle \bigwedge P. \neg (\forall x. P(x)) \longleftrightarrow (\exists x. \neg P(x)) \rangle$
 $\langle \bigwedge P. \neg (\exists x. P(x)) \longleftrightarrow (\forall x. \neg P(x)) \rangle$
 $\langle \text{proof} \rangle$

lemmas *demorgans* = *demorgans1 demorgans2*

lemma *nnf-simps*:

$\langle (P \longrightarrow Q) \longleftrightarrow (\neg P \vee Q) \rangle$
 $\langle \neg (P \longrightarrow Q) \longleftrightarrow (P \wedge \neg Q) \rangle$
 $\langle (P \longleftrightarrow Q) \longleftrightarrow (\neg P \vee Q) \wedge (\neg Q \vee P) \rangle$
 $\langle \neg (P \longleftrightarrow Q) \longleftrightarrow (P \vee Q) \wedge (\neg P \vee \neg Q) \rangle$
 $\langle \text{proof} \rangle$

13.1.2 Pushing in the existential quantifiers

lemma *ex-simps*:

$\langle (\exists x. P) \longleftrightarrow P \rangle$
 $\langle \bigwedge P Q. (\exists x. P(x) \wedge Q) \longleftrightarrow (\exists x. P(x)) \wedge Q \rangle$
 $\langle \bigwedge P Q. (\exists x. P \wedge Q(x)) \longleftrightarrow P \wedge (\exists x. Q(x)) \rangle$
 $\langle \bigwedge P Q. (\exists x. P(x) \vee Q(x)) \longleftrightarrow (\exists x. P(x)) \vee (\exists x. Q(x)) \rangle$
 $\langle \bigwedge P Q. (\exists x. P(x) \vee Q) \longleftrightarrow (\exists x. P(x)) \vee Q \rangle$
 $\langle \bigwedge P Q. (\exists x. P \vee Q(x)) \longleftrightarrow P \vee (\exists x. Q(x)) \rangle$
 $\langle \text{proof} \rangle$

13.1.3 Pushing in the universal quantifiers

lemma *all-simps*:

```

⟨(∀ x. P) ⟷ P⟩
⟨∧ P Q. (∀ x. P(x) ∧ Q(x)) ⟷ (∀ x. P(x)) ∧ (∀ x. Q(x))⟩
⟨∧ P Q. (∀ x. P(x) ∧ Q) ⟷ (∀ x. P(x)) ∧ Q⟩
⟨∧ P Q. (∀ x. P ∧ Q(x)) ⟷ P ∧ (∀ x. Q(x))⟩
⟨∧ P Q. (∀ x. P(x) ∨ Q) ⟷ (∀ x. P(x)) ∨ Q⟩
⟨∧ P Q. (∀ x. P ∨ Q(x)) ⟷ P ∨ (∀ x. Q(x))⟩
⟨proof⟩

```

lemmas *mini-simps = demorgans nnf-simps ex-simps all-simps*

⟨ML⟩

end

14 First-Order Logic: the 'if' example

theory *If*
imports *FOL*
begin

definition *if* :: $\langle [o, o, o] \Rightarrow o \rangle$
where $\langle \text{if}(P, Q, R) \equiv P \wedge Q \vee \neg P \wedge R \rangle$

lemma *ifI*: $\langle \llbracket P \Rightarrow Q; \neg P \Rightarrow R \rrbracket \Rightarrow \text{if}(P, Q, R) \rangle$
 ⟨proof⟩

lemma *ifE*: $\langle \llbracket \text{if}(P, Q, R); \llbracket P; Q \rrbracket \Rightarrow S; \llbracket \neg P; R \rrbracket \Rightarrow S \rrbracket \Rightarrow S \rangle$
 ⟨proof⟩

lemma *if-commute*: $\langle \text{if}(P, \text{if}(Q, A, B), \text{if}(Q, C, D)) \longleftrightarrow \text{if}(Q, \text{if}(P, A, C), \text{if}(P, B, D)) \rangle$
 ⟨proof⟩

Trying again from the beginning in order to use *blast*

declare *ifI* [*intro!*]
declare *ifE* [*elim!*]

lemma *if-commute*: $\langle \text{if}(P, \text{if}(Q, A, B), \text{if}(Q, C, D)) \longleftrightarrow \text{if}(Q, \text{if}(P, A, C), \text{if}(P, B, D)) \rangle$
 ⟨proof⟩

lemma $\langle \text{if}(\text{if}(P, Q, R), A, B) \longleftrightarrow \text{if}(P, \text{if}(Q, A, B), \text{if}(R, A, B)) \rangle$
 ⟨proof⟩

Trying again from the beginning in order to prove from the definitions

lemma $\langle \text{if}(\text{if}(P, Q, R), A, B) \longleftrightarrow \text{if}(P, \text{if}(Q, A, B), \text{if}(R, A, B)) \rangle$
 ⟨proof⟩

An invalid formula. High-level rules permit a simpler diagnosis.

lemma $\langle \text{if}(\text{if}(P, Q, R), A, B) \longleftrightarrow \text{if}(P, \text{if}(Q, A, B), \text{if}(R, B, A)) \rangle$
 $\langle \text{proof} \rangle$

Trying again from the beginning in order to prove from the definitions.

lemma $\langle \text{if}(\text{if}(P, Q, R), A, B) \longleftrightarrow \text{if}(P, \text{if}(Q, A, B), \text{if}(R, B, A)) \rangle$
 $\langle \text{proof} \rangle$

end