Analysis

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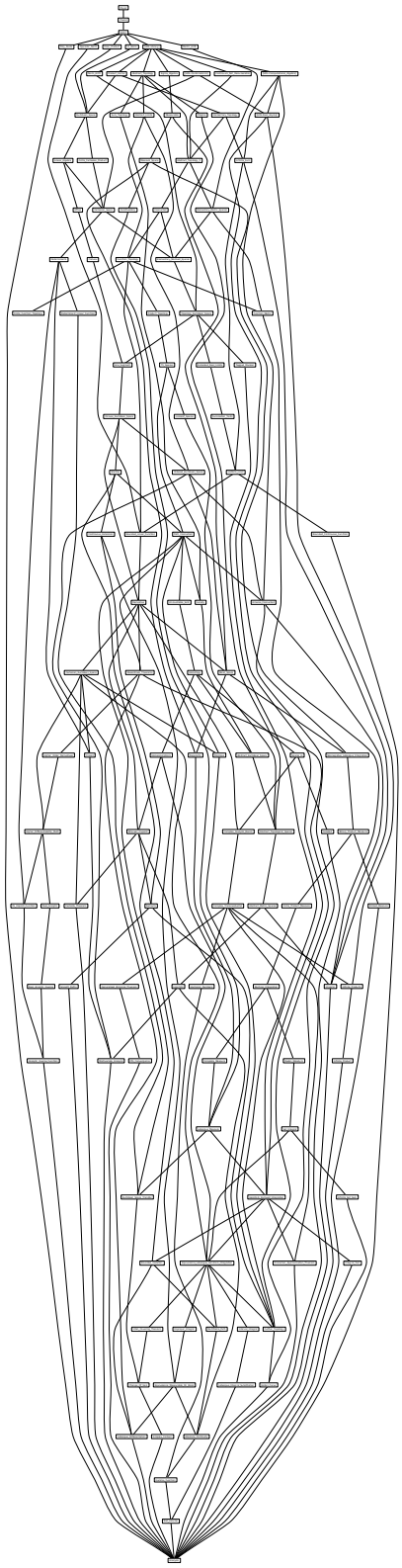
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Chapter 1

Linear Algebra

theory *L2_Norm*
imports *Complex_Main*
begin

1.1 L2 Norm

definition *L2_set* :: $('a \Rightarrow real) \Rightarrow 'a\ set \Rightarrow real$ **where**
$L2_set\ f\ A = sqrt\ (\sum i{\in}A.\ (f\ i)^2)$

proposition *L2_set_triangle_ineq*:
$L2_set\ (\lambda i.\ f\ i + g\ i)\ A \leq L2_set\ f\ A + L2_set\ g\ A$

end

1.2 Inner Product Spaces and Gradient Derivative

theory *Inner_Product*
imports *Complex_Main*
begin

1.2.1 Real inner product spaces

class *real_inner* = *real_vector* + *sgn_div_norm* + *dist_norm* + *uniformity_dist*
+ *open_uniformity* +
fixes *inner* :: $'a \Rightarrow 'a \Rightarrow real$
assumes *inner_commute*: $inner\ x\ y = inner\ y\ x$
and *inner_add_left*: $inner\ (x + y)\ z = inner\ x\ z + inner\ y\ z$
and *inner_scaleR_left* [*simp*]: $inner\ (scaleR\ r\ x)\ y = r * (inner\ x\ y)$
and *inner_ge_zero* [*simp*]: $0 \leq inner\ x\ x$
and *inner_eq_zero_iff* [*simp*]: $inner\ x\ x = 0 \longleftrightarrow x = 0$
and *norm_eq_sqrt_inner*: $norm\ x = sqrt\ (inner\ x\ x)$
begin

1.2.2 Class instances

instantiation *real* :: *real_inner*
begin

instantiation *complex* :: *real_inner*
begin

1.2.3 Gradient derivative

definition
gderiv :: ['*a*::*real_inner* ⇒ *real*, '*a*, '*a*] ⇒ *bool*
(‹(‹*notation*=‹*mixfix GDERIV*››*GDERIV* (_)/ (_)/ :> (_))› [*1000*, *1000*, *60*] *60*)
where
GDERIV f x :> *D* ⟷ *FDERIV f x* :> (λ*h*. *inner h D*)

end

1.3 Cartesian Products as Vector Spaces

theory *Product_Vector*
imports
Complex_Main
HOL−Library.Product_Plus
begin

1.3.1 Product is a Module

lemma *scale_prod*: *scale x* (*a*, *b*) = (*s1 x a*, *s2 x b*)

sublocale *p*: *module scale*

1.3.2 Product is a Real Vector Space

instantiation *prod* :: (*real_vector*, *real_vector*) *real_vector*
begin

proposition *scaleR_Pair* [*simp*]: *scaleR r* (*a*, *b*) = (*scaleR r a*, *scaleR r b*)

1.3.3 Product is a Metric Space

class *uniform_topological_monoid_add* = *topological_monoid_add* + *uniform_space*
+
assumes *uniformly_continuous_add'*:
filterlim (λ((*a*,*b*), (*c*,*d*)). (*a* + *c*, *b* + *d*)) *uniformity* (*uniformity* ×$_F$ *uniformity*)

class *uniform_topological_group_add* = *topological_group_add* + *uniform_topological_monoid_add*
+
assumes *uniformly_continuous_uminus'*: *filterlim* (λ(*a*, *b*). (−*a*, −*b*)) *uniformity*
uniformity
begin

instantiation *prod* :: (*metric_space*, *metric_space*) *metric_space*
begin

proposition *dist_Pair_Pair*: *dist* (*a*, *b*) (*c*, *d*) = *sqrt* ((*dist a c*)^2 + (*dist b d*)^2)

1.3.4 Product is a Complete Metric Space

instance *prod* :: (*complete_space*, *complete_space*) *complete_space*

1.3.5 Product is a Normed Vector Space

instantiation *prod* :: (*real_normed_vector*, *real_normed_vector*) *real_normed_vector*
begin

proposition *norm_Pair*: *norm* (*a*, *b*) = *sqrt* ((*norm a*)^2 + (*norm b*)^2)

instance *prod* :: (*banach*, *banach*) *banach*

proposition *has_derivative_Pair* [*derivative_intros*]:
assumes *f*: (*f has_derivative f'*) (*at x within s*)
and *g*: (*g has_derivative g'*) (*at x within s*)
shows ((λ*x*. (*f x*, *g x*)) *has_derivative* (λ*h*. (*f' h*, *g' h*))) (*at x within s*)

1.3.6 Product is Finite Dimensional

proposition *dim_Times*:
assumes *vs1.subspace S vs2.subspace T*
shows *p.dim*(*S* × *T*) = *vs1.dim S* + *vs2.dim T*

end

1.4 Finite-Dimensional Inner Product Spaces

theory *Euclidean_Space*
imports
L2_Norm
Inner_Product
Product_Vector
begin

1.4.1 Type class of Euclidean spaces

class *euclidean_space* = *real_inner* +
fixes *Basis* :: *'a set*
assumes *nonempty_Basis* [*simp*]: *Basis* ≠ {}
assumes *finite_Basis* [*simp*]: *finite Basis*
assumes *inner_Basis*:
⟦*u* ∈ *Basis*; *v* ∈ *Basis*⟧ ⟹ *inner u v* = (*if u* = *v then 1 else 0*)
assumes *euclidean_all_zero_iff*:
(∀ *u*∈*Basis*. *inner x u* = *0*) ⟷ (*x* = *0*)

1.4.2 Class instances

instantiation *real* :: *euclidean_space*
begin
instantiation *complex* :: *euclidean_space*
begin
instantiation *prod* :: (*real_inner*, *real_inner*) *real_inner*
begin

instantiation *prod* :: (*euclidean_space*, *euclidean_space*) *euclidean_space*
begin

1.4.3 Locale instances

end

1.5 Elementary Linear Algebra on Euclidean Spaces

theory *Linear_Algebra*
imports
Euclidean_Space
HOL−Library.Infinite_Set
begin

1.5.1 Substandard Basis

1.5.2 Orthogonality

definition (**in** *real_inner*) *orthogonal x y* $\longleftrightarrow$ $x \cdot y = 0$

1.5.3 Orthogonality of a transformation

definition *orthogonal_transformation f* $\longleftrightarrow$ *linear f* $\wedge$ $(\forall\, v\ w.\ f\ v \cdot f\ w = v \cdot w)$

1.5.4 Bilinear functions

definition
bilinear :: ('*a*::*real_vector* $\Rightarrow$ '*b*::*real_vector* $\Rightarrow$ '*c*::*real_vector*) $\Rightarrow$ *bool* **where**
bilinear f $\longleftrightarrow$ $(\forall\, x.\ linear\ (\lambda y.\ f\ x\ y)) \wedge (\forall\, y.\ linear\ (\lambda x.\ f\ x\ y))$

1.5.5 Adjoints

definition *adjoint* :: (('*a*::*real_inner*) $\Rightarrow$ ('*b*::*real_inner*)) $\Rightarrow$ '*b* $\Rightarrow$ '*a* **where**
adjoint f = $(SOME\ f'.\ \forall\, x\ y.\ f\ x \cdot y = x \cdot f'\ y)$

1.5.6 Infinity norm

definition *infnorm* (*x*::'*a*::*euclidean_space*) = *Sup* $\{|x \cdot b|\ |b.\ b \in Basis\}$

1.5.7 Collinearity

definition *collinear* :: '*a*::*real_vector set* $\Rightarrow$ *bool*
where *collinear S* $\longleftrightarrow$ $(\exists\, u.\ \forall\, x \in S.\ \forall\ y \in S.\ \exists\, c.\ x - y = c *_R u)$

1.5.8 Properties of special hyperplanes

proposition *dim_hyperplane*:
fixes *a* :: '*a*::*euclidean_space*
assumes $a \neq 0$
shows *dim* $\{x.\ a \cdot x = 0\}$ = $DIM('a) - 1$

1.5.9 Orthogonal bases and Gram-Schmidt process

proposition *Gram_Schmidt_step*:
fixes *S* :: '*a*::*euclidean_space set*
assumes *S*: *pairwise orthogonal S* **and** *x*: $x \in span\ S$

shows *orthogonal x* $(a - (\sum b{\in}S.\ (b \cdot a\ /\ (b \cdot b))\ *_R\ b))$

proposition *orthogonal_extension*:
fixes S :: *'a*::*euclidean_space set*
assumes S: *pairwise orthogonal S*
obtains U **where** *pairwise orthogonal* $(S \cup U)$ *span* $(S \cup U) =$ *span* $(S \cup T)$

1.5.10 Decomposing a vector into parts in orthogonal subspaces

proposition *orthonormal_basis_subspace*:
fixes S :: *'a* :: *euclidean_space set*
assumes *subspace S*
obtains B **where** $B \subseteq S$ *pairwise orthogonal B*
and $\bigwedge x.\ x \in B \Longrightarrow$ *norm x = 1*
and *independent B card B = dim S span B = S*

proposition *dim_orthogonal_sum*:
fixes A :: *'a*::*euclidean_space set*
assumes $\bigwedge x\ y.\ [\![x \in A;\ y \in B]\!] \Longrightarrow x \cdot y = 0$
shows *dim*$(A \cup B) =$ *dim A* + *dim B*

1.5.11 Linear functions are (uniformly) continuous on any set

end

1.6 Affine Sets

theory *Affine*
imports *Linear_Algebra*
begin

1.6.1 Affine set and affine hull

definition *affine* :: *'a*::*real_vector set* ⇒ *bool*
where *affine S* $\longleftrightarrow (\forall\, x{\in}S.\ \forall\, y{\in}S.\ \forall\, u\ v.\ u + v = 1 \longrightarrow u *_R x + v *_R y \in S)$

1.6.2 Affine Dependence

definition *affine_dependent* :: *'a::real_vector set* ⇒ *bool*
where *affine_dependent S* ⟷ (∃ *x*∈*S*. *x* ∈ *affine hull* (*S* − {*x*}))

proposition *affine_dependent_explicit*:
affine_dependent p ⟷
(∃ *S U*. *finite S* ∧ *S* ⊆ *p* ∧ *sum U S* = *0* ∧ (∃ *v*∈*S*. *U v* ≠ *0*) ∧ *sum* (λ*v*. *U v* *$_R$ *v*) *S* = *0*)

proposition *extend_to_affine_basis*:
fixes *S V* :: *'n::real_vector set*
assumes ¬ *affine_dependent S S* ⊆ *V*
obtains *T* **where** ¬ *affine_dependent T S* ⊆ *T T* ⊆ *V affine hull T* = *affine hull V*

1.6.3 Affine Dimension of a Set

definition *aff_dim* :: (*'a::euclidean_space*) *set* ⇒ *int*
where *aff_dim V* =
(*SOME d* :: *int*.
∃ *B*. *affine hull B* = *affine hull V* ∧ ¬ *affine_dependent B* ∧ *of_nat* (*card B*) = *d* + *1*)

end

1.7 Convex Sets and Functions

theory *Convex*
imports
Affine *HOL−Library.Set_Algebras* *HOL−Library.FuncSet*
begin

1.7.1 Convex Sets

definition *convex* :: *'a::real_vector set* ⇒ *bool*
where *convex s* ⟷ (∀ *x*∈*s*. ∀ *y*∈*s*. ∀ *u*≥*0*. ∀ *v*≥*0*. *u* + *v* = *1* ⟶ *u* *$_R$ *x* + *v* *$_R$ *y* ∈ *s*)

1.7.2 Convex Functions on a Set

definition *convex_on* :: *'a::real_vector set* ⇒ (*'a* ⇒ *real*) ⇒ *bool*
where *convex_on S f* ⟷ *convex S* ∧
(∀ *x*∈*S*. ∀ *y*∈*S*. ∀ *u*≥*0*. ∀ *v*≥*0*. *u* + *v* = *1* ⟶ *f* (*u* *$_R$ *x* + *v* *$_R$ *y*) ≤ *u* ∗ *f x* + *v* ∗ *f y*)

definition *concave_on* :: *'a::real_vector set* ⇒ (*'a* ⇒ *real*) ⇒ *bool*
where *concave_on S f* ≡ *convex_on S* (λ*x*. − *f x*)

1.7.3 Convexity of the generalised binomial

1.7.4 Some inequalities: Applications of convexity

1.7.5 Misc related lemmas

1.7.6 Cones

definition *cone* :: *'a::real_vector set* ⇒ *bool*
where *cone s* ⟷ (∀ *x*∈*s*. ∀ *c*≥*0*. *c* ∗*R x* ∈ *s*)

proposition *cone_hull_expl*: *cone hull S* = {*c* ∗*R x* | *c x*. *c* ≥ *0* ∧ *x* ∈ *S*}
(**is** *?lhs* = *?rhs*)

1.7.7 Convex hull

proposition *convex_hull_indexed*:
fixes *S* :: *'a::real_vector set*
shows *convex hull S* =
{*y*. ∃ *k u x*. (∀ *i*∈{*1*::*nat* .. *k*}. *0* ≤ *u i* ∧ *x i* ∈ *S*) ∧
(*sum u* {*1*..*k*} = *1*) ∧ (∑ *i* = *1*..*k*. *u i* ∗*R x i*) = *y*}
(**is** *?xyz* = *?hull*)

1.7.8 Caratheodory's theorem

theorem *caratheodory*:
convex hull p =
{*x*::*'a*::*euclidean_space*. ∃ *S*. *finite S* ∧ *S* ⊆ *p* ∧ *card S* ≤ *DIM*(*'a*) + *1* ∧ *x* ∈ *convex hull S*}

1.8 Conic sets and conic hull

1.9 Convex cones and corresponding hulls

1.9.1 Radon's theorem

theorem *Radon*:
assumes *affine_dependent c*

obtains $M\ P$ **where** $M \subseteq c\ P \subseteq c\ M \cap P = \{\}$ $(\mathit{convex\ hull}\ M) \cap (\mathit{convex\ hull}\ P) \neq \{\}$

1.9.2 Helly's theorem

theorem *Helly*:
fixes $\mathcal{F} :: 'a{::}\mathit{euclidean_space\ set\ set}$
assumes $\mathit{card}\ \mathcal{F} \geq \mathit{DIM}('a) + 1\ \forall s{\in}\mathcal{F}.\ \mathit{convex}\ s$
and $\bigwedge t.\ [\![t{\subseteq}\mathcal{F};\ \mathit{card}\ t = \mathit{DIM}('a) + 1]\!] \Longrightarrow \bigcap t \neq \{\}$
shows $\bigcap \mathcal{F} \neq \{\}$

1.9.3 Epigraphs of convex functions

definition $\mathit{epigraph}\ S\ (f :: _ \Rightarrow \mathit{real}) = \{xy.\ \mathit{fst}\ xy \in S \wedge f\ (\mathit{fst}\ xy) \leq \mathit{snd}\ xy\}$

end

1.10 Definition of Finite Cartesian Product Type

theory *Finite_Cartesian_Product*
imports
Euclidean_Space
L2_Norm
HOL−Library.Numeral_Type
HOL−Library.Countable_Set
HOL−Library.FuncSet
begin

1.10.1 Cardinality of vectors

proposition *CARD_vec* [*simp*]:
$\mathit{CARD}('a\,\widehat{}\,'b) = \mathit{CARD}('a)\ \widehat{}\ \mathit{CARD}('b)$
instantiation *vec* :: (*zero*, *finite*) *zero*
begin

instantiation *vec* :: (*plus*, *finite*) *plus*
begin

instantiation *vec* :: (*minus*, *finite*) *minus*
begin

instantiation *vec* :: (*uminus*, *finite*) *uminus*
begin
instantiation *vec* :: (*times*, *finite*) *times*
begin

instantiation *vec* :: (*one*, *finite*) *one*
begin

instantiation *vec* :: (*ord*, *finite*) *ord*
begin

1.10.2 Real vector space

definition *scaleR* ≡ (λ *r x*. (χ *i*. *scaleR r* (*x*$*i*)))

1.10.3 Topological space

definition [*code drop*]:
open (*S* :: ('*a* ^ '*b*) *set*) ⟷
(∀ *x*∈*S*. ∃ *A*. (∀ *i*. *open* (*A i*) ∧ *x*$*i* ∈ *A i*) ∧
(∀ *y*. (∀ *i*. *y*$*i* ∈ *A i*) ⟶ *y* ∈ *S*))

1.10.4 Metric space

definition
dist x y = *L2_set* (λ*i*. *dist* (*x*$*i*) (*y*$*i*)) *UNIV*

definition
(*uniformity* :: (('*a*^'*b*::_) × ('*a*^'*b*::_)) *filter*) =
(*INF e*∈{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

proposition *dist_vec_nth_le*: *dist* (*x* $ *i*) (*y* $ *i*) ≤ *dist x y*

1.10.5 Normed vector space

definition *norm x* = *L2_set* (λ*i*. *norm* (*x*$*i*)) *UNIV*

definition *sgn* (*x*::'*a*^'*b*) = *scaleR* (*inverse* (*norm x*)) *x*

1.10.6 Inner product space

definition *inner x y* = *sum* (λ*i*. *inner* (*x*$*i*) (*y*$*i*)) *UNIV*

1.10.7 Euclidean space

definition *axis k x* = (*χ i. if i* = *k then x else 0*)

definition *Basis* = (⋃ *i.* ⋃ *u*∈*Basis.* {*axis i u*})

proposition *DIM_cart* [*simp*]: *DIM*('*a*^'*b*) = *CARD*('*b*) ∗ *DIM*('*a*)

1.10.8 Matrix operations

definition *map_matrix*::('*a* ⇒ '*b*) ⇒ (('*a*, '*i*::*finite*)*vec*, '*j*::*finite*) *vec* ⇒ (('*b*, '*i*)*vec*, '*j*) *vec* **where**
map_matrix f x = (*χ i j. f* (*x* $ *i* $ *j*))

definition *matrix_matrix_mult* :: ('*a*::*semiring_1*) ^'*n*^'*m* ⇒ '*a* ^'*p*^'*n* ⇒ '*a* ^ '*p* ^'*m*
(**infixl** ‹∗∗› 70)
where *m* ∗∗ *m'* == (*χ i j. sum* (λ*k.* ((*m*$*i*)$*k*) ∗ ((*m'*$*k*)$*j*)) (*UNIV* :: '*n set*)) ::'*a* ^ '*p* ^'*m*

definition *matrix_vector_mult* :: ('*a*::*semiring_1*) ^'*n*^'*m* ⇒ '*a* ^'*n* ⇒ '*a* ^ '*m*
(**infixl** ‹∗*v*› 70)
where *m* ∗*v x* ≡ (*χ i. sum* (λ*j.* ((*m*$*i*)$*j*) ∗ (*x*$*j*)) (*UNIV* ::'*n set*)) :: '*a*^'*m*

definition *vector_matrix_mult* :: '*a* ^ '*m* ⇒ ('*a*::*semiring_1*) ^'*n*^'*m* ⇒ '*a* ^'*n*
(**infixl** ‹*v*∗› 70)
where *v v*∗ *m* == (*χ j. sum* (λ*i.* ((*v*$*i*) ∗ (*m*$*i*)$*j*)) (*UNIV* :: '*m set*)) :: '*a*^'*n*
definition *matrix* :: ('*a*::{*plus*,*times*, *one*, *zero*}^'*m* ⇒ '*a* ^ '*n*) ⇒ '*a*^'*m*^'*n*
where *matrix f* = (*χ i j.* (*f*(*axis j 1*))$*i*)

1.10.9 Inverse matrices (not necessarily square)

definition
invertible(*A*::'*a*::*semiring_1*^'*n*^'*m*) ⟷ (∃ *A'*::'*a*^'*m*^'*n. A* ∗∗ *A'* = *mat 1* ∧ *A'* ∗∗ *A* = *mat 1*)

definition
matrix_inv(*A*:: '*a*::*semiring_1*^'*n*^'*m*) =
(*SOME A'*::'*a*^'*m*^'*n. A* ∗∗ *A'* = *mat 1* ∧ *A'* ∗∗ *A* = *mat 1*)

end

1.11 Linear Algebra on Finite Cartesian Products

theory *Cartesian_Space*
imports
HOL−*Combinatorics.Transposition*
Finite_Cartesian_Product

Linear_Algebra
begin

1.11.1 Some interesting theorems and interpretations

1.11.2 Rank of a matrix

definition *rank* :: *'a::field^'n^'m=>nat*
where *row_rank_def_gen*: *rank A ≡ vec.dim(rows A)*

1.11.3 Orthogonality of a matrix

definition *orthogonal_matrix* (*Q::'a::semiring_1^'n^'n*) ⟷
*transpose Q ** Q = mat 1 ∧ Q ** transpose Q = mat 1*

proposition *orthogonal_matrix_mul*:
fixes *A* :: *real ^'n^'n*
assumes *orthogonal_matrix A orthogonal_matrix B*
shows *orthogonal_matrix*(*A ** B*)

proposition *orthogonal_transformation_matrix*:
fixes *f*:: *real^'n ⇒ real^'n*
shows *orthogonal_transformation f* ⟷ *linear f ∧ orthogonal_matrix*(*matrix f*)
(**is** *?lhs* ⟷ *?rhs*)

1.11.4 Finding an Orthogonal Matrix

proposition *orthogonal_matrix_exists_basis*:
fixes *a* :: *real^'n*
assumes *norm a = 1*
obtains *A* **where** *orthogonal_matrix A A *v* (*axis k 1*) = *a*

proposition *orthogonal_transformation_exists*:
fixes *a b* :: *real^'n*
assumes *norm a = norm b*
obtains *f* **where** *orthogonal_transformation f f a = b*

1.11.5 Scaling and isometry

proposition *scaling_linear*:
fixes f :: *'a::real_inner* ⇒ *'a::real_inner*
assumes *f0*: $f\ 0 = 0$
and *fd*: $\forall\, x\ y.\ dist\ (f\ x)\ (f\ y) = c * dist\ x\ y$
shows *linear f*
proposition *orthogonal_transformation_isometry*:
orthogonal_transformation f ⟷ $f(0::'a::real_inner) = (0::'a) \wedge (\forall\, x\ y.\ dist(f\ x)\ (f\ y) = dist\ x\ y)$

1.11.6 Induction on matrix row operations

end

1.12 Traces and Determinants of Square Matrices

theory *Determinants*
imports
HOL−Combinatorics.Permutations
Cartesian_Space
begin

1.12.1 Trace

definition *trace* :: *'a::semiring_1^'n^'n* ⇒ *'a*
where $trace\ A = sum\ (\lambda i.\ ((A\$i)\$i))\ (UNIV::'n\ set)$

Definition of determinant

definition *det*:: *'a::comm_ring_1^'n^'n* ⇒ *'a* **where**
$det\ A =$
$sum\ (\lambda p.\ of_int\ (sign\ p) * prod\ (\lambda i.\ A\$i\$p\ i)\ (UNIV :: 'n\ set))$
$\{p.\ p\ permutes\ (UNIV :: 'n\ set)\}$
proposition *det_diagonal*:
fixes A :: *'a::comm_ring_1^'n^'n*
assumes *ld*: $\bigwedge i\ j.\ i \neq j \Longrightarrow A\$i\$j = 0$
shows $det\ A = prod\ (\lambda i.\ A\$i\$i)\ (UNIV::'n\ set)$

proposition *det_matrix_scaleR* [*simp*]: $det\ (matrix\ (((*_R)\ r)) :: real^'n^'n) = r\ \hat{}\ CARD('n::finite)$

proposition *det_mul*:
fixes $A\ B$ :: *'a::comm_ring_1^'n^'n*
shows $det\ (A ** B) = det\ A * det\ B$

1.12.2 Relation to invertibility

proposition *invertible_det_nz*:
fixes *A*::*'a*::{*field*}^*'n*^*'n*
shows *invertible A* ⟷ *det A* ≠ *0*

Invertibility of matrices and corresponding linear functions

1.12.3 Cramer's rule

proposition *cramer_lemma*:
fixes *A* :: *'a*::{*field*}^*'n*^*'n*
shows *det*((χ *i j*. *if j* = *k then* (*A* ∗*v x*)$*i else A*$*i*$*j*):: *'a*::{*field*}^*'n*^*'n*) = *x*$*k* ∗ *det A*

proposition *cramer*:
fixes *A* ::*'a*::{*field*}^*'n*^*'n*
assumes *d0*: *det A* ≠ *0*
shows *A* ∗*v x* = *b* ⟷ *x* = (χ *k*. *det*(χ *i j*. *if j*=*k then b*$*i else A*$*i*$*j*) / *det A*)

proposition *det_orthogonal_matrix*:
fixes *Q*:: *'a*::*linordered_idom*^*'n*^*'n*
assumes *oQ*: *orthogonal_matrix Q*
shows *det Q* = *1* ∨ *det Q* = − *1*

proposition *orthogonal_transformation_det* [*simp*]:
fixes *f* :: *real*^*'n* ⇒ *real*^*'n*
shows *orthogonal_transformation f* ⟹ |*det* (*matrix f*)| = *1*

1.12.4 Rotation, reflection, rotoinversion

definition *rotation_matrix Q* ⟷ *orthogonal_matrix Q* ∧ *det Q* = *1*
definition *rotoinversion_matrix Q* ⟷ *orthogonal_matrix Q* ∧ *det Q* = − *1*

end

1.13 Operators involving abstract topology

theory *Abstract_Topology*
imports
Complex_Main
HOL−Library.Set_Idioms
HOL−Library.FuncSet
begin

1.13.1 General notion of a topology as a value

definition *istopology* :: $('a\ set \Rightarrow bool) \Rightarrow bool$ **where**
istopology $L \equiv (\forall S\ T.\ L\ S \longrightarrow L\ T \longrightarrow L\ (S \cap T)) \wedge (\forall \mathcal{K}.\ (\forall K{\in}\mathcal{K}.\ L\ K) \longrightarrow L\ (\bigcup \mathcal{K}))$

typedef $'a\ topology = \{L{::}('a\ set) \Rightarrow bool.\ istopology\ L\}$
morphisms *openin topology*
proposition *openin_clauses*:
fixes $U :: 'a\ topology$
shows
openin $U\ \{\}$
$\bigwedge S\ T.\ openin\ U\ S \Longrightarrow openin\ U\ T \Longrightarrow openin\ U\ (S{\cap}T)$
$\bigwedge K.\ (\forall S \in K.\ openin\ U\ S) \Longrightarrow openin\ U\ (\bigcup K)$
definition *closedin* :: $'a\ topology \Rightarrow 'a\ set \Rightarrow bool$ **where**
closedin $U\ S \longleftrightarrow S \subseteq topspace\ U \wedge openin\ U\ (topspace\ U - S)$

1.13.2 The discrete topology

1.13.3 Subspace topology

definition *subtopology* :: $'a\ topology \Rightarrow 'a\ set \Rightarrow 'a\ topology$
where *subtopology* $U\ V = topology\ (\lambda T.\ \exists S.\ T = S \cap V \wedge openin\ U\ S)$

1.13.4 The canonical topology from the underlying type class

abbreviation *euclidean* :: $'a{::}topological_space\ topology$
where *euclidean* $\equiv$ *topology open*

1.13.5 Basic "localization" results are handy for connectedness.

1.13.6 Derived set (set of limit points)

1.13.7 Closure with respect to a topological space

1.13.8 Frontier with respect to topological space

1.13.9 Locally finite collections

1.13.10 Continuous maps

lemma *continuous_map_alt*:
continuous_map T1 T2 f
= ((∀ U. openin T2 U ⟶ openin T1 (f −‘ U ∩ topspace T1)) ∧ f ∈ topspace T1 → topspace T2)

1.13.11 Open and closed maps (not a priori assumed continuous)

1.13.12 Quotient maps

1.13.13 Separated Sets

1.13.14 Homeomorphisms

1.13.15 Relation of homeomorphism between topological spaces

1.13.16 Connected topological spaces

1.13.17 Compact sets

proposition *compact_space_fip*:
compact_space X ⟷
(∀𝒰. (∀ C∈𝒰. closedin X C) ∧ (∀ℱ. finite ℱ ∧ ℱ ⊆ 𝒰 ⟶ ⋂ℱ ≠ {}) ⟶ ⋂𝒰 ≠ {})
*(***is** *_ = ?rhs)*

corollary *compactin_fip*:
compactin X S ⟷
S ⊆ topspace X ∧
(∀𝒰. (∀ C∈𝒰. closedin X C) ∧ (∀ℱ. finite ℱ ∧ ℱ ⊆ 𝒰 ⟶ S ∩ ⋂ℱ ≠ {}) ⟶ S ∩ ⋂𝒰 ≠ {})

corollary *compact_space_imp_nest*:
fixes *C* :: *nat* ⇒ *'a set*
assumes *compact_space X* **and** *clo*: ⋀*n. closedin X (C n)*
and *ne*: ⋀*n. C n ≠ {}* **and** *dec*: *decseq C*
shows *(⋂ n. C n) ≠ {}*

1.13.18 Embedding maps

1.13.19 Retraction and section maps

1.13.20 Continuity

1.13.21 The topology generated by some (open) subsets

1.13.22 Topology bases and sub-bases

1.13.23 Continuity via bases/subbases, hence upper and lower semicontinuity

1.13.24 Pullback topology

definition *pullback_topology*::(*'a set*) ⇒ (*'a* ⇒ *'b*) ⇒ (*'b topology*) ⇒ (*'a topology*)
where *pullback_topology A f T* = *topology* (λ*S*. ∃ *U*. *openin T U* ∧ *S* = *f*−‘*U* ∩ *A*)

proposition *continuous_map_pullback* [*intro*]:
assumes *continuous_map T1 T2 g*
shows *continuous_map* (*pullback_topology A f T1*) *T2* (*g o f*)

proposition *continuous_map_pullback'* [*intro*]:
assumes *continuous_map T1 T2* (*f o g*) *topspace T1* ⊆ *g*−‘*A*
shows *continuous_map T1* (*pullback_topology A f T2*) *g*

1.13.25 Proper maps (not a priori assumed continuous)

1.13.26 Perfect maps (proper, continuous and surjective)

end

1.14 *F*-Sigma and *G*-Delta sets in a Topological Space

theory *FSigma*
imports *Abstract_Topology*
begin

end

1.15 Disjoint sum of arbitarily many spaces

theory *Sum_Topology*
  imports *Abstract_Topology*
begin

end

Chapter 2

Topology

theory *Elementary_Topology*
imports
HOL−Library.Set_Idioms
HOL−Library.Disjoint_Sets
Product_Vector
begin

2.1 Elementary Topology

2.1.1 Topological Basis

definition *topological_basis B* $\longleftrightarrow$
$(\forall\, b{\in}B.\ open\ b) \wedge (\forall\, x.\ open\ x \longrightarrow (\exists\, B'.\ B' \subseteq B \wedge \bigcup B' = x))$

2.1.2 Countable Basis

locale *countable_basis* = *topological_space p* **for** *p*::*'a set* $\Rightarrow$ *bool* +
fixes *B* :: *'a set set*
assumes *is_basis*: *topological_basis B*
and *countable_basis*: *countable B*
begin

class *second_countable_topology* = *topological_space* +
assumes *ex_countable_subbasis*:
$\exists\, B$::*'a set set*. *countable B* $\wedge$ *open* = *generate_topology B*
begin

proposition *Lindelof*:
fixes $\mathcal{F}$ :: *'a*::*second_countable_topology set set*
assumes $\mathcal{F}$: $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow open\ S$
obtains $\mathcal{F}'$ **where** $\mathcal{F}' \subseteq \mathcal{F}$ *countable* $\mathcal{F}'$ $\bigcup \mathcal{F}' = \bigcup \mathcal{F}$

2.1.3 Polish spaces

class *polish_space* = *complete_space* + *second_countable_topology*

2.1.4 Limit Points

definition (**in** *topological_space*) *islimpt*:: *'a* ⇒ *'a set* ⇒ *bool* (**infixr** ‹*islimpt*› *60*)
where *x islimpt S* ⟷ (∀ *T*. *x*∈*T* ⟶ *open T* ⟶ (∃ *y*∈*S*. *y*∈*T* ∧ *y*≠*x*))

2.1.5 Interior of a Set

definition *interior* :: (*'a*::*topological_space*) *set* ⇒ *'a set* **where**
interior S = ⋃{*T*. *open T* ∧ *T* ⊆ *S*}

2.1.6 Closure of a Set

definition *closure* :: (*'a*::*topological_space*) *set* ⇒ *'a set* **where**
closure S = *S* ∪ {*x* . *x islimpt S*}

2.1.7 Frontier (also known as boundary)

definition *frontier* :: (*'a*::*topological_space*) *set* ⇒ *'a set* **where**
frontier S = *closure S* − *interior S*

2.1.8 Limits

2.1.9 Compactness

proposition *Heine_Borel_imp_Bolzano_Weierstrass*:
assumes *compact S*
and *infinite T*
and *T* ⊆ *S*
shows ∃ *x* ∈ *S*. *x islimpt T*

definition *countably_compact* :: (*'a*::*topological_space*) *set* ⇒ *bool* **where**
countably_compact U ⟷
(∀ *A*. *countable A* ⟶ (∀ *a*∈*A*. *open a*) ⟶ *U* ⊆ ⋃ *A*
⟶ (∃ *T*⊆*A*. *finite T* ∧ *U* ⊆ ⋃ *T*))

proposition *countably_compact_imp_compact_second_countable*:
countably_compact U ⟹ *compact* (*U* :: *'a* :: *second_countable_topology set*)

definition *seq_compact* :: *'a::topological_space set* ⇒ *bool* **where**
seq_compact S ⟷
(∀ *f*. (∀ *n*. *f n* ∈ *S*) ⟶ (∃ *l*∈*S*. ∃ *r*::*nat*⇒*nat*. *strict_mono r* ∧ (*f* ∘ *r*) ⟶ *l*))

proposition *Bolzano_Weierstrass_imp_seq_compact*:
fixes *S* :: *'a::{t1_space, first_countable_topology} set*
shows (⋀*T*. ⟦*infinite T*; *T* ⊆ *S*⟧ ⟹∃ *x* ∈ *S*. *x islimpt T*) ⟹ *seq_compact S*

2.1.10 Continuity

2.1.11 Homeomorphisms

definition *homeomorphism S T f g* ⟷
(∀ *x*∈*S*. (*g*(*f x*) = *x*)) ∧ (*f* ‘ *S* = *T*) ∧ *continuous_on S f* ∧
(∀ *y*∈*T*. (*f*(*g y*) = *y*)) ∧ (*g* ‘ *T* = *S*) ∧ *continuous_on T g*

definition *homeomorphic* :: *'a::topological_space set* ⇒ *'b::topological_space set* ⇒ *bool*
(**infixr** ‹*homeomorphic*› *60*)
where *s homeomorphic t* ≡ (∃ *f g*. *homeomorphism s t f g*)

end
theory *Abstract_Limits*
imports
Abstract_Topology
begin

2.1.12 nhdsin and atin

2.1.13 Limits in a topological space

2.1.14 Pointwise continuity in topological spaces

2.1.15 Combining theorems for continuous functions into the reals

end

2.2 Non-Denumerability of the Continuum

theory *Continuum_Not_Denumerable*
imports
Complex_Main

HOL−Library.Countable_Set
begin

theorem *real_non_denum*: $\nexists f :: nat \Rightarrow real.\ surj\ f$

corollary *complex_non_denum*: $\nexists f :: nat \Rightarrow complex.\ surj\ f$

end

2.3 Abstract Topology 2

theory *Abstract_Topology_2*
imports
Elementary_Topology Abstract_Topology Continuum_Not_Denumerable
HOL−Library.Indicator_Function
HOL−Library.Equipollence
begin

2.3.1 Closure

corollary *infinite_openin*:
fixes $S :: 'a :: t1_space\ set$
shows $[\![openin\ (top_of_set\ U)\ S;\ x \in S;\ x\ islimpt\ U]\!] \Longrightarrow infinite\ S$

2.3.2 Frontier

2.3.3 Compactness

2.3.4 Continuity

2.3.5 Retractions

definition *retraction* :: $('a::topological_space)\ set \Rightarrow 'a\ set \Rightarrow ('a \Rightarrow 'a) \Rightarrow bool$
where *retraction S T r* $\longleftrightarrow$
$T \subseteq S \wedge continuous_on\ S\ r \wedge r \in S \to T \wedge (\forall x{\in}T.\ r\ x = x)$

definition *retract_of* (**infixl** ‹*retract'_of*› *50*) **where**
T retract_of S $\longleftrightarrow$ $(\exists r.\ retraction\ S\ T\ r)$

2.3.6 Retractions on a topological space

2.3.7 Paths and path-connectedness

2.3.8 Connected components

2.3.9 Combining theorems for continuous functions into the reals

2.3.10 A few cardinality results

end

Chapter 3

Connected Components

theory *Connected*
imports
Abstract_Topology_2
begin

3.0.1 Connected components, considered as a connectedness relation or a set

definition *connected_component S x y* $\equiv \exists T.$ *connected* $T \wedge T \subseteq S \wedge x \in T \wedge y \in T$

3.0.2 The set of connected components of a set

definition *components*:: *'a*::*topological_space set* $\Rightarrow$ *'a set set*
where *components S* $\equiv$ *connected_component_set S ' S*

3.0.3 Lemmas about components

proposition *component_diff_connected*:
fixes *S* :: *'a*::*metric_space set*
assumes *connected S connected U S* $\subseteq$ *U* **and** *C*: *C* $\in$ *components* (*U* $-$ *S*)
shows *connected*(*U* $-$ *C*)

end

theory *Function_Topology*
imports

Elementary_Topology
Abstract_Limits
Connected
begin

3.1 Function Topology

3.1.1 The product topology

definition *product_topology*::('*i* ⇒ ('*a topology*)) ⇒ ('*i set*) ⇒ (('*i* ⇒ '*a*) *topology*)
where *product_topology T I* =
topology_generated_by {(Π$_E$ *i*∈*I*. *X i*) |*X*. (∀ *i*. *openin* (*T i*) (*X i*)) ∧ *finite* {*i*. *X i* ≠ *topspace* (*T i*)}}
proposition *product_topology*:
product_topology X I =
topology
(*arbitrary union_of*
((*finite intersection_of*
(λ*F*. ∃ *i U*. *F* = {*f*. *f i* ∈ *U*} ∧ *i* ∈ *I* ∧ *openin* (*X i*) *U*))
relative_to (Π$_E$ *i*∈*I*. *topspace* (*X i*))))
(**is** _ = *topology* (_ *union_of* ((_ *intersection_of* ?Ψ) *relative_to* *?TOP*)))

proposition *product_topology_open_contains_basis*:
assumes *openin* (*product_topology T I*) *U x* ∈ *U*
shows ∃ *X*. *x* ∈ (Π$_E$ *i*∈*I*. *X i*) ∧ (∀ *i*. *openin* (*T i*) (*X i*)) ∧ *finite* {*i*. *X i* ≠ *topspace* (*T i*)} ∧ (Π$_E$ *i*∈*I*. *X i*) ⊆ *U*

corollary *openin_product_topology_alt*:
openin (*product_topology X I*) *S* ⟷
(∀ *x* ∈ *S*. ∃ *U*. *finite* {*i* ∈ *I*. *U i* ≠ *topspace*(*X i*)} ∧
(∀ *i* ∈ *I*. *openin* (*X i*) (*U i*)) ∧ *x* ∈ *Pi$_E$ I U* ∧ *Pi$_E$ I U* ⊆ *S*)

corollary *closedin_product_topology*:
closedin (*product_topology X I*) (*PiE I S*) ⟷ *PiE I S* = {} ∨ (∀ *i* ∈ *I*. *closedin* (*X i*) (*S i*))

corollary *closedin_product_topology_singleton*:
f ∈ *extensional I* ⟹ *closedin* (*product_topology X I*) {*f*} ⟷ (∀ *i* ∈ *I*. *closedin* (*X i*) {*f i*})

Powers of a single topological space as a topological space, using type classes

instantiation *fun* :: (*type*, *topological_space*) *topological_space*
begin

definition *open_fun_def*:
open U = *openin* (*product_topology* (λ*i*. *euclidean*) *UNIV*) *U*

proposition *product_topology_basis'*:
fixes $x::'i \Rightarrow 'a$ **and** $U::'i \Rightarrow ('b::topological_space)\ set$
assumes *finite* I $\bigwedge i.\ i \in I \Longrightarrow open\ (U\ i)$
shows *open* $\{f.\ \forall i{\in}I.\ f\ (x\ i) \in U\ i\}$

Topological countability for product spaces

proposition *product_topology_countable_basis*:
shows $\exists K::((('a::countable \Rightarrow 'b::second_countable_topology)\ set)\ set).$
topological_basis $K \wedge$ *countable* $K \wedge$
$(\forall k{\in}K.\ \exists X.\ (k = Pi_E\ UNIV\ X) \wedge (\forall i.\ open\ (X\ i)) \wedge finite\ \{i.\ X\ i \neq UNIV\})$

3.1.2 The Alexander subbase theorem

theorem *Alexander_subbase*:
assumes X: *topology* (*arbitrary union_of* (*finite intersection_of* $(\lambda x.\ x \in \mathcal{B})$ *relative_to* $\bigcup \mathcal{B}$)) $= X$
and *fin*: $\bigwedge C.\ [\![C \subseteq \mathcal{B};\ \bigcup C = topspace\ X]\!] \Longrightarrow \exists C'.\ finite\ C' \wedge C' \subseteq C \wedge \bigcup C' = topspace\ X$
shows *compact_space* X

corollary *Alexander_subbase_alt*:
assumes $U \subseteq \bigcup \mathcal{B}$
and *fin*: $\bigwedge C.\ [\![C \subseteq \mathcal{B};\ U \subseteq \bigcup C]\!] \Longrightarrow \exists C'.\ finite\ C' \wedge C' \subseteq C \wedge U \subseteq \bigcup C'$
and X: *topology*
(*arbitrary union_of*
(*finite intersection_of* $(\lambda x.\ x \in \mathcal{B})$ *relative_to* U)) $= X$
shows *compact_space* X

proposition *continuous_map_componentwise*:
continuous_map X (*product_topology* $Y\ I$) $f \longleftrightarrow$
$f\ `\ (topspace\ X) \subseteq extensional\ I \wedge (\forall k \in I.\ continuous_map\ X\ (Y\ k)\ (\lambda x.\ f\ x\ k))$
(**is** *?lhs* $\longleftrightarrow$ _ $\wedge$ *?rhs*)

proposition *open_map_product_projection*:
assumes $i \in I$
shows *open_map* (*product_topology* $Y\ I$) $(Y\ i)$ $(\lambda f.\ f\ i)$

3.1.3 Open Pi-sets in the product topology

proposition *openin_PiE_gen*:

openin (*product_topology X I*) (*PiE I S*) ⟷
PiE I S = {} ∨
finite {*i* ∈ *I*. *S i* ≠ *topspace* (*X i*)} ∧ (∀ *i* ∈ *I*. *openin* (*X i*) (*S i*))
(**is** *?lhs* ⟷ _ ∨ *?rhs*)

corollary *openin_PiE*:
finite I ⟹ *openin* (*product_topology X I*) (*PiE I S*) ⟷ *PiE I S* = {} ∨ (∀ *i* ∈ *I*. *openin* (*X i*) (*S i*))

proposition *compact_space_product_topology*:
compact_space(*product_topology X I*) ⟷
(*product_topology X I*) = *trivial_topology* ∨ (∀ *i* ∈ *I*. *compact_space*(*X i*))
(**is** *?lhs* = *?rhs*)

corollary *compactin_PiE*:
compactin (*product_topology X I*) (*PiE I S*) ⟷
PiE I S = {} ∨ (∀ *i* ∈ *I*. *compactin* (*X i*) (*S i*))

3.1.4 Relationship with connected spaces, paths, etc.

proposition *connected_space_product_topology*:
connected_space(*product_topology X I*) ⟷
(∃ *i* ∈ *I*. *X i* = *trivial_topology*) ∨ (∀ *i* ∈ *I*. *connected_space*(*X i*))
(**is** *?lhs* ⟷ *?eq* ∨ *?rhs*)

3.1.5 Projections from a function topology to a component

3.1.6 Limits

end

3.2 The binary product topology

theory *Product_Topology*
imports *Function_Topology*
begin

3.3 Product Topology

3.3.1 Definition

3.3.2 Continuity

proposition *compact_space_prod_topology*:
compact_space(*prod_topology X Y*) ⟷ (*prod_topology X Y*) = *trivial_topology*
∨ *compact_space X* ∧ *compact_space Y*

3.3.3 Homeomorphic maps

proposition *connected_space_prod_topology*:
connected_space(*prod_topology X Y*) ⟷
(*prod_topology X Y*) = *trivial_topology* ∨ *connected_space X* ∧ *connected_space*
Y (**is** *?lhs=?rhs*)

end

3.4 T1 and Hausdorff spaces

theory *T1_Spaces*
imports *Product_Topology*
begin

3.5 T1 spaces with equivalences to many naturally "nice" properties.

proposition *t1_space_product_topology*:
t1_space (*product_topology X I*)
⟷ (*product_topology X I*) = *trivial_topology* ∨ (∀ *i* ∈ *I*. *t1_space* (*X i*))

3.5.1 Hausdorff Spaces

end

3.6 Lindelöf spaces

theory *Lindelof_Spaces*
imports *T1_Spaces*
begin

end

Chapter 4

Functional Analysis

theory *Metric_Arith*
  imports *HOL.Real_Vector_Spaces*
begin
theorem *metric_eq_thm* [*THEN HOL.eq_reflection*]:
  $x \in s \Longrightarrow y \in s \Longrightarrow x = y \longleftrightarrow (\forall a{\in}s.\ dist\ x\ a = dist\ y\ a)$

end

Chapter 5

Elementary Metric Spaces

theory *Elementary_Metric_Spaces*
  imports
    Abstract_Topology_2
    Metric_Arith
begin

5.1 Open and closed balls

definition *ball* :: *'a*::*metric_space* ⇒ *real* ⇒ *'a set*
  where *ball x ε* = {*y*. *dist x y* < *ε*}

definition *cball* :: *'a*::*metric_space* ⇒ *real* ⇒ *'a set*
  where *cball x ε* = {*y*. *dist x y* ≤ *ε*}

definition *sphere* :: *'a*::*metric_space* ⇒ *real* ⇒ *'a set*
  where *sphere x ε* = {*y*. *dist x y* = *ε*}

5.2 Limit Points

5.3 Perfect Metric Spaces

5.4 Finite and discrete

5.5 Interior

5.6 Frontier

5.7 Limits

proposition *Lim*: $(f \longrightarrow l)$ *net* $\longleftrightarrow$ *trivial_limit net* $\vee$ $(\forall \varepsilon{>}0.$ *eventually* $(\lambda x.$ *dist* $(f\ x)\ l < \varepsilon)$ *net*)
proposition *Lim_within_le*: $(f \longrightarrow l)$(*at a within S*) $\longleftrightarrow$
$(\forall \varepsilon{>}0.\ \exists \delta{>}0.\ \forall x{\in}S.\ 0 <$ *dist x a* $\wedge$ *dist x a* $\leq \delta \longrightarrow$ *dist* $(f\ x)\ l < \varepsilon)$

proposition *Lim_within*: $(f \longrightarrow l)$ $($*at a within S*$) \longleftrightarrow$
$(\forall \varepsilon > 0.\ \exists \delta{>}0.\ \forall x \in S.\ 0 <$ *dist x a* $\wedge$ *dist x a* $< \delta \longrightarrow$ *dist* $(f\ x)\ l < \varepsilon)$

corollary *Lim_withinI* [*intro?*]:
assumes $\bigwedge \varepsilon.\ \varepsilon > 0 \Longrightarrow \exists \delta{>}0.\ \forall x \in S.\ 0 <$ *dist x a* $\wedge$ *dist x a* $< \delta \longrightarrow$ *dist* $(f\ x)\ l \leq \varepsilon$
shows $(f \longrightarrow l)$ $($*at a within S*$)$

proposition *Lim_at*: $(f \longrightarrow l)$ $($*at a*$) \longleftrightarrow$
$(\forall \varepsilon > 0.\ \exists \delta{>}0.\ \forall x.\ 0 <$ *dist x a* $\wedge$ *dist x a* $< \delta \longrightarrow$ *dist* $(f\ x)\ l < \varepsilon)$

5.8 Continuity

proposition *continuous_within_eps_delta*:
continuous $($*at x within s*$)\ f \longleftrightarrow (\forall \varepsilon{>}0.\ \exists \delta{>}0.\ \forall x'{\in}\ s.$ *dist* $x'\ x < \delta$ *–>* *dist* $(f\ x')\ (f\ x) < \varepsilon)$

corollary *continuous_at_eps_delta*:
continuous $($*at x*$)\ f \longleftrightarrow (\forall \varepsilon > 0.\ \exists \delta > 0.\ \forall x'.$ *dist* $x'\ x < \delta \longrightarrow$ *dist* $(f\ x')\ (f\ x) < \varepsilon)$

corollary *continuous_at_ball*:
continuous $($*at x*$)\ f \longleftrightarrow (\forall \varepsilon{>}0.\ \exists \delta{>}0.\ f\ `\ (ball\ x\ \delta) \subseteq ball\ (f\ x)\ \varepsilon)$

5.9 Closure and Limit Characterization

5.10 Boundedness

definition (**in** *metric_space*) *bounded* :: *'a set* $\Rightarrow$ *bool*
where *bounded S* $\longleftrightarrow (\exists x\ \varepsilon.\ \forall y{\in}S.$ *dist x y* $\leq \varepsilon)$

5.11 Compactness

proposition *seq_compact_imp_totally_bounded*:
assumes *seq_compact S*
shows $\forall \varepsilon{>}0.\ \exists k.\ \textit{finite}\ k \wedge k \subseteq S \wedge S \subseteq (\bigcup x{\in}k.\ \textit{ball}\ x\ \varepsilon)$
proposition *seq_compact_imp_Heine_Borel*:
fixes $S :: 'a :: \textit{metric_space set}$
assumes *seq_compact S*
shows *compact S*

proposition *compact_eq_seq_compact_metric*:
$\textit{compact}\ (S :: 'a{::}\textit{metric_space set}) \longleftrightarrow \textit{seq_compact}\ S$

proposition *compact_def*: — this is the definition of compactness in HOL Light
$\textit{compact}\ (S :: 'a{::}\textit{metric_space set}) \longleftrightarrow$
$(\forall f.\ (\forall n.\ f\ n \in S) \longrightarrow (\exists l{\in}S.\ \exists r{::}\textit{nat}{\Rightarrow}\textit{nat}.\ \textit{strict_mono}\ r \wedge (f \circ r) \longrightarrow l))$
proposition *compact_eq_Bolzano_Weierstrass*:
fixes $S :: 'a{::}\textit{metric_space set}$
shows $\textit{compact}\ S \longleftrightarrow (\forall T.\ \textit{infinite}\ T \wedge T \subseteq S \longrightarrow (\exists x \in S.\ x\ \textit{islimpt}\ T))$

proposition *Bolzano_Weierstrass_imp_bounded*:
$(\bigwedge T.\ [\![\textit{infinite}\ T;\ T \subseteq S]\!] \Longrightarrow (\exists x \in S.\ x\ \textit{islimpt}\ T)) \Longrightarrow \textit{bounded}\ S$

5.12 Banach fixed point theorem

theorem *Banach_fix*:
assumes *S*: $\textit{complete}\ S\ S \neq \{\}$
and *c*: $0 \leq c\ c < 1$
and *f*: $f\ `\ S \subseteq S$
and *lipschitz*: $\bigwedge x\ y.\ [\![x{\in}S;\ y{\in}S]\!] \Longrightarrow \textit{dist}\ (f\ x)\ (f\ y) \leq c * \textit{dist}\ x\ y$
shows $\exists!x{\in}S.\ f\ x = x$

5.13 Edelstein fixed point theorem

theorem *Edelstein_fix*:
fixes $S :: 'a{::}\textit{metric_space set}$
assumes *S*: $\textit{compact}\ S\ S \neq \{\}$
and *gs*: $(g\ `\ S) \subseteq S$
and *dist*: $\bigwedge x\ y.\ [\![x{\in}S;\ y{\in}S]\!] \Longrightarrow x \neq y \longrightarrow \textit{dist}\ (g\ x)\ (g\ y) < \textit{dist}\ x\ y$
shows $\exists!x{\in}S.\ g\ x = x$

5.14 The diameter of a set

definition $\textit{diameter} :: 'a{::}\textit{metric_space set} \Rightarrow \textit{real}$ **where**

diameter S = (if S = {} then 0 else SUP (x,y)∈S×S. dist x y)

proposition *Lebesgue_number_lemma*:
assumes *compact S* $\mathcal{C} \neq \{\}$ $S \subseteq \bigcup \mathcal{C}$ **and** *ope*: $\bigwedge B.\ B \in \mathcal{C} \Longrightarrow$ *open B*
obtains δ **where** $0 < \delta$ $\bigwedge T.\ [\![T \subseteq S;$ *diameter T* $< \delta]\!] \Longrightarrow \exists B \in \mathcal{C}.\ T \subseteq B$

5.15 Metric spaces with the Heine-Borel property

class *heine_borel = metric_space +*
assumes *bounded_imp_convergent_subsequence*:
bounded (range f) $\Longrightarrow \exists l\ r.$ *strict_mono* $(r::nat \Rightarrow nat) \wedge ((f \circ r) \longrightarrow l)$ *sequentially*

proposition *bounded_closed_imp_seq_compact*:
fixes $S::'a::$*heine_borel set*
assumes *bounded S*
and *closed S*
shows *seq_compact S*

instance *real* :: *heine_borel*

instance *prod* :: (*heine_borel*, *heine_borel*) *heine_borel*

5.16 Completeness

proposition (**in** *metric_space*) *completeI*:
assumes $\bigwedge f.\ \forall n.\ f\ n \in s \Longrightarrow$ *Cauchy f* $\Longrightarrow \exists l \in s.\ f \longrightarrow l$
shows *complete s*

proposition (**in** *metric_space*) *completeE*:
assumes *complete s* **and** $\forall n.\ f\ n \in s$ **and** *Cauchy f*
obtains *l* **where** $l \in s$ **and** $f \longrightarrow l$

proposition *compact_eq_totally_bounded*:
compact S $\longleftrightarrow$ *complete S* $\wedge$ $(\forall \varepsilon > 0.\ \exists k.$ *finite k* $\wedge\ S \subseteq (\bigcup x \in k.$ *ball x* $\varepsilon))$
(**is** _ $\longleftrightarrow$ *?rhs*)

5.17 Cauchy continuity

5.18 Properties of Balls and Spheres

5.19 Distance from a Set

5.20 Infimum Distance

definition *infdist x A* = (*if A* = {} *then 0 else INF a*∈*A. dist x a*)

5.21 Separation between Points and Sets

proposition *separate_point_closed*:
fixes *S* :: *'a*::*heine_borel set*
assumes *closed S* **and** *a* ∉ *S*
shows ∃ *δ*>*0*. ∀ *x*∈*S*. *δ* ≤ *dist a x*

proposition *separate_compact_closed*:
fixes *S T* :: *'a*::*heine_borel set*
assumes *compact S*
and *T*: *closed T S* ∩ *T* = {}
shows ∃ *δ*>*0*. ∀ *x*∈*S*. ∀ *y*∈*T*. *δ* ≤ *dist x y*

proposition *separate_closed_compact*:
fixes *S T* :: *'a*::*heine_borel set*
assumes *S*: *closed S*
and *T*: *compact T*
and *dis*: *S* ∩ *T* = {}
shows ∃ *δ*>*0*. ∀ *x*∈*S*. ∀ *y*∈*T*. *δ* ≤ *dist x y*

proposition *compact_in_open_separated*:
fixes *A*::*'a*::*heine_borel set*
assumes *A*: *A* ≠ {} *compact A*
assumes *open B*
assumes *A* ⊆ *B*
obtains *ε* **where** *ε* > *0* {*x*. *infdist x A* ≤ *ε*} ⊆ *B*

5.22 Uniform Continuity

5.23 Continuity on a Compact Domain Implies Uniform Continuity

corollary *compact_uniformly_continuous*:
fixes *f* :: *'a* :: *metric_space* ⇒ *'b* :: *metric_space*

assumes f: *continuous_on S f* **and** S: *compact S*
shows *uniformly_continuous_on S f*

5.24 With Abstract Topology (TODO: move and remove dependency?)

5.25 Closed Nest

5.26 Consequences for Real Numbers

5.27 The infimum of the distance between two sets

definition *setdist* :: *'a::metric_space set* ⇒ *'a set* ⇒ *real* **where**
setdist S T ≡
(*if S* = {} ∨ *T* = {} *then 0*
else Inf {*dist x y*| *x y*. *x* ∈ *S* ∧ *y* ∈ *T*})

proposition *setdist_attains_inf*:
assumes *compact B B* ≠ {}
obtains *y* **where** *y* ∈ *B setdist A B* = *infdist y A*

5.28 Diameter Lemma

end

5.29 Elementary Normed Vector Spaces

theory *Elementary_Normed_Spaces*
imports
HOL−Library.FuncSet
Elementary_Metric_Spaces Cartesian_Space
Connected
begin

5.29.1 Orthogonal Transformation of Balls

5.29.2 Support

5.29.3 Intervals

5.29.4 Limit Points

5.29.5 Balls and Spheres in Normed Spaces

corollary *compact_sphere* [*simp*]:
fixes *a* :: *'a*::{*real_normed_vector*,*perfect_space*,*heine_borel*}
shows *compact* (*sphere a r*)

corollary *bounded_sphere* [*simp*]:
fixes *a* :: *'a*::{*real_normed_vector*,*perfect_space*,*heine_borel*}
shows *bounded* (*sphere a r*)

corollary *closed_sphere* [*simp*]:
fixes *a* :: *'a*::{*real_normed_vector*,*perfect_space*,*heine_borel*}
shows *closed* (*sphere a r*)

5.29.6 Filters

5.29.7 Trivial Limits

5.29.8 Limits

proposition *Lim_at_infinity*: $(f \longrightarrow l)$ *at_infinity* $\longleftrightarrow$ $(\forall e{>}0.\ \exists b.\ \forall x.\ norm\ x \geq b \longrightarrow dist\ (f\ x)\ l < e)$

corollary *Lim_at_infinityI* [*intro?*]:
assumes $\bigwedge e.\ e > 0 \Longrightarrow \exists B.\ \forall x.\ norm\ x \geq B \longrightarrow dist\ (f\ x)\ l \leq e$
shows $(f \longrightarrow l)$ *at_infinity*

5.29.9 Boundedness

corollary *cobounded_imp_unbounded*:
fixes *S* :: *'a*::{*real_normed_vector*, *perfect_space*} *set*
shows *bounded* $(-\ S) \Longrightarrow \neg$ *bounded S*

5.29.10 Normed spaces with the Heine-Borel property

5.29.11 Intersecting chains of compact sets and the Baire property

proposition *bounded_closed_chain*:
fixes $\mathcal{F}$ :: *'a*::*heine_borel set set*
assumes $B \in \mathcal{F}$ *bounded B* **and** $\mathcal{F}$: $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *closed S* **and** $\{\} \notin \mathcal{F}$
and *chain*: $\bigwedge S\ T.\ S \in \mathcal{F} \wedge T \in \mathcal{F} \Longrightarrow S \subseteq T \vee T \subseteq S$

shows $\bigcap \mathcal{F} \neq \{\}$

corollary *compact_chain*:
fixes $\mathcal{F}$ :: *'a*::*heine_borel set set*
assumes $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *compact S* $\{\} \notin \mathcal{F}$
$\bigwedge S\ T.\ S \in \mathcal{F} \wedge T \in \mathcal{F} \Longrightarrow S \subseteq T \vee T \subseteq S$
shows $\bigcap\ \mathcal{F} \neq \{\}$
theorem *Baire*:
fixes *S*::*'a*::{*real_normed_vector*,*heine_borel*} *set*
assumes *closed S countable* $\mathcal{G}$
and *ope*: $\bigwedge T.\ T \in \mathcal{G} \Longrightarrow$ *openin* (*top_of_set S*) $T \wedge S \subseteq$ *closure T*
shows $S \subseteq$ *closure*$(\bigcap \mathcal{G})$

5.29.12 Continuity

proposition *homeomorphic_ball_UNIV*:
fixes *a* ::*'a*::*real_normed_vector*
assumes $0 < r$ **shows** *ball a r homeomorphic* (*UNIV*:: *'a set*)

5.29.13 Connected Normed Spaces

end

5.30 Linear Decision Procedure for Normed Spaces

theory *Norm_Arith*
imports *HOL−Library.Sum_of_Squares*
begin

method_setup *norm* = ‹
Scan.succeed (*SIMPLE_METHOD' o NormArith.norm_arith_tac*)
› *prove simple linear statements about vector norms*

proposition *dist_triangle_add*:
fixes *x y x' y'* :: *'a*::*real_normed_vector*
shows *dist* $(x + y)$ $(x' + y') \leq$ *dist x x' + dist y y'*

end

Chapter 6

Vector Analysis

theory *Topology_Euclidean_Space*
imports
Elementary_Normed_Spaces
Linear_Algebra
Norm_Arith
begin

6.1 Elementary Topology in Euclidean Space

6.1.1 Boxes

abbreviation *One* :: *'a*::*euclidean_space* **where**
$One \equiv \sum Basis$

definition (**in** *euclidean_space*) *eucl_less* (**infix** ‹<*e*› *50*) **where**
eucl_less a b $\longleftrightarrow$ $(\forall i{\in}Basis.\ a \cdot i < b \cdot i)$

definition *box_eucl_less*: *box a b* = $\{x.\ a <e\ x \wedge x <e\ b\}$
definition *cbox a b* = $\{x.\ \forall i{\in}Basis.\ a \cdot i \leq x \cdot i \wedge x \cdot i \leq b \cdot i\}$

corollary *open_countable_Union_open_box*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *open S*
obtains $\mathcal{D}$ **where** *countable* $\mathcal{D}$ $\mathcal{D} \subseteq Pow\ S$ $\bigwedge X.\ X \in \mathcal{D} \Longrightarrow \exists a\ b.\ X = box\ a\ b$ $\bigcup \mathcal{D} = S$

corollary *open_countable_Union_open_cbox*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *open S*
obtains $\mathcal{D}$ **where** *countable* $\mathcal{D}$ $\mathcal{D} \subseteq Pow\ S$ $\bigwedge X.\ X \in \mathcal{D} \Longrightarrow \exists a\ b.\ X = cbox\ a\ b$ $\bigcup \mathcal{D} = S$

6.1.2 General Intervals

definition *is_interval* (*s*::(′*a*::*euclidean_space*) *set*) ⟷
(∀ *a*∈*s*. ∀ *b*∈*s*. ∀ *x*. (∀ *i*∈*Basis*. ((*a*•*i* ≤ *x*•*i* ∧ *x*•*i* ≤ *b*•*i*) ∨ (*b*•*i* ≤ *x*•*i* ∧ *x*•*i* ≤ *a*•*i*)))
⟶ *x* ∈ *s*)

6.1.3 Limit Component Bounds

6.1.4 Class Instances

instance *euclidean_space* ⊆ *heine_borel*

instance *euclidean_space* ⊆ *banach*

6.1.5 Compact Boxes

proposition *is_interval_compact*:
is_interval S ∧ *compact S* ⟷ (∃ *a b*. *S* = *cbox a b*) (**is** *?lhs* = *?rhs*)

proposition *tendsto_componentwise_iff*:
fixes *f* :: _ ⇒ ′*b*::*euclidean_space*
shows (*f* ⟶ *l*) *F* ⟷ (∀ *i* ∈ *Basis*. ((λ*x*. (*f x* • *i*)) ⟶ (*l* • *i*)) *F*)
(**is** *?lhs* = *?rhs*)

corollary *continuous_componentwise*:
continuous F f ⟷ (∀ *i* ∈ *Basis*. *continuous F* (λ*x*. (*f x* • *i*)))

corollary *continuous_on_componentwise*:
fixes *S* :: ′*a* :: *t2_space set*
shows *continuous_on S f* ⟷ (∀ *i* ∈ *Basis*. *continuous_on S* (λ*x*. (*f x* • *i*)))

6.1.6 Separability

proposition *separable*:
fixes *S* :: ′*a*::{*metric_space*, *second_countable_topology*} *set*
obtains *T* **where** *countable T T* ⊆ *S S* ⊆ *closure T*
proposition *open_surjective_linear_image*:
fixes *f* :: ′*a*::*real_normed_vector* ⇒ ′*b*::*euclidean_space*
assumes *open A linear f surj f*
shows *open*(*f* ‘ *A*)

corollary *open_bijective_linear_image_eq*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *linear f bij f*
shows *open*(f ‘ A) ⟷ *open A*

corollary *interior_bijective_linear_image*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *linear f bij f*
shows *interior* (f ‘ S) = f ‘ *interior S*
proposition *injective_imp_isometric*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes s: *closed s subspace s*
and f: *bounded_linear f* ∀ x∈s. $f\ x = 0 \longrightarrow x = 0$
shows ∃ e>0. ∀ x∈s. *norm* ($f\ x$) ≥ e ∗ *norm x*

proposition *closed_injective_image_subspace*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *subspace s bounded_linear f* ∀ x∈s. $f\ x = 0 \longrightarrow x = 0$ *closed s*
shows *closed*(f ‘ s)

6.1.7 Set Distance

corollary *setdist_gt_0_compact_closed*:
assumes S: *compact S* **and** T: *closed T*
shows *setdist S T* > 0 ⟷ ($S \neq \{\} \wedge T \neq \{\} \wedge S \cap T = \{\}$)

end

6.2 Line Segment

theory *Line_Segment*
imports
Convex
Topology_Euclidean_Space
begin

corollary *component_complement_connected*:
fixes S :: *'a::real_normed_vector set*
assumes *connected S C* ∈ *components* (−S)
shows *connected*(−C)

proposition *clopen*:
fixes S :: *'a* :: *real_normed_vector set*
shows *closed S* ∧ *open S* ⟷ $S = \{\} \vee S = UNIV$

corollary *compact_open*:

fixes $S :: 'a :: euclidean_space\ set$
shows $compact\ S \wedge open\ S \longleftrightarrow S = \{\}$

corollary *finite_imp_not_open*:
fixes $S :: 'a::\{real_normed_vector, perfect_space\}\ set$
shows $[\![finite\ S;\ open\ S]\!] \Longrightarrow S=\{\}$

corollary *empty_interior_finite*:
fixes $S :: 'a::\{real_normed_vector, perfect_space\}\ set$
shows $finite\ S \Longrightarrow interior\ S = \{\}$

6.2.1 Midpoint

definition $midpoint :: 'a::real_vector \Rightarrow 'a \Rightarrow 'a$
where $midpoint\ a\ b = (inverse\ (2::real)) *_R (a + b)$

6.2.2 Open and closed segments

definition $closed_segment :: 'a::real_vector \Rightarrow 'a \Rightarrow 'a\ set$
where $closed_segment\ a\ b = \{(1 - u) *_R a + u *_R b \mid u::real.\ 0 \leq u \wedge u \leq 1\}$

definition $open_segment :: 'a::real_vector \Rightarrow 'a \Rightarrow 'a\ set$ **where**
$open_segment\ a\ b \equiv closed_segment\ a\ b - \{a,b\}$

proposition *dist_decreases_open_segment*:
fixes $a :: 'a :: euclidean_space$
assumes $x \in open_segment\ a\ b$
shows $dist\ c\ x < dist\ c\ a \vee dist\ c\ x < dist\ c\ b$

corollary *open_segment_furthest_le*:
fixes $a\ b\ x\ y :: 'a::euclidean_space$
assumes $x \in open_segment\ a\ b$
shows $norm\ (y - x) < norm\ (y - a) \vee\ \ norm\ (y - x) < norm\ (y - b)$

corollary *dist_decreases_closed_segment*:
fixes $a :: 'a :: euclidean_space$
assumes $x \in closed_segment\ a\ b$
shows $dist\ c\ x \leq dist\ c\ a \vee dist\ c\ x \leq dist\ c\ b$

corollary *segment_furthest_le*:
fixes $a\ b\ x\ y :: 'a::euclidean_space$
assumes $x \in closed_segment\ a\ b$
shows $norm\ (y - x) \leq norm\ (y - a) \vee\ \ norm\ (y - x) \leq norm\ (y - b)$

6.2.3 Betweenness

definition *between* = $(\lambda(a,b)\ x.\ x \in$ *closed_segment a b*$)$

end

6.3 Convex Sets and Functions on (Normed) Euclidean Spaces

theory *Convex_Euclidean_Space*
imports
Convex Topology_Euclidean_Space Line_Segment
begin
corollary *empty_interior_lowdim*:
fixes S :: $'n$::*euclidean_space set*
shows *dim* $S <$ *DIM* $('n) \Longrightarrow$ *interior* $S = \{\}$

corollary *aff_dim_nonempty_interior*:
fixes S :: $'a$::*euclidean_space set*
shows *interior* $S \neq \{\} \Longrightarrow$ *aff_dim* $S =$ *DIM*$('a)$

6.3.1 Relative interior of a set

definition *rel_interior* $S =$
$\{x.\ \exists\, T.$ *openin* $($*top_of_set* $($*affine hull* $S))\ T \wedge x \in T \wedge T \subseteq S\}$
definition *rel_open* $S \longleftrightarrow$ *rel_interior* $S = S$

6.3.2 Closest point of a convex set is unique, with a continuous projection

definition *closest_point* :: $'a$::$\{$*real_inner*,*heine_borel*$\}$ *set* $\Rightarrow$ $'a \Rightarrow 'a$
where *closest_point* $S\ a = (SOME\ x.\ x \in S \wedge (\forall\, y{\in}S.$ *dist* $a\ x \leq$ *dist* $a\ y))$

proposition *closest_point_in_rel_interior*:
assumes *closed* S $S \neq \{\}$ **and** x: $x \in$ *affine hull* S
shows *closest_point* $S\ x \in$ *rel_interior* $S \longleftrightarrow x \in$ *rel_interior* S

6.3.3 The relative frontier of a set

definition *rel_frontier* $S =$ *closure* $S -$ *rel_interior* S

end

Chapter 7

Unsorted

theory *Starlike*
imports
Convex_Euclidean_Space
Line_Segment
begin

7.1 Relative frontier of a convex set

proposition *ray_to_rel_frontier*:
fixes a :: *'a*::*real_inner*
assumes *bounded S*
and *a*: $a \in$ *rel_interior S*
and *aff*: $(a + l) \in$ *affine hull S*
and $l \neq 0$
obtains d **where** $0 < d$ $(a + d *_R l) \in$ *rel_frontier S*
$\bigwedge e.\ [\![0 \leq e; e < d]\!] \Longrightarrow (a + e *_R l) \in$ *rel_interior S*

corollary *ray_to_frontier*:
fixes a :: *'a*::*euclidean_space*
assumes *bounded S*
and *a*: $a \in$ *interior S*
and $l \neq 0$
obtains d **where** $0 < d$ $(a + d *_R l) \in$ *frontier S*
$\bigwedge e.\ [\![0 \leq e; e < d]\!] \Longrightarrow (a + e *_R l) \in$ *interior S*

proposition *rel_frontier_not_sing*:
fixes a :: *'a*::*euclidean_space*
assumes *bounded S*
shows *rel_frontier S* $\neq \{a\}$

7.2 Coplanarity, and collinearity in terms of affine hull

definition *coplanar* **where**
coplanar S ≡ ∃ *u v w*. *S* ⊆ *affine hull* {*u*,*v*,*w*}

7.3 Connectedness of the intersection of a chain

proposition *connected_chain*:
fixes $\mathcal{F}$ :: *'a* :: *euclidean_space set set*
assumes *cc*: ⋀*S*. *S* ∈ $\mathcal{F}$ ⟹ *compact S* ∧ *connected S*
and *linear*: ⋀*S T*. *S* ∈ $\mathcal{F}$ ∧ *T* ∈ $\mathcal{F}$ ⟹ *S* ⊆ *T* ∨ *T* ⊆ *S*
shows *connected*(⋂$\mathcal{F}$)

7.4 Proper maps, including projections out of compact sets

proposition *proper_map*:
fixes *f* :: *'a*::*heine_borel* ⇒ *'b*::*heine_borel*
assumes *closedin* (*top_of_set S*) *K*
and *com*: ⋀*U*. ⟦*U* ⊆ *T*; *compact U*⟧ ⟹ *compact* (*S* ∩ *f* −‘ *U*)
and *f* ‘ *S* ⊆ *T*
shows *closedin* (*top_of_set T*) (*f* ‘ *K*)

7.5 Closure of conic hulls

proposition *closedin_conic_hull*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *compact T* 0 ∉ *T T* ⊆ *S*
shows *closedin* (*top_of_set* (*conic hull S*)) (*conic hull T*)

corollary *affine_hull_convex_Int_open*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *convex S open T S* ∩ *T* ≠ {}
shows *affine hull* (*S* ∩ *T*) = *affine hull S*

corollary *affine_hull_affine_Int_nonempty_interior*:

fixes S :: $'a$::*real_normed_vector set*
assumes *affine* S $S \cap$ *interior* $T \neq \{\}$
shows *affine hull* $(S \cap T)$ = *affine hull* S

corollary *affine_hull_affine_Int_open*:
fixes S :: $'a$::*real_normed_vector set*
assumes *affine* S *open* T $S \cap T \neq \{\}$
shows *affine hull* $(S \cap T)$ = *affine hull* S

corollary *affine_hull_convex_Int_openin*:
fixes S :: $'a$::*real_normed_vector set*
assumes *convex* S *openin* (*top_of_set* (*affine hull* S)) T $S \cap T \neq \{\}$
shows *affine hull* $(S \cap T)$ = *affine hull* S

corollary *affine_hull_openin*:
fixes S :: $'a$::*real_normed_vector set*
assumes *openin* (*top_of_set* (*affine hull* T)) S $S \neq \{\}$
shows *affine hull* S = *affine hull* T

corollary *affine_hull_open*:
fixes S :: $'a$::*real_normed_vector set*
assumes *open* S $S \neq \{\}$
shows *affine hull* S = *UNIV*

proposition *aff_dim_eq_hyperplane*:
fixes S :: $'a$::*euclidean_space set*
shows *aff_dim* S = *DIM*($'a$) − *1* $\longleftrightarrow$ $(\exists\, a\ b.\ a \neq 0 \wedge$ *affine hull* $S = \{x.\ a \cdot x = b\})$
(**is** *?lhs* = *?rhs*)

corollary *aff_dim_hyperplane* [*simp*]:
fixes a :: $'a$::*euclidean_space*
shows $a \neq 0 \Longrightarrow$ *aff_dim* $\{x.\ a \cdot x = r\}$ = *DIM*($'a$) − *1*

proposition *aff_dim_sums_Int*:
assumes *affine* S
and *affine* T
and $S \cap T \neq \{\}$
shows *aff_dim* $\{x + y |\ x\ y.\ x \in S \wedge y \in T\}$ = (*aff_dim* S + *aff_dim* T) − *aff_dim*$(S \cap T)$

7.6 Lower-dimensional affine subsets are nowhere dense

proposition *dense_complement_subspace*:
fixes $S :: 'a :: euclidean_space\ set$
assumes *dim_less*: $dim\ T < dim\ S$ **and** $subspace\ S$ **shows** $closure(S - T) = S$

7.7 Paracompactness

proposition *paracompact*:
fixes $S :: 'a :: \{metric_space, second_countable_topology\}\ set$
assumes $S \subseteq \bigcup \mathcal{C}$ **and** *opC*: $\bigwedge T.\ T \in \mathcal{C} \Longrightarrow open\ T$
obtains $\mathcal{C}'$ **where** $S \subseteq \bigcup \mathcal{C}'$
and $\bigwedge U.\ U \in \mathcal{C}' \Longrightarrow open\ U \wedge (\exists T.\ T \in \mathcal{C} \wedge U \subseteq T)$
and $\bigwedge x.\ x \in S$
$\Longrightarrow \exists V.\ open\ V \wedge x \in V \wedge finite\ \{U.\ U \in \mathcal{C}' \wedge (U \cap V \neq \{\})\}$

corollary *paracompact_closedin*:
fixes $S :: 'a :: \{metric_space, second_countable_topology\}\ set$
assumes *cin*: $closedin\ (top_of_set\ U)\ S$
and *oin*: $\bigwedge T.\ T \in \mathcal{C} \Longrightarrow openin\ (top_of_set\ U)\ T$
and $S \subseteq \bigcup \mathcal{C}$
obtains $\mathcal{C}'$ **where** $S \subseteq \bigcup \mathcal{C}'$
and $\bigwedge V.\ V \in \mathcal{C}' \Longrightarrow openin\ (top_of_set\ U)\ V \wedge (\exists T.\ T \in \mathcal{C} \wedge V \subseteq T)$
and $\bigwedge x.\ x \in U$
$\Longrightarrow \exists V.\ openin\ (top_of_set\ U)\ V \wedge x \in V \wedge$
$finite\ \{X.\ X \in \mathcal{C}' \wedge (X \cap V \neq \{\})\}$

7.8 Covering an open set by a countable chain of compact sets

proposition *open_Union_compact_subsets*:
fixes $S :: 'a::euclidean_space\ set$
assumes $open\ S$
obtains C **where** $\bigwedge n.\ compact(C\ n)\ \bigwedge n.\ C\ n \subseteq S$
$\bigwedge n.\ C\ n \subseteq interior(C(Suc\ n))$
$\bigcup (range\ C) = S$
$\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow \exists N.\ \forall n \geq N.\ K \subseteq (C\ n)$

7.9 Orthogonal complement

definition *orthogonal_comp* (‹(‹*open_block notation=‹postfix* ⊥››_⊥)› [*80*] *80*)

where *orthogonal_comp* $W \equiv \{x.\ \forall\, y \in W.\ \mathit{orthogonal}\ y\ x\}$

proposition *subspace_orthogonal_comp*: *subspace* $(W^{\perp})$

proposition *subspace_sum_orthogonal_comp*:
fixes $U :: 'a :: \mathit{euclidean_space}\ \mathit{set}$
assumes *subspace* U
shows $U + U^{\perp} = \mathit{UNIV}$

end

7.10 Path-Connectedness

theory *Path_Connected*
imports
Starlike T1_Spaces
begin

7.10.1 Paths and Arcs

definition *path* :: $(\mathit{real} \Rightarrow 'a{::}\mathit{topological_space}) \Rightarrow \mathit{bool}$
where $\mathit{path}\ g \equiv \mathit{continuous_on}\ \{0..1\}\ g$

definition *pathstart* :: $(\mathit{real} \Rightarrow 'a{::}\mathit{topological_space}) \Rightarrow 'a$
where $\mathit{pathstart}\ g \equiv g\ 0$

definition *pathfinish* :: $(\mathit{real} \Rightarrow 'a{::}\mathit{topological_space}) \Rightarrow 'a$
where $\mathit{pathfinish}\ g \equiv g\ 1$

definition *path_image* :: $(\mathit{real} \Rightarrow 'a{::}\mathit{topological_space}) \Rightarrow 'a\ \mathit{set}$
where $\mathit{path_image}\ g \equiv g\ `\ \{0\ ..\ 1\}$

definition *reversepath* :: $(\mathit{real} \Rightarrow 'a{::}\mathit{topological_space}) \Rightarrow \mathit{real} \Rightarrow 'a$
where $\mathit{reversepath}\ g \equiv (\lambda x.\ g(1 - x))$

definition *joinpaths* :: $(\mathit{real} \Rightarrow 'a{::}\mathit{topological_space}) \Rightarrow (\mathit{real} \Rightarrow 'a) \Rightarrow \mathit{real} \Rightarrow 'a$
(**infixr** ‹+++› 75)
where $g1\ +++\ g2 \equiv (\lambda x.\ \mathit{if}\ x \leq 1/2\ \mathit{then}\ g1\ (2 * x)\ \mathit{else}\ g2\ (2 * x - 1))$

definition *loop_free* :: $(\mathit{real} \Rightarrow 'a{::}\mathit{topological_space}) \Rightarrow \mathit{bool}$
where $\mathit{loop_free}\ g \equiv \forall\, x{\in}\{0..1\}.\ \forall\, y{\in}\{0..1\}.\ g\ x = g\ y \longrightarrow x = y \vee x = 0 \wedge y = 1 \vee x = 1 \wedge y = 0$

definition *simple_path* :: $(\mathit{real} \Rightarrow 'a{::}\mathit{topological_space}) \Rightarrow \mathit{bool}$
where $\mathit{simple_path}\ g \equiv \mathit{path}\ g \wedge \mathit{loop_free}\ g$

definition *arc* :: $(\mathit{real} \Rightarrow 'a :: \mathit{topological_space}) \Rightarrow \mathit{bool}$
where $\mathit{arc}\ g \equiv \mathit{path}\ g \wedge \mathit{inj_on}\ g\ \{0..1\}$

7.10.2 Subpath

definition *subpath* :: *real* ⇒ *real* ⇒ (*real* ⇒ *'a*) ⇒ *real* ⇒ *'a*::*real_normed_vector*
where *subpath a b g* ≡ λ*x*. *g*((*b* − *a*) ∗ *x* + *a*)

7.10.3 Shift Path to Start at Some Given Point

definition *shiftpath* :: *real* ⇒ (*real* ⇒ *'a*::*topological_space*) ⇒ *real* ⇒ *'a*
where *shiftpath a f* = (λ*x*. *if* (*a* + *x*) ≤ *1 then f* (*a* + *x*) *else f* (*a* + *x* − *1*))

7.10.4 Straight-Line Paths

definition *linepath* :: *'a*::*real_normed_vector* ⇒ *'a* ⇒ *real* ⇒ *'a*
where *linepath a b* = (λ*x*. (*1* − *x*) ∗$_R$ *a* + *x* ∗$_R$ *b*)
proposition *injective_eq_1d_open_map_UNIV*:
fixes *f* :: *real* ⇒ *real*
assumes *contf*: *continuous_on S f* **and** *S*: *is_interval S*
shows *inj_on f S* ⟷ (∀ *T*. *open T* ∧ *T* ⊆ *S* ⟶ *open*(*f* ‘ *T*))
(**is** *?lhs* = *?rhs*)

7.10.5 Path component

definition *path_component S x y* ≡
(∃ *g*. *path g* ∧ *path_image g* ⊆ *S* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*)

abbreviation
path_component_set S x ≡ *Collect* (*path_component S x*)

7.10.6 Path connectedness of a space

definition *path_connected S* ⟷
(∀ *x*∈*S*. ∀ *y*∈*S*. ∃ *g*. *path g* ∧ *path_image g* ⊆ *S* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*)

7.10.7 Path components

7.10.8 Paths and path-connectedness

7.10.9 Path components

7.10.10 Sphere is path-connected

corollary *connected_punctured_universe*:

$2 \leq DIM('N::euclidean_space) \Longrightarrow connected(- \{a::'N\})$

proposition *path_connected_sphere*:
fixes $a :: 'a :: euclidean_space$
assumes $2 \leq DIM('a)$
shows *path_connected*(*sphere a r*)

corollary *path_connected_complement_bounded_convex*:
fixes $S :: 'a :: euclidean_space\ set$
assumes *bounded S convex S* **and** *2*: $2 \leq DIM('a)$
shows *path_connected* $(- S)$

proposition *connected_open_delete*:
assumes *open S connected S* **and** *2*: $2 \leq DIM('N::euclidean_space)$
shows $connected(S - \{a::'N\})$

corollary *path_connected_open_delete*:
assumes *open S connected S* **and** *2*: $2 \leq DIM('N::euclidean_space)$
shows $path_connected(S - \{a::'N\})$

corollary *path_connected_punctured_ball*:
$2 \leq DIM('N::euclidean_space) \Longrightarrow path_connected(ball\ a\ r - \{a::'N\})$

corollary *connected_punctured_ball*:
$2 \leq DIM('N::euclidean_space) \Longrightarrow connected(ball\ a\ r - \{a::'N\})$

corollary *connected_open_delete_finite*:
fixes $S\ T::'a::euclidean_space\ set$
assumes *S*: *open S connected S* **and** *2*: $2 \leq DIM('a)$ **and** *finite T*
shows $connected(S - T)$

7.10.11 Every annulus is a connected set

proposition *path_connected_annulus*:
fixes $a :: 'N::euclidean_space$
assumes $2 \leq DIM('N)$
shows $path_connected\ \{x.\ r1 < norm(x - a) \wedge norm(x - a) < r2\}$
$path_connected\ \{x.\ r1 < norm(x - a) \wedge norm(x - a) \leq r2\}$
$path_connected\ \{x.\ r1 \leq norm(x - a) \wedge norm(x - a) < r2\}$
$path_connected\ \{x.\ r1 \leq norm(x - a) \wedge norm(x - a) \leq r2\}$

proposition *connected_annulus*:
fixes $a :: 'N::euclidean_space$
assumes $2 \leq DIM('N::euclidean_space)$
shows $connected\ \{x.\ r1 < norm(x - a) \wedge norm(x - a) < r2\}$
$connected\ \{x.\ r1 < norm(x - a) \wedge norm(x - a) \leq r2\}$

connected $\{x.\ r1 \leq norm(x - a) \wedge norm(x - a) < r2\}$
connected $\{x.\ r1 \leq norm(x - a) \wedge norm(x - a) \leq r2\}$

corollary *open_components*:
fixes S :: *'a*::*real_normed_vector set*
shows $[\![open\ u;\ S \in components\ u]\!] \Longrightarrow open\ S$

proposition *components_open_unique*:
fixes S :: *'a*::*real_normed_vector set*
assumes *pairwise disjnt A* $\bigcup A = S$
$\bigwedge X.\ X \in A \Longrightarrow open\ X \wedge connected\ X \wedge X \neq \{\}$
shows *components* $S = A$

7.10.12 The *inside* and *outside* of a Set

The inside comprises the points in a bounded connected component of the set's complement. The outside comprises the points in unbounded connected component of the complement.

definition *inside* **where**
inside $S \equiv \{x.\ (x \notin S) \wedge bounded(connected_component_set\ (-S)\ x)\}$

definition *outside* **where**
outside $S \equiv -S \cap \{x.\ \neg\ bounded(connected_component_set\ (-S)\ x)\}$

7.10.13 Lipschitz projections

proposition *inverse_lipschitz_convex_spherical_projection_explicit*:
fixes $x\ y$ :: *'a*::*euclidean_space*
assumes *convex S r>0* $0 \in S$
and *reli*: *ball 0 r* $\cap$ *affine hull S* $\subseteq$ *rel_interior S*
and *S*: $S \subseteq$ *cball 0 R*
and *x*: $x \in$ *rel_frontier S* **and** *y*: $y \in$ *rel_frontier S*
shows $r\ /\ R^2 * dist\ x\ y \leq dist\ ((1\ /\ norm\ x) *_R x)\ ((1\ /\ norm\ y) *_R y)$

corollary *inverse_lipschitz_convex_spherical_projection*:
fixes S :: *'a*::*euclidean_space set*
assumes *convex S bounded S* **and** *0*: $0 \in$ *rel_interior S*
shows $\exists B>0.\ \forall x\ y.\ x \in$ *rel_frontier S* $\longrightarrow$ $y \in$ *rel_frontier S* $\longrightarrow$
$B * dist\ x\ y \leq dist\ ((1\ /\ norm\ x) *_R x)\ ((1\ /\ norm\ y) *_R y)$

proposition *lipschitz_convex_spherical_projection_explicit*:
fixes r :: *real* **and** S :: *'a*::*euclidean_space set*
assumes *convex S* $0 < r$ $0 \in S$
and *ball_sub*: *ball 0 r* $\cap$ *affine hull S* $\subseteq$ *rel_interior S*
and *x_rf*: $x \in$ *rel_frontier S* **and** *y_rf*: $y \in$ *rel_frontier S*
shows $dist\ (x /_R norm\ x)\ (y /_R norm\ y) \leq inverse\ r * dist\ x\ y$

corollary *lipschitz_convex_spherical_projection*:
fixes S :: *'a::euclidean_space set*
assumes *convex* S $0 \in$ *rel_interior* S
shows $\exists B.\ \forall x\ y.\ x \in$ *rel_frontier* $S \longrightarrow y \in$ *rel_frontier* $S \longrightarrow$
dist $(x\ /_R$ *norm* $x)$ $(y\ /_R$ *norm* $y) \leq B *$ *dist* $x\ y$

7.10.14 Condition for an open map's image to contain a ball

proposition *ball_subset_open_map_image*:
fixes f :: *'a::heine_borel* $\Rightarrow$ *'b* :: $\{$*real_normed_vector*,*heine_borel*$\}$
assumes *contf*: *continuous_on* $($*closure* $S)$ f
and *oint*: *open* $(f$ ‘ *interior* $S)$
and *le_no*: $\bigwedge z.\ z \in$ *frontier* $S \Longrightarrow r \leq$ *norm*$(f\ z - f\ a)$
and *bounded* S $a \in S$ $0 < r$
shows *ball* $(f\ a)$ $r \subseteq f$ ‘ S

proposition *embedding_map_into_euclideanreal*:
assumes *path_connected_space* X
shows *embedding_map* X *euclideanreal* $f \longleftrightarrow$
continuous_map X *euclideanreal* $f \wedge$ *inj_on* f $($*topspace* $X)$

end

7.11 Neighbourhood bases and Locally path-connected spaces

theory *Locally*
imports
Path_Connected Function_Topology Sum_Topology
begin

7.11.1 Neighbourhood Bases

7.11.2 Locally path-connected spaces

7.11.3 Locally connected spaces

7.11.4 Dimension of a topological space

end

7.12 Some Uncountable Sets

theory *Uncountable_Sets*
imports *Path_Connected Continuum_Not_Denumerable*
begin

end

7.13 Homotopy of Maps

theory *Homotopy*
imports *Path_Connected Product_Topology Uncountable_Sets*
begin

definition *homotopic_with*
where
homotopic_with P X Y f g ≡
(∃ *h*. *continuous_map* (*prod_topology* (*top_of_set* {*0..1*::*real*}) *X*) *Y h* ∧
(∀ *x*. *h*(*0*, *x*) = *f x*) ∧
(∀ *x*. *h*(*1*, *x*) = *g x*) ∧
(∀ *t* ∈ {*0..1*}. *P*(λ*x*. *h*(*t*,*x*))))

proposition *homotopic_with*:
assumes ⋀*h k*. (⋀*x*. *x* ∈ *topspace X* ⟹ *h x* = *k x*) ⟹ (*P h* ⟷ *P k*)
shows *homotopic_with P X Y p q* ⟷
(∃ *h*. *continuous_map* (*prod_topology* (*subtopology euclideanreal* {*0..1*}) *X*) *Y h* ∧
(∀ *x* ∈ *topspace X*. *h*(*0*,*x*) = *p x*) ∧
(∀ *x* ∈ *topspace X*. *h*(*1*,*x*) = *q x*) ∧
(∀ *t* ∈ {*0..1*}. *P*(λ*x*. *h*(*t*, *x*))))

7.13.1 Homotopy with P is an equivalence relation

proposition *homotopic_with_trans*:
assumes *homotopic_with P X Y f g homotopic_with P X Y g h*
shows *homotopic_with P X Y f h*

7.13.2 Continuity lemmas

corollary *homotopic_compose*:
assumes *homotopic_with* (λ*x*. *True*) *X Y f f′ homotopic_with* (λ*x*. *True*) *Y Z g g′*
shows *homotopic_with* (λ*x*. *True*) *X Z* (*g* ∘ *f*) (*g′* ∘ *f′*)

proposition *homotopic_with_compose_continuous_right*:
⟦*homotopic_with_canon* (λ*f*. *p* (*f* ∘ *h*)) *X Y f g*; *continuous_on W h*; *h* ∈ *W* → *X*⟧
⟹ *homotopic_with_canon p W Y* (*f* ∘ *h*) (*g* ∘ *h*)

proposition *homotopic_with_compose_continuous_left*:
⟦*homotopic_with_canon* (λ*f*. *p* (*h* ∘ *f*)) *X Y f g*; *continuous_on Y h*; *h* ∈ *Y* → *Z*⟧
⟹ *homotopic_with_canon p X Z* (*h* ∘ *f*) (*h* ∘ *g*)

proposition *homotopic_with_eq*:
assumes *h*: *homotopic_with P X Y f g*
and *f′*: ⋀*x*. *x* ∈ *topspace X* ⟹ *f′ x* = *f x*
and *g′*: ⋀*x*. *x* ∈ *topspace X* ⟹ *g′ x* = *g x*
and *P*: (⋀*h k*. (⋀*x*. *x* ∈ *topspace X* ⟹ *h x* = *k x*) ⟹ *P h* ⟷ *P k*)
shows *homotopic_with P X Y f′ g′*

7.13.3 Homotopy of paths, maintaining the same endpoints

definition *homotopic_paths* :: [*'a set*, *real* ⇒ *'a*, *real* ⇒ *'a*::*topological_space*] ⇒ *bool*
where
homotopic_paths S p q ≡
homotopic_with_canon (λ*r*. *pathstart r* = *pathstart p* ∧ *pathfinish r* = *pathfinish p*) {*0*..*1*} *S p q*

proposition *homotopic_paths_imp_pathstart*:
homotopic_paths S p q ⟹ *pathstart p* = *pathstart q*

proposition *homotopic_paths_imp_pathfinish*:
homotopic_paths S p q ⟹ *pathfinish p* = *pathfinish q*

proposition *homotopic_paths_refl* [*simp*]: *homotopic_paths S p p* ⟷ *path p* ∧ *path_image p* ⊆ *S*

proposition *homotopic_paths_sym*: *homotopic_paths S p q* ⟹ *homotopic_paths S q p*

proposition *homotopic_paths_sym_eq*: *homotopic_paths S p q* ⟷ *homotopic_paths S q p*

proposition *homotopic_paths_trans* [*trans*]:

assumes *homotopic_paths S p q homotopic_paths S q r*
shows *homotopic_paths S p r*

proposition *homotopic_paths_eq*:
⟦*path p*; *path_image p* ⊆ *S*; ⋀*t*. *t* ∈ {*0..1*} ⟹ *p t* = *q t*⟧ ⟹ *homotopic_paths S p q*

proposition *homotopic_paths_reparametrize*:
assumes *path p*
and *pips*: *path_image p* ⊆ *S*
and *contf*: *continuous_on* {*0..1*} *f*
and *f01* :*f* ∈ {*0..1*} → {*0..1*}
and [*simp*]: *f*(*0*) = *0 f*(*1*) = *1*
and *q*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *q*(*t*) = *p*(*f t*)
shows *homotopic_paths S p q*

proposition *homotopic_paths_reversepath*:
homotopic_paths S (*reversepath p*) (*reversepath q*) ⟷ *homotopic_paths S p q*

proposition *homotopic_paths_join*:
⟦*homotopic_paths S p p*′; *homotopic_paths S q q*′; *pathfinish p* = *pathstart q*⟧ ⟹ *homotopic_paths S* (*p* +++ *q*) (*p*′ +++ *q*′)

proposition *homotopic_paths_continuous_image*:
⟦*homotopic_paths S f g*; *continuous_on S h*; *h* ∈ *S* → *t*⟧ ⟹ *homotopic_paths t* (*h* ∘ *f*) (*h* ∘ *g*)

7.13.4 Group properties for homotopy of paths

So taking equivalence classes under homotopy would give the fundamental group

proposition *homotopic_paths_rid*:
assumes *path p path_image p* ⊆ *S*
shows *homotopic_paths S* (*p* +++ *linepath* (*pathfinish p*) (*pathfinish p*)) *p*

proposition *homotopic_paths_lid*:
⟦*path p*; *path_image p* ⊆ *S*⟧ ⟹ *homotopic_paths S* (*linepath* (*pathstart p*) (*pathstart p*) +++ *p*) *p*

proposition *homotopic_paths_assoc*:
⟦*path p*; *path_image p* ⊆ *S*; *path q*; *path_image q* ⊆ *S*; *path r*; *path_image r* ⊆ *S*; *pathfinish p* = *pathstart q*;
pathfinish q = *pathstart r*⟧
⟹ *homotopic_paths S* (*p* +++ (*q* +++ *r*)) ((*p* +++ *q*) +++ *r*)

proposition *homotopic_paths_rinv*:

assumes *path p path_image p* $\subseteq$ *S*
shows *homotopic_paths S* (*p +++ reversepath p*) (*linepath* (*pathstart p*) (*pathstart p*))

proposition *homotopic_paths_linv*:
assumes *path p path_image p* $\subseteq$ *S*
shows *homotopic_paths S* (*reversepath p +++ p*) (*linepath* (*pathfinish p*) (*pathfinish p*))

7.13.5 Homotopy of loops without requiring preservation of endpoints

definition *homotopic_loops* :: *'a*::*topological_space set* $\Rightarrow$ (*real* $\Rightarrow$ *'a*) $\Rightarrow$ (*real* $\Rightarrow$ *'a*) $\Rightarrow$ *bool* **where**
homotopic_loops S p q $\equiv$
homotopic_with_canon (λr. *pathfinish r* = *pathstart r*) {*0..1*} *S p q*

proposition *homotopic_loops_imp_loop*:
homotopic_loops S p q $\Longrightarrow$ *pathfinish p* = *pathstart p* $\wedge$ *pathfinish q* = *pathstart q*

proposition *homotopic_loops_imp_path*:
homotopic_loops S p q $\Longrightarrow$ *path p* $\wedge$ *path q*

proposition *homotopic_loops_imp_subset*:
homotopic_loops S p q $\Longrightarrow$ *path_image p* $\subseteq$ *S* $\wedge$ *path_image q* $\subseteq$ *S*

proposition *homotopic_loops_refl*:
homotopic_loops S p p $\longleftrightarrow$
path p $\wedge$ *path_image p* $\subseteq$ *S* $\wedge$ *pathfinish p* = *pathstart p*

proposition *homotopic_loops_sym*: *homotopic_loops S p q* $\Longrightarrow$ *homotopic_loops S q p*

proposition *homotopic_loops_sym_eq*: *homotopic_loops S p q* $\longleftrightarrow$ *homotopic_loops S q p*

proposition *homotopic_loops_trans*:
$[\![$*homotopic_loops S p q*; *homotopic_loops S q r*$]\!] \Longrightarrow$ *homotopic_loops S p r*

proposition *homotopic_loops_subset*:
$[\![$*homotopic_loops S p q*; *S* $\subseteq$ *t*$]\!] \Longrightarrow$ *homotopic_loops t p q*

proposition *homotopic_loops_eq*:
$[\![$*path p*; *path_image p* $\subseteq$ *S*; *pathfinish p* = *pathstart p*; $\bigwedge t$. *t* $\in$ {*0..1*} $\Longrightarrow$ *p*(*t*) = *q*(*t*)$]\!]$
$\Longrightarrow$ *homotopic_loops S p q*

proposition *homotopic_loops_continuous_image*:
⟦*homotopic_loops S f g*; *continuous_on S h*; *h* ∈ *S* → *t*⟧ ⟹ *homotopic_loops t* (*h* ∘ *f*) (*h* ∘ *g*)

7.13.6 Relations between the two variants of homotopy

proposition *homotopic_paths_imp_homotopic_loops*:
⟦*homotopic_paths S p q*; *pathfinish p* = *pathstart p*; *pathfinish q* = *pathstart p*⟧
⟹ *homotopic_loops S p q*

proposition *homotopic_loops_imp_homotopic_paths_null*:
assumes *homotopic_loops S p* (*linepath a a*)
shows *homotopic_paths S p* (*linepath* (*pathstart p*) (*pathstart p*))

proposition *homotopic_loops_conjugate*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *path p path q* **and** *pip*: *path_image p* ⊆ *S* **and** *piq*: *path_image q* ⊆ *S*
and *pq*: *pathfinish p* = *pathstart q* **and** *qloop*: *pathfinish q* = *pathstart q*
shows *homotopic_loops S* (*p* +++ *q* +++ *reversepath p*) *q*

7.13.7 Homotopy and subpaths

proposition *homotopic_join_subpaths*:
⟦*path g*; *path_image g* ⊆ *S*; *u* ∈ {*0..1*}; *v* ∈ {*0..1*}; *w* ∈ {*0..1*}⟧
⟹ *homotopic_paths S* (*subpath u v g* +++ *subpath v w g*) (*subpath u w g*)

7.13.8 Simply connected sets

defined as "all loops are homotopic (as loops)

definition *simply_connected* **where**
simply_connected S ≡
∀ *p q*. *path p* ∧ *pathfinish p* = *pathstart p* ∧ *path_image p* ⊆ *S* ∧
path q ∧ *pathfinish q* = *pathstart q* ∧ *path_image q* ⊆ *S*
⟶ *homotopic_loops S p q*

proposition *simply_connected_Times*:
fixes *S* :: *'a*::*real_normed_vector set* **and** *T* :: *'b*::*real_normed_vector set*
assumes *S*: *simply_connected S* **and** *T*: *simply_connected T*
shows *simply_connected*(*S* × *T*)

7.13.9 Contractible sets

definition *contractible* **where**
contractible S ≡ ∃ *a*. *homotopic_with_canon* (λ*x*. *True*) *S S id* (λ*x*. *a*)

proposition *contractible_imp_simply_connected*:
fixes S :: _::*real_normed_vector set*
assumes *contractible* S **shows** *simply_connected* S

corollary *contractible_imp_connected*:
fixes S :: _::*real_normed_vector set*
shows *contractible* $S \Longrightarrow$ *connected* S

7.13.10 Starlike sets

definition *starlike* $S \longleftrightarrow (\exists\, a{\in}S.\ \forall\, x{\in}S.\ \mathit{closed_segment}\ a\ x \subseteq S)$

7.13.11 The slotted complex plane

7.13.12 Contractible sets

7.13.13 Local versions of topological properties in general

definition *locally* :: $('a{::}\mathit{topological_space}\ \mathit{set} \Rightarrow \mathit{bool}) \Rightarrow 'a\ \mathit{set} \Rightarrow \mathit{bool}$
where
locally $P\ S \equiv$
$\forall\, w\ x.\ \mathit{openin}\ (\mathit{top_of_set}\ S)\ w \wedge x \in w$
$\longrightarrow (\exists\, U\ V.\ \mathit{openin}\ (\mathit{top_of_set}\ S)\ U \wedge P\ V \wedge x \in U \wedge U \subseteq V \wedge V \subseteq w)$

proposition *homeomorphism_locally_imp*:
fixes S :: $'a{::}\mathit{metric_space}\ \mathit{set}$ **and** T :: $'b{::}\mathit{t2_space}\ \mathit{set}$
assumes S: *locally* $P\ S$ **and** *hom*: *homeomorphism* $S\ T\ f\ g$
and Q: $\bigwedge S\ S'.\ [\![P\ S;\ \mathit{homeomorphism}\ S\ S'\ f\ g]\!] \Longrightarrow Q\ S'$
shows *locally* $Q\ T$

7.13.14 An induction principle for connected sets

proposition *connected_induction*:
assumes *connected* S
and *opD*: $\bigwedge T\ a.\ [\![\mathit{openin}\ (\mathit{top_of_set}\ S)\ T;\ a \in T]\!] \Longrightarrow \exists\, z.\ z \in T \wedge P\ z$
and *opI*: $\bigwedge a.\ a \in S$
$\Longrightarrow \exists\, T.\ \mathit{openin}\ (\mathit{top_of_set}\ S)\ T \wedge a \in T \wedge$
$(\forall\, x \in T.\ \forall\, y \in T.\ P\ x \wedge P\ y \wedge Q\ x \longrightarrow Q\ y)$
and *etc*: $a \in S\ b \in S\ P\ a\ P\ b\ Q\ a$

shows $Q\ b$

7.13.15 Basic properties of local compactness

proposition *locally_compact*:
fixes $S :: 'a :: metric_space\ set$
shows
$locally\ compact\ S \longleftrightarrow$
$(\forall x \in S.\ \exists u\ v.\ x \in u \wedge u \subseteq v \wedge v \subseteq S \wedge$
$openin\ (top_of_set\ S)\ u \wedge compact\ v)$
(**is** *?lhs* = *?rhs*)

7.13.16 Sura-Bura's results about compact components of sets

proposition *Sura_Bura_compact*:
fixes $S :: 'a::euclidean_space\ set$
assumes *compact S* **and** C: $C \in components\ S$
shows $C = \bigcap \{T.\ C \subseteq T \wedge openin\ (top_of_set\ S)\ T \wedge$
$closedin\ (top_of_set\ S)\ T\}$
(**is** $C = \bigcap ?\mathcal{T}$)

corollary *Sura_Bura_clopen_subset*:
fixes $S :: 'a::euclidean_space\ set$
assumes S: *locally compact S* **and** C: $C \in components\ S$ **and** *compact C*
and U: *open U* $C \subseteq U$
obtains K **where** $openin\ (top_of_set\ S)\ K$ *compact K* $C \subseteq K\ K \subseteq U$

corollary *Sura_Bura_clopen_subset_alt*:
fixes $S :: 'a::euclidean_space\ set$
assumes S: *locally compact S* **and** C: $C \in components\ S$ **and** *compact C*
and *opeSU*: $openin\ (top_of_set\ S)\ U$ **and** $C \subseteq U$
obtains K **where** $openin\ (top_of_set\ S)\ K$ *compact K* $C \subseteq K\ K \subseteq U$

corollary *Sura_Bura*:
fixes $S :: 'a::euclidean_space\ set$
assumes *locally compact S* $C \in components\ S$ *compact C*
shows $C = \bigcap \{K.\ C \subseteq K \wedge compact\ K \wedge openin\ (top_of_set\ S)\ K\}$
(**is** C = *?rhs*)

7.13.17 Special cases of local connectedness and path connectedness

proposition *locally_path_connected*:
locally path_connected S ⟷
(∀ *V x*. *openin* (*top_of_set S*) *V* ∧ *x* ∈ *V*
⟶ (∃ *U*. *openin* (*top_of_set S*) *U* ∧ *path_connected U* ∧ *x* ∈ *U* ∧ *U* ⊆ *V*))

proposition *locally_path_connected_open_path_component*:
locally path_connected S ⟷
(∀ *t x*. *openin* (*top_of_set S*) *t* ∧ *x* ∈ *t*
⟶ *openin* (*top_of_set S*) (*path_component_set t x*))

proposition *locally_connected_im_kleinen*:
locally connected S ⟷
(∀ *v x*. *openin* (*top_of_set S*) *v* ∧ *x* ∈ *v*
⟶ (∃ *u*. *openin* (*top_of_set S*) *u* ∧
x ∈ *u* ∧ *u* ⊆ *v* ∧
(∀ *y*. *y* ∈ *u* ⟶ (∃ *c*. *connected c* ∧ *c* ⊆ *v* ∧ *x* ∈ *c* ∧ *y* ∈ *c*))))
(**is** *?lhs* = *?rhs*)

proposition *locally_path_connected_im_kleinen*:
locally path_connected S ⟷
(∀ *v x*. *openin* (*top_of_set S*) *v* ∧ *x* ∈ *v*
⟶ (∃ *u*. *openin* (*top_of_set S*) *u* ∧
x ∈ *u* ∧ *u* ⊆ *v* ∧
(∀ *y*. *y* ∈ *u* ⟶ (∃ *p*. *path p* ∧ *path_image p* ⊆ *v* ∧
pathstart p = *x* ∧ *pathfinish p* = *y*))))
(**is** *?lhs* = *?rhs*)

7.13.18 Relations between components and path components

proposition *locally_connected_quotient_image*:
assumes *lcS*: *locally connected S*
and *oo*: ⋀*T*. *T* ⊆ *f* ‘ *S*
⟹ *openin* (*top_of_set S*) (*S* ∩ *f* −‘ *T*) ⟷
openin (*top_of_set* (*f* ‘ *S*)) *T*
shows *locally connected* (*f* ‘ *S*)
proposition *locally_path_connected_quotient_image*:
assumes *lcS*: *locally path_connected S*
and *oo*: ⋀*T*. *T* ⊆ *f* ‘ *S*
⟹ *openin* (*top_of_set S*) (*S* ∩ *f* −‘ *T*) ⟷ *openin* (*top_of_set* (*f* ‘ *S*)) *T*
shows *locally path_connected* (*f* ‘ *S*)

7.13.19 Existence of isometry between subspaces of same dimension

proposition *isometries_subspaces*:
fixes $S :: 'a::euclidean_space\ set$
and $T :: 'b::euclidean_space\ set$
assumes S: *subspace* S
and T: *subspace* T
and d: *dim* S = *dim* T
obtains $f\ g$ **where** *linear* f *linear* g $f\ `\ S = T$ $g\ `\ T = S$
$\bigwedge x.\ x \in S \Longrightarrow norm(f\ x) = norm\ x$
$\bigwedge x.\ x \in T \Longrightarrow norm(g\ x) = norm\ x$
$\bigwedge x.\ x \in S \Longrightarrow g(f\ x) = x$
$\bigwedge x.\ x \in T \Longrightarrow f(g\ x) = x$

corollary *isometry_subspaces*:
fixes $S :: 'a::euclidean_space\ set$
and $T :: 'b::euclidean_space\ set$
assumes S: *subspace* S
and T: *subspace* T
and d: *dim* S = *dim* T
obtains f **where** *linear* f $f\ `\ S = T$ $\bigwedge x.\ x \in S \Longrightarrow norm(f\ x) = norm\ x$

corollary *isomorphisms_UNIV_UNIV*:
assumes $DIM('M) = DIM('N)$
obtains $f::'M::euclidean_space \Rightarrow 'N::euclidean_space$ **and** g
where *linear* f *linear* g
$\bigwedge x.\ norm(f\ x) = norm\ x$ $\bigwedge y.\ norm(g\ y) = norm\ y$
$\bigwedge x.\ g\ (f\ x) = x$ $\bigwedge y.\ f(g\ y) = y$

7.13.20 Retracts, in a general sense, preserve (co)homotopic triviality)

locale *Retracts* =
fixes $S\ h\ t\ k$
assumes *conth*: *continuous_on* $S\ h$
and *imh*: $h\ `\ S = t$
and *contk*: *continuous_on* $t\ k$
and *imk*: $k \in t \to S$
and *idhk*: $\bigwedge y.\ y \in t \Longrightarrow h(k\ y) = y$

begin

7.13.21 Homotopy equivalence

7.13.22 Homotopy equivalence of topological spaces.

definition *homotopy_equivalent_space*
(**infix** ‹*homotopy'_equivalent'_space*› *50*)
where *X homotopy_equivalent_space Y* ≡
(∃*f g*. *continuous_map X Y f* ∧
continuous_map Y X g ∧
homotopic_with (λ*x*. *True*) *X X* (*g* ∘ *f*) *id* ∧
homotopic_with (λ*x*. *True*) *Y Y* (*f* ∘ *g*) *id*)

7.13.23 Contractible spaces

corollary *contractible_space_euclideanreal*: *contractible_space euclideanreal*

abbreviation *homotopy_eqv* :: *'a*::*topological_space set* ⇒ *'b*::*topological_space set* ⇒ *bool*
(**infix** ‹*homotopy'_eqv*› *50*)
where *S homotopy_eqv T* ≡ *top_of_set S homotopy_equivalent_space top_of_set T*

corollary *bounded_path_connected_Compl_real*:
fixes *S* :: *real set*
assumes *bounded S path_connected*(− *S*) **shows** *S* = {}
proposition *path_connected_convex_diff_countable*:
fixes *U* :: *'a*::*euclidean_space set*
assumes *convex U* ¬ *collinear U countable S*
shows *path_connected*(*U* − *S*)

corollary *connected_convex_diff_countable*:
fixes *U* :: *'a*::*euclidean_space set*
assumes *convex U* ¬ *collinear U countable S*
shows *connected*(*U* − *S*)

proposition *path_connected_openin_diff_countable*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *connected S* **and** *ope*: *openin* (*top_of_set* (*affine hull S*)) *S*
and ¬ *collinear S countable T*
shows *path_connected*(*S* − *T*)

corollary *connected_openin_diff_countable*:
fixes $S :: 'a::euclidean_space\ set$
assumes *connected* S **and** *ope*: *openin* (*top_of_set* (*affine hull* S)) S
and $\neg$ *collinear* S *countable* T
shows *connected*$(S - T)$

corollary *path_connected_open_diff_countable*:
fixes $S :: 'a::euclidean_space\ set$
assumes $2 \leq DIM('a)$ *open* S *connected* S *countable* T
shows *path_connected*$(S - T)$

corollary *connected_open_diff_countable*:
fixes $S :: 'a::euclidean_space\ set$
assumes $2 \leq DIM('a)$ *open* S *connected* S *countable* T
shows *connected*$(S - T)$

7.13.24 Nullhomotopic mappings

proposition *nullhomotopic_from_sphere_extension*:
fixes $f :: 'M::euclidean_space \Rightarrow 'a::real_normed_vector$
shows $(\exists\, c.\ homotopic_with_canon\ (\lambda x.\ True)\ (sphere\ a\ r)\ S\ f\ (\lambda x.\ c)) \longleftrightarrow$
$(\exists\, g.\ continuous_on\ (cball\ a\ r)\ g \wedge g\ `\ (cball\ a\ r) \subseteq S \wedge$
$(\forall\, x \in sphere\ a\ r.\ g\ x = f\ x))$
(**is** *?lhs* = *?rhs*)

end

7.14 Euclidean space and n-spheres, as subtopologies of n-dimensional space

theory *Abstract_Euclidean_Space*
imports *Homotopy Locally*
begin

7.14.1 Euclidean spaces as abstract topologies

7.14.2 n-dimensional spheres

proposition *contractible_space_upper_hemisphere*:
assumes $k \leq n$

shows *contractible_space*(*subtopology* (*nsphere n*) {*x*. *x k* ≥ *0*})

corollary *contractible_space_lower_hemisphere*:
assumes $k \leq n$
shows *contractible_space*(*subtopology* (*nsphere n*) {*x*. *x k* ≤ *0*})

proposition *nullhomotopic_nonsurjective_sphere_map*:
assumes *f*: *continuous_map* (*nsphere p*) (*nsphere p*) *f*
and *fim*: *f* ‘ (*topspace*(*nsphere p*)) ≠ *topspace*(*nsphere p*)
obtains *a* **where** *homotopic_with* (λ*x*. *True*) (*nsphere p*) (*nsphere p*) *f* (λ*x*. *a*)

end

7.15 Various Forms of Topological Spaces

theory *Abstract_Topological_Spaces*
imports *Lindelof_Spaces Locally Abstract_Euclidean_Space Sum_Topology FSigma*
begin

7.15.1 Connected topological spaces

7.15.2 The notion of "separated between" (complement of "connected between)"

7.15.3 Connected components

7.15.4 Monotone maps (in the general topological sense)

proposition *connected_space_monotone_quotient_map_preimage*:
assumes *f*: *monotone_map X Y f quotient_map X Y f* **and** *connected_space Y*
shows *connected_space X*

7.15.5 Other countability properties

7.15.6 Neigbourhood bases EXTRAS

7.15.7 T_0 spaces and the Kolmogorov quotient

proposition *t0_space_product_topology*:
t0_space (*product_topology X I*) ⟷ *product_topology X I* = *trivial_topology*
∨ (∀ *i* ∈ *I*. *t0_space* (*X i*))
(**is** *?lhs=?rhs*)

7.15.8 Kolmogorov quotients

7.15.9 Closed diagonals and graphs

7.15.10 KC spaces, those where all compact sets are closed.

proposition *kc_space_prod_topology_left*:
assumes *X*: *kc_space X* **and** *Y*: *Hausdorff_space Y*
shows *kc_space* (*prod_topology X Y*)

7.15.11 Technical results about proper maps, perfect maps, etc

7.15.12 Regular spaces

proposition *regular_space_continuous_proper_map_image*:
assumes *regular_space X* **and** *contf*: *continuous_map X Y f* **and** *pmapf*: *proper_map X Y f*
and *fim*: *f ' (topspace X) = topspace Y*
shows *regular_space Y*

proposition *regular_space_perfect_map_image_eq*:
assumes *Hausdorff_space X* **and** *perf*: *perfect_map X Y f*

shows *regular_space X* $\longleftrightarrow$ *regular_space Y* (**is** *?lhs=?rhs*)

7.15.13 Locally compact spaces

proposition *quotient_map_prod_right*:
assumes *loc*: *locally_compact_space Z*
and *reg*: *Hausdorff_space Z* $\vee$ *regular_space Z*
and *f*: *quotient_map X Y f*
shows *quotient_map* (*prod_topology Z X*) (*prod_topology Z Y*) (λ(*x*,*y*). (*x*,*f y*))

7.15.14 Special characterizations of classes of functions into and out of R

7.15.15 Normal spaces

7.15.16 Hereditary topological properties

7.15.17 Limits in a topological space

7.15.18 Quasi-components

7.15.19 Additional quasicomponent and continuum properties like Boundary Bumping

7.15.20 Compactly generated spaces (k-spaces)

end

7.16 Abstract Metric Spaces

theory *Abstract_Metric_Spaces*
imports *Elementary_Metric_Spaces Abstract_Limits Abstract_Topological_Spaces*
begin

7.16.1 Metric topology

7.16.2 Bounded sets

7.16.3 Subspace of a metric space

7.16.4 Abstract type of metric spaces

7.16.5 The discrete metric

7.16.6 Metrizable spaces

7.16.7 Limits at a point in a topological space

7.16.8 Normal spaces and metric spaces

7.16.9 Topological limitin in metric spaces

7.16.10 Cauchy sequences and complete metric spaces

7.16.11 Totally bounded subsets of metric spaces

7.16.12 Compactness in metric spaces

7.16.13 Continuous functions on metric spaces

7.16.14 Completely metrizable spaces

7.16.15 Product metric

7.16.16 More sequential characterizations in a metric space

7.16.17 Three strong notions of continuity for metric spaces

7.16.18 Isometries

7.16.19 "Capped" equivalent bounded metrics and general product metrics

proposition *metrizable_space_product_topology*:
metrizable_space (*product_topology X I*) $\longleftrightarrow$
(*product_topology X I*) = *trivial_topology* $\vee$
countable $\{i \in I.\ \neg\ (\exists\, a.\ topspace(X\ i) \subseteq \{a\})\}$ $\wedge$
$(\forall\, i \in I.\ metrizable_space\ (X\ i))$

proposition *completely_metrizable_space_product_topology*:
completely_metrizable_space (*product_topology X I*) $\longleftrightarrow$
(*product_topology X I*) = *trivial_topology* $\vee$
countable $\{i \in I.\ \neg\ (\exists\, a.\ topspace(X\ i) \subseteq \{a\})\}$ $\wedge$
$(\forall\, i \in I.\ completely_metrizable_space\ (X\ i))$

end

7.17 Infinite sums

theory *Infinite_Sum*
imports
Elementary_Topology
HOL−Library.Extended_Nonnegative_Real
HOL−Library.Complex_Order
HOL−Computational_Algebra.Formal_Power_Series
begin

7.17.1 Definition and syntax

7.17.2 General properties

7.17.3 Absolute convergence

7.17.4 Extended reals and nats

7.17.5 Real numbers

7.17.6 Complex numbers

class *complete_uniform_space* = *uniform_space* +
  assumes *cauchy_filter_convergent'*: *cauchy_filter* (*F* :: *'a filter*) ⟹ *F* ≠ *bot* ⟹ *convergent_filter F*

theorem (**in** *uniform_space*) *controlled_sequences_convergent_imp_complete*:
  fixes *U* :: *nat* ⇒ (*'a* × *'a*) *set*
  assumes *gen*: *countably_generated_filter* (*uniformity* :: (*'a* × *'a*) *filter*)
  assumes *U*: ⋀*n*. *eventually* (λ*z*. *z* ∈ *U n*) *uniformity*
  assumes *conv*: ⋀(*u* :: *nat* ⇒ *'a*). (⋀*N m n*. *N* ≤ *m* ⟹ *N* ≤ *n* ⟹ (*u m*, *u n*) ∈ *U N*) ⟹ *convergent u*
  shows *class.complete_uniform_space open uniformity*

theorem (**in** *uniform_space*) *controlled_seq_imp_Cauchy_seq*:
  fixes *U* :: *nat* ⇒ (*'a* × *'a*) *set*
  assumes *U*: ⋀*P*. *eventually P uniformity* ⟹ (∃ *n*. ∀ *x*∈*U n*. *P x*)
  assumes *controlled*: ⋀*N m n*. *N* ≤ *m* ⟹ *N* ≤ *n* ⟹ (*f m*, *f n*) ∈ *U N*
  shows *Cauchy f*

theorem (**in** *uniform_space*) *Cauchy_seq_convergent_imp_complete*:
  fixes *U* :: *nat* ⇒ (*'a* × *'a*) *set*
  assumes *gen*: *countably_generated_filter* (*uniformity* :: (*'a* × *'a*) *filter*)
  assumes *conv*: ⋀(*u* :: *nat* ⇒ *'a*). *Cauchy u* ⟹ *convergent u*
  shows *class.complete_uniform_space open uniformity*

7.17.7 Infinite sums of formal power series

end

7.18 Ordered Euclidean Space

theory *Ordered_Euclidean_Space*
imports
Convex_Euclidean_Space Abstract_Limits
HOL−Library.Product_Order
beginclass *ordered_euclidean_space* = *ord* + *inf* + *sup* + *abs* + *Inf* + *Sup* + *euclidean_space* +
assumes *eucl_le*: $x \leq y \longleftrightarrow (\forall i{\in}Basis.\ x \cdot i \leq y \cdot i)$
assumes *eucl_less_le_not_le*: $x < y \longleftrightarrow x \leq y \wedge \neg\ y \leq x$
assumes *eucl_inf*: $inf\ x\ y = (\sum i{\in}Basis.\ inf\ (x \cdot i)\ (y \cdot i) *_R i)$
assumes *eucl_sup*: $sup\ x\ y = (\sum i{\in}Basis.\ sup\ (x \cdot i)\ (y \cdot i) *_R i)$
assumes *eucl_Inf*: $Inf\ X = (\sum i{\in}Basis.\ (INF\ x{\in}X.\ x \cdot i) *_R i)$
assumes *eucl_Sup*: $Sup\ X = (\sum i{\in}Basis.\ (SUP\ x{\in}X.\ x \cdot i) *_R i)$
assumes *eucl_abs*: $|x| = (\sum i{\in}Basis.\ |x \cdot i| *_R i)$
begin

proposition *compact_attains_Inf_componentwise*:
fixes *b*::*'a*::*ordered_euclidean_space*
assumes $b \in Basis$ **assumes** $X \neq \{\}$ *compact X*
obtains *x* **where** $x \in X\ x \cdot b = Inf\ X \cdot b\ \bigwedge y.\ y \in X \Longrightarrow x \cdot b \leq y \cdot b$

proposition
compact_attains_Sup_componentwise:
fixes *b*::*'a*::*ordered_euclidean_space*
assumes $b \in Basis$ **assumes** $X \neq \{\}$ *compact X*
obtains *x* **where** $x \in X\ x \cdot b = Sup\ X \cdot b\ \bigwedge y.\ y \in X \Longrightarrow y \cdot b \leq x \cdot b$
proposition
fixes *a* :: *'a*::*ordered_euclidean_space*
shows *cbox_interval*: $cbox\ a\ b = \{a..b\}$
and *interval_cbox*: $\{a..b\} = cbox\ a\ b$
and *eucl_le_atMost*: $\{x.\ \forall i{\in}Basis.\ x \cdot i <= a \cdot i\} = \{..a\}$
and *eucl_le_atLeast*: $\{x.\ \forall i{\in}Basis.\ a \cdot i <= x \cdot i\} = \{a..\}$

instantiation *vec* :: (*ordered_euclidean_space*, *finite*) *ordered_euclidean_space*
begin

definition $inf\ x\ y = (\chi\ i.\ inf\ (x\ \$\ i)\ (y\ \$\ i))$
definition $sup\ x\ y = (\chi\ i.\ sup\ (x\ \$\ i)\ (y\ \$\ i))$
definition $Inf\ X = (\chi\ i.\ (INF\ x{\in}X.\ x\ \$\ i))$
definition $Sup\ X = (\chi\ i.\ (SUP\ x{\in}X.\ x\ \$\ i))$
definition $|x| = (\chi\ i.\ |x\ \$\ i|)$

end

7.19 Arcwise-Connected Sets

theory *Arcwise_Connected*
imports *Path_Connected Ordered_Euclidean_Space HOL−Computational_Algebra.Primes*
begin

7.19.1 The Brouwer reduction theorem

theorem *Brouwer_reduction_theorem_gen*:
fixes $S :: 'a::euclidean_space\ set$
assumes $closed\ S\ \varphi\ S$
and φ: $\bigwedge F.\ [\![\bigwedge n.\ closed(F\ n);\ \bigwedge n.\ \varphi(F\ n);\ \bigwedge n.\ F(Suc\ n) \subseteq F\ n]\!] \Longrightarrow \varphi(\bigcap(range\ F))$
obtains T **where** $T \subseteq S\ closed\ T\ \varphi\ T\ \bigwedge U.\ [\![U \subseteq S;\ closed\ U;\ \varphi\ U]\!] \Longrightarrow \neg\ (U \subset T)$

corollary *Brouwer_reduction_theorem*:
fixes $S :: 'a::euclidean_space\ set$
assumes $compact\ S\ \varphi\ S\ S \neq \{\}$
and φ: $\bigwedge F.\ [\![\bigwedge n.\ compact(F\ n);\ \bigwedge n.\ F\ n \neq \{\};\ \bigwedge n.\ \varphi(F\ n);\ \bigwedge n.\ F(Suc\ n) \subseteq F\ n]\!] \Longrightarrow \varphi(\bigcap(range\ F))$
obtains T **where** $T \subseteq S\ compact\ T\ T \neq \{\}\ \varphi\ T$
$\bigwedge U.\ [\![U \subseteq S;\ closed\ U;\ U \neq \{\};\ \varphi\ U]\!] \Longrightarrow \neg\ (U \subset T)$

7.19.2 Density of points with dyadic rational coordinates

proposition *closure_dyadic_rationals*:
$closure\ (\bigcup k.\ \bigcup f \in Basis \to \mathbb{Z}.\ \{\ \sum i :: 'a :: euclidean_space \in Basis.\ (f\ i\ /\ 2\hat{}k) *_R i\ \}) = UNIV$

corollary *closure_rational_coordinates*:
$closure\ (\bigcup f \in Basis \to \mathbb{Q}.\ \{\ \sum i :: 'a :: euclidean_space \in Basis.\ f\ i *_R i\ \}) = UNIV$

theorem *homeomorphic_monotone_image_interval*:
fixes $f :: real \Rightarrow 'a::\{real_normed_vector, complete_space\}$
assumes $cont_f$: $continuous_on\ \{0..1\}\ f$
and $conn$: $\bigwedge y.\ connected\ (\{0..1\} \cap f -` \{y\})$
and f_1not0: $f\ 1 \neq f\ 0$
shows $(f\ `\ \{0..1\})\ homeomorphic\ \{0..1::real\}$

theorem *path_contains_arc*:
fixes p :: *real* ⇒ *'a*::{*complete_space*,*real_normed_vector*}
assumes *path p* **and** *a*: *pathstart p* = *a* **and** *b*: *pathfinish p* = *b* **and** $a \neq b$
obtains *q* **where** *arc q path_image q* ⊆ *path_image p pathstart q* = *a pathfinish q* = *b*

corollary *path_connected_arcwise*:
fixes *S* :: *'a*::{*complete_space*,*real_normed_vector*} *set*
shows *path_connected S* ⟷
(∀ *x* ∈ *S*. ∀ *y* ∈ *S*. *x* ≠ *y* ⟶ (∃ *g*. *arc g* ∧ *path_image g* ⊆ *S* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*))
(**is** *?lhs* = *?rhs*)

corollary *arc_connected_trans*:
fixes *g* :: *real* ⇒ *'a*::{*complete_space*,*real_normed_vector*}
assumes *arc g arc h pathfinish g* = *pathstart h pathstart g* ≠ *pathfinish h*
obtains *i* **where** *arc i path_image i* ⊆ *path_image g* ∪ *path_image h*
pathstart i = *pathstart g pathfinish i* = *pathfinish h*

7.19.3 Accessibility of frontier points

end

7.20 Metrics on product spaces

theory *Function_Metric*
imports
Function_Topology
Elementary_Metric_Spaces
begininstantiation *fun* :: (*countable*, *metric_space*) *metric_space*
begin

definition *dist_fun_def*:
dist x y = (∑ *n*. (*1*/*2*)^*n* ∗ *min* (*dist* (*x* (*from_nat n*)) (*y* (*from_nat n*))) *1*)

definition *uniformity_fun_def*:
(*uniformity*::((*'a* ⇒ *'b*) × (*'a* ⇒ *'b*)) *filter*) = (*INF* *e*∈{*0*<..}. *principal* {(*x*, *y*). *dist* (*x*::(*'a*⇒*'b*)) *y* < *e*})
end

7.21 The Urysohn lemma, its consequences and other advanced material about metric spaces

theory *Urysohn*
imports *Abstract_Metric_Spaces Infinite_Sum Arcwise_Connected Function_Metric*
begin

7.21.1 Urysohn lemma and Tietze's theorem

proposition *Urysohn_lemma*:
fixes $a\ b :: real$
assumes *normal_space X closedin X S closedin X T disjnt S T* $a \leq b$
obtains f **where** *continuous_map X* (*top_of_set* $\{a..b\}$) $f\ f\ `\ S \subseteq \{a\}\ f\ `\ T \subseteq \{b\}$

theorem *Tietze_extension_closed_real_interval*:
assumes *normal_space X* **and** *closedin X S*
and *contf*: *continuous_map* (*subtopology X S*) *euclideanreal f*
and *fim*: $f\ `\ S \subseteq \{a..b\}$ **and** $a \leq b$
obtains g
where *continuous_map X euclideanreal g*
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x\ g\ `\ topspace\ X \subseteq \{a..b\}$

theorem *Tietze_extension_realinterval*:
assumes *XS*: *normal_space X closedin X S* **and** *T*: *is_interval T T* $\neq \{\}$
and *contf*: *continuous_map* (*subtopology X S*) *euclideanreal f*
and $f\ `\ S \subseteq T$
obtains g **where** *continuous_map X euclideanreal g* $g\ `\ topspace\ X \subseteq T\ \bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

7.21.2 Urysohn's metrisation theorem

theorem *Urysohn_metrization*:
fixes $X :: \langle 'a\ topology \rangle$
assumes ‹*regular_space X*› **and** ‹*second_countable X*› **and** ‹*t1_space X*› **shows** ‹*metrizable_space X*›

7.21.3 Random metric space stuff

7.21.4 Hereditarily normal spaces

7.21.5 Completely regular spaces

proposition *locally_compact_regular_imp_completely_regular_space*:
assumes *locally_compact_space X Hausdorff_space X* ∨ *regular_space X*
shows *completely_regular_space X*

proposition *completely_regular_space_product_topology*:
completely_regular_space (*product_topology X I*) ⟷
(∃ *i*∈*I*. *X i* = *trivial_topology*) ∨ (∀ *i* ∈ *I*. *completely_regular_space* (*X i*))
(**is** *?lhs* ⟷ *?rhs*)

7.21.6 More generally, the k-ification functor

7.21.7 One-point compactifications and the Alexandroff extension construction

proposition *kc_space_one_point_compactification_gen*:
assumes *compact_space X*
shows *kc_space X* ⟷
openin X (*topspace X* − {*a*}) ∧ (∀ *K*. *compactin X K* ∧ *a*∉*K* ⟶ *closedin X K*) ∧
k_space (*subtopology X* (*topspace X* − {*a*})) ∧ *kc_space* (*subtopology X* (*topspace X* − {*a*}))
(**is** *?lhs* ⟷ *?rhs*)

proposition *istopology_Alexandroff_open*: *istopology* (*Alexandroff_open X*)

proposition *regular_space_one_point_compactification*:
assumes *compact_space X* **and** *ope*: *openin X* (*topspace X* − {*a*})
and §: ⋀*K*. ⟦*compactin* (*subtopology X* (*topspace X* − {*a*})) *K*; *closedin* (*subtopology X* (*topspace X* − {*a*})) *K*⟧ ⟹ *closedin X K*
shows *regular_space X* ⟷
regular_space (*subtopology X* (*topspace X* − {*a*})) ∧ *locally_compact_space* (*subtopology X* (*topspace X* − {*a*}))
(**is** *?lhs* ⟷ *?rhs*)

proposition *Hausdorff_space_one_point_compactification_asymmetric_prod*:
assumes *compact_space X*
shows *Hausdorff_space X* ⟷
kc_space (*prod_topology X* (*subtopology X* (*topspace X* − {*a*}))) ∧
k_space (*prod_topology X* (*subtopology X* (*topspace X* − {*a*}))) (**is** *?lhs* ⟷ *?rhs*)

7.21.8 Extending continuous maps "pointwise" in a regular space

7.21.9 Extending Cauchy continuous functions to the closure

7.21.10 Metric space of bounded functions

7.21.11 Metric space of continuous bounded functions

7.21.12 Existence of completion for any metric space M as a subspace of M=>R

7.21.13 Contractions

7.21.14 The Baire Category Theorem

7.21.15 Sierpinski-Hausdorff type results about countable closed unions

7.21.16 The Tychonoff embedding

7.21.17 Urysohn and Tietze analogs for completely regular spaces

7.21.18 Size bounds on connected or path-connected spaces

7.21.19 Lavrentiev extension etc

7.21.20 Embedding in products and hence more about completely metrizable spaces

7.21.21 Theorems from Kuratowski

7.21.22 A perfect set in common cases must have at least the cardinality of the continuum

proposition *Kuratowski_component_number_invariance_aux*:
assumes *compact_space X* **and** *HsX*: *Hausdorff_space X*
and *lcX*: *locally_connected_space X* **and** *hnX*: *hereditarily normal_space X*
and *hom*: (*subtopology X S*) *homeomorphic_space* (*subtopology X T*)
and *leXS*: {..<*n*::*nat*} ≲ *connected_components_of* (*subtopology X* (*topspace X* − *S*))
assumes §: ⋀*S T*.
⟦*closedin X S*; *closedin X T*; (*subtopology X S*) *homeomorphic_space* (*subtopology X T*);
{..<*n*::*nat*} ≲ *connected_components_of* (*subtopology X* (*topspace X* − *S*))⟧
⟹ {..<*n*::*nat*} ≲ *connected_components_of* (*subtopology X* (*topspace X* − *T*))
shows {..<*n*::*nat*} ≲ *connected_components_of* (*subtopology X* (*topspace X* − *T*))

theorem *Kuratowski_component_number_invariance*:
assumes *compact_space X Hausdorff_space X locally_connected_space X hereditarily normal_space X*
shows ((∀ *S T n*.
closedin X S ∧ *closedin X T* ∧
(*subtopology X S*) *homeomorphic_space* (*subtopology X T*)
⟶ (*connected_components_of*
(*subtopology X* (*topspace X* − *S*)) ≈ {..<*n*::*nat*} ⟷
connected_components_of
(*subtopology X* (*topspace X* − *T*)) ≈ {..<*n*::*nat*})) ⟷
(∀ *S T n*.
(*subtopology X S*) *homeomorphic_space* (*subtopology X T*)
⟶ (*connected_components_of*
(*subtopology X* (*topspace X* − *S*)) ≈ {..<*n*::*nat*} ⟷
connected_components_of
(*subtopology X* (*topspace X* − *T*)) ≈ {..<*n*::*nat*})))
(**is** *?lhs* = *?rhs*)

end
theory *Sparse_In*
imports *Homotopy*

begin

7.21.23 A set of points sparse in another set

7.21.24 Co-sparseness filter

end
theory *Isolated*
imports *Elementary_Metric_Spaces Sparse_In*

begin

7.21.25 Isolate and discrete

end

7.22 Operator Norm

theory *Operator_Norm*
imports *Complex_Main*
begin

definition
onorm :: (*'a*::*real_normed_vector* ⇒ *'b*::*real_normed_vector*) ⇒ *real* **where**
onorm f = (*SUP x*. *norm* (*f x*) / *norm x*)

proposition *onorm_bound*:
assumes *0* ≤ *b* **and** ⋀*x*. *norm* (*f x*) ≤ *b* ∗ *norm x*
shows *onorm f* ≤ *b*

end

7.23 Limits on the Extended Real Number Line

theory *Extended_Real_Limits*
imports
Topology_Euclidean_Space
HOL−Library.Extended_Real
HOL−Library.Extended_Nonnegative_Real
HOL−Library.Indicator_Function
begin

7.23.1 Extended-Real.thy

Continuity of addition

Continuity of multiplication

Continuity of division

7.23.2 Extended-Nonnegative-Real.thy

7.23.3 monoset

7.23.4 Relate extended reals and the indicator function

end

7.24 Radius of Convergence and Summation Tests

theory *Summation_Tests*
imports
Complex_Main
HOL−Library.Discrete_Functions
HOL−Library.Extended_Real
HOL−Library.Liminf_Limsup
Extended_Real_Limits
begin

7.24.1 Convergence tests for infinite sums

theorem *root_test_convergence′*:
fixes $f :: nat \Rightarrow 'a :: banach$
defines $l \equiv limsup\ (\lambda n.\ ereal\ (root\ n\ (norm\ (f\ n))))$
assumes *l*: $l < 1$
shows *summable f*

theorem *root_test_divergence*:
fixes $f :: nat \Rightarrow 'a :: banach$
defines $l \equiv limsup\ (\lambda n.\ ereal\ (root\ n\ (norm\ (f\ n))))$
assumes *l*: $l > 1$
shows $\neg summable\ f$

theorem *condensation_test*:
assumes *mono*: $\bigwedge m.\ 0 < m \Longrightarrow f\ (Suc\ m) \leq f\ m$

assumes *nonneg*: $\bigwedge n.\ f\ n \geq 0$
shows *summable* $f \longleftrightarrow$ *summable* $(\lambda n.\ 2\hat{}n * f\ (2\hat{}n))$

theorem *summable_complex_powr_iff*:
assumes $Re\ s < -1$
shows *summable* $(\lambda n.\ exp\ (of_real\ (ln\ (of_nat\ n)) * s))$
theorem *kummers_test_convergence*:
fixes $f\ p :: nat \Rightarrow real$
assumes *pos_f*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
assumes *nonneg_p*: *eventually* $(\lambda n.\ p\ n \geq 0)$ *sequentially*
defines $l \equiv liminf\ (\lambda n.\ ereal\ (p\ n * f\ n\ /\ f\ (Suc\ n) - p\ (Suc\ n)))$
assumes *l*: $l > 0$
shows *summable f*

theorem *kummers_test_divergence*:
fixes $f\ p :: nat \Rightarrow real$
assumes *pos_f*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
assumes *pos_p*: *eventually* $(\lambda n.\ p\ n > 0)$ *sequentially*
assumes *divergent_p*: $\neg summable\ (\lambda n.\ inverse\ (p\ n))$
defines $l \equiv limsup\ (\lambda n.\ ereal\ (p\ n * f\ n\ /\ f\ (Suc\ n) - p\ (Suc\ n)))$
assumes *l*: $l < 0$
shows $\neg summable\ f$
theorem *ratio_test_convergence*:
fixes $f :: nat \Rightarrow real$
assumes *pos_f*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
defines $l \equiv liminf\ (\lambda n.\ ereal\ (f\ n\ /\ f\ (Suc\ n)))$
assumes *l*: $l > 1$
shows *summable f*

theorem *ratio_test_divergence*:
fixes $f :: nat \Rightarrow real$
assumes *pos_f*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
defines $l \equiv limsup\ (\lambda n.\ ereal\ (f\ n\ /\ f\ (Suc\ n)))$
assumes *l*: $l < 1$
shows $\neg summable\ f$
theorem *raabes_test_convergence*:
fixes $f :: nat \Rightarrow real$
assumes *pos*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
defines $l \equiv liminf\ (\lambda n.\ ereal\ (of_nat\ n * (f\ n\ /\ f\ (Suc\ n) - 1)))$
assumes *l*: $l > 1$
shows *summable f*

theorem *raabes_test_divergence*:
fixes $f :: nat \Rightarrow real$
assumes *pos*: *eventually* $(\lambda n.\ f\ n > 0)$ *sequentially*
defines $l \equiv limsup\ (\lambda n.\ ereal\ (of_nat\ n * (f\ n\ /\ f\ (Suc\ n) - 1)))$
assumes *l*: $l < 1$
shows $\neg summable\ f$

7.24.2 Radius of convergence

definition *conv_radius* :: (*nat* ⇒ *'a* :: *banach*) ⇒ *ereal* **where**
conv_radius f = *inverse* (*limsup* (λ*n*. *ereal* (*root n* (*norm* (*f n*)))))

theorem *abs_summable_in_conv_radius*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *real_normed_div_algebra*}
assumes *ereal* (*norm z*) < *conv_radius f*
shows *summable* (λ*n*. *norm* (*f n* ∗ *z* ^ *n*))

theorem *not_summable_outside_conv_radius*:
fixes *f* :: *nat* ⇒ *'a* :: {*banach*, *real_normed_div_algebra*}
assumes *ereal* (*norm z*) > *conv_radius f*
shows ¬*summable* (λ*n*. *f n* ∗ *z* ^ *n*)

end

7.25 Uniform Limit and Uniform Convergence

theory *Uniform_Limit*
imports *Connected Summation_Tests Infinite_Sum*
begin

7.25.1 Definition

definition *uniformly_on* :: *'a set* ⇒ (*'a* ⇒ *'b*::*metric_space*) ⇒ (*'a* ⇒ *'b*) *filter*
where *uniformly_on S l* = (*INF e*∈{*0* <..}. *principal* {*f*. ∀*x*∈*S*. *dist* (*f x*) (*l x*) < *e*})

abbreviation
uniform_limit S f l ≡ *filterlim f* (*uniformly_on S l*)

proposition *uniform_limit_iff*:
uniform_limit S f l F ⟷ (∀ *e*>*0*. ∀ *F n in F*. ∀*x*∈*S*. *dist* (*f n x*) (*l x*) < *e*)

7.25.2 Exchange limits

proposition *swap_uniform_limit'*:
assumes *f*: ∀ *F n in F*. (*f n* ⟶ *g n*) *G*
assumes *g*: (*g* ⟶ *l*) *F*
assumes *uc*: *uniform_limit S f h F*
assumes *ev*: ∀ *F x in G*. *x* ∈ *S*
assumes ¬*trivial_limit F*

shows $(h \longrightarrow l)\ G$

corollary *swap_uniform_limit*:
assumes $\forall_F\ n\ in\ F.\ (f\ n \longrightarrow g\ n)\ (at\ x\ within\ S)$
assumes $(g \longrightarrow l)\ F\ uniform_limit\ S\ f\ h\ F\ \neg trivial_limit\ F$
shows $(h \longrightarrow l)\ (at\ x\ within\ S)$

7.25.3 Uniform limit theorem

theorem *uniform_limit_theorem*:
assumes *c*: $\forall_F\ n\ in\ F.\ continuous_on\ A\ (f\ n)$
assumes *ul*: $uniform_limit\ A\ f\ l\ F$
assumes $\neg\ trivial_limit\ F$
shows $continuous_on\ A\ l$

7.25.4 Comparison Test

7.25.5 Weierstrass M-Test

proposition *Weierstrass_m_test_ev*:
fixes $f :: _ \Rightarrow _ \Rightarrow _ :: banach$
assumes $eventually\ (\lambda n.\ \forall x{\in}A.\ norm\ (f\ n\ x) \leq M\ n)\ sequentially$
assumes $summable\ M$
shows $uniform_limit\ A\ (\lambda n\ x.\ \sum i{<}n.\ f\ i\ x)\ (\lambda x.\ suminf\ (\lambda i.\ f\ i\ x))\ sequentially$

7.25.6 Power series and uniform convergence

proposition *powser_uniformly_convergent*:
fixes $a :: nat \Rightarrow 'a{::}\{real_normed_div_algebra, banach\}$
assumes $r < conv_radius\ a$
shows $uniformly_convergent_on\ (cball\ \xi\ r)\ (\lambda n\ x.\ \sum i{<}n.\ a\ i * (x - \xi)\ \hat{}\ i)$

7.25.7 Tannery's Theorem

end

7.26 Bounded Linear Function

theory *Bounded_Linear_Function*
imports
Topology_Euclidean_Space
Operator_Norm
Uniform_Limit

Function_Topology

begin

7.26.1 Type of bounded linear functions

typedef (**overloaded**) ('*a*, '*b*) *blinfun* (‹(‹*notation*=‹*infix* $\Rightarrow_L$›› _ $\Rightarrow_L$ /_)› [*22*, *21*] *21*) =
{*f*::'*a*::*real_normed_vector*$\Rightarrow$'*b*::*real_normed_vector*. *bounded_linear f*}
morphisms *blinfun_apply Blinfun*

7.26.2 Type class instantiations

instantiation *blinfun* :: (*real_normed_vector*, *real_normed_vector*) *real_normed_vector*
begin

lift_definition *norm_blinfun* :: '*a* $\Rightarrow_L$ '*b* $\Rightarrow$ *real* **is** *onorm*
lift_definition *zero_blinfun* :: '*a* $\Rightarrow_L$ '*b* **is** $\lambda x.$ *0*

lift_definition *plus_blinfun* :: '*a* $\Rightarrow_L$ '*b* $\Rightarrow$ '*a* $\Rightarrow_L$ '*b* $\Rightarrow$ '*a* $\Rightarrow_L$ '*b*
is $\lambda f\ g\ x.\ f\ x + g\ x$

lift_definition *scaleR_blinfun*::*real* $\Rightarrow$ '*a* $\Rightarrow_L$ '*b* $\Rightarrow$ '*a* $\Rightarrow_L$ '*b* **is** $\lambda r\ f\ x.\ r *_R f\ x$

7.26.3 The strong operator topology on continuous linear operators

definition *strong_operator_topology*::('*a*::*real_normed_vector* $\Rightarrow_L$'*b*::*real_normed_vector*) *topology*
where *strong_operator_topology* = *pullback_topology UNIV blinfun_apply euclidean*

end

7.27 Derivative

theory *Derivative*
imports
Bounded_Linear_Function

Line_Segment
Convex_Euclidean_Space
begin

7.27.1 Derivatives

proposition *has_derivative_within'*:
(*f has_derivative f'*)(*at x within s*) ⟷
bounded_linear f' ∧
(∀ *e*>*0*. ∃ *d*>*0*. ∀ *x'*∈*s*. *0* < *norm* (*x'* − *x*) ∧ *norm* (*x'* − *x*) < *d* ⟶
norm (*f x'* − *f x* − *f'*(*x'* − *x*)) / *norm* (*x'* − *x*) < *e*)

7.27.2 Differentiability

definition
differentiable_on :: ('*a*::*real_normed_vector* ⇒ '*b*::*real_normed_vector*) ⇒ '*a set* ⇒ *bool*
(**infix** ‹*differentiable'_on*› *50*)
where *f differentiable_on s* ⟷ (∀ *x*∈*s*. *f differentiable* (*at x within s*))

7.27.3 Frechet derivative and Jacobian matrix

proposition *frechet_derivative_works*:
f differentiable net ⟷ (*f has_derivative* (*frechet_derivative f net*)) *net*

7.27.4 Differentiability implies continuity

proposition *differentiable_imp_continuous_within*:
f differentiable (*at x within s*) ⟹ *continuous* (*at x within s*) *f*

7.27.5 The chain rule

proposition *diff_chain_within*[*derivative_intros*]:
assumes (*f has_derivative f'*) (*at x within s*)
and (*g has_derivative g'*) (*at* (*f x*) *within* (*f* ‘ *s*))
shows ((*g* ∘ *f*) *has_derivative* (*g'* ∘ *f'*))(*at x within s*)

7.27.6 Uniqueness of derivative

The general result is a bit messy because we need approachability of the limit point from any direction. But OK for nontrivial intervals etc.

proposition *frechet_derivative_unique_within*:

fixes f :: $'a$::*euclidean_space* ⇒ $'b$::*real_normed_vector*
assumes *1*: (f *has_derivative* f') (*at* x *within* S)
and *2*: (f *has_derivative* f'') (*at* x *within* S)
and S: $\bigwedge i\ e.\ [\![i{\in}Basis;\ e{>}0]\!] \Longrightarrow \exists\, d.\ 0 < |d| \wedge |d| < e \wedge (x + d *_R i) \in S$
shows $f' = f''$

proposition *frechet_derivative_unique_within_closed_interval*:
fixes f::$'a$::*euclidean_space* ⇒ $'b$::*real_normed_vector*
assumes *ab*: $\bigwedge i.\ i{\in}Basis \Longrightarrow a{\cdot}i < b{\cdot}i$
and x: $x \in cbox\ a\ b$
and (f *has_derivative* f') (*at* x *within cbox* a b)
and (f *has_derivative* f'') (*at* x *within cbox* a b)
shows $f' = f''$

7.27.7 Derivatives of local minima and maxima are zero

7.27.8 One-dimensional mean value theorem

7.27.9 More general bound theorems

proposition *differentiable_bound_general*:
fixes f :: *real* ⇒ $'a$::*real_normed_vector*
assumes $a < b$
and *f_cont*: *continuous_on* {a..b} f
and *phi_cont*: *continuous_on* {a..b} φ
and f': $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow (f$ *has_vector_derivative* $f'\ x)$ (*at* x)
and *phi'*: $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow (\varphi$ *has_vector_derivative* $\varphi'\ x)$ (*at* x)
and *bnd*: $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow$ *norm* $(f'\ x) \le \varphi'\ x$
shows *norm* $(f\ b - f\ a) \le \varphi\ b - \varphi\ a$

7.27.10 Differentiability of inverse function (most basic form)

proposition *has_derivative_inverse*:
fixes f :: $'a$::*real_normed_vector* ⇒ $'b$::*real_normed_vector*
assumes *compact* S
and $x \in S$
and *fx*: $f\ x \in$ *interior* (f ‘ S)
and *continuous_on* S f
and *gf*: $\bigwedge y.\ y \in S \Longrightarrow g\ (f\ y) = y$
and B: (f *has_derivative* f') (*at* x) *bounded_linear* g' $g' \circ f' = id$
shows (g *has_derivative* g') (*at* ($f\ x$))
proposition *has_derivative_locally_injective*:
fixes f :: $'n$::*euclidean_space* ⇒ $'m$::*euclidean_space*
assumes $a \in S$
and *open* S
and *bling*: *bounded_linear* g'
and $g' \circ f'\ a = id$
and *derf*: $\bigwedge x.\ x \in S \Longrightarrow (f$ *has_derivative* $f'\ x)$ (*at* x)

and $\bigwedge e.\ e > 0 \Longrightarrow \exists\, d{>}0.\ \forall\, x.\ dist\ a\ x < d \longrightarrow onorm\ (\lambda v.\ f'\ x\ v - f'\ a\ v) < e$
obtains r **where** $r > 0\ ball\ a\ r \subseteq S\ inj_on\ f\ (ball\ a\ r)$

7.27.11 Uniformly convergent sequence of derivatives

proposition *has_derivative_sequence*:
fixes $f :: nat \Rightarrow 'a{::}real_normed_vector \Rightarrow 'b{::}banach$
assumes *convex S*
and *derf*: $\bigwedge n\ x.\ x \in S \Longrightarrow ((f\ n)\ has_derivative\ (f'\ n\ x))\ (at\ x\ within\ S)$
and *nle*: $\bigwedge e.\ e > 0 \Longrightarrow \forall_F\ n\ in\ sequentially.\ \forall\, x{\in}S.\ \forall\, h.\ norm\ (f'\ n\ x\ h - g'\ x\ h) \leq e * norm\ h$
and $x0 \in S$
and *lim*: $((\lambda n.\ f\ n\ x0) \longrightarrow l)\ sequentially$
shows $\exists\, g.\ \forall\, x{\in}S.\ (\lambda n.\ f\ n\ x) \longrightarrow g\ x \wedge (g\ has_derivative\ g'(x))\ (at\ x\ within\ S)$

7.27.12 Differentiation of a series

proposition *has_derivative_series*:
fixes $f :: nat \Rightarrow 'a{::}real_normed_vector \Rightarrow 'b{::}banach$
assumes *convex S*
and $\bigwedge n\ x.\ x \in S \Longrightarrow ((f\ n)\ has_derivative\ (f'\ n\ x))\ (at\ x\ within\ S)$
and $\bigwedge e.\ e{>}0 \Longrightarrow \forall_F\ n\ in\ sequentially.\ \forall\, x{\in}S.\ \forall\, h.\ norm\ (sum\ (\lambda i.\ f'\ i\ x\ h)\ \{..{<}n\} - g'\ x\ h) \leq e * norm\ h$
and $x \in S$
and $(\lambda n.\ f\ n\ x)\ sums\ l$
shows $\exists\, g.\ \forall\, x{\in}S.\ (\lambda n.\ f\ n\ x)\ sums\ (g\ x) \wedge (g\ has_derivative\ g'\ x)\ (at\ x\ within\ S)$

7.27.13 Derivative as a vector

proposition *vector_derivative_works*:
$f\ differentiable\ net \longleftrightarrow (f\ has_vector_derivative\ (vector_derivative\ f\ net))\ net$
(**is** $?l = ?r$)

corollary *vector_differentiable*:
$f\ differentiable\ net \longleftrightarrow (\exists f'.\ (f\ has_vector_derivative\ f')\ net)$

7.27.14 Field differentiability

definition *field_differentiable* :: $['a \Rightarrow 'a{::}real_normed_field,\ 'a\ filter] \Rightarrow bool$
(**infixr** ‹(*field'_differentiable*)› *50*)
where $f\ field_differentiable\ F \equiv \exists f'.\ (f\ has_field_derivative\ f')\ F$

7.27.15 Field derivative

definition *deriv* :: (*'a* ⇒ *'a*::*real_normed_field*) ⇒ *'a* ⇒ *'a* **where**
deriv f x ≡ *SOME D*. *DERIV f x* :> *D*

proposition *field_differentiable_derivI*:
f field_differentiable (*at x*) ⟹ (*f has_field_derivative deriv f x*) (*at x*)

corollary *kth_deriv_monomial*:
fixes *x* :: *real*
assumes *kn*: *k* ≤ *n*
shows (*deriv* ^^ *k*) (λ*t*. *t*^*n*) *x*
= (*of_nat* (*fact n*) / *of_nat* (*fact* (*n* − *k*))) ∗ *x*^(*n* − *k*)

corollary *kth_deriv_monomial_zero*:
assumes *kn*: *k* > *n*
shows (*deriv* ^^ *k*) (λ*t*::*real*. *t*^*n*) *x* = *0*

corollary *kth_deriv_power2_ge3*:
fixes *x* :: *real*
assumes *k* ≥ *3*
shows (*deriv* ^^ *k*) *power2 x* = *0*

corollary *kth_deriv_power2_0*: (*deriv* ^^ *0*) *power2 x* = *x*^*2*

corollary *kth_deriv_power2_1*:
fixes *x* :: *real*
shows (*deriv* ^^ *1*) *power2 x* = *2* ∗ *x*

corollary *kth_deriv_power2_2*:
fixes *x* :: *real*
shows (*deriv* ^^ *2*) *power2 x* = *2*

corollary *kth_deriv_power2_cases*:
fixes *x* :: *real*
shows (*deriv* ^^ *k*) *power2 x* = (*if k* = *0 then x*^*2 else if k* = *1 then 2*∗*x else if k* = *2 then 2 else 0*)

7.27.16 Derivative Transferring

7.27.17 Relation between convexity and derivative

proposition *convex_on_imp_above_tangent*:
assumes *convex*: *convex_on A f* **and** *connected*: *connected A*
assumes *c*: *c* ∈ *interior A* **and** *x* : *x* ∈ *A*
assumes *deriv*: (*f has_field_derivative f'*) (*at c within A*)
shows *f x* − *f c* ≥ *f'* ∗ (*x* − *c*)

7.27.18 Partial derivatives

proposition *has_derivative_partialsI*:
fixes *f*::*'a*::*real_normed_vector* ⇒ *'b*::*real_normed_vector* ⇒ *'c*::*real_normed_vector*
assumes *fx*: ((λ*x*. *f x y*) *has_derivative fx*) (*at x within X*)
assumes *fy*: ⋀*x y*. *x* ∈ *X* ⟹ *y* ∈ *Y* ⟹ ((λ*y*. *f x y*) *has_derivative blinfun_apply* (*fy x y*)) (*at y within Y*)
assumes *fy_cont*[*unfolded continuous_within*]: *continuous* (*at* (*x*, *y*) *within X* × *Y*) (λ(*x*, *y*). *fy x y*)
assumes *y* ∈ *Y convex Y*
shows ((λ(*x*, *y*). *f x y*) *has_derivative* (λ(*tx*, *ty*). *fx tx* + *fy x y ty*)) (*at* (*x*, *y*) *within X* × *Y*)

7.27.19 Clairaut's theorem (Symmetry of mixed partials)

7.27.20 The Inverse Function Theorem

theorem *inverse_function_theorem*:
fixes *f*::*'a*::*euclidean_space* ⇒ *'a*
and *f'*::*'a* ⇒ (*'a* ⇒$_L$ *'a*)
assumes *open U*
and *derf*: ⋀*x*. *x* ∈ *U* ⟹ (*f has_derivative* (*blinfun_apply* (*f' x*))) (*at x*)
and *contf*: *continuous_on U f'*
and *x0* ∈ *U*
and *invf*: *invf* o_L *f' x0* = *id_blinfun*
obtains *U' V g g'* **where** *open U' U'* ⊆ *U x0* ∈ *U' open V f x0* ∈ *V homeomorphism U' V f g*
⋀*y*. *y* ∈ *V* ⟹ (*g has_derivative* (*g' y*)) (*at y*)
⋀*y*. *y* ∈ *V* ⟹ *g' y* = *inv* (*blinfun_apply* (*f'*(*g y*)))
⋀*y*. *y* ∈ *V* ⟹ *bij* (*blinfun_apply* (*f'*(*g y*)))

7.27.21 The concept of continuously differentiable

definition *C1_differentiable_on* :: (*real* ⇒ *'a*::*real_normed_vector*) ⇒ *real set* ⇒ *bool*
(**infix** ‹*C1'_differentiable'_on*› *50*)
where
f C1_differentiable_on S ⟷
(∃ *D*. (∀ *x* ∈ *S*. (*f has_vector_derivative* (*D x*)) (*at x*)) ∧ *continuous_on S D*)

definition *piecewise_C1_differentiable_on*
(**infixr** ‹*piecewise'_C1'_differentiable'_on*› *50*)
where *f piecewise_C1_differentiable_on i* ≡
continuous_on i f ∧

$(\exists S.\ \mathit{finite}\ S \wedge (f\ \mathit{C1_differentiable_on}\ (i - S)))$

end

7.28 Finite Cartesian Products of Euclidean Spaces

theory *Cartesian_Euclidean_Space*
imports *Derivative*
begin

7.28.1 Closures and interiors of halfspaces

7.28.2 Bounds on components etc. relative to operator norm

7.28.3 Convex Euclidean Space

7.28.4 Arbitrarily good rational approximations

proposition *matrix_rational_approximation*:
fixes $A :: \mathit{real}^{\frown\prime}n^{\frown\prime}m$
assumes $e > 0$
obtains B **where** $\bigwedge i\ j.\ B\$i\$j \in \mathbb{Q}\ \mathit{onorm}(\lambda x.\ (A - B) *v\ x) < e$

7.28.5 Derivative

definition $\mathit{jacobian}\ f\ \mathit{net} = \mathit{matrix}(\mathit{frechet_derivative}\ f\ \mathit{net})$

proposition *jacobian_works*:
$(f::(\mathit{real}^{\frown\prime}a) \Rightarrow (\mathit{real}^{\frown\prime}b))\ \mathit{differentiable}\ \mathit{net} \longleftrightarrow$
$(f\ \mathit{has_derivative}\ (\lambda h.\ (\mathit{jacobian}\ f\ \mathit{net}) *v\ h))\ \mathit{net}$ (**is** *?lhs* = *?rhs*)

7.28.6 Hessian chain rule

7.28.7 Jacobian

proposition *differential_zero_maxmin_cart*:
fixes $f::\mathit{real}^{\frown\prime}a \Rightarrow \mathit{real}^{\frown\prime}b$
assumes $0 < e\ ((\forall y \in \mathit{ball}\ x\ e.\ (f\ y)\$k \leq (f\ x)\$k) \vee (\forall y \in \mathit{ball}\ x\ e.\ (f\ x)\$k \leq (f\ y)\$k))$
$f\ \mathit{differentiable}\ (\mathit{at}\ x)$
shows $\mathit{jacobian}\ f\ (\mathit{at}\ x)\ \$\ k = 0$

7.28.8 Gradient chain rule

corollary *GDERIV_compose'*:
fixes $g :: real^{\prime}m::finite \Rightarrow real$
and $F :: real^{\prime}n::finite \Rightarrow real^{\prime}m$
assumes *Gg*: $GDERIV\ g\ (F\ x) :> dg$
and *DF*: $F\ differentiable\ (at\ x)$
shows $GDERIV\ (\lambda y.\ g\ (F\ y))\ x :> transpose\ (jacobian\ F\ (at\ x)) *v\ dg$

end

7.29 Complex Analysis Basics

theory *Complex_Analysis_Basics*
imports *Derivative HOL−Library.Nonpos_Ints Uncountable_Sets Sparse_In*
begin

7.29.1 Holomorphic functions

definition $holomorphic_on :: [complex \Rightarrow complex, complex\ set] \Rightarrow bool$
(**infixl** ‹(*holomorphic'_on*)› 50)
where $f\ holomorphic_on\ s \equiv \forall x{\in}s.\ f\ field_differentiable\ (at\ x\ within\ s)$

named_theorems *holomorphic_intros structural introduction rules for holomorphic_on*

7.29.2 Holomorphic implies real-C^∞

7.29.3 Analyticity on a set

definition *analytic_on* (**infixl** ‹(*analytic'_on*)› 50)
where $f\ analytic_on\ S \equiv \forall x \in S.\ \exists \varepsilon.\ 0 < \varepsilon \wedge f\ holomorphic_on\ (ball\ x\ \varepsilon)$

named_theorems *analytic_intros introduction rules for proving analyticity*

end

7.30 Complex Transcendental Functions

theory *Complex_Transcendental*
imports
Complex_Analysis_Basics Summation_Tests HOL−Library.Periodic_Fun
begin

7.30.1 Möbius transformations

definition $moebius\ a\ b\ c\ d \equiv (\lambda z.\ (a*z+b)\ /\ (c*z+d :: {}'a :: field))$

theorem *moebius_inverse*:
assumes $a * d \neq b * c$ $c * z + d \neq 0$
shows *moebius d* $(-b)$ $(-c)$ *a* (*moebius a b c d z*) $= z$

7.30.2 Euler and de Moivre formulas

theorem *exp_Euler*: $exp(\mathrm{i} * z) = cos(z) + \mathrm{i} * sin(z)$

theorem *Euler*: $exp(z) = of_real(exp(Re\ z)) *$
$(of_real(cos(Im\ z)) + \mathrm{i} * of_real(sin(Im\ z)))$

7.30.3 The argument of a complex number (HOL Light version)

definition *is_Arg* :: [*complex*,*real*] ⇒ *bool*
where *is_Arg z r* ≡ $z = of_real(norm\ z) * exp(\mathrm{i} * of_real\ r)$

definition *Arg2pi* :: *complex* ⇒ *real*
where *Arg2pi z* ≡ *if z = 0 then 0 else THE t. 0 ≤ t ∧ t < 2∗pi ∧ is_Arg z t*

7.30.4 The principal branch of the Complex logarithm

instantiation *complex* :: *ln*
begin

definition *ln_complex* :: *complex* ⇒ *complex*
where *ln_complex* ≡ λ*z*. *THE w*. *exp w = z & −pi < Im*(*w*) *& Im*(*w*) ≤ *pi*
theorem *Ln_series*:
fixes *z* :: *complex*
assumes *norm z < 1*
shows (λ*n*. (−1)^*Suc n / of_nat n ∗ z*^*n*) *sums ln* (*1 + z*) (**is** (λ*n*. *?f n ∗ z*^*n*) *sums* _)

corollary *norm_Ln_prod_le*:
fixes *f* :: *'a* ⇒ *complex*
assumes ⋀*x*. *x* ∈ *A* ⟹ *f x* ≠ *0*
shows *cmod* (*Ln* (*prod f A*)) ≤ (∑ *x* ∈ *A*. *cmod* (*Ln* (*f x*)))

7.30.5 The Argument of a Complex Number

lemma *Arg_def*:
shows *Arg z* = (*if z = 0 then 0 else Im* (*Ln z*))

7.30.6 The Unwinding Number and the Ln product Formula

definition *unwinding* :: *complex* ⇒ *int* **where**
unwinding z ≡ *THE k. of_int k* = (*z* − *Ln*(*exp z*)) / (*of_real*(2∗*pi*) ∗ i)

7.30.7 Characterisation of *Im* (*Ln z*) (Wenda Li)

7.30.8 Complex arctangent

definition *Arctan* :: *complex* ⇒ *complex* **where**
Arctan ≡ λ*z*. (i/2) ∗ *Ln*((1 − i∗*z*) / (1 + i∗*z*))

theorem *Arctan_series*:
assumes *z*: *norm* (*z* :: *complex*) < 1
defines *g* ≡ λ*n*. *if odd n then* −i∗i^*n* / *n else* 0
defines *h* ≡ λ*z n*. (−1)^*n* / *of_nat* (2∗*n*+1) ∗ (*z*::*complex*)^(2∗*n*+1)
shows (λ*n*. *g n* ∗ *z*^*n*) *sums Arctan z*
and *h z sums Arctan z*
theorem *ln_series_quadratic*:
assumes *x*: *x* > (0::*real*)
shows (λ*n*. (2∗((*x* − 1) / (*x* + 1)) ^(2∗*n*+1) / *of_nat* (2∗*n*+1))) *sums ln x*

7.30.9 Inverse Sine

definition *Arcsin* :: *complex* ⇒ *complex* **where**
Arcsin ≡ λ*z*. −i ∗ *Ln*(i ∗ *z* + *csqrt*(1 − *z*²))

7.30.10 Inverse Cosine

definition *Arccos* :: *complex* ⇒ *complex* **where**
Arccos ≡ λ*z*. −i ∗ *Ln*(*z* + i ∗ *csqrt*(1 − *z*²))

7.30.11 Roots of unity

theorem *complex_root_unity*:
fixes *j*::*nat*
assumes *n* ≠ 0
shows *exp*(2 ∗ *of_real pi* ∗ i ∗ *of_nat j* / *of_nat n*)^*n* = 1

corollary *bij_betw_roots_unity*:
bij_betw (λ*j*. *exp*(2 ∗ *of_real pi* ∗ i ∗ *of_nat j* / *of_nat n*))

$\{..<n\}$ $\{exp(2 * of_real\ pi * \mathrm{i} * of_nat\ j\ /\ of_nat\ n) \mid j.\ j < n\}$

7.30.12 Normalisation of angles

7.30.13 Convexity of circular sectors in the complex plane

7.30.14 Complex cones

end

Chapter 8

Measure and Integration Theory

theory *Sigma_Algebra*
imports
Complex_Main
HOL−Library.Countable_Set
HOL−Library.FuncSet
HOL−Library.Indicator_Function
HOL−Library.Extended_Nonnegative_Real
HOL−Library.Disjoint_Sets
begin

8.1 Sigma Algebra

8.1.1 Families of sets

locale *subset_class* =
fixes Ω :: *'a set* **and** M :: *'a set set*
assumes *space_closed*: $M \subseteq Pow\ \Omega$
locale *semiring_of_sets* = *subset_class* +
assumes *empty_sets*[*iff*]: $\{\} \in M$
assumes *Int*[*intro*]: $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow a \cap b \in M$
assumes *Diff_cover*:
$\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow \exists\, C \subseteq M.\ finite\ C \wedge disjoint\ C \wedge a - b = \bigcup C$
locale *ring_of_sets* = *semiring_of_sets* +
assumes *Un* [*intro*]: $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow a \cup b \in M$
locale *algebra* = *ring_of_sets* +
assumes *top* [*iff*]: $\Omega \in M$

proposition *algebra_iff_Un*:
algebra $\Omega\ M \longleftrightarrow$
$M \subseteq Pow\ \Omega \wedge$
$\{\} \in M \wedge$
$(\forall\, a \in M.\ \Omega - a \in M) \wedge$

$(\forall\, a \in M.\ \forall\ b \in M.\ a \cup b \in M)$ (**is** _ $\longleftrightarrow$ *?Un*)

proposition *algebra_iff_Int*:
  algebra Ω $M \longleftrightarrow$
    $M \subseteq Pow\ \Omega$ & $\{\} \in M$ &
    $(\forall\, a \in M.\ \Omega - a \in M)$ &
    $(\forall\, a \in M.\ \forall\ b \in M.\ a \cap b \in M)$ (**is** _ $\longleftrightarrow$ *?Int*)
locale *sigma_algebra* = *algebra* +
  assumes *countable_nat_UN* [*intro*]: $\bigwedge A.\ range\ A \subseteq M \Longrightarrow (\bigcup i{::}nat.\ A\ i) \in M$

Sigma algebras can naturally be created as the closure of any set of M with regard to the properties just postulated.

inductive_set *sigma_sets* :: *'a set* $\Rightarrow$ *'a set set* $\Rightarrow$ *'a set set*
  for *sp* :: *'a set* **and** *A* :: *'a set set*
  where
    Basic[*intro*, *simp*]: $a \in A \Longrightarrow a \in sigma_sets\ sp\ A$
  | *Empty*: $\{\} \in sigma_sets\ sp\ A$
  | *Compl*: $a \in sigma_sets\ sp\ A \Longrightarrow sp - a \in sigma_sets\ sp\ A$
  | *Union*: $(\bigwedge i{::}nat.\ a\ i \in sigma_sets\ sp\ A) \Longrightarrow (\bigcup i.\ a\ i) \in sigma_sets\ sp\ A$
definition *closed_cdi* :: *'a set* $\Rightarrow$ *'a set set* $\Rightarrow$ *bool* **where**
  closed_cdi Ω $M \longleftrightarrow$
   $M \subseteq Pow\ \Omega$ &
   $(\forall\, s \in M.\ \Omega - s \in M)$ &
   $(\forall\, A.\ (range\ A \subseteq M)$ & $(A\ 0 = \{\})$ & $(\forall\, n.\ A\ n \subseteq A\ (Suc\ n)) \longrightarrow$
       $(\bigcup i.\ A\ i) \in M)$ &
   $(\forall\, A.\ (range\ A \subseteq M)$ & $disjoint_family\ A \longrightarrow (\bigcup i{::}nat.\ A\ i) \in M)$

locale *Dynkin_system* = *subset_class* +
  assumes *space*: $\Omega \in M$
    and *compl*[*intro*!]: $\bigwedge A.\ A \in M \Longrightarrow \Omega - A \in M$
    and *UN*[*intro*!]: $\bigwedge A.\ disjoint_family\ A \Longrightarrow range\ A \subseteq M$
                $\Longrightarrow (\bigcup i{::}nat.\ A\ i) \in M$
definition *Int_stable* :: *'a set set* $\Rightarrow$ *bool* **where**
Int_stable $M \longleftrightarrow (\forall\ a \in M.\ \forall\ b \in M.\ a \cap b \in M)$
definition *Dynkin* :: *'a set* $\Rightarrow$ *'a set set* $\Rightarrow$ *'a set set* **where**
  Dynkin Ω $M = \ (\bigcap \{D.\ Dynkin_system\ \Omega\ D \wedge M \subseteq D\})$

The reason to introduce Dynkin-systems is the following induction rules for σ-algebras generated by a generator closed under intersection.

proposition *sigma_sets_induct_disjoint*[*consumes 3*, *case_names basic empty compl union*]:
  assumes *Int_stable G*
    and *closed*: $G \subseteq Pow\ \Omega$
    and *A*: $A \in sigma_sets\ \Omega\ G$
  assumes *basic*: $\bigwedge A.\ A \in G \Longrightarrow P\ A$
    and *empty*: $P\ \{\}$
    and *compl*: $\bigwedge A.\ A \in sigma_sets\ \Omega\ G \Longrightarrow P\ A \Longrightarrow P\ (\Omega - A)$

and *union*: $\bigwedge A.$ *disjoint_family* $A \Longrightarrow$ *range* $A \subseteq$ *sigma_sets* Ω $G \Longrightarrow (\bigwedge i.$ P $(A$ $i)) \Longrightarrow P$ $(\bigcup i{::}nat.$ A $i)$
shows P A

8.1.2 Measure type

definition *positive* :: $'a$ *set set* $\Rightarrow$ ($'a$ *set* $\Rightarrow$ *ennreal*) $\Rightarrow$ *bool* **where**
positive M $\mu \longleftrightarrow \mu$ $\{\} = 0$

definition *countably_additive* :: $'a$ *set set* $\Rightarrow$ ($'a$ *set* $\Rightarrow$ *ennreal*) $\Rightarrow$ *bool* **where**
countably_additive M $f \longleftrightarrow$
($\forall A.$ *range* $A \subseteq M \longrightarrow$ *disjoint_family* $A \longrightarrow (\bigcup i.$ A $i) \in M \longrightarrow$
$(\sum i.$ f $(A$ $i)) = f$ $(\bigcup i.$ A $i))$

definition *measure_space* :: $'a$ *set* $\Rightarrow$ $'a$ *set set* $\Rightarrow$ ($'a$ *set* $\Rightarrow$ *ennreal*) $\Rightarrow$ *bool* **where**
measure_space Ω A $\mu \longleftrightarrow$
sigma_algebra Ω $A \wedge$ *positive* A $\mu \wedge$ *countably_additive* A μ

typedef $'a$ *measure* =
$\{(\Omega{::}'a$ *set*, A, $\mu).$ $(\forall a{\in}{-}A.$ μ $a = 0) \wedge$ *measure_space* Ω A μ $\}$

definition *space* :: $'a$ *measure* $\Rightarrow$ $'a$ *set* **where**
space $M =$ *fst* (*Rep_measure* M)

definition *sets* :: $'a$ *measure* $\Rightarrow$ $'a$ *set set* **where**
sets $M =$ *fst* (*snd* (*Rep_measure* M))

definition *emeasure* :: $'a$ *measure* $\Rightarrow$ $'a$ *set* $\Rightarrow$ *ennreal* **where**
emeasure $M =$ *snd* (*snd* (*Rep_measure* M))

definition *measure* :: $'a$ *measure* $\Rightarrow$ $'a$ *set* $\Rightarrow$ *real* **where**
measure M $A =$ *enn2real* (*emeasure* M A)

definition *measure_of* :: $'a$ *set* $\Rightarrow$ $'a$ *set set* $\Rightarrow$ ($'a$ *set* $\Rightarrow$ *ennreal*) $\Rightarrow$ $'a$ *measure*
where
measure_of Ω A $\mu \equiv$
Abs_measure (Ω, *if* $A \subseteq$ *Pow* Ω *then sigma_sets* Ω A *else* $\{\{\}, \Omega\}$,
$\lambda a.$ *if* $a \in$ *sigma_sets* Ω $A \wedge$ *measure_space* Ω (*sigma_sets* Ω A) μ *then* μ a *else* 0)
proposition *emeasure_measure_of*:
assumes M: $M =$ *measure_of* Ω A μ
assumes *ms*: $A \subseteq$ *Pow* Ω *positive* (*sets* M) μ *countably_additive* (*sets* M) μ
assumes X: $X \in$ *sets* M
shows *emeasure* M $X = \mu$ X
definition *measurable* :: $'a$ *measure* $\Rightarrow$ $'b$ *measure* $\Rightarrow$ ($'a \Rightarrow 'b$) *set*
(**infixr** $\langle\rightarrow_M\rangle$ *60*) **where**
measurable A $B = \{f \in$ *space* $A \rightarrow$ *space* $B.$ $\forall y \in$ *sets* $B.$ $f -` y \cap$ *space* $A \in$ *sets*

$A\}$
definition *count_space* :: *'a set* ⇒ *'a measure* **where**
count_space Ω = *measure_of* Ω (*Pow* Ω) (λ*A*. *if finite A then of_nat* (*card A*) *else* ∞)

8.1.3 The smallest σ-algebra regarding a function

definition *vimage_algebra* :: *'a set* ⇒ (*'a* ⇒ *'b*) ⇒ *'b measure* ⇒ *'a measure* **where**
vimage_algebra X f M = *sigma X* {*f* −‘ *A* ∩ *X* | *A*. *A* ∈ *sets M*}

end

8.2 Measurability Prover

theory *Measurable*
imports
Sigma_Algebra
HOL−Library.Order_Continuity
begin

method_setup *measurable* = ‹ *Scan.lift* (*Scan.succeed* (*METHOD o Measurable.measurable_tac*)) ›
measurability prover

simproc_setup *measurable* (*A* ∈ *sets M* | *f* ∈ *measurable M N*) =
‹*K Measurable.proc*›

end

8.3 Measure Spaces

theory *Measure_Space*
imports
Measurable HOL−Library.Extended_Nonnegative_Real
begin

8.3.1 μ-null sets

definition *null_sets* :: *'a measure* ⇒ *'a set set* **where**
null_sets M = {*N*∈*sets M*. *emeasure M N* = *0*}

8.3.2 The almost everywhere filter (i.e. quantifier)

definition *ae_filter* :: *'a measure* ⇒ *'a filter* **where**
ae_filter M = (*INF N*∈*null_sets M*. *principal* (*space M* − *N*))

8.3.3 σ-finite Measures

locale *sigma_finite_measure* =
fixes M :: $'a$ *measure*
assumes *sigma_finite_countable*:
$\exists A::'a\ set\ set.\ countable\ A \wedge A \subseteq sets\ M \wedge (\bigcup A) = space\ M \wedge (\forall a{\in}A.\ emeasure\ M\ a \neq \infty)$

8.3.4 Measure space induced by distribution of $(\rightarrow_M)$-functions

definition *distr* :: $'a$ *measure* $\Rightarrow$ $'b$ *measure* $\Rightarrow$ ($'a \Rightarrow 'b$) $\Rightarrow$ $'b$ *measure* **where**
distr M N f =
measure_of (*space N*) (*sets N*) ($\lambda A.$ *emeasure M* (*f* $-`$ *A* $\cap$ *space M*))

proposition *distr_distr*:
$g \in measurable\ N\ L \Longrightarrow f \in measurable\ M\ N \Longrightarrow distr\ (distr\ M\ N\ f)\ L\ g = distr\ M\ L\ (g \circ f)$

8.3.5 Set of measurable sets with finite measure

definition *fmeasurable* :: $'a$ *measure* $\Rightarrow$ $'a$ *set set* **where**
fmeasurable M = $\{A{\in}sets\ M.\ emeasure\ M\ A < \infty\}$

8.3.6 Measure spaces with $emeasure\ M\ (space\ M) < \infty$

locale *finite_measure* = *sigma_finite_measure M* **for** *M* +
assumes *finite_emeasure_space*: *emeasure M* (*space M*) $\neq$ *top*

8.3.7 Scaling a measure

definition *scale_measure* :: *ennreal* $\Rightarrow$ $'a$ *measure* $\Rightarrow$ $'a$ *measure* **where**
scale_measure r M = *measure_of* (*space M*) (*sets M*) ($\lambda A.$ *r* $*$ *emeasure M A*)

8.3.8 Complete lattice structure on measures

proposition *unsigned_Hahn_decomposition*:
assumes [*simp*]: *sets N* = *sets M* **and** [*measurable*]: *A* $\in$ *sets M*
and [*simp*]: *emeasure M A* $\neq$ *top emeasure N A* $\neq$ *top*
shows $\exists\ Y{\in}sets\ M.\ Y \subseteq A \wedge (\forall\ X{\in}sets\ M.\ X \subseteq Y \longrightarrow N\ X \leq M\ X) \wedge (\forall\ X{\in}sets\ M.\ X \subseteq A \longrightarrow X \cap Y = \{\} \longrightarrow M\ X \leq N\ X)$

Define a lexicographical order on *measure*, in the order space, sets and measure. The parts of the lexicographical order are point-wise ordered.

instantiation *measure* :: (*type*) *order_bot*
begin

definition *less_measure* :: *'a measure* ⇒ *'a measure* ⇒ *bool* **where**
less_measure M N ⟷ (*M* ≤ *N* ∧ ¬ *N* ≤ *M*)

definition *bot_measure* :: *'a measure* **where**
bot_measure = *sigma* {} {}

proposition *le_measure*: *sets M* = *sets N* ⟹ *M* ≤ *N* ⟷ (∀ *A*∈*sets M*. *emeasure M A* ≤ *emeasure N A*)

definition *sup_measure'* :: *'a measure* ⇒ *'a measure* ⇒ *'a measure* **where**
sup_measure' A B =
measure_of (*space A*) (*sets A*)
(λ*X*. *SUP Y*∈*sets A*. *emeasure A* (*X* ∩ *Y*) + *emeasure B* (*X* ∩ − *Y*))

definition *sup_lexord* :: *'a* ⇒ *'a* ⇒ (*'a* ⇒ *'b*::*order*) ⇒ *'a* ⇒ *'a* ⇒ *'a* **where**
sup_lexord A B k s c =
(*if k A* = *k B then c else*
if ¬ *k A* ≤ *k B* ∧ ¬ *k B* ≤ *k A then s else*
if k B ≤ *k A then A else B*)

instantiation *measure* :: (*type*) *semilattice_sup*
begin

definition *sup_measure* :: *'a measure* ⇒ *'a measure* ⇒ *'a measure* **where**
sup_measure A B =
sup_lexord A B space (*sigma* (*space A* ∪ *space B*) {})
(*sup_lexord A B sets* (*sigma* (*space A*) (*sets A* ∪ *sets B*)) (*sup_measure' A B*))

definition
Sup_lexord :: (*'a* ⇒ *'b*::*complete_lattice*) ⇒ (*'a set* ⇒ *'a*) ⇒ (*'a set* ⇒ *'a*) ⇒ *'a set* ⇒ *'a*
where
Sup_lexord k c s A =
(*let U* = (*SUP a*∈*A*. *k a*)
in if ∃ *a*∈*A*. *k a* = *U then c* {*a*∈*A*. *k a* = *U*} *else s A*)

instantiation *measure* :: (*type*) *complete_lattice*
begin

definition *Sup_measure'* :: *'a measure set* ⇒ *'a measure* **where**
Sup_measure' M =
measure_of (⋃ *a*∈*M*. *space a*) (⋃ *a*∈*M*. *sets a*)
(λ*X*. (*SUP P*∈{*P*. *finite P* ∧ *P* ⊆ *M* }. *sup_measure.F id P X*))

definition *Sup_measure* :: *'a measure set* ⇒ *'a measure* **where**
Sup_measure =
Sup_lexord space
(*Sup_lexord sets Sup_measure'*
(λ*U*. *sigma* (⋃ *u*∈*U*. *space u*) (⋃ *u*∈*U*. *sets u*)))
(λ*U*. *sigma* (⋃ *u*∈*U*. *space u*) {})

definition *Inf_measure* :: *'a measure set* ⇒ *'a measure* **where**
Inf_measure A = *Sup* {*x*. ∀ *a*∈*A*. *x* ≤ *a*}

definition *inf_measure* :: *'a measure* ⇒ *'a measure* ⇒ *'a measure* **where**
inf_measure a b = *Inf* {*a*, *b*}

definition *top_measure* :: *'a measure* **where**
top_measure = *Inf* {}

end

8.4 Borel Space

theory *Borel_Space*
imports
Measurable Derivative Ordered_Euclidean_Space Extended_Real_Limits
begin

proposition *open_prod_generated*: *open* = *generate_topology* {*A* × *B* | *A B*. *open A* ∧ *open B*}

proposition *mono_on_imp_deriv_nonneg*:
assumes *mono*: *mono_on A f* **and** *deriv*: (*f has_real_derivative D*) (*at x*)
assumes *x* ∈ *interior A*
shows *D* ≥ *0*

proposition *mono_on_ctble_discont*:
fixes *f* :: *real* ⇒ *real*
fixes *A* :: *real set*
assumes *mono_on A f*
shows *countable* {*a*∈*A*. ¬ *continuous* (*at a within A*) *f*}

8.4.1 Generic Borel spaces

definition (**in** *topological_space*) *borel* :: *'a measure* **where**
borel = *sigma UNIV* {*S*. *open S*}

theorem *second_countable_borel_measurable*:
fixes $X :: 'a::second_countable_topology\ set\ set$
assumes *eq*: $open = generate_topology\ X$
shows $borel = sigma\ UNIV\ X$

proposition *borel_eq_countable_basis*:
fixes $B::'a::topological_space\ set\ set$
assumes $countable\ B$
assumes $topological_basis\ B$
shows $borel = sigma\ UNIV\ B$

8.4.2 Borel spaces on order topologies

8.4.3 Borel spaces on topological monoids

8.4.4 Borel spaces on Euclidean spaces

8.4.5 Borel measurable operators

lemma *borel_measurable_complex_iff*:
$f \in borel_measurable\ M \longleftrightarrow$
$(\lambda x.\ Re\ (f\ x)) \in borel_measurable\ M \wedge (\lambda x.\ Im\ (f\ x)) \in borel_measurable\ M$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)

8.4.6 Borel space on the extended reals

theorem *borel_measurable_ereal_iff_real*:
fixes $f :: 'a \Rightarrow ereal$
shows $f \in borel_measurable\ M \longleftrightarrow$
$((\lambda x.\ real_of_ereal\ (f\ x)) \in borel_measurable\ M \wedge f -` \{\infty\} \cap space\ M \in sets\ M \wedge f -` \{-\infty\} \cap space\ M \in sets\ M)$

8.4.7 Borel space on the extended non-negative reals

definition [*simp*]: $is_borel\ f\ M \longleftrightarrow f \in borel_measurable\ M$

8.4.8 LIMSEQ is borel measurable

proposition *measurable_limit* [*measurable*]:
fixes $f::nat \Rightarrow 'a \Rightarrow 'b::first_countable_topology$
assumes [*measurable*]: $\bigwedge n::nat.\ f\ n \in borel_measurable\ M$
shows $Measurable.pred\ M\ (\lambda x.\ (\lambda n.\ f\ n\ x) \longrightarrow c)$

end

8.5 Lebesgue Integration for Nonnegative Functions

theory *Nonnegative_Lebesgue_Integration*
imports *Measure_Space Borel_Space*
begin

8.5.1 Simple function

definition *simple_function M g* $\longleftrightarrow$
$\quad$ *finite* (*g* ‘ *space M*) $\wedge$
$\quad$ ($\forall x \in g$ ‘ *space M*. *g* $-$‘ $\{x\} \cap$ *space M* $\in$ *sets M*)

lemma *borel_measurable_implies_simple_function_sequence*:
fixes *u* :: $'a \Rightarrow$ *ennreal*
assumes *u*[*measurable*]: *u* $\in$ *borel_measurable M*
shows $\exists f$. *incseq f* $\wedge$ ($\forall i$. ($\forall x$. *f i x* $<$ *top*) $\wedge$ *simple_function M* (*f i*)) $\wedge$ *u* = (*SUP i*. *f i*)

lemma *simple_function_induct*
$\quad$ [*consumes 1*, *case_names cong set mult add*, *induct set*: *simple_function*]:
fixes *u* :: $'a \Rightarrow$ *ennreal*
assumes *u*: *simple_function M u*
assumes *cong*: $\bigwedge f\ g$. *simple_function M f* $\Longrightarrow$ *simple_function M g* $\Longrightarrow$ (*AE x in M*. *f x* = *g x*) $\Longrightarrow$ *P f* $\Longrightarrow$ *P g*
assumes *set*: $\bigwedge A$. *A* $\in$ *sets M* $\Longrightarrow$ *P* (*indicator A*)
assumes *mult*: $\bigwedge u\ c$. *P u* $\Longrightarrow$ *P* (λx. *c* $*$ *u x*)
assumes *add*: $\bigwedge u\ v$. *P u* $\Longrightarrow$ *P v* $\Longrightarrow$ *P* (λx. *v x* + *u x*)
shows *P u*

lemma *borel_measurable_induct*
$\quad$ [*consumes 1*, *case_names cong set mult add seq*, *induct set*: *borel_measurable*]:
fixes *u* :: $'a \Rightarrow$ *ennreal*
assumes *u*: *u* $\in$ *borel_measurable M*
assumes *cong*: $\bigwedge f\ g$. *f* $\in$ *borel_measurable M* $\Longrightarrow$ *g* $\in$ *borel_measurable M* $\Longrightarrow$ ($\bigwedge x$. *x* $\in$ *space M* $\Longrightarrow$ *f x* = *g x*) $\Longrightarrow$ *P g* $\Longrightarrow$ *P f*
assumes *set*: $\bigwedge A$. *A* $\in$ *sets M* $\Longrightarrow$ *P* (*indicator A*)
assumes *mult'*: $\bigwedge u\ c$. *c* $<$ *top* $\Longrightarrow$ *u* $\in$ *borel_measurable M* $\Longrightarrow$ ($\bigwedge x$. *x* $\in$ *space M* $\Longrightarrow$ *u x* $<$ *top*) $\Longrightarrow$ *P u* $\Longrightarrow$ *P* (λx. *c* $*$ *u x*)
assumes *add*: $\bigwedge u\ v$. *u* $\in$ *borel_measurable M* $\Longrightarrow$ ($\bigwedge x$. *x* $\in$ *space M* $\Longrightarrow$ *u x* $<$ *top*) $\Longrightarrow$ *P u* $\Longrightarrow$ *v* $\in$ *borel_measurable M* $\Longrightarrow$ ($\bigwedge x$. *x* $\in$ *space M* $\Longrightarrow$ *v x* $<$ *top*) $\Longrightarrow$ ($\bigwedge x$. *x* $\in$ *space M* $\Longrightarrow$ *u x* = 0 $\vee$ *v x* = 0) $\Longrightarrow$ *P v* $\Longrightarrow$ *P* (λx. *v x* + *u x*)
assumes *seq*: $\bigwedge U$. ($\bigwedge i$. *U i* $\in$ *borel_measurable M*) $\Longrightarrow$ ($\bigwedge i\ x$. *x* $\in$ *space M* $\Longrightarrow$ *U i x* $<$ *top*) $\Longrightarrow$ ($\bigwedge i$. *P* (*U i*)) $\Longrightarrow$ *incseq U* $\Longrightarrow$ *u* = (*SUP i*. *U i*) $\Longrightarrow$ *P* (*SUP i*. *U i*)

shows $P\ u$

8.5.2 Simple integral

definition *simple_integral* :: $'a$ *measure* $\Rightarrow$ ($'a \Rightarrow$ *ennreal*) $\Rightarrow$ *ennreal* (‹$integral^S$›)
where
$integral^S\ M\ f = (\sum x \in f\ `\ space\ M.\ x * emeasure\ M\ (f -` \{x\} \cap space\ M))$

8.5.3 Integral on nonnegative functions

definition *nn_integral* :: $'a$ *measure* $\Rightarrow$ ($'a \Rightarrow$ *ennreal*) $\Rightarrow$ *ennreal* (‹$integral^N$›)
where
$integral^N\ M\ f = (SUP\ g \in \{g.\ simple_function\ M\ g \wedge g \leq f\}.\ integral^S\ M\ g)$

theorem *nn_integral_monotone_convergence_SUP_AE*:
assumes f: $\bigwedge i.\ AE\ x\ in\ M.\ f\ i\ x \leq f\ (Suc\ i)\ x\ \bigwedge i.\ f\ i \in borel_measurable\ M$
shows $(\int^+ x.\ (SUP\ i.\ f\ i\ x)\ \partial M) = (SUP\ i.\ integral^N\ M\ (f\ i))$

theorem *nn_integral_suminf*:
assumes f: $\bigwedge i.\ f\ i \in borel_measurable\ M$
shows $(\int^+ x.\ (\sum i.\ f\ i\ x)\ \partial M) = (\sum i.\ integral^N\ M\ (f\ i))$

theorem *nn_integral_Markov_inequality*:
assumes u: $(\lambda x.\ u\ x * indicator\ A\ x) \in borel_measurable\ M$ **and** $A \in sets\ M$
shows $(emeasure\ M)\ (\{x \in A.\ 1 \leq c * u\ x\}) \leq c * (\int^+ x.\ u\ x * indicator\ A\ x\ \partial M)$
(**is** $(emeasure\ M)\ ?A \leq _ * ?PI$)

theorem *nn_integral_monotone_convergence_INF_AE*:
fixes $f :: nat \Rightarrow 'a \Rightarrow ennreal$
assumes f: $\bigwedge i.\ AE\ x\ in\ M.\ f\ (Suc\ i)\ x \leq f\ i\ x$
and [*measurable*]: $\bigwedge i.\ f\ i \in borel_measurable\ M$
and *fin*: $(\int^+ x.\ f\ i\ x\ \partial M) < \infty$
shows $(\int^+ x.\ (INF\ i.\ f\ i\ x)\ \partial M) = (INF\ i.\ integral^N\ M\ (f\ i))$

theorem *nn_integral_liminf*:
fixes $u :: nat \Rightarrow 'a \Rightarrow ennreal$
assumes u: $\bigwedge i.\ u\ i \in borel_measurable\ M$
shows $(\int^+ x.\ liminf\ (\lambda n.\ u\ n\ x)\ \partial M) \leq liminf\ (\lambda n.\ integral^N\ M\ (u\ n))$

theorem *nn_integral_limsup*:
fixes $u :: nat \Rightarrow 'a \Rightarrow ennreal$
assumes [*measurable*]: $\bigwedge i.\ u\ i \in borel_measurable\ M\ w \in borel_measurable\ M$
assumes *bounds*: $\bigwedge i.\ AE\ x\ in\ M.\ u\ i\ x \leq w\ x$ **and** w: $(\int^+ x.\ w\ x\ \partial M) < \infty$
shows $limsup\ (\lambda n.\ integral^N\ M\ (u\ n)) \leq (\int^+ x.\ limsup\ (\lambda n.\ u\ n\ x)\ \partial M)$

theorem *nn_integral_dominated_convergence*:
assumes [*measurable*]:

$\bigwedge i.\ u\ i \in borel_measurable\ M\ u' \in borel_measurable\ M\ w \in borel_measurable\ M$
and *bound*: $\bigwedge j.\ AE\ x\ in\ M.\ u\ j\ x \leq w\ x$
and *w*: $(\int^+ x.\ w\ x\ \partial M) < \infty$
and *u'*: $AE\ x\ in\ M.\ (\lambda i.\ u\ i\ x) \longrightarrow u'\ x$
shows $(\lambda i.\ (\int^+ x.\ u\ i\ x\ \partial M)) \longrightarrow (\int^+ x.\ u'\ x\ \partial M)$

theorem *nn_integral_lfp*:
assumes *sets*[*simp*]: $\bigwedge s.\ sets\ (M\ s) = sets\ N$
assumes *f*: *sup_continuous f*
assumes *g*: *sup_continuous g*
assumes *meas*: $\bigwedge F.\ F \in borel_measurable\ N \Longrightarrow f\ F \in borel_measurable\ N$
assumes *step*: $\bigwedge F\ s.\ F \in borel_measurable\ N \Longrightarrow integral^N\ (M\ s)\ (f\ F) = g\ (\lambda s.\ integral^N\ (M\ s)\ F)\ s$
shows $(\int^+ \omega.\ lfp\ f\ \omega\ \partial M\ s) = lfp\ g\ s$

theorem *nn_integral_gfp*:
assumes *sets*[*simp*]: $\bigwedge s.\ sets\ (M\ s) = sets\ N$
assumes *f*: *inf_continuous f* **and** *g*: *inf_continuous g*
assumes *meas*: $\bigwedge F.\ F \in borel_measurable\ N \Longrightarrow f\ F \in borel_measurable\ N$
assumes *bound*: $\bigwedge F\ s.\ F \in borel_measurable\ N \Longrightarrow (\int^+ x.\ f\ F\ x\ \partial M\ s) < \infty$
assumes *non_zero*: $\bigwedge s.\ emeasure\ (M\ s)\ (space\ (M\ s)) \neq 0$
assumes *step*: $\bigwedge F\ s.\ F \in borel_measurable\ N \Longrightarrow integral^N\ (M\ s)\ (f\ F) = g\ (\lambda s.\ integral^N\ (M\ s)\ F)\ s$
shows $(\int^+ \omega.\ gfp\ f\ \omega\ \partial M\ s) = gfp\ g\ s$

8.5.4 Integral under concrete measures

definition *density* :: *'a measure* ⇒ (*'a* ⇒ *ennreal*) ⇒ *'a measure* **where**
density M f = *measure_of* (*space M*) (*sets M*) (λ*A*. $\int^+ x.\ f\ x * indicator\ A\ x\ \partial M$)

lemma *nn_integral_density*:
assumes *f*: $f \in borel_measurable\ M$
assumes *g*: $g \in borel_measurable\ M$
shows $integral^N\ (density\ M\ f)\ g = (\int^+ x.\ f\ x * g\ x\ \partial M)$
definition *point_measure* :: *'a set* ⇒ (*'a* ⇒ *ennreal*) ⇒ *'a measure* **where**
point_measure A f = *density* (*count_space A*) *f*
definition *uniform_measure M A* = *density M* (λ*x*. *indicator A x* / *emeasure M A*)
definition *uniform_count_measure A* = *point_measure A* (λ*x*. *1* / *card A*)

end

8.6 Binary Product Measure

theory *Binary_Product_Measure*

imports *Nonnegative_Lebesgue_Integration*
begin

8.6.1 Binary products

definition *pair_measure* (**infixr** ‹$\bigotimes_M$› *80*) **where**
$A \bigotimes_M B = measure_of\ (space\ A \times space\ B)$
$\{a \times b \mid a\ b.\ a \in sets\ A \wedge b \in sets\ B\}$
$(\lambda X.\ \int^+ x.\ (\int^+ y.\ indicator\ X\ (x,y)\ \partial B)\ \partial A)$

proposition (**in** *sigma_finite_measure*) *emeasure_pair_measure_Times*:
assumes *A*: $A \in sets\ N$ **and** *B*: $B \in sets\ M$
shows $emeasure\ (N \bigotimes_M M)\ (A \times B) = emeasure\ N\ A * emeasure\ M\ B$

8.6.2 Binary products of σ-finite emeasure spaces

proposition (**in** *pair_sigma_finite*) *sigma_finite_up_in_pair_measure_generator*:
defines $E \equiv \{A \times B \mid A\ B.\ A \in sets\ M1 \wedge B \in sets\ M2\}$
shows $\exists F{::}nat \Rightarrow ('a \times 'b)\ set.\ range\ F \subseteq E \wedge incseq\ F \wedge (\bigcup i.\ F\ i) = space\ M1 \times space\ M2 \wedge$
$(\forall i.\ emeasure\ (M1 \bigotimes_M M2)\ (F\ i) \neq \infty)$

8.6.3 Fubinis theorem

proposition (**in** *pair_sigma_finite*) *nn_integral_snd*:
assumes *f*[*measurable*]: $f \in borel_measurable\ (M1 \bigotimes_M M2)$
shows $(\int^+ y.\ (\int^+ x.\ f\ (x,\ y)\ \partial M1)\ \partial M2) = integral^N\ (M1 \bigotimes_M M2)\ f$

theorem (**in** *pair_sigma_finite*) *Fubini*:
assumes *f*: $f \in borel_measurable\ (M1 \bigotimes_M M2)$
shows $(\int^+ y.\ (\int^+ x.\ f\ (x,\ y)\ \partial M1)\ \partial M2) = (\int^+ x.\ (\int^+ y.\ f\ (x,\ y)\ \partial M2)\ \partial M1)$

theorem (**in** *pair_sigma_finite*) *Fubini'*:
assumes *f*: $case_prod\ f \in borel_measurable\ (M1 \bigotimes_M M2)$
shows $(\int^+ y.\ (\int^+ x.\ f\ x\ y\ \partial M1)\ \partial M2) = (\int^+ x.\ (\int^+ y.\ f\ x\ y\ \partial M2)\ \partial M1)$

8.6.4 Products on counting spaces, densities and distributions

proposition *sigma_prod*:
assumes *X_cover*: $\exists E \subseteq A.\ countable\ E \wedge X = \bigcup E$ **and** *A*: $A \subseteq Pow\ X$
assumes *Y_cover*: $\exists E \subseteq B.\ countable\ E \wedge Y = \bigcup E$ **and** *B*: $B \subseteq Pow\ Y$

shows *sigma X A* $\bigotimes_M$ *sigma Y B* = *sigma* (*X* × *Y*) {*a* × *b* | *a b*. *a* ∈ *A* ∧ *b* ∈ *B*}
(**is** *?P* = *?S*)

proposition *sets_pair_eq*:
assumes *Ea*: *Ea* ⊆ *Pow* (*space A*) *sets A* = *sigma_sets* (*space A*) *Ea*
and *Ca*: *countable Ca Ca* ⊆ *Ea* ⋃ *Ca* = *space A*
and *Eb*: *Eb* ⊆ *Pow* (*space B*) *sets B* = *sigma_sets* (*space B*) *Eb*
and *Cb*: *countable Cb Cb* ⊆ *Eb* ⋃ *Cb* = *space B*
shows *sets* (*A* $\bigotimes_M$ *B*) = *sets* (*sigma* (*space A* × *space B*) { *a* × *b* | *a b*. *a* ∈ *Ea* ∧ *b* ∈ *Eb* })
(**is** _ = *sets* (*sigma ?Ω ?E*))

proposition *borel_prod*:
(*borel* $\bigotimes_M$ *borel*) = (*borel* :: (′*a*::*second_countable_topology* × ′*b*::*second_countable_topology*) *measure*)
(**is** *?P* = *?B*)

proposition *pair_measure_count_space*:
assumes *A*: *finite A* **and** *B*: *finite B*
shows *count_space A* $\bigotimes_M$ *count_space B* = *count_space* (*A* × *B*) (**is** *?P* = *?C*)

theorem *pair_measure_density*:
assumes *f*: *f* ∈ *borel_measurable M1*
assumes *g*: *g* ∈ *borel_measurable M2*
assumes *sigma_finite_measure M2 sigma_finite_measure* (*density M2 g*)
shows *density M1 f* $\bigotimes_M$ *density M2 g* = *density* (*M1* $\bigotimes_M$ *M2*) (λ(*x*,*y*). *f x* ∗ *g y*) (**is** *?L* = *?R*)

proposition *nn_integral_fst_count_space*:
(∫⁺ *x*. ∫⁺ *y*. *f* (*x*, *y*) ∂*count_space UNIV* ∂*count_space UNIV*) = *integral*^{N} (*count_space UNIV*) *f*
(**is** *?lhs* = *?rhs*)

proposition *nn_integral_snd_count_space*:
(∫⁺ *y*. ∫⁺ *x*. *f* (*x*, *y*) ∂*count_space UNIV* ∂*count_space UNIV*) = *integral*^{N} (*count_space UNIV*) *f*
(**is** *?lhs* = *?rhs*)

8.6.5 Product of Borel spaces

theorem *borel_Times*:
fixes *A* :: ′*a*::*topological_space set* **and** *B* :: ′*b*::*topological_space set*
assumes *A*: *A* ∈ *sets borel* **and** *B*: *B* ∈ *sets borel*
shows *A* × *B* ∈ *sets borel*

end

8.7 Finite Product Measure

theory *Finite_Product_Measure*
imports *Binary_Product_Measure Function_Topology*
begin

8.7.1 Finite product spaces

definition *prod_emb* **where**
prod_emb I M K X = $(\lambda x.\ restrict\ x\ K) -` X \cap (\Pi_E\ i{\in}I.\ space\ (M\ i))$

definition *PiM* :: *'i set* ⇒ (*'i* ⇒ *'a measure*) ⇒ (*'i* ⇒ *'a*) *measure* **where**
PiM I M = *extend_measure* $(\Pi_E\ i{\in}I.\ space\ (M\ i))$
$\{(J,\ X).\ (J \neq \{\} \vee I = \{\}) \wedge finite\ J \wedge J \subseteq I \wedge X \in (\Pi\ j{\in}J.\ sets\ (M\ j))\}$
$(\lambda(J,\ X).\ prod_emb\ I\ M\ J\ (\Pi_E\ j{\in}J.\ X\ j))$
$(\lambda(J,\ X).\ \prod j{\in}J \cup \{i{\in}I.\ emeasure\ (M\ i)\ (space\ (M\ i)) \neq 1\}.\ if\ j \in J\ then\ emeasure\ (M\ j)\ (X\ j)\ else\ emeasure\ (M\ j)\ (space\ (M\ j)))$

definition *prod_algebra* :: *'i set* ⇒ (*'i* ⇒ *'a measure*) ⇒ (*'i* ⇒ *'a*) *set set* **where**
prod_algebra I M = $(\lambda(J,\ X).\ prod_emb\ I\ M\ J\ (\Pi_E\ j{\in}J.\ X\ j))$ ‘
$\{(J,\ X).\ (J \neq \{\} \vee I = \{\}) \wedge finite\ J \wedge J \subseteq I \wedge X \in (\Pi\ j{\in}J.\ sets\ (M\ j))\}$

proposition *prod_algebra_mono*:
assumes *space*: $\bigwedge i.\ i \in I \Longrightarrow space\ (E\ i) = space\ (F\ i)$
assumes *sets*: $\bigwedge i.\ i \in I \Longrightarrow sets\ (E\ i) \subseteq sets\ (F\ i)$
shows *prod_algebra I E* ⊆ *prod_algebra I F*
proposition *prod_algebra_cong*:
assumes *I* = *J* **and** $(\bigwedge i.\ i \in I \Longrightarrow sets\ (M\ i) = sets\ (N\ i))$
shows *prod_algebra I M* = *prod_algebra J N*

proposition *sets_PiM_single*: *sets* (*PiM I M*) =
sigma_sets $(\Pi_E\ i{\in}I.\ space\ (M\ i))$ $\{\{f{\in}\Pi_E\ i{\in}I.\ space\ (M\ i).\ f\ i \in A\} \mid i\ A.\ i \in I \wedge A \in sets\ (M\ i)\}$
(**is** _ = *sigma_sets ?Ω ?R*)

proposition *sets_PiM_sigma*:
assumes *Ω_cover*: $\bigwedge i.\ i \in I \Longrightarrow \exists S{\subseteq}E\ i.\ countable\ S \wedge \Omega\ i = \bigcup S$
assumes *E*: $\bigwedge i.\ i \in I \Longrightarrow E\ i \subseteq Pow\ (\Omega\ i)$
assumes *J*: $\bigwedge j.\ j \in J \Longrightarrow finite\ j \bigcup J = I$
defines $P \equiv \{\{f{\in}(\Pi_E\ i{\in}I.\ \Omega\ i).\ \forall i{\in}j.\ f\ i \in A\ i\} \mid A\ j.\ j \in J \wedge A \in Pi\ j\ E\}$
shows *sets* $(\Pi_M\ i{\in}I.\ sigma\ (\Omega\ i)\ (E\ i)) = sets\ (sigma\ (\Pi_E\ i{\in}I.\ \Omega\ i)\ P)$

proposition *measurable_PiM*:
assumes *space*: $f \in space\ N \to (\Pi_E\ i{\in}I.\ space\ (M\ i))$
assumes *sets*: $\bigwedge X\ J.\ J \neq \{\} \vee I = \{\} \Longrightarrow finite\ J \Longrightarrow J \subseteq I \Longrightarrow (\bigwedge i.\ i \in J$

$\Longrightarrow X\ i \in sets\ (M\ i)) \Longrightarrow$
$f -` prod_emb\ I\ M\ J\ (Pi_E\ J\ X) \cap space\ N \in sets\ N$
shows $f \in measurable\ N\ (PiM\ I\ M)$

proposition *measurable_fun_upd*:
assumes *I*: $I = J \cup \{i\}$
assumes *f*[*measurable*]: $f \in measurable\ N\ (PiM\ J\ M)$
assumes *h*[*measurable*]: $h \in measurable\ N\ (M\ i)$
shows $(\lambda x.\ (f\ x)\ (i := h\ x)) \in measurable\ N\ (PiM\ I\ M)$

proposition *measure_eqI_PiM_finite*:
assumes [*simp*]: *finite I sets P* = *PiM I M sets Q* = *PiM I M*
assumes *eq*: $\bigwedge A.\ (\bigwedge i.\ i \in I \Longrightarrow A\ i \in sets\ (M\ i)) \Longrightarrow P\ (Pi_E\ I\ A) = Q\ (Pi_E\ I\ A)$
assumes *A*: *range A* $\subseteq$ *prod_algebra I M* $(\bigcup i.\ A\ i) = space\ (PiM\ I\ M)\ \bigwedge i{::}nat.\ P\ (A\ i) \neq \infty$
shows $P = Q$

proposition *measure_eqI_PiM_infinite*:
assumes [*simp*]: *sets P* = *PiM I M sets Q* = *PiM I M*
assumes *eq*: $\bigwedge A\ J.\ finite\ J \Longrightarrow J \subseteq I \Longrightarrow (\bigwedge i.\ i \in J \Longrightarrow A\ i \in sets\ (M\ i)) \Longrightarrow$
$P\ (prod_emb\ I\ M\ J\ (Pi_E\ J\ A)) = Q\ (prod_emb\ I\ M\ J\ (Pi_E\ J\ A))$
assumes *A*: *finite_measure P*
shows $P = Q$

proposition (**in** *finite_product_sigma_finite*) *sigma_finite_pairs*:
$\exists F{::}'i \Rightarrow nat \Rightarrow 'a\ set.$
$(\forall i{\in}I.\ range\ (F\ i) \subseteq sets\ (M\ i)) \wedge$
$(\forall k.\ \forall i{\in}I.\ emeasure\ (M\ i)\ (F\ i\ k) \neq \infty) \wedge incseq\ (\lambda k.\ \Pi_E\ i{\in}I.\ F\ i\ k) \wedge$
$(\bigcup k.\ \Pi_E\ i{\in}I.\ F\ i\ k) = space\ (PiM\ I\ M)$

lemma (**in** *product_sigma_finite*) *distr_merge*:
assumes *IJ*[*simp*]: $I \cap J = \{\}$ **and** *fin*: *finite I finite J*
shows $distr\ (Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M)\ (Pi_M\ (I \cup J)\ M)\ (merge\ I\ J) = Pi_M\ (I \cup J)\ M$
(**is** *?D* = *?P*)

proposition (**in** *product_sigma_finite*) *product_nn_integral_fold*:
assumes *IJ*: $I \cap J = \{\}$ *finite I finite J*
and *f*[*measurable*]: $f \in borel_measurable\ (Pi_M\ (I \cup J)\ M)$
shows $integral^N\ (Pi_M\ (I \cup J)\ M)\ f = (\int^+ x.\ (\int^+ y.\ f\ (merge\ I\ J\ (x, y))\ \partial(Pi_M\ J\ M))\ \partial(Pi_M\ I\ M))$
(**is** *?lhs* = *?rhs*)

proposition (**in** *product_sigma_finite*) *product_nn_integral_insert*:
assumes *I*[*simp*]: *finite I* $i \notin I$
and *f*: $f \in borel_measurable\ (Pi_M\ (insert\ i\ I)\ M)$
shows $integral^N\ (Pi_M\ (insert\ i\ I)\ M)\ f = (\int^+ x.\ (\int^+ y.\ f\ (x(i := y))\ \partial(M\ i))$

$\partial(Pi_M\ I\ M))$

proposition (**in** *product_sigma_finite*) *product_nn_integral_pair*:
assumes [*measurable*]: *case_prod f* ∈ *borel_measurable* (*M x* ⨂$_M$ *M y*)
assumes *xy*: *x* ≠ *y*
shows (∫$^+\sigma$. *f* (*σ x*) (*σ y*) *∂PiM* {*x*, *y*} *M*) = (∫^+z. *f* (*fst z*) (*snd z*) *∂*(*M x* ⨂$_M$ *M y*))

8.7.2 Measurability

proposition *sets_PiM_equal_borel*:
sets (*Pi*$_M$ *UNIV* (λ*i*::('*a*::*countable*). *borel*::('*b*::*second_countable_topology measure*))) = *sets borel*

end

8.8 Caratheodory Extension Theorem

theory *Caratheodory*
imports *Measure_Space*
begin

8.8.1 Characterizations of Measures

definition *outer_measure_space* **where**
outer_measure_space M f ⟷ *positive M f* ∧ *increasing M f* ∧ *countably_subadditive M f*

Lambda Systems

definition *lambda_system* :: '*a set* ⇒ '*a set set* ⇒ ('*a set* ⇒ *ennreal*) ⇒ '*a set set*
where
lambda_system Ω *M f* = {*l* ∈ *M*. ∀ *x* ∈ *M*. *f* (*l* ∩ *x*) + *f* ((Ω − *l*) ∩ *x*) = *f x*}

proposition (**in** *sigma_algebra*) *lambda_system_caratheodory*:
assumes *oms*: *outer_measure_space M f*
and *A*: *range A* ⊆ *lambda_system* Ω *M f*
and *disj*: *disjoint_family A*
shows (⋃ *i*. *A i*) ∈ *lambda_system* Ω *M f* ∧ (∑ *i*. *f* (*A i*)) = *f* (⋃ *i*. *A i*)

proposition (**in** *sigma_algebra*) *caratheodory_lemma*:
assumes *oms*: *outer_measure_space M f*
defines *L* ≡ *lambda_system* Ω *M f*
shows *measure_space* Ω *L f*

definition *outer_measure* :: *'a set set* ⇒ (*'a set* ⇒ *ennreal*) ⇒ *'a set* ⇒ *ennreal* **where**
outer_measure M f X =
(*INF A*∈{*A*. *range A* ⊆ *M* ∧ *disjoint_family A* ∧ *X* ⊆ (⋃ *i*. *A i*)}. ∑ *i*. *f* (*A i*))

8.8.2 Caratheodory's theorem

theorem (**in** *ring_of_sets*) *caratheodory'*:
assumes *posf*: *positive M f* **and** *ca*: *countably_additive M f*
shows ∃ *μ* :: *'a set* ⇒ *ennreal*. (∀ *s* ∈ *M*. *μ s* = *f s*) ∧ *measure_space* Ω (*sigma_sets* Ω *M*) *μ*

8.8.3 Volumes

definition *volume* :: *'a set set* ⇒ (*'a set* ⇒ *ennreal*) ⇒ *bool* **where**
volume M f ⟷
(*f* {} = *0*) ∧ (∀ *a*∈*M*. *0* ≤ *f a*) ∧
(∀ *C*⊆*M*. *disjoint C* ⟶ *finite C* ⟶ ⋃ *C* ∈ *M* ⟶ *f* (⋃ *C*) = (∑ *c*∈*C*. *f c*))

proposition *volume_finite_additive*:
assumes *volume M f*
assumes *A*: ⋀*i*. *i* ∈ *I* ⟹ *A i* ∈ *M* *disjoint_family_on A I* *finite I* ⋃(*A* ‘ *I*) ∈ *M*
shows *f* (⋃(*A* ‘ *I*)) = (∑ *i*∈*I*. *f* (*A i*))

proposition (**in** *semiring_of_sets*) *extend_volume*:
assumes *volume M μ*
shows ∃ *μ'*. *volume generated_ring μ'* ∧ (∀ *a*∈*M*. *μ' a* = *μ a*)

Caratheodory on semirings

theorem (**in** *semiring_of_sets*) *caratheodory*:
assumes *pos*: *positive M μ* **and** *ca*: *countably_additive M μ*
shows ∃ *μ'* :: *'a set* ⇒ *ennreal*. (∀ *s* ∈ *M*. *μ' s* = *μ s*) ∧ *measure_space* Ω (*sigma_sets* Ω *M*) *μ'*

proposition *extend_measure_caratheodory_pair*:
fixes *G* :: *'i* ⇒ *'j* ⇒ *'a set*
assumes *M*: *M* = *extend_measure* Ω {(*a*, *b*). *P a b*} (λ(*a*, *b*). *G a b*) (λ(*a*, *b*). *μ a b*)
assumes *P i j*
assumes *semiring*: *semiring_of_sets* Ω {*G a b* | *a b*. *P a b*}
assumes *empty*: ⋀*i j*. *P i j* ⟹ *G i j* = {} ⟹ *μ i j* = *0*
assumes *inj*: ⋀*i j k l*. *P i j* ⟹ *P k l* ⟹ *G i j* = *G k l* ⟹ *μ i j* = *μ k l*
assumes *nonneg*: ⋀*i j*. *P i j* ⟹ *0* ≤ *μ i j*
assumes *add*: ⋀*A*::*nat* ⇒ *'i*. ⋀*B*::*nat* ⇒ *'j*. ⋀*j k*.

$(\bigwedge n.\ P\ (A\ n)\ (B\ n)) \Longrightarrow P\ j\ k \Longrightarrow$ *disjoint_family* $(\lambda n.\ G\ (A\ n)\ (B\ n)) \Longrightarrow$
$(\bigcup i.\ G\ (A\ i)\ (B\ i)) = G\ j\ k \Longrightarrow (\sum n.\ \mu\ (A\ n)\ (B\ n)) = \mu\ j\ k$
shows *emeasure* $M\ (G\ i\ j) = \mu\ i\ j$

end

8.9 Bochner Integration for Vector-Valued Functions

theory *Bochner_Integration*
imports *Finite_Product_Measure*
beginproposition *borel_measurable_implies_sequence_metric*:
fixes $f :: 'a \Rightarrow 'b :: \{metric_space, second_countable_topology\}$
assumes [*measurable*]: $f \in borel_measurable\ M$
shows $\exists F.\ (\forall i.\ simple_function\ M\ (F\ i)) \wedge (\forall x{\in}space\ M.\ (\lambda i.\ F\ i\ x) \longrightarrow f\ x) \wedge$
$(\forall i.\ \forall x{\in}space\ M.\ dist\ (F\ i\ x)\ z \leq 2 * dist\ (f\ x)\ z)$

definition *simple_bochner_integral* :: $'a\ measure \Rightarrow ('a \Rightarrow 'b{::}real_vector) \Rightarrow 'b$
where
simple_bochner_integral $M\ f = (\sum y{\in}f`space\ M.\ measure\ M\ \{x{\in}space\ M.\ f\ x = y\} *_R\ y)$

proposition *simple_bochner_integral_partition*:
assumes f: *simple_bochner_integrable* $M\ f$ **and** g: *simple_function* $M\ g$
assumes *sub*: $\bigwedge x\ y.\ x \in space\ M \Longrightarrow y \in space\ M \Longrightarrow g\ x = g\ y \Longrightarrow f\ x = f\ y$
assumes v: $\bigwedge x.\ x \in space\ M \Longrightarrow f\ x = v\ (g\ x)$
shows *simple_bochner_integral* $M\ f = (\sum y{\in}g\ `\ space\ M.\ measure\ M\ \{x{\in}space\ M.\ g\ x = y\} *_R\ v\ y)$
(**is** $_ = ?r$)

proposition *has_bochner_integral_implies_finite_norm*:
has_bochner_integral $M\ f\ x \Longrightarrow (\int^+ x.\ norm\ (f\ x)\ \partial M) < \infty$

proposition *has_bochner_integral_norm_bound*:
assumes i: *has_bochner_integral* $M\ f\ x$
shows $norm\ x \leq (\int^+ x.\ norm\ (f\ x)\ \partial M)$

definition *lebesgue_integral* (‹$integral^L$›) **where**
$integral^L\ M\ f = (if\ \exists x.\ has_bochner_integral\ M\ f\ x\ then\ THE\ x.\ has_bochner_integral\ M\ f\ x\ else\ 0)$

proposition *nn_integral_dominated_convergence_norm*:
fixes $u' :: _ \Rightarrow _{::}\{real_normed_vector, second_countable_topology\}$
assumes [*measurable*]:
$\bigwedge i.\ u\ i \in borel_measurable\ M\ u' \in borel_measurable\ M\ w \in borel_measurable\ M$
and *bound*: $\bigwedge j.\ AE\ x\ in\ M.\ norm\ (u\ j\ x) \leq w\ x$

and w: $(\int^+ x.\ w\ x\ \partial M) < \infty$
and u': $AE\ x\ in\ M.\ (\lambda i.\ u\ i\ x) \longrightarrow u'\ x$
shows $(\lambda i.\ (\int^+ x.\ norm\ (u'\ x - u\ i\ x)\ \partial M)) \longrightarrow 0$

proposition *integrableI_bounded*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second_countable_topology*}
assumes f[*measurable*]: $f \in borel_measurable\ M$ **and** *fin*: $(\int^+ x.\ norm\ (f\ x)\ \partial M) < \infty$
shows *integrable M f*

proposition *nn_integral_eq_integral*:
assumes f: *integrable M f*
assumes *nonneg*: $AE\ x\ in\ M.\ 0 \le f\ x$
shows $(\int^+ x.\ f\ x\ \partial M) = integral^L\ M\ f$

proposition *integral_norm_bound* [*simp*]:
fixes $f :: _ \Rightarrow 'a$ :: {*banach*, *second_countable_topology*}
shows $norm\ (integral^L\ M\ f) \le (\int x.\ norm\ (f\ x)\ \partial M)$

proposition *integral_abs_bound* [*simp*]:
fixes $f :: 'a \Rightarrow real$ **shows** $abs\ (\int x.\ f\ x\ \partial M) \le (\int x.\ |f\ x|\ \partial M)$

proposition *integrable_induct*[*consumes 1*, *case_names base add lim*, *induct pred*: *integrable*]:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second_countable_topology*}
assumes *integrable M f*
assumes *base*: $\bigwedge A\ c.\ A \in sets\ M \Longrightarrow emeasure\ M\ A < \infty \Longrightarrow P\ (\lambda x.\ indicator\ A\ x *_R c)$
assumes *add*: $\bigwedge f\ g.\ integrable\ M\ f \Longrightarrow P\ f \Longrightarrow integrable\ M\ g \Longrightarrow P\ g \Longrightarrow P\ (\lambda x.\ f\ x + g\ x)$
assumes *lim*: $\bigwedge f\ s.\ (\bigwedge i.\ integrable\ M\ (s\ i)) \Longrightarrow (\bigwedge i.\ P\ (s\ i)) \Longrightarrow$
$(\bigwedge x.\ x \in space\ M \Longrightarrow (\lambda i.\ s\ i\ x) \longrightarrow f\ x) \Longrightarrow$
$(\bigwedge i\ x.\ x \in space\ M \Longrightarrow norm\ (s\ i\ x) \le 2 * norm\ (f\ x)) \Longrightarrow integrable\ M\ f \Longrightarrow P\ f$
shows $P\ f$

theorem *integral_Markov_inequality*:
assumes [*measurable*]: *integrable M u* **and** $AE\ x\ in\ M.\ 0 \le u\ x$ $0 < (c$::*real*$)$
shows $(emeasure\ M)\ \{x \in space\ M.\ u\ x \ge c\} \le (1/c) * (\int x.\ u\ x\ \partial M)$

theorem *integral_Markov_inequality_measure*:
assumes [*measurable*]: *integrable M u* **and** $A \in sets\ M$ **and** $AE\ x\ in\ M.\ 0 \le u\ x$ $0 < (c$::*real*$)$
shows $measure\ M\ \{x \in space\ M.\ u\ x \ge c\} \le (\int x.\ u\ x\ \partial M)\ /\ c$

theorem (**in** *finite_measure*) *second_moment_method*:
assumes [*measurable*]: $f \in M \to_M borel$
assumes *integrable M* $(\lambda x.\ f\ x\ \hat{}\ 2)$

defines $\mu \equiv$ *lebesgue_integral M f*
assumes $a > 0$
shows *measure M* $\{x \in$ *space M*. $|f\,x| \geq a\} \leq$ *lebesgue_integral M* $(\lambda x.\ f\,x\ \widehat{}\ 2)$ / a^2
proof −
have *integrable*: *integrable M f*
using *assms* **by** (*blast dest*: *square_integrable_imp_integrable*)
have $\{x \in$ *space M*. $|f\,x| \geq a\} = \{x \in$ *space M*. $f\,x\ \widehat{}\ 2 \geq a^2\}$
using ‹$a > 0$› **by** (*simp flip*: *abs_le_square_iff*)
hence *measure M* $\{x \in$ *space M*. $|f\,x| \geq a\}$ = *measure M* $\{x \in$ *space M*. $f\,x\ \widehat{}\ 2 \geq a^2\}$
by *simp*
also have ... ≤ *lebesgue_integral M* $(\lambda x.\ f\,x\ \widehat{}\ 2)$ / a^2
using *assms* **by** (*intro integral_Markov_inequality_measure*) *auto*
finally show *?thesis* **.**
qed

proposition *tendsto_L1_int*:
fixes *u* :: _ ⇒ _ ⇒ *'b*::{*banach*, *second_countable_topology*}
assumes [*measurable*]: $\bigwedge n.$ *integrable M* $(u\ n)$ *integrable M f*
and $((\lambda n.\ (\int^+ x.\ norm(u\ n\ x - f\,x)\ \partial M)) \longrightarrow 0)\ F$
shows $((\lambda n.\ (\int x.\ u\ n\ x\ \partial M)) \longrightarrow (\int x.\ f\,x\ \partial M))\ F$
proposition *tendsto_L1_AE_subseq*:
fixes *u* :: *nat* ⇒ *'a* ⇒ *'b*::{*banach*, *second_countable_topology*}
assumes [*measurable*]: $\bigwedge n.$ *integrable M* $(u\ n)$
and $(\lambda n.\ (\int x.\ norm(u\ n\ x)\ \partial M)) \longrightarrow 0$
shows $\exists r{::}nat \Rightarrow nat.$ *strict_mono r* $\wedge$ $(AE\ x\ in\ M.\ (\lambda n.\ u\ (r\ n)\ x) \longrightarrow 0)$

8.9.1 Restricted measure spaces

8.9.2 Measure spaces with an associated density

8.9.3 Distributions

8.9.4 Lebesgue integration on *count_space*

8.9.5 Point measure

proposition *integrable_point_measure_finite*:
fixes *g* :: *'a* ⇒ *'b*::{*banach*, *second_countable_topology*} **and** *f* :: *'a* ⇒ *real*
assumes *finite A*
shows *integrable* (*point_measure A f*) *g*

8.9.6 Lebesgue integration on *null_measure*

8.9.7 Legacy lemmas for the real-valued Lebesgue integral

theorem *real_lebesgue_integral_def*:

assumes *f*[*measurable*]: *integrable M f*
shows *integral*^{L} *M f* = *enn2real* ($\int^+ x.$ *f x* ∂M) − *enn2real* ($\int^+ x.$ *ennreal* (− *f x*) ∂M)

theorem *real_integrable_def*:
integrable M f $\longleftrightarrow$ *f* ∈ *borel_measurable M* ∧
($\int^+ x.$ *ennreal* (*f x*) ∂M) ≠ ∞ ∧ ($\int^+ x.$ *ennreal* (− *f x*) ∂M) ≠ ∞

8.9.8 Product measure

proposition (**in** *sigma_finite_measure*) *borel_measurable_lebesgue_integral*[*measurable* (*raw*)]:
fixes *f* :: _ ⇒ _ ⇒ _::{*banach*, *second_countable_topology*}
assumes *f*[*measurable*]: *case_prod f* ∈ *borel_measurable* (*N* $\bigotimes_M$ *M*)
shows ($\lambda x.$ $\int y.$ *f x y* ∂M) ∈ *borel_measurable N*

theorem (**in** *pair_sigma_finite*) *Fubini_integrable*:
fixes *f* :: _ ⇒ _::{*banach*, *second_countable_topology*}
assumes *f*[*measurable*]: *f* ∈ *borel_measurable* (*M1* $\bigotimes_M$ *M2*)
and *integ1*: *integrable M1* ($\lambda x.$ $\int$ *y. norm* (*f* (*x*, *y*)) ∂*M2*)
and *integ2*: *AE x in M1. integrable M2* ($\lambda y.$ *f* (*x*, *y*))
shows *integrable* (*M1* $\bigotimes_M$ *M2*) *f*

proposition (**in** *pair_sigma_finite*) *integral_fst'*:
fixes *f* :: _ ⇒ _::{*banach*, *second_countable_topology*}
assumes *f*: *integrable* (*M1* $\bigotimes_M$ *M2*) *f*
shows ($\int x.$ ($\int y.$ *f* (*x*, *y*) ∂*M2*) ∂*M1*) = *integral*^{L} (*M1* $\bigotimes_M$ *M2*) *f*

proposition (**in** *pair_sigma_finite*) *Fubini_integral*:
fixes *f* :: _ ⇒ _ ⇒ _ :: {*banach*, *second_countable_topology*}
assumes *f*: *integrable* (*M1* $\bigotimes_M$ *M2*) (*case_prod f*)
shows ($\int y.$ ($\int x.$ *f x y* ∂*M1*) ∂*M2*) = ($\int x.$ ($\int y.$ *f x y* ∂*M2*) ∂*M1*)

end

8.10 Complete Measures

theory *Complete_Measure*
imports *Bochner_Integration*
begin

locale *complete_measure* =
fixes *M* :: *'a measure*
assumes *complete*: $\bigwedge$*A B*. *B* ⊆ *A* ⟹ *A* ∈ *null_sets M* ⟹ *B* ∈ *sets M*

definition
split_completion M A p = (*if A* ∈ *sets M then p* = (*A*, {}) *else*

$\exists N'.\ A = fst\ p \cup snd\ p \wedge fst\ p \cap snd\ p = \{\} \wedge fst\ p \in sets\ M \wedge snd\ p \subseteq N' \wedge N' \in null_sets\ M$)

definition
main_part M A = *fst* (*Eps* (*split_completion M A*))

definition
null_part M A = *snd* (*Eps* (*split_completion M A*))

definition *completion* :: *'a measure* ⇒ *'a measure* **where**
completion M = *measure_of* (*space M*) { $S \cup N$ |*S N N'*. $S \in sets\ M \wedge N' \in null_sets\ M \wedge N \subseteq N'$ }
(*emeasure M* ∘ *main_part M*)

lemma *sets_completion*:
sets (*completion M*) = { $S \cup N$ |*S N N'*. $S \in sets\ M \wedge N' \in null_sets\ M \wedge N \subseteq N'$ }

lemma *measurable_completion*: $f \in M \rightarrow_M N \Longrightarrow f \in completion\ M \rightarrow_M N$

lemma *split_completion*:
assumes $A \in sets\ (completion\ M)$
shows *split_completion M A* (*main_part M A*, *null_part M A*)

lemma *emeasure_completion*[*simp*]:
assumes *S*: $S \in sets\ (completion\ M)$
shows *emeasure* (*completion M*) *S* = *emeasure M* (*main_part M S*)

lemma *completion_ex_borel_measurable*:
fixes *g* :: *'a* ⇒ *ennreal*
assumes *g*: $g \in borel_measurable\ (completion\ M)$
shows $\exists g' \in borel_measurable\ M.\ (AE\ x\ in\ M.\ g\ x = g'\ x)$

locale *semifinite_measure* =
fixes *M* :: *'a measure*
assumes *semifinite*:
$\bigwedge A.\ A \in sets\ M \Longrightarrow emeasure\ M\ A = \infty \Longrightarrow \exists B \in sets\ M.\ B \subseteq A \wedge emeasure\ M\ B < \infty$

locale *locally_determined_measure* = *semifinite_measure* +
assumes *locally_determined*:
$\bigwedge A.\ A \subseteq space\ M \Longrightarrow (\bigwedge B.\ B \in sets\ M \Longrightarrow emeasure\ M\ B < \infty \Longrightarrow A \cap B \in sets\ M) \Longrightarrow A \in sets\ M$

locale *cld_measure* =
complete_measure M + *locally_determined_measure M* **for** *M* :: *'a measure*

definition *outer_measure_of* :: *'a measure* ⇒ *'a set* ⇒ *ennreal*
where *outer_measure_of M A* = (*INF B* ∈ {$B \in sets\ M.\ A \subseteq B$}. *emeasure M*

B)

definition *measurable_envelope* :: *'a measure* ⇒ *'a set* ⇒ *'a set* ⇒ *bool*
where *measurable_envelope M A E* ⟷
(*A* ⊆ *E* ∧ *E* ∈ *sets M* ∧ (∀ *F*∈*sets M*. *emeasure M* (*F* ∩ *E*) = *outer_measure_of M* (*F* ∩ *A*)))

lemma *measurable_envelope_eq2*:
assumes *A* ⊆ *E E* ∈ *sets M emeasure M E* < ∞
shows *measurable_envelope M A E* ⟷ (*emeasure M E* = *outer_measure_of M A*)

proposition (**in** *complete_measure*) *fmeasurable_inner_outer*:
S ∈ *fmeasurable M* ⟷
(∀ *e*>*0*. ∃ *T*∈*fmeasurable M*. ∃ *U*∈*fmeasurable M*. *T* ⊆ *S* ∧ *S* ⊆ *U* ∧ |*measure M T* − *measure M U*| < *e*)
(**is** _ ⟷ *?approx*)

end

8.11 Regularity of Measures

theory *Regularity*
imports *Measure_Space Borel_Space*
begin

theorem
fixes *M*::*'a*::{*second_countable_topology*, *complete_space*} *measure*
assumes *sb*: *sets M* = *sets borel*
assumes *emeasure M* (*space M*) ≠ ∞
assumes *B* ∈ *sets borel*
shows *inner_regular*: *emeasure M B* =
(*SUP K* ∈ {*K*. *K* ⊆ *B* ∧ *compact K*}. *emeasure M K*) (**is** *?inner B*)
and *outer_regular*: *emeasure M B* =
(*INF U* ∈ {*U*. *B* ⊆ *U* ∧ *open U*}. *emeasure M U*) (**is** *?outer B*)

end

8.12 Lebesgue Measure

theory *Lebesgue_Measure*
imports
Finite_Product_Measure
Caratheodory
Complete_Measure
Summation_Tests
Regularity
begin

8.12.1 Measures defined by monotonous functions

definition *interval_measure* :: (*real* ⇒ *real*) ⇒ *real measure* **where**
interval_measure F =
extend_measure UNIV {(*a*, *b*). *a* ≤ *b*} (λ(*a*, *b*). {*a*<..*b*}) (λ(*a*, *b*). *ennreal* (*F b* − *F a*))

lemma *emeasure_interval_measure_Ioc*:
assumes *a* ≤ *b*
assumes *mono_F*: ⋀*x y*. *x* ≤ *y* ⟹ *F x* ≤ *F y*
assumes *right_cont_F* : ⋀*a*. *continuous* (*at_right a*) *F*
shows *emeasure* (*interval_measure F*) {*a*<..*b*} = *F b* − *F a*

lemma *sets_interval_measure* [*simp*, *measurable_cong*]:
sets (*interval_measure F*) = *sets borel*

lemma *sigma_finite_interval_measure*:
assumes *mono_F*: ⋀*x y*. *x* ≤ *y* ⟹ *F x* ≤ *F y*
assumes *right_cont_F* : ⋀*a*. *continuous* (*at_right a*) *F*
shows *sigma_finite_measure* (*interval_measure F*)

8.12.2 Lebesgue-Borel measure

definition *lborel* :: ('*a* :: *euclidean_space*) *measure* **where**
lborel = *distr* (Π$_M$ *b*∈*Basis*. *interval_measure* (λ*x*. *x*)) *borel* (λ*f*. ∑ *b*∈*Basis*. *f b* ∗$_R$ *b*)

abbreviation *lebesgue* :: '*a*::*euclidean_space measure*
where *lebesgue* ≡ *completion lborel*

abbreviation *lebesgue_on* :: '*a set* ⇒ '*a*::*euclidean_space measure*
where *lebesgue_on* Ω ≡ *restrict_space* (*completion lborel*) Ω

8.12.3 Borel measurability

lemma *emeasure_lborel_cbox*[*simp*]:
assumes [*simp*]: ⋀*b*. *b* ∈ *Basis* ⟹ *l* ⋅ *b* ≤ *u* ⋅ *b*
shows *emeasure lborel* (*cbox l u*) = (∏ *b*∈*Basis*. (*u* − *l*) ⋅ *b*)

8.12.4 Affine transformation on the Lebesgue-Borel

lemma *lborel_eqI*:
fixes *M* :: '*a*::*euclidean_space measure*

assumes *emeasure_eq*: $\bigwedge l\ u.\ (\bigwedge b.\ b \in Basis \Longrightarrow l \cdot b \leq u \cdot b) \Longrightarrow emeasure\ M\ (box\ l\ u) = (\prod b{\in}Basis.\ (u - l) \cdot b)$
assumes *sets_eq*: $sets\ M = sets\ borel$
shows $lborel = M$

lemma *lborel_affine_euclidean*:
fixes $c :: 'a{::}euclidean_space \Rightarrow real$ **and** t
defines $T\ x \equiv t + (\sum j{\in}Basis.\ (c\ j * (x \cdot j)) *_R j)$
assumes c: $\bigwedge j.\ j \in Basis \Longrightarrow c\ j \neq 0$
shows $lborel = density\ (distr\ lborel\ borel\ T)\ (\lambda_.\ (\prod j{\in}Basis.\ |c\ j|))$ (**is** $_ = ?D$)

lemma *lborel_integral_real_affine*:
fixes $f :: real \Rightarrow 'a :: \{banach, second_countable_topology\}$ **and** $c :: real$
assumes c: $c \neq 0$ **shows** $(\int x.\ f\ x\ \partial\ lborel) = |c| *_R (\int x.\ f\ (t + c * x)\ \partial lborel)$

corollary *lebesgue_real_affine*:
$c \neq 0 \Longrightarrow lebesgue = density\ (distr\ lebesgue\ lebesgue\ (\lambda x.\ t + c * x))\ (\lambda_.\ ennreal\ (abs\ c))$

lemma *lborel_prod*:
$lborel \bigotimes_M lborel = (lborel :: ('a{::}euclidean_space \times 'b{::}euclidean_space)\ measure)$

8.12.5 Lebesgue measurable sets

abbreviation *lmeasurable* :: $'a{::}euclidean_space\ set\ set$
where
$lmeasurable \equiv fmeasurable\ lebesgue$

lemma *lmeasurable_iff_integrable*:
$S \in lmeasurable \longleftrightarrow integrable\ lebesgue\ (indicator\ S :: 'a{::}euclidean_space \Rightarrow real)$

8.12.6 Lebesgue measurability of ordinate sets

8.12.7 A nice lemma for negligibility proofs

proposition *starlike_negligible_bounded_gmeasurable*:
fixes $S :: 'a :: euclidean_space\ set$
assumes S: $S \in sets\ lebesgue$ **and** $bounded\ S$
and *eq1*: $\bigwedge c\ x.\ [\![(c *_R x) \in S;\ 0 \leq c;\ x \in S]\!] \Longrightarrow c = 1$
shows $S \in null_sets\ lebesgue$

corollary *starlike_negligible_compact*:
$compact\ S \Longrightarrow (\bigwedge c\ x.\ [\![(c *_R x) \in S;\ 0 \leq c;\ x \in S]\!] \Longrightarrow c = 1) \Longrightarrow S \in null_sets\ lebesgue$

proposition *outer_regular_lborel_le*:
assumes *B*[*measurable*]: $B \in$ *sets borel* **and** $0 <$ (*e*::*real*)
obtains *U* **where** *open U B* $\subseteq U$ **and** *emeasure lborel* $(U - B) \leq e$

lemma *outer_regular_lborel*:
assumes *B*: $B \in$ *sets borel* **and** $0 <$ (*e*::*real*)
obtains *U* **where** *open U B* $\subseteq U$ *emeasure lborel* $(U - B) < e$

8.12.8 *F_sigma* and *G_delta* sets.

inductive *fsigma* :: *'a*::*topological_space set* $\Rightarrow$ *bool* **where**
$(\bigwedge n$::*nat*. *closed* $(F\ n)) \Longrightarrow$ *fsigma* $(\bigcup (F$ ' *UNIV*$))$

inductive *gdelta* :: *'a*::*topological_space set* $\Rightarrow$ *bool* **where**
$(\bigwedge n$::*nat*. *open* $(F\ n)) \Longrightarrow$ *gdelta* $(\bigcap (F$ ' *UNIV*$))$

end

8.13 Tagged Divisions for Henstock-Kurzweil Integration

theory *Tagged_Division*
imports *Topology_Euclidean_Space*
begin

8.13.1 Some useful lemmas about intervals

8.13.2 Bounds on intervals where they exist

definition *interval_upperbound* :: (*'a*::*euclidean_space*) *set* $\Rightarrow$ *'a*
where *interval_upperbound s* $= (\sum i \in Basis.\ (SUP\ x \in s.\ x \cdot i) *_R\ i)$

definition *interval_lowerbound* :: (*'a*::*euclidean_space*) *set* $\Rightarrow$ *'a*
where *interval_lowerbound s* $= (\sum i \in Basis.\ (INF\ x \in s.\ x \cdot i) *_R\ i)$

8.13.3 The notion of a gauge — simply an open set containing the point

definition *gauge* $\gamma \longleftrightarrow (\forall x.\ x \in \gamma\ x \wedge$ *open* $(\gamma\ x))$

8.13.4 Attempt a systematic general set of "offset" results for components

8.13.5 Divisions

definition *division_of* (**infixl** ‹*division'_of*› *40*)
where
s division_of i ⟷
finite s ∧
(∀ *K*∈*s*. *K* ⊆ *i* ∧ *K* ≠ {} ∧ (∃ *a b*. *K* = *cbox a b*)) ∧
(∀ *K1*∈*s*. ∀ *K2*∈*s*. *K1* ≠ *K2* ⟶ *interior*(*K1*) ∩ *interior*(*K2*) = {}) ∧
(⋃ *s* = *i*)

proposition *partial_division_extend_interval*:
assumes *p division_of* (⋃ *p*) (⋃ *p*) ⊆ *cbox a b*
obtains *q* **where** *p* ⊆ *q q division_of cbox a* (*b*::'*a*::*euclidean_space*)

proposition *division_union_intervals_exists*:
assumes *cbox a b* ≠ {}
obtains *p* **where** (*insert* (*cbox a b*) *p*) *division_of* (*cbox a b* ∪ *cbox c d*)

8.13.6 Tagged (partial) divisions

definition *tagged_partial_division_of* (**infixr** ‹*tagged'_partial'_division'_of*› *40*)
where *s tagged_partial_division_of i* ⟷
finite s ∧
(∀ *x K*. (*x*, *K*) ∈ *s* ⟶ *x* ∈ *K* ∧ *K* ⊆ *i* ∧ (∃ *a b*. *K* = *cbox a b*)) ∧
(∀ *x1 K1 x2 K2*. (*x1*, *K1*) ∈ *s* ∧ (*x2*, *K2*) ∈ *s* ∧ (*x1*, *K1*) ≠ (*x2*, *K2*) ⟶
interior K1 ∩ *interior K2* = {})

definition *tagged_division_of* (**infixr** ‹*tagged'_division'_of*› *40*)
where *s tagged_division_of i* ⟷ *s tagged_partial_division_of i* ∧ (⋃{*K*. ∃ *x*. (*x*,*K*) ∈ *s*} = *i*)

8.13.7 Functions closed on boxes: morphisms from boxes to monoids

Using additivity of lifted function to encode definedness. **definition** *lift_option* :: ('*a* ⇒ '*b* ⇒ '*c*) ⇒ '*a option* ⇒ '*b option* ⇒ '*c option*
where
lift_option f a' b' = *Option.bind a'* (λ*a*. *Option.bind b'* (λ*b*. *Some* (*f a b*)))

lemma *comm_monoid_lift_option*:
assumes *comm_monoid f z*
shows *comm_monoid* (*lift_option f*) (*Some z*)

Misc

Division points **definition** *division_points* (*k*::(*'a*::*euclidean_space*) *set*) *d* =
{(*j*,*x*). *j* ∈ *Basis* ∧ (*interval_lowerbound k*)·*j* < *x* ∧ *x* < (*interval_upperbound k*)·*j* ∧
(∃ *i*∈*d*. (*interval_lowerbound i*)·*j* = *x* ∨ (*interval_upperbound i*)·*j* = *x*)}

Operative
proposition *tagged_division*:
assumes *d tagged_division_of* (*cbox a b*)
shows *F* (λ(_, *l*). *g l*) *d* = *g* (*cbox a b*)

8.13.8 Special case of additivity we need for the FTC

8.13.9 Fine-ness of a partition w.r.t. a gauge

definition *fine* (**infixr** ‹*fine*› *46*)
where *d fine s* ⟷ (∀(*x*,*k*) ∈ *s*. *k* ⊆ *d x*)

8.13.10 Some basic combining lemmas

8.13.11 General bisection principle for intervals; might be useful elsewhere

8.13.12 Cousin's lemma

8.13.13 A technical lemma about "refinement" of division

Covering lemma

proposition *covering_lemma*:
assumes *S* ⊆ *cbox a b box a b* ≠ {} *gauge g*
obtains *D* **where**
countable D ⋃*D* ⊆ *cbox a b*
⋀*K*. *K* ∈ *D* ⟹ *interior K* ≠ {} ∧ (∃ *c d*. *K* = *cbox c d*)
pairwise (λ*A B*. *interior A* ∩ *interior B* = {}) *D*
⋀*K*. *K* ∈ *D* ⟹ ∃ *x* ∈ *S* ∩ *K*. *K* ⊆ *g x*
⋀*u v*. *cbox u v* ∈ *D* ⟹ ∃ *n*. ∀ *i* ∈ *Basis*. *v* · *i* − *u* · *i* = (*b* · *i* − *a* · *i*) / *2*^*n*
S ⊆ ⋃*D*

8.13.14 Division filter

definition *division_filter* :: *'a*::*euclidean_space set* ⇒ (*'a* × *'a set*) *set filter*
where *division_filter s* = (*INF g*∈{*g*. *gauge g*}. *principal* {*p*. *p tagged_division_of s* ∧ *g fine p*})

proposition *eventually_division_filter*:
(∀$_F$ *p in division_filter s*. *P p*) ⟷
(∃ *g*. *gauge g* ∧ (∀ *p*. *p tagged_division_of s* ∧ *g fine p* ⟶ *P p*))

end

8.14 Henstock-Kurzweil Gauge Integration in Many Dimensions

theory *Henstock_Kurzweil_Integration*
imports
Lebesgue_Measure Tagged_Division HOL−Real_Asymp.Real_Asymp

begin

8.14.1 Content (length, area, volume, etc.) of an interval

8.14.2 Gauge integral

8.14.3 Basic theorems about integrals

corollary *integral_mult_left* [*simp*]:
fixes *c*:: *'a*::{*real_normed_algebra*,*division_ring*}
shows *integral S* (λ*x*. *f x* ∗ *c*) = *integral S f* ∗ *c*

corollary *integral_mult_right* [*simp*]:
fixes *c*:: *'a*::{*real_normed_field*}
shows *integral S* (λ*x*. *c* ∗ *f x*) = *c* ∗ *integral S f*

corollary *integral_divide* [*simp*]:
fixes *z* :: *'a*::*real_normed_field*
shows *integral S* (λ*x*. *f x* / *z*) = *integral S* (λ*x*. *f x*) / *z*

8.14.4 Cauchy-type criterion for integrability

proposition *integrable_Cauchy*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::{*real_normed_vector*,*complete_space*}
shows *f integrable_on cbox a b* ⟷
(∀ *e*>*0*. ∃γ. *gauge* γ ∧
(∀ *D1 D2*. *D1 tagged_division_of* (*cbox a b*) ∧ γ *fine D1* ∧
D2 tagged_division_of (*cbox a b*) ∧ γ *fine D2* ⟶
norm ((∑(*x*,*K*)∈*D1*. *content K* ∗$_R$ *f x*) − (∑(*x*,*K*)∈*D2*. *content K* ∗$_R$ *f x*)) < *e*))
(**is** *?l* = (∀ *e*>*0*. ∃γ. *?P e* γ))

8.14.5 Additivity of integral on abutting intervals

proposition *has_integral_split*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::real_normed_vector$
assumes *fi*: $(f\ has_integral\ i)\ (cbox\ a\ b \cap \{x.\ x \cdot k \leq c\})$
and *fj*: $(f\ has_integral\ j)\ (cbox\ a\ b \cap \{x.\ x \cdot k \geq c\})$
and *k*: $k \in Basis$
shows $(f\ has_integral\ (i + j))\ (cbox\ a\ b)$

8.14.6 A sort of converse, integrability on subintervals

8.14.7 Bounds on the norm of Riemann sums and the integral itself

corollary *integrable_bound*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::real_normed_vector$
assumes $0 \leq B$
and $f\ integrable_on\ (cbox\ a\ b)$
and $\bigwedge x.\ x \in cbox\ a\ b \Longrightarrow norm\ (f\ x) \leq B$
shows $norm\ (integral\ (cbox\ a\ b)\ f) \leq B * content\ (cbox\ a\ b)$

8.14.8 Similar theorems about relationship among components

8.14.9 Uniform limit of integrable functions is integrable

8.14.10 Negligible sets

proposition *negligible_standard_hyperplane*[*intro*]:
fixes $k :: 'a::euclidean_space$
assumes *k*: $k \in Basis$
shows $negligible\ \{x.\ x \cdot k = c\}$

corollary *negligible_standard_hyperplane_cart*:
fixes $k :: 'a::finite$
shows $negligible\ \{x.\ x\$k = (0::real)\}$

proposition *has_integral_negligible*:
fixes $f :: 'b::euclidean_space \Rightarrow 'a::real_normed_vector$

assumes *negs*: *negligible S*
and $\bigwedge x.\ x \in (T - S) \Longrightarrow f\ x = 0$
shows (*f has_integral 0*) *T*

8.14.11 Some other trivialities about negligible sets

8.14.12 Finite case of the spike theorem is quite commonly needed

corollary *has_integral_bound_real*:
fixes *f* :: *real* ⇒ *'b*::*real_normed_vector*
assumes $0 \le B$ *finite S*
and (*f has_integral i*) $\{a..b\}$
and $\bigwedge x.\ x \in \{a..b\} - S \Longrightarrow$ *norm* $(f\ x) \le B$
shows *norm i* $\le B *$ *content* $\{a..b\}$

8.14.13 In particular, the boundary of an interval is negligible

8.14.14 Integrability of continuous functions

8.14.15 Specialization of additivity to one dimension

8.14.16 A useful lemma allowing us to factor out the content size

8.14.17 Fundamental theorem of calculus

theorem *fundamental_theorem_of_calculus*:
fixes *f* :: *real* ⇒ *'a*::*banach*
assumes $a \le b$
and *vecd*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow$ (*f has_vector_derivative f' x*) (*at x within* $\{a..b\}$)
shows (*f' has_integral* (*f b* − *f a*)) $\{a..b\}$

8.14.18 Taylor series expansion

8.14.19 Only need trivial subintervals if the interval itself is trivial

proposition *division_of_nontrivial*:
fixes $\mathcal{D}$ :: *'a*::*euclidean_space set set*
assumes *sdiv*: $\mathcal{D}$ *division_of* (*cbox a b*)
and *cont0*: *content* (*cbox a b*) ≠ *0*
shows $\{k.\ k \in \mathcal{D} \wedge$ *content k* ≠ *0*$\}$ *division_of* (*cbox a b*)

8.14.20 Integrability on subintervals

8.14.21 Combining adjacent intervals in 1 dimension

8.14.22 Reduce integrability to "local" integrability

8.14.23 Second FTC or existence of antiderivative

8.14.24 Combined fundamental theorem of calculus

8.14.25 General "twiddling" for interval-to-interval function image

8.14.26 Special case of a basic affine transformation

8.14.27 Special case of stretching coordinate axes separately

8.14.28 even more special cases

8.14.29 Stronger form of FCT; quite a tedious proof

theorem *fundamental_theorem_of_calculus_interior*:
fixes f :: *real* $\Rightarrow$ *'a*::*real_normed_vector*
assumes $a \leq b$
and *contf*: *continuous_on* $\{a..b\}$ f
and *derf*: $\bigwedge x.\ x \in \{a <..< b\} \Longrightarrow$ (f *has_vector_derivative* f' x) (*at* x)
shows (f' *has_integral* (f b $-$ f a)) $\{a..b\}$

proposition *indefinite_integral_continuous_left*:
fixes f:: *real* $\Rightarrow$ *'a*::*banach*
assumes *intf*: f *integrable_on* $\{a..b\}$ **and** $a < c$ $c \leq b$ $e > 0$
obtains d **where** $d > 0$
and $\forall t.\ c - d < t \wedge t \leq c \longrightarrow$ *norm* (*integral* $\{a..c\}$ f $-$ *integral* $\{a..t\}$ f) $<$ e

theorem *integral_has_vector_derivative'*:
fixes f :: *real* $\Rightarrow$ *'b*::*banach*
assumes *continuous_on* $\{a..b\}$ f
and $x \in \{a..b\}$
shows (($\lambda u.$ *integral* $\{u..b\}$ f) *has_vector_derivative* $-$ f x) (*at* x *within* $\{a..b\}$)

8.14.30 Integrating characteristic function of an interval

corollary *has_integral_restrict_UNIV*:
fixes f :: *'n*::*euclidean_space* $\Rightarrow$ *'a*::*banach*
shows (($\lambda x.$ *if* $x \in s$ *then* f x *else* 0) *has_integral* i) *UNIV* $\longleftrightarrow$ (f *has_integral* i) s

8.14.31 Integrals on set differences

corollary *integral_spike_set*:
fixes f :: $'n$::*euclidean_space* ⇒ $'a$::*banach*
assumes *negligible* $\{x \in S - T.\ f\,x \neq 0\}$ *negligible* $\{x \in T - S.\ f\,x \neq 0\}$
shows *integral* $S\ f$ = *integral* $T\ f$

8.14.32 More lemmas that are useful later

8.14.33 Continuity of the integral (for a 1-dimensional interval)

8.14.34 A straddling criterion for integrability

8.14.35 Adding integrals over several sets

8.14.36 Also tagged divisions

8.14.37 Henstock's lemma

8.14.38 Monotone convergence (bounded interval first)

8.14.39 differentiation under the integral sign

8.14.40 Exchange uniform limit and integral

8.14.41 Compute a double integral using iterated integrals and switching the order of integration

theorem *integral_swap_continuous*:
fixes f :: [$'a$::*euclidean_space*, $'b$::*euclidean_space*] ⇒ $'c$::*banach*
assumes *continuous_on* (*cbox* (*a*,*c*) (*b*,*d*)) (λ(*x*,*y*). *f x y*)
shows *integral* (*cbox a b*) (λ*x*. *integral* (*cbox c d*) (*f x*)) =
integral (*cbox c d*) (λ*y*. *integral* (*cbox a b*) (λ*x*. *f x y*))

8.14.42 Definite integrals for exponential and power function

8.14.43 Adaption to ordered Euclidean spaces and the Cartesian Euclidean space

end

Chapter 9

Kronecker's Theorem with Applications

theory *Kronecker_Approximation_Theorem*

imports *Complex_Transcendental Henstock_Kurzweil_Integration*
HOL−Real_Asymp.Real_Asymp

begin

9.1 Dirichlet's Approximation Theorem

theorem *Dirichlet_approx_simult*:
fixes $\vartheta :: nat \Rightarrow real$ **and** $N\ n :: nat$
assumes $N > 0$
obtains $q\ p$ **where** $0<q\ q \leq int\ (N\hat{}n)$
and $\bigwedge i.\ i<n \Longrightarrow |of_int\ q * \vartheta\ i - of_int(p\ i)| < 1/N$
corollary *Dirichlet_approx*:
fixes ϑ:: *real* **and** N:: *nat*
assumes $N > 0$
obtains $h\ k$ **where** $0 < k\ k \leq int\ N\ |of_int\ k * \vartheta - of_int\ h| < 1/N$
corollary *Dirichlet_approx_coprime*:
fixes ϑ:: *real* **and** N:: *nat*
assumes $N > 0$
obtains $h\ k$ **where** *coprime* $h\ k\ 0 < k\ k \leq int\ N\ |of_int\ k * \vartheta - of_int\ h| < 1/N$
theorem *infinite_approx_set*:
assumes *infinite* (*approx_set* ϑ)
shows $\exists h\ k.\ (h,k) \in approx_set\ \vartheta \wedge k > K$
theorem *rational_iff_finite_approx_set*:
shows $\vartheta \in \mathbb{Q} \longleftrightarrow$ *finite* (*approx_set* ϑ)

9.2 Kronecker's Approximation Theorem: the One-dimensional Case

theorem *Kronecker_approx_1_explicit*:
fixes ϑ :: *real*
assumes $\vartheta \notin \mathbb{Q}$ **and** α: $0 \leq \alpha$ $\alpha \leq 1$ **and** $\varepsilon > 0$
obtains k **where** $k{>}0$ $|frac(real\ k * \vartheta) - \alpha| < \varepsilon$
corollary *Kronecker_approx_1*:
fixes ϑ :: *real*
assumes $\vartheta \notin \mathbb{Q}$
shows *closure* (*range* (λn. *frac* (*real* $n * \vartheta$))) = $\{0..1\}$ (**is** *?C* = _)
corollary *sequence_of_fractional_parts_is_dense*:
fixes ϑ :: *real*
assumes $\vartheta \notin \mathbb{Q}$ $\varepsilon > 0$
obtains h k **where** $k > 0$ $|of_int\ k * \vartheta - of_int\ h - \alpha| < \varepsilon$

9.3 Extension of Kronecker's Theorem to Simultaneous Approximation

9.3.1 Towards Lemma 1

9.3.2 Towards Lemma 2

9.3.3 Towards lemma 3

9.3.4 And finally Kroncker's theorem itself

theorem *Kronecker_thm_1*:
fixes α ϑ:: *nat* $\Rightarrow$ *real* **and** n:: *nat*
assumes *indp*: *module.independent* (λr. ($*$) (*real_of_int r*)) (ϑ ' $\{..<n\}$)
and *injϑ*: *inj_on* ϑ $\{..<n\}$ **and** $\varepsilon > 0$
obtains t h **where** $\bigwedge i.\ i < n \Longrightarrow |t * \vartheta\ i - of_int\ (h\ i) - \alpha\ i| < \varepsilon$
corollary *Kronecker_thm_2*:
fixes α ϑ :: *nat* $\Rightarrow$ *real* **and** n :: *nat*
assumes *indp*: *module.independent* ($\lambda r\ x$. *of_int* $r * x$) (ϑ ' $\{..n\}$)
and *injϑ*: *inj_on* ϑ $\{..n\}$ **and** [*simp*]: ϑ $n = 1$ **and** $\varepsilon > 0$
obtains k m **where** $\bigwedge i.\ i < n \Longrightarrow |of_int\ k * \vartheta\ i - of_int\ (m\ i) - \alpha\ i| < \varepsilon$

end

9.4 Bernstein-Weierstrass and Stone-Weierstrass

theory *Weierstrass_Theorems*
imports *Uniform_Limit Path_Connected Derivative*
begin

9.4.1 Bernstein polynomials

definition *Bernstein* :: [*nat*,*nat*,*real*] ⇒ *real* **where**
Bernstein n k x ≡ *of_nat* (*n choose k*) ∗ *x*^*k* ∗ (*1* − *x*)^(*n* − *k*)

9.4.2 Explicit Bernstein version of the 1D Weierstrass approximation theorem

theorem *Bernstein_Weierstrass*:
fixes *f* :: *real* ⇒ *real*
assumes *contf*: *continuous_on* {*0..1*} *f* **and** *e*: *0* < *e*
shows ∃ *N*. ∀ *n x*. *N* ≤ *n* ∧ *x* ∈ {*0..1*}
⟶ |*f x* − (∑ *k*≤*n*. *f*(*k*/*n*) ∗ *Bernstein n k x*)| < *e*

9.4.3 General Stone-Weierstrass theorem

definition *normf* :: (*'a*::*t2_space* ⇒ *real*) ⇒ *real*
where *normf f* ≡ *SUP x*∈*S*. |*f x*|
proposition (**in** *function_ring_on*) *Stone_Weierstrass_basic*:
assumes *f*: *continuous_on S f* **and** *e*: *e* > *0*
shows ∃ *g* ∈ *R*. ∀ *x*∈*S*. |*f x* − *g x*| < *e*

theorem (**in** *function_ring_on*) *Stone_Weierstrass*:
assumes *f*: *continuous_on S f*
shows ∃ *F*∈*UNIV* → *R*. *LIM n sequentially*. *F n* :> *uniformly_on S f*
corollary *Stone_Weierstrass_HOL*:
fixes *R* :: (*'a*::*t2_space* ⇒ *real*) *set* **and** *S* :: *'a set*
assumes *compact S* ⋀*c*. *P*(λ*x*. *c*::*real*)
⋀*f*. *P f* ⟹ *continuous_on S f*
⋀*f g*. *P*(*f*) ∧ *P*(*g*) ⟹ *P*(λ*x*. *f x* + *g x*) ⋀*f g*. *P*(*f*) ∧ *P*(*g*) ⟹ *P*(λ*x*. *f x* ∗ *g x*)
⋀*x y*. *x* ∈ *S* ∧ *y* ∈ *S* ∧ *x* ≠ *y* ⟹ ∃*f*. *P*(*f*) ∧ *f x* ≠ *f y*
continuous_on S f
0 < *e*
shows ∃ *g*. *P*(*g*) ∧ (∀ *x* ∈ *S*. |*f x* − *g x*| < *e*)

9.4.4 Polynomial functions

definition *polynomial_function* :: ('*a*::*real_normed_vector* ⇒ '*b*::*real_normed_vector*) ⇒ *bool*
where
polynomial_function p ≡ (∀ *f*. *bounded_linear f* ⟶ *real_polynomial_function* (*f o p*))

9.4.5 Stone-Weierstrass theorem for polynomial functions

theorem *Stone_Weierstrass_polynomial_function*:
fixes *f* :: '*a*::*euclidean_space* ⇒ '*b*::*euclidean_space*
assumes *S*: *compact S*
and *f*: *continuous_on S f*
and *e*: *0* < *e*
shows ∃ *g*. *polynomial_function g* ∧ (∀ *x* ∈ *S*. *norm*(*f x* − *g x*) < *e*)

proposition *Stone_Weierstrass_uniform_limit*:
fixes *f* :: '*a*::*euclidean_space* ⇒ '*b*::*euclidean_space*
assumes *S*: *compact S*
and *f*: *continuous_on S f*
obtains *g* **where** *uniform_limit S g f sequentially* ⋀*n*. *polynomial_function* (*g n*)

9.4.6 Polynomial functions as paths

proposition *connected_open_polynomial_connected*:
fixes *S* :: '*a*::*euclidean_space set*
assumes *S*: *open S connected S*
and *x* ∈ *S y* ∈ *S*
shows ∃ *g*. *polynomial_function g* ∧ *path_image g* ⊆ *S* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*

theorem *Stone_Weierstrass_polynomial_function_subspace*:
fixes *f* :: '*a*::*euclidean_space* ⇒ '*b*::*euclidean_space*
assumes *compact S*
and *contf*: *continuous_on S f*
and *0* < *e*
and *subspace T f* ‘ *S* ⊆ *T*
obtains *g* **where** *polynomial_function g g* ‘ *S* ⊆ *T*
⋀*x*. *x* ∈ *S* ⟹ *norm*(*f x* − *g x*) < *e*

end

9.5 Higher-Order Derivatives and C^k Functions on the Real Line

theory *Higher_Order_Derivatives*
imports *Weierstrass_Theorems*
begin

9.5.1 Higher-Order Derivatives and $C^k(U)$ Smoothness

9.6 Auxiliary Facts

9.6.1 Combinatorics

9.7 Higher-Order Differentiability

9.7.1 Definitions

9.7.2 Basic Facts

corollary *k_times_differentiable_at_Suc_le_deriv*:
assumes *f* (*Suc k*)−*times_differentiable_at a*
and $m \leq k$
shows ((*deriv* ^^ *m*) *f has_derivative* (λ*h*. (*deriv* ^^ *Suc m*) *f a* ∗ *h*)) (*at a*)
and ((*deriv* ^^ *m*) *f has_real_derivative* (*deriv* ^^ *Suc m*) *f a*) (*at a*)

corollary *k_times_differentiable_ball_has_derivative_chain*:
assumes *diff_ball*: ∀ *z*. |*z* − *x0*| < ε ⟶ *f n*−*times_differentiable_at z*
shows ∀ *i*<*n*. ∀ *z*. |*z* − *x0*| < ε
⟶ ((*deriv* ^^ *i*) *f has_derivative* (λ*h*. (*deriv* ^^ *Suc i*) *f z* ∗ *h*)) (*at z*)

9.7.3 Continuity corollaries

9.7.4 Set-wise Higher-Order Derivatives

9.7.5 Linearity of Higher Differentiability

corollary *kth_deriv_commute_and_shift_dualE*:
assumes $k \leq m$
and *f m*−*times_differentiable_at a*
shows ((*deriv* ^^ (*m* − *k*)) *f*) *k*−*times_differentiable_at a*

corollary *kth_deriv_commute_and_shiftE*:
assumes $k \leq m$
and *f m*−*times_differentiable_at a*
shows ((*deriv* ^^ *k*) *f*) (*m* − *k*)−*times_differentiable_at a*

corollary *kth_deriv_const_eq*:

fixes $x :: real$
assumes $k > 0$
shows $(deriv\ \widehat{}\ k)\ (\lambda_.\ c)\ x = 0$

corollary *kth_deriv_const_cases*:
$(deriv\ \widehat{}\ k)\ (\lambda t{::}real.\ c)\ x = (if\ k = 0\ then\ c\ else\ 0)$

corollary *k_times_differentiable_at_constE*:
$k_times_differentiable_at\ m\ (\lambda_.\ c)\ x$

corollary *kth_deriv_id_eq'*:
fixes $x :: real$
shows
$(deriv\ \widehat{}\ Suc\ m)\ (\lambda t.\ t)\ x = (if\ m = 0\ then\ 1\ else\ 0)$

corollary *kth_deriv_id_ge2_at*:
assumes $k \geq 2$
shows $(deriv\ \widehat{}\ k)\ (\lambda t{::}real.\ t)\ x = 0$

corollary *kth_deriv_id_1_eq*:
$(deriv\ \widehat{}\ Suc\ 0)\ (\lambda t.\ t)\ x = (1 :: real)$

corollary *kth_deriv_id_eq*:
assumes $m > 0$
shows $(deriv\ \widehat{}\ Suc\ m)\ (\lambda t.\ t)\ x = (0 :: real)$

corollary *k_times_differentiable_at_idE*:
$(\lambda t.\ t)\ k{-}times_differentiable_at\ x$

corollary *kth_deriv_cmult_eq*:
assumes $f\ k{-}times_differentiable_at\ x$
and $(deriv\ \widehat{}\ k)\ f = f'$
shows $(deriv\ \widehat{}\ k)\ (\lambda y.\ c * f\ y)\ x = c * f'\ x$

corollary *kth_deriv_cmultE*:
assumes $f\ k{-}times_differentiable_at\ x$
shows $k_times_differentiable_at\ k\ (\lambda z.\ c * f\ z)\ x$

corollary *kth_derivative_uminus*:
assumes $f\ k{-}times_differentiable_at\ x$
shows $(deriv\ \widehat{}\ k)\ (\lambda t.\ -\ f\ t)\ x = -\ (deriv\ \widehat{}\ k)\ f\ x$

corollary *kth_deriv_uminus_eq*:
assumes $f\ k{-}times_differentiable_at\ x$
and $(deriv\ \widehat{}\ k)\ f = f'$
shows $(deriv\ \widehat{}\ k)\ (\lambda t.\ -\ f\ t)\ x = -\ f'\ x$

corollary *kth_derivative_uminusE*:

assumes f k−*times_differentiable_at* x
shows $(\lambda t.\ -\ f\ t)$ k−*times_differentiable_at* x

corollary *kth_deriv_add_eq*:
assumes f k−*times_differentiable_at* x
and g k−*times_differentiable_at* x
assumes $(deriv\ \widehat{}\widehat{}\ k)\ f = f'$
assumes $(deriv\ \widehat{}\widehat{}\ k)\ g = g'$
shows $(deriv\ \widehat{}\widehat{}\ k)\ (\lambda y.\ f\ y + g\ y)\ x\ =\ f'(x) + g'(x)$

corollary *kth_deriv_addE*:
assumes f k−*times_differentiable_at* x
and g k−*times_differentiable_at* x
shows $(\lambda y.\ f\ y + g\ y)$ k−*times_differentiable_at* x

corollary *kth_deriv_sub_eq*:
assumes f k−*times_differentiable_at* x
and g k−*times_differentiable_at* x
and $(deriv\ \widehat{}\widehat{}\ k)\ f = f'$
and $(deriv\ \widehat{}\widehat{}\ k)\ g = g'$
shows $(deriv\ \widehat{}\widehat{}\ k)\ (\lambda t.\ f\ t\ -\ g\ t)\ x = f'\ x\ -\ g'\ x$

corollary *kth_deriv_subE*:
assumes f k−*times_differentiable_at* x
and g k−*times_differentiable_at* x
shows $(\lambda y.\ f\ y\ -\ g\ y)$ k−*times_differentiable_at* x

corollary *Leibniz_prod_eq*:
fixes $F\ G :: nat \Rightarrow real \Rightarrow real$
assumes *fCk*: f k−*times_differentiable_at* x
and *gCk*: g k−*times_differentiable_at* x
and *Ffam*: $\bigwedge j.\ j \leq k \Longrightarrow (deriv\ \widehat{}\widehat{}\ j)\ f = F\ j$
and *Gfam*: $\bigwedge j.\ j \leq k \Longrightarrow (deriv\ \widehat{}\widehat{}\ j)\ g = G\ j$
shows $(deriv\ \widehat{}\widehat{}\ k)\ (\lambda y.\ f\ y * g\ y)\ x$
$= (\sum\ j{\leq}k.\ of_nat\ (k\ choose\ j) * F\ j\ x * G\ (k\ -\ j)\ x)$

corollary *kth_deriv_multE*:
fixes $f\ g :: real \Rightarrow real$ **and** $k :: nat$ **and** $x :: real$
assumes *fCk*: f k−*times_differentiable_at* x
and *gCk*: g k−*times_differentiable_at* x
shows $(\lambda y.\ f\ y * g\ y)$ k−*times_differentiable_at* x

corollary *kth_deriv_sum_upto_eq*:
fixes $F\ H :: nat \Rightarrow real \Rightarrow real$
fixes $k\ n :: nat$ **and** $x :: real$
assumes *diff*: $\bigwedge i.\ i \leq n \Longrightarrow (F\ i)$ k−*times_differentiable_at* x
and *fam*: $\bigwedge i.\ i \leq n \Longrightarrow (deriv\ \widehat{}\widehat{}\ k)\ (F\ i) = H\ i$
shows $(deriv\ \widehat{}\widehat{}\ k)\ (\lambda y.\ \sum\ i{\leq}n.\ F\ i\ y)\ x$
$= (\sum\ i{\leq}n.\ H\ i\ x)$

corollary *kth_deriv_pow_fun_eq*:
fixes $F :: nat \Rightarrow real \Rightarrow real$
assumes *fCk*: $f\ m{-}times_differentiable_at\ x$
and *fam*: $\bigwedge j.\ j \leq m \Longrightarrow (deriv\ \widehat{\ }\widehat{\ }\ j)\ f = F\ j$
shows
$(deriv\ \widehat{\ }\widehat{\ }\ m)\ (\lambda t.\ (f\ t)\widehat{\ }(Suc\ r))\ x$
$= (\sum\ j{\leq}m.\ of_nat\ (m\ choose\ j) * F\ j\ x$
$* (deriv\ \widehat{\ }\widehat{\ }\ (m - j))\ (\lambda t.\ (f\ t)\widehat{\ }r)\ x)$

9.7.6 Derivative Formulas for Shifted Monomials

corollary *kth_deriv_cmult_pow* [*kderivs*]:
$(deriv\ \widehat{\ }\widehat{\ }\ k)\ (\lambda t{::}real.\ (c * t)\ \widehat{\ }\ n)\ x = (c\ \widehat{\ }\ n) * (deriv\ \widehat{\ }\widehat{\ }\ k)\ (\lambda t.\ t\ \widehat{\ }\ n)\ x$

9.7.7 Relationship between Differentiability at a Point and $C^k(U)$

9.7.8 Closure of C^k

9.8 Taylor's Theorem with Peano Remainder

9.8.1 Real Polynomial Functions: Closure under Differentiation

9.8.2 Taylor Polynomials and Peano Remainders

theorem *Taylor_k_times_differentiable*:
$\forall t.\ a \leq t \longrightarrow t \leq b \longrightarrow f\ n{-}times_differentiable_at\ t$
$\Longrightarrow [\![0 < n;\ a \leq c;\ c \leq b;\ a \leq x;\ x \leq b;\ x \neq c]\!]$
$\Longrightarrow \exists \xi.\ (if\ x < c\ then\ x < \xi \wedge \xi < c\ else\ c < \xi \wedge \xi < x) \wedge$
$f\ x = (\sum m{<}n.\ ((deriv\widehat{\ }\widehat{\ }m)\ f)\ c\ /\ fact\ m * (x - c)\ \widehat{\ }\ m)$
$+ ((deriv\widehat{\ }\widehat{\ }n)\ f)\ \xi\ /\ fact\ n * (x - c)\ \widehat{\ }\ n$

corollary *Taylor_as_limit*:
assumes *npos*: $0 < n$
and *cAB*: $c \in \{a..b\}$
and *cont*: $isCont\ ((deriv\ \widehat{\ }\widehat{\ }\ n)\ f)\ c$
and *diff*: $\bigwedge t.\ t \in \{a..b\} \Longrightarrow f\ n{-}times_differentiable_at\ t$
shows $((\lambda x.$
$(f\ x - (\sum m{\leq}n.\ ((deriv\ \widehat{\ }\widehat{\ }\ m)\ f)\ c\ /\ fact\ m * (x - c)\ \widehat{\ }\ m))$
$/\ (x - c)\ \widehat{\ }\ n) \longrightarrow 0)\ (at\ c\ within\ \{a..b\})$

9.8.3 Derivatives of the Taylor Polynomial and Peano Remainder

9.8.4 Taylor's Theorem with Peano Remainder

theorem *Taylor_Peano_remainder*:
assumes *f (Suc n)−times_differentiable_at x0*
shows *((λx. peano_remainder (n+1) f x0 x / (x−x0) ^ (n+1)) ⟶ 0) (at x0)*

corollary *Taylor_Peano*:
assumes *f (Suc n)−times_differentiable_at a*
obtains *h :: real ⇒ real*
where *((λx. h x) ⟶ 0) (at a)*
and *f x = (∑ i≤(n+1). (deriv ^^ i) f a/fact i ∗ (x−a) ^ i) + h x ∗ (x−a)^(n+1)*

end

9.9 Higher-Order Differentiability in Several Variables

theory *Higher_Differentiability_Multi*
imports *Higher_Order_Derivatives Cartesian_Euclidean_Space*
begin

9.9.1 Multi-dimensional k-times Fréchet differentiability at a point

9.9.2 Set-wise k-times Fréchet differentiability

9.9.3 C^k at a point (with continuity)

9.9.4 C^k on an open set

9.9.5 Relationships between the three notions

corollary *Ck_on_imp_k_times_Fr_on*:
Ck_on k f U ⟹ k_times_Fr_differentiable_on k f U

9.9.6 Basic properties of C^k at a point and on an open set

9.9.7 Bridge to the one-dimensional theory

9.9.8 Basic closure properties

9.9.9 Gradient for $real^n \Rightarrow real$

9.9.10 Hessian for $real^n \Rightarrow real$

9.9.11 Connecting C^k to the Hessian

theorem *clairaut_hessian_symmetric*:
fixes f :: $real^{\wedge}{}'n$::*finite* $\Rightarrow$ *real*
assumes *open U*
and $x \in U$
and *Ck_on 2 f U*
shows *transpose* $(\nabla^2\ f\ x) = \nabla^2\ f\ x$
corollary *mixed_partials_commute*:
fixes f :: $real^{\wedge}{}'n$::*finite* $\Rightarrow$ *real*
assumes *open U* **and** $x \in U$ **and** *Ck_on 2 f U*
shows $\nabla^2\ f\ x\ \$\ i\ \$\ j = \nabla^2\ f\ x\ \$\ j\ \$\ i$

9.9.12 Basic algebra of gradients

9.9.13 Constants and affine maps

9.9.14 Hessian: constants and affine maps

9.9.15 Coordinate formulas

9.9.16 Hessian algebra at the predicate level

9.9.17 Linearity of the Hessian on C^2 maps

9.9.18 Outer product of vectors

9.9.19 Gradient product rule

9.9.20 Hessian product rule

9.9.21 Component closure

9.9.22 Affine composition (special case)

9.9.23 Summary of the hierarchy

end

9.10 The C^1 inverse and implicit function theorems

theory *Ck_Implicit_Function*

imports *Higher_Differentiability_Multi HOL−Analysis.Determinants*
begin

9.11 From C^1 to the inverse-function-theorem interface

theorem *Ck1_on_imp_C1_interface*:
fixes $G :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes $Ck_on\ (Suc\ 0)\ G\ W$
shows $(\forall z{\in}W.\ (G\ has_derivative\ blinfun_apply\ (Dblinfun\ G\ z))\ (at\ z))$
$\wedge\ continuous_on\ W\ (Dblinfun\ G)$

9.11.1 The C^1 local inverse

theorem *C1_local_inverse*:
fixes $G :: 'a::euclidean_space \Rightarrow 'a$
assumes *C1*: $Ck_on\ (Suc\ 0)\ G\ W$
and *pW*: $z0 \in W$
and *reg*: $\exists B.\ (G\ has_derivative\ B)\ (at\ z0) \wedge bij\ B$
obtains $U'\ V\ H$ **where**
$open\ U'$ **and** $z0 \in U'$ **and** $U' \subseteq W$
and $open\ V$ **and** $G\ z0 \in V$
and $homeomorphism\ U'\ V\ G\ H$
and $\bigwedge y.\ y \in V \Longrightarrow H\ differentiable\ (at\ y)$

9.11.2 The C^1 implicit function theorem

theorem *C1_implicit_function*:
fixes $F :: ('a::euclidean_space \times 'b::euclidean_space) \Rightarrow 'b$
assumes *C1*: $Ck_on\ (Suc\ 0)\ F\ W$
and *pW*: $(x0, y0) \in W$
and *F0*: $F\ (x0, y0) = 0$
and *reg*: $\exists L.\ ((\lambda y.\ F\ (x0, y))\ has_derivative\ L)\ (at\ y0) \wedge bij\ L$
obtains $U\ g$ **where**
$open\ U$ **and** $x0 \in U$ **and** $g\ x0 = y0$
and $\bigwedge x.\ x \in U \Longrightarrow g\ differentiable\ (at\ x)$
and $\forall x{\in}U.\ (x, g\ x) \in W \wedge F\ (x, g\ x) = 0$

9.11.3 Further C^k closure properties

9.11.4 Cramer: a linear system with C^k data has a C^k solution

9.11.5 The C^k implicit function theorem

theorem *Ck_implicit_function*:

fixes $F :: ('a::euclidean_space \times (real^{\frown}'n::finite)) \Rightarrow real^{\frown}'n$
assumes *Ck*: $Ck_on\ k\ F\ W$
and *k1*: $1 \leq k$
and pW: $(x0,\ y0) \in W$
and *F0*: $F\ (x0,\ y0) = 0$
and *reg*: $\exists L.\ ((\lambda y.\ F\ (x0,\ y))\ has_derivative\ L)\ (at\ y0) \wedge bij\ L$
obtains $U\ g$ **where**
$open\ U$ **and** $x0 \in U$ **and** $g\ x0 = y0$
and $Ck_on\ k\ g\ U$
and $\forall x{\in}U.\ (x,\ g\ x) \in W \wedge F\ (x,\ g\ x) = 0$

end

9.12 Real-Analytic (C^{ω}) Functions in Several Variables

theory *Real_Analytic*
imports *Higher_Differentiability_Multi Infinite_Sum*
begin

9.12.1 C^∞ smoothness (infinitely many continuous derivatives)

9.12.2 One-dimensional real analyticity via the Peano/Taylor expansion

9.12.3 Multivariate real analyticity (local convergent power series)

9.12.4 Multivariate power-series differentiation

9.12.5 Per-term Fréchet derivative

9.12.6 One-dimensional power-series infrastructure

9.12.7 Coefficient bound

9.12.8 Directional-derivative monomial norm bound

9.12.9 A degree-weighted geometric majorant is summable on *ra_idx*

9.12.10 *ra_idx* is countably infinite

9.12.11 Per-coordinate monomial / derivative bounds

9.12.12 Multivariate geometric majorant (single ratio)

9.12.13 Per-coordinate coefficient bound

9.12.14 Uniform majorant for the differentiated series near an interior point

9.12.15 Term-by-term differentiation

theorem *ra_power_series_has_derivative*:
fixes f :: *'a::euclidean_space* ⇒ *'b::banach*
assumes r: $0 < r$
and *series*: $\bigwedge y.\ dist\ y\ x0 < r \Longrightarrow$
$((\lambda\alpha.\ ra_monomial\ (y - x0)\ \alpha *_R c\ \alpha)\ has_sum\ f\ y)\ ra_idx$
and x: $dist\ x\ x0 < r$
shows
$(f\ has_derivative$
$(\lambda v.\ infsum\ (\lambda\alpha.\ ra_Dmonomial\ (x - x0)\ \alpha\ v *_R c\ \alpha)\ ra_idx))$
$(at\ x)$

9.12.16 Incrementing a multi-index stays in the index set

9.12.17 Decrement is the inverse of *ra_inc* on the support slice

9.12.18 Splitting a monomial off one basis direction

9.12.19 The per-basis LHS and RHS pieces

9.12.20 Per-basis reindexing of has-sum

9.12.21 The reindexing identity

9.12.22 Analytic implies infinitely differentiable

theorem *real_analytic_imp_Cinfinity*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::banach$
assumes *A*: *real_analytic_on f U*
shows *Cinfinity_on f U*

9.12.23 Closure properties

9.12.24 Nowhere-dense zeros in one dimension

theorem *real_analytic_1d_nowhere_dense_zeros*:
fixes $f :: real \Rightarrow real$
assumes *ana*: *real_analytic_on f U* **and** *conn*: *connected U*
and *ex*: $\exists x{\in}U.\ f\,x \neq 0$
shows $interior\ (closure\ \{x \in U.\ f\,x = 0\}) = \{\}$

9.12.25 Convergent power series in one variable

9.12.26 Degree of a multi-index and the monomial scaling identity

9.12.27 Real-analytic functions are continuous

9.12.28 The affine slice of a real-analytic function is real-analytic (at a point)

9.12.29 Propagating a zero of an analytic function along a segment

9.12.30 Nowhere-dense zeros in several variables

theorem *real_analytic_nowhere_dense_zeros*:
fixes $f :: 'a::euclidean_space \Rightarrow real$
assumes *ana*: *real_analytic_on f U* **and** *conn*: *connected U*
and *ex*: $\exists x{\in}U.\ f\,x \neq 0$
shows *interior* (*closure* $\{x \in U.\ f\,x = 0\}$) = {}

9.12.31 Explicit multi-index operations

9.12.32 Quantitative series-on-a-ball representation

9.12.33 Majorant norm: submultiplicative under the Cauchy product

9.12.34 Step (a): power lift of a majorant bound

9.12.35 Step (b): coefficient bound gives a majorant bound

9.12.36 Majle helpers: domination, scaling, triangle

9.12.37 Combined series and majorant bookkeeping

9.12.38 Step (c): composition

9.12.39 Step (c): dominated Fubini sum of series families

9.12.40 Final composition theorem

9.12.41 Closure under linear maps, pairing, and finite sums

9.12.42 Locality of real-analyticity

9.13 A C^∞ Function That Is Not Analytic

9.13.1 Derivatives of the flat function

9.13.2 Smooth but not analytic

theorem *exp_bump_Cinfinity*: *Cinfinity_on exp_bump UNIV*

theorem *exp_bump_not_real_analytic*: ¬ *real_analytic_on exp_bump* (*ball 0 1*)

end

9.14 The formal-series local inverse of a real-analytic map (majorant method)

theory *Real_Analytic_Inverse*
imports *Real_Analytic*
begin

9.14.1 A fixed enumeration of the Euclidean basis

9.14.2 Unit multi-indices and the concrete identity coefficient family

9.14.3 Cauchy-product support helpers

9.14.4 Valuation (vanishing below a degree) under Cauchy products

9.14.5 Degree locality of Cauchy products and powers

9.14.6 Canonical monomial-composition coefficients along the basis list

9.14.7 Convolution algebra on scalar degree sequences

9.14.8 Per-degree ℓ^1 profiles of coefficient families

9.14.9 The composition operator *ra_inverse_step* and the formal fixed point

theorem *ra_inverse_coeffs_fixed_point*:
fixes *bphi* :: (*'a*::*euclidean_space* ⇒ *nat*) ⇒ *'a*
assumes *g*: *γ* ∈ *ra_idx*
shows *ra_inverse_coeffs bphi γ* = *ra_inverse_step bphi* (*ra_inverse_coeffs bphi*) *γ*

corollary *ra_inverse_coeffs_coeff_equation_le_degree*:
fixes *bphi* :: (*'a*::*euclidean_space* ⇒ *nat*) ⇒ *'a*
assumes *g*: *γ* ∈ *ra_idx*

and *low*: $\bigwedge\beta.\ \beta \in ra_idx \Longrightarrow ra_deg\ \beta < 2 \Longrightarrow bphi\ \beta = 0$
shows *ra_inverse_coeffs bphi* $\gamma =$
ra_coeff_id $\gamma + (\sum \beta{\in}ra_deg_range\ 0\ (ra_deg\ \gamma).\ ra_mono_coeffs\ (ra_inverse_coeffs\ bphi)\ \beta\ \gamma *_R bphi\ \beta)$

corollary *ra_inverse_coeffs_coeff_equation_infsum*:
fixes *bphi* :: ('a::*euclidean_space* ⇒ *nat*) ⇒ 'a
assumes *g*: $\gamma \in ra_idx$
and *low*: $\bigwedge\beta.\ \beta \in ra_idx \Longrightarrow ra_deg\ \beta < 2 \Longrightarrow bphi\ \beta = 0$
shows *ra_inverse_coeffs bphi* $\gamma =$
ra_coeff_id $\gamma + (\sum_\infty \beta{\in}(ra_idx::('a \Rightarrow nat)\ set).\ ra_mono_coeffs\ (ra_inverse_coeffs\ bphi)\ \beta\ \gamma *_R bphi\ \beta)$

9.14.10 The majorant bootstrap for the fixed point

theorem *ra_inverse_coeffs_majorized*:
fixes *bphi* :: ('a::*euclidean_space* ⇒ *nat*) ⇒ 'a
defines $n \equiv real\ (card\ (Basis :: 'a\ set))$
assumes *t0*: $0 < t$ **and** *majb*: *ra_majorized* $t\ (\lambda\beta.\ norm\ (bphi\ \beta))\ M$
and *s0*: $0 < \sigma$
and *s1*: $\sigma \le t\ /\ (4 * n)$
and *s2*: $\sigma \le t^2\ /\ (8 * (M + 1) * n)$
shows *ra_majorized* $\sigma\ (\lambda\gamma.\ norm\ (ra_inverse_coeffs\ bphi\ \gamma))\ (2 * n * \sigma)$

corollary *ra_inverse_coeffs_power_series_has_sum*:
fixes *bphi* :: ('a::*euclidean_space* ⇒ *nat*) ⇒ 'a
defines $n \equiv real\ (card\ (Basis :: 'a\ set))$
assumes *t0*: $0 < t$ **and** *majb*: *ra_majorized* $t\ (\lambda\beta.\ norm\ (bphi\ \beta))\ M$
and *s0*: $0 < \sigma$
and *s1*: $\sigma \le t\ /\ (4 * n)$
and *s2*: $\sigma \le t^2\ /\ (8 * (M + 1) * n)$
and *hle*: $norm\ h \le \sigma$
shows $((\lambda\gamma.\ ra_monomial\ h\ \gamma *_R ra_inverse_coeffs\ bphi\ \gamma)$
has_sum $(\sum_\infty \gamma{\in}(ra_idx::('a \Rightarrow nat)\ set).\ ra_monomial\ h\ \gamma *_R ra_inverse_coeffs\ bphi\ \gamma))$
$(ra_idx::('a \Rightarrow nat)\ set)$

corollary *ra_inverse_coeffs_power_series_converges_near_zero*:
fixes *bphi* :: ('a::*euclidean_space* ⇒ *nat*) ⇒ 'a
assumes *t0*: $0 < t$
and *majb*: *ra_majorized* $t\ (\lambda\beta.\ norm\ (bphi\ \beta))\ M$
obtains σ **where** $0 < \sigma$
and $\bigwedge h::'a.\ norm\ h < \sigma \Longrightarrow$
$((\lambda\gamma.\ ra_monomial\ h\ \gamma *_R ra_inverse_coeffs\ bphi\ \gamma)$
has_sum $(\sum_\infty \gamma{\in}(ra_idx::('a \Rightarrow nat)\ set).\ ra_monomial\ h\ \gamma *_R ra_inverse_coeffs\ bphi\ \gamma))$
$(ra_idx::('a \Rightarrow nat)\ set)$

theorem *ra_inverse_coeffs_formal_inverse_data*:
fixes *bphi* :: (*'a*::*euclidean_space* ⇒ *nat*) ⇒ *'a*
assumes *t0*: *0* < *t*
and *majb*: *ra_majorized t* (λβ. *norm* (*bphi* β)) *M*
and *low*: ⋀β. β ∈ *ra_idx* ⟹ *ra_deg* β < *2* ⟹ *bphi* β = *0*
obtains σ **where** *0* < σ
and ⋀*h*::*'a*. *norm h* < σ ⟹
((λγ. *ra_monomial h* γ ∗$_R$ *ra_inverse_coeffs bphi* γ)
has_sum (∑ ∞γ∈(*ra_idx*::(*'a* ⇒ *nat*) *set*).
ra_monomial h γ ∗$_R$ *ra_inverse_coeffs bphi* γ))
(*ra_idx*::(*'a* ⇒ *nat*) *set*)
and ⋀γ. γ ∈ *ra_idx* ⟹
ra_inverse_coeffs bphi γ =
ra_coeff_id γ + (∑ ∞β∈(*ra_idx*::(*'a* ⇒ *nat*) *set*).
ra_mono_coeffs (*ra_inverse_coeffs bphi*) β γ ∗$_R$ *bphi* β)

corollary *ra_inverse_coeffs_component_series_on_near_zero*:
fixes *bphi* :: (*'a*::*euclidean_space* ⇒ *nat*) ⇒ *'a*
assumes *t0*: *0* < *t*
and *majb*: *ra_majorized t* (λβ. *norm* (*bphi* β)) *M*
obtains σ **where** *0* < σ
and ⋀*b*. *b* ∈ *Basis* ⟹
ra_series_on (*0*::*'a*) σ (λγ. *ra_inverse_coeffs bphi* γ ∙ *b*)
(λ*h*. (∑ ∞γ∈(*ra_idx*::(*'a* ⇒ *nat*) *set*).
ra_monomial h γ ∗$_R$ *ra_inverse_coeffs bphi* γ) ∙ *b*)

corollary *ra_inverse_coeffs_mono_coeffs_series_on_near_zero*:
fixes *bphi* :: (*'a*::*euclidean_space* ⇒ *nat*) ⇒ *'a*
assumes *t0*: *0* < *t*
and *majb*: *ra_majorized t* (λβ. *norm* (*bphi* β)) *M*
obtains σ **where** *0* < σ
and ⋀β. *ra_series_on* (*0*::*'a*) σ (*ra_mono_coeffs* (*ra_inverse_coeffs bphi*) β)
(λ*h*. *ra_monomial*
(∑ ∞γ∈(*ra_idx*::(*'a* ⇒ *nat*) *set*).
ra_monomial h γ ∗$_R$ *ra_inverse_coeffs bphi* γ) β)

corollary *ra_inverse_coeffs_mono_coeffs_majorized*:
fixes *bphi* :: (*'a*::*euclidean_space* ⇒ *nat*) ⇒ *'a*
defines *n* ≡ *real* (*card* (*Basis* :: *'a set*))
assumes *t0*: *0* < *t* **and** *majb*: *ra_majorized t* (λβ. *norm* (*bphi* β)) *M*
and *s0*: *0* < σ
and *s1*: σ ≤ *t* / (*4* ∗ *n*)
and *s2*: σ ≤ t^2 / (*8* ∗ (*M* + *1*) ∗ *n*)
shows *ra_majorized* σ (*ra_mono_coeffs* (*ra_inverse_coeffs bphi*) β) ((*2* ∗ *n* ∗ σ) ^ *ra_deg* β)

corollary *ra_inverse_coeffs_right_inverse_at_scale*:
fixes *bphi* :: (*'a*::*euclidean_space* ⇒ *nat*) ⇒ *'a*

defines $n \equiv real\ (card\ (Basis :: 'a\ set))$
assumes *t0*: $0 < t$ **and** *majb*: $ra_majorized\ t\ (\lambda\beta.\ norm\ (bphi\ \beta))\ M$
and *low*: $\bigwedge\beta.\ \beta \in ra_idx \Longrightarrow ra_deg\ \beta < 2 \Longrightarrow bphi\ \beta = 0$
and *s0*: $0 < \sigma$
and *s1*: $\sigma \leq t\ /\ (4 * n)$
and *s2*: $\sigma \leq t^2\ /\ (8 * (M + 1) * n)$
shows $\forall h.\ dist\ h\ (0::'a) < \sigma \longrightarrow$
$((\sum_\infty \gamma \in (ra_idx::('a \Rightarrow nat)\ set).$
$ra_monomial\ h\ \gamma *_R ra_inverse_coeffs\ bphi\ \gamma)$
$- (\sum_\infty \beta \in (ra_idx::('a \Rightarrow nat)\ set).$
$ra_monomial$
$(\sum_\infty \gamma \in (ra_idx::('a \Rightarrow nat)\ set).$
$ra_monomial\ h\ \gamma *_R ra_inverse_coeffs\ bphi\ \gamma)\ \beta *_R bphi\ \beta))$
$= h$

theorem *normalized_analytic_formal_right_inverse*:
fixes $f :: 'a::euclidean_space \Rightarrow 'a$
assumes *ana*: *real_analytic_on f U*
and *zero_U*: $0 \in U$
and *f0*: $f\ 0 = 0$
and *der0*: $(f\ has_derivative\ id)\ (at\ 0)$
obtains σ *bphi* **where**
$0 < \sigma$
real_analytic_on
$(\lambda h::'a.\ \sum_\infty \gamma \in (ra_idx::('a \Rightarrow nat)\ set).$
$ra_monomial\ h\ \gamma *_R ra_inverse_coeffs\ bphi\ \gamma)$
$(ball\ (0::'a)\ (\sigma\ /\ 2))$
$\bigwedge h::'a.\ norm\ h < \sigma \Longrightarrow$
$((\lambda\gamma.\ ra_monomial\ h\ \gamma *_R ra_inverse_coeffs\ bphi\ \gamma)$
$has_sum\ (\sum_\infty \gamma \in (ra_idx::('a \Rightarrow nat)\ set).$
$ra_monomial\ h\ \gamma *_R ra_inverse_coeffs\ bphi\ \gamma))$
$(ra_idx::('a \Rightarrow nat)\ set)$
$\bigwedge h::'a.\ norm\ h < \sigma \Longrightarrow$
$f\ (\sum_\infty \gamma \in (ra_idx::('a \Rightarrow nat)\ set).$
$ra_monomial\ h\ \gamma *_R ra_inverse_coeffs\ bphi\ \gamma) = h$

end

9.15 The real-analytic inverse function theorem

theory *Analytic_Inverse_Function*
imports *Real_Analytic_Inverse Ck_Implicit_Function*
begin

9.15.1 The C^1 inverse function theorem, with derivative data

9.15.2 Derivatives of real-analytic maps

9.15.3 Upgrading the C^1 local inverse to real-analytic

9.15.4 The real-analytic local inverse theorem

theorem *real_analytic_local_inverse_normalized*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'a*
assumes *ana*: *real_analytic_on f U*
and *U*: *open U*
and *zero_U*: *0* ∈ *U*
and *f0*: *f 0* = *0*
and *der0*: (*f has_derivative id*) (*at 0*)
obtains *U' V g* **where**
open U' 0 ∈ *U' U'* ⊆ *U open V 0* ∈ *V*
homeomorphism U' V f g
real_analytic_on g V

theorem *real_analytic_local_inverse*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'a*
assumes *ana*: *real_analytic_on f U*
and *U*: *open U*
and *x0*: *x0* ∈ *U*
and *reg*: ∃ *L*. (*f has_derivative L*) (*at x0*) ∧ *bij L*
obtains *U' V g* **where**
open U' x0 ∈ *U' U'* ⊆ *U open V f x0* ∈ *V*
homeomorphism U' V f g
real_analytic_on g V
— Reduce to the normalised case by an affine change of variables.

9.16 The real-analytic implicit function theorem

theorem *real_analytic_implicit_function*:
fixes *F* :: (*'a*::*euclidean_space* × *'b*::*euclidean_space*) ⇒ *'b*
assumes *ana*: *real_analytic_on F W*
and *Wopen*: *open W*
and *pW*: (*x0*, *y0*) ∈ *W*
and *F0*: *F* (*x0*, *y0*) = *0*
and *reg*: ∃ *L*. ((λ*y*. *F* (*x0*, *y*)) *has_derivative L*) (*at y0*) ∧ *bij L*
obtains *U g* **where**
open U **and** *x0* ∈ *U* **and** *g x0* = *y0*
and *real_analytic_on g U*
and ∀ *x*∈*U*. (*x*, *g x*) ∈ *W* ∧ *F* (*x*, *g x*) = *0*

end
theory *Bounded_Variation*
imports *Henstock_Kurzweil_Integration Starlike*

begin

9.17 Set variation

9.18 Bounded variation for functions

9.18.1 Closure and subset properties

9.18.2 Other fundamental properties

9.18.3 Composition and monotonicity

9.18.4 Bounded variation implies bounded

corollary *has_bounded_variation_on_imp_bounded_on_interval*:
assumes *has_bounded_variation_on f* {*a*..*b*}
shows *bounded* (*f* ‘ {*a*..*b*})

9.18.5 Increasing/decreasing functions

9.18.6 Darboux decomposition

9.18.7 One-sided limits of monotone and BV functions

9.18.8 Continuity of vector variation

end

9.19 Radon-Nikodým Derivative

theory *Radon_Nikodym*
imports *Bochner_Integration*
begin

definition *diff_measure* :: *'a measure* ⇒ *'a measure* ⇒ *'a measure*

where
diff_measure M N = measure_of (space M) (sets M) (λA. emeasure M A − emeasure N A)
proposition (**in** *sigma_finite_measure*) *obtain_positive_integrable_function*:
obtains *f*::*'a* ⇒ *real* **where**
f ∈ borel_measurable M
⋀*x*. *f x > 0*
⋀*x*. *f x ≤ 1*
integrable M f

9.19.1 Absolutely continuous

definition *absolutely_continuous* :: *'a measure* ⇒ *'a measure* ⇒ *bool* **where**
absolutely_continuous M N ⟷ *null_sets M ⊆ null_sets N*

9.19.2 Existence of the Radon-Nikodym derivative

proposition
(**in** *finite_measure*) *Radon_Nikodym_finite_measure*:
assumes *finite_measure N* **and** *sets_eq*[*simp*]: *sets N = sets M*
assumes *absolutely_continuous M N*
shows ∃*f ∈ borel_measurable M*. *density M f = N*

proposition (**in** *finite_measure*) *Radon_Nikodym_finite_measure_infinite*:
assumes *absolutely_continuous M N* **and** *sets_eq*: *sets N = sets M*
shows ∃*f*∈*borel_measurable M*. *density M f = N*

theorem (**in** *sigma_finite_measure*) *Radon_Nikodym*:
assumes *ac*: *absolutely_continuous M N* **assumes** *sets_eq*: *sets N = sets M*
shows ∃*f ∈ borel_measurable M*. *density M f = N*

9.19.3 Uniqueness of densities

proposition (**in** *sigma_finite_measure*) *density_unique*:
assumes *f*: *f ∈ borel_measurable M*
assumes *f'*: *f' ∈ borel_measurable M*
assumes *density_eq*: *density M f = density M f'*
shows *AE x in M*. *f x = f' x*

9.19.4 Radon-Nikodym derivative

definition *RN_deriv* :: *'a measure* ⇒ *'a measure* ⇒ *'a* ⇒ *ennreal* **where**
RN_deriv M N =
(*if* ∃*f*. *f ∈ borel_measurable M ∧ density M f = N*
then SOME f. *f ∈ borel_measurable M ∧ density M f = N*

else (λ_. *0*))

proposition (**in** *sigma_finite_measure*) *real_RN_deriv*:
assumes *finite_measure N*
assumes *ac*: *absolutely_continuous M N sets N = sets M*
obtains *D* **where** *D* ∈ *borel_measurable M*
and *AE x in M*. *RN_deriv M N x = ennreal* (*D x*)
and *AE x in N*. *0* < *D x*
and ⋀*x*. *0* ≤ *D x*

end

Chapter 10

Integrals over a Set

theory *Set_Integral*
imports *Radon_Nikodym*
begin

10.1 Notation

definition *set_borel_measurable M A f* ≡ (λ*x*. *indicator A x* ∗$_R$ *f x*) ∈ *borel_measurable M*

definition *set_integrable M A f* ≡ *integrable M* (λ*x*. *indicator A x* ∗$_R$ *f x*)

definition *set_lebesgue_integral M A f* ≡ *lebesgue_integral M* (λ*x*. *indicator A x* ∗$_R$ *f x*)

10.2 Basic properties

proposition *set_borel_measurable_subset*:
fixes *f* :: _ ⇒ _ :: {*banach*, *second_countable_topology*}
assumes [*measurable*]: *set_borel_measurable M A f B* ∈ *sets M* **and** *B* ⊆ *A*
shows *set_borel_measurable M B f*

10.3 Complex integrals

10.4 NN Set Integrals

proposition *nn_integral_disjoint_family*:
assumes [*measurable*]: $f \in$ *borel_measurable* M $\bigwedge(n{::}nat).\ B\ n \in$ *sets* M
and *disjoint_family* B
shows $(\int^+ x \in (\bigcup n.\ B\ n).\ f\ x\ \partial M) = (\sum n.\ (\int^+ x \in B\ n.\ f\ x\ \partial M))$

10.5 Scheffé's lemma

proposition *Scheffe_lemma1*:
assumes $\bigwedge n.$ *integrable* M $(F\ n)$ *integrable* $M\ f$
AE x *in* $M.\ (\lambda n.\ F\ n\ x) \longrightarrow f\ x$
limsup $(\lambda n.\ \int^+ x.\ norm(F\ n\ x)\ \partial M) \leq (\int^+ x.\ norm(f\ x)\ \partial M)$
shows $(\lambda n.\ \int^+ x.\ norm(F\ n\ x - f\ x)\ \partial M) \longrightarrow 0$

proposition *Scheffe_lemma2*:
fixes $F{::}nat \Rightarrow 'a \Rightarrow 'b{::}\{banach,\ second_countable_topology\}$
assumes $\bigwedge\ n{::}nat.\ F\ n \in$ *borel_measurable* M *integrable* $M\ f$
AE x *in* $M.\ (\lambda n.\ F\ n\ x) \longrightarrow f\ x$
$\bigwedge n.\ (\int^+ x.\ norm(F\ n\ x)\ \partial M) \leq (\int^+ x.\ norm(f\ x)\ \partial M)$
shows $(\lambda n.\ \int^+ x.\ norm(F\ n\ x - f\ x)\ \partial M) \longrightarrow 0$

10.6 Convergence of integrals over an interval

proposition *tendsto_set_lebesgue_integral_at_top*:
fixes f :: *real* $\Rightarrow 'a{::}\{banach,\ second_countable_topology\}$
assumes *sets*: $\bigwedge b.\ b \geq a \Longrightarrow \{a..b\} \in$ *sets* M
and *int*: *set_integrable* $M\ \{a..\}\ f$
shows $((\lambda b.$ *set_lebesgue_integral* $M\ \{a..b\}\ f) \longrightarrow$ *set_lebesgue_integral* M $\{a..\}\ f)$ *at_top*

proposition *tendsto_set_lebesgue_integral_at_bot*:
fixes f :: *real* $\Rightarrow 'a{::}\{banach,\ second_countable_topology\}$
assumes *sets*: $\bigwedge a.\ a \leq b \Longrightarrow \{a..b\} \in$ *sets* M
and *int*: *set_integrable* $M\ \{..b\}\ f$
shows $((\lambda a.$ *set_lebesgue_integral* $M\ \{a..b\}\ f) \longrightarrow$ *set_lebesgue_integral* M $\{..b\}\ f)$ *at_bot*

theorem *integral_Markov_inequality'*:
fixes $u :: 'a \Rightarrow$ *real*
assumes [*measurable*]: *set_integrable* $M\ A\ u$ **and** $A \in$ *sets* M
assumes *AE* x *in* $M.\ x \in A \longrightarrow u\ x \geq 0$ **and** $0 < (c{::}real)$

shows *emeasure M* {*x*∈*A*. *u x* ≥ *c*} ≤ (*1*/*c*::*real*) * (∫ *x*∈*A*. *u x* ∂*M*)

theorem *integral_Markov_inequality'_measure*:
assumes [*measurable*]: *set_integrable M A u* **and** *A* ∈ *sets M*
and *AE x in M*. *x* ∈ *A* ⟶ *0* ≤ *u x 0* < (*c*::*real*)
shows *measure M* {*x*∈*A*. *u x* ≥ *c*} ≤ (∫ *x*∈*A*. *u x* ∂*M*) / *c*

theorem (**in** *finite_measure*) *Chernoff_ineq_ge*:
assumes *s*: *s* > *0*
assumes *integrable*: *set_integrable M A* (λ*x*. *exp* (*s* * *f x*)) **and** *A* ∈ *sets M*
shows *measure M* {*x*∈*A*. *f x* ≥ *a*} ≤ *exp* (−*s* * *a*) * (∫ *x*∈*A*. *exp* (*s* * *f x*) ∂*M*)
proof −
have {*x*∈*A*. *f x* ≥ *a*} = {*x*∈*A*. *exp* (*s* * *f x*) ≥ *exp* (*s* * *a*)}
using *s* **by** *auto*
also have *measure M* ... ≤ *set_lebesgue_integral M A* (λ*x*. *exp* (*s* * *f x*)) / *exp* (*s* * *a*)
by (*intro integral_Markov_inequality'_measure assms*) *auto*
finally show *?thesis*
by (*simp add*: *exp_minus field_simps*)
qed

theorem (**in** *finite_measure*) *Chernoff_ineq_le*:
assumes *s*: *s* > *0*
assumes *integrable*: *set_integrable M A* (λ*x*. *exp* (−*s* * *f x*)) **and** *A* ∈ *sets M*
shows *measure M* {*x*∈*A*. *f x* ≤ *a*} ≤ *exp* (*s* * *a*) * (∫ *x*∈*A*. *exp* (−*s* * *f x*) ∂*M*)
proof −
have {*x*∈*A*. *f x* ≤ *a*} = {*x*∈*A*. *exp* (−*s* * *f x*) ≥ *exp* (−*s* * *a*)}
using *s* **by** *auto*
also have *measure M* ... ≤ *set_lebesgue_integral M A* (λ*x*. *exp* (−*s* * *f x*)) / *exp* (−*s* * *a*)
by (*intro integral_Markov_inequality'_measure assms*) *auto*
finally show *?thesis*
by (*simp add*: *exp_minus field_simps*)
qed

10.7 Integrable Simple Functions

lemma *integrable_simple_function_induct*[*consumes 2*, *case_names cong indicator add*, *induct set*: *simple_function*]:
fixes *f* :: *'a* ⇒ *'b* :: {*second_countable_topology*, *banach*}
assumes *f*: *simple_function M f emeasure M* {*y* ∈ *space M*. *f y* ≠ *0*} ≠ ∞
assumes *cong*: ⋀*f g*. *simple_function M f* ⟹ *emeasure M* {*y* ∈ *space M*. *f y* ≠ *0*} ≠ ∞
⟹ *simple_function M g* ⟹ *emeasure M* {*y* ∈ *space M*. *g y* ≠ *0*} ≠ ∞
⟹ (⋀*x*. *x* ∈ *space M* ⟹ *f x* = *g x*) ⟹ *P f* ⟹ *P g*
assumes *indicator*: ⋀*A y*. *A* ∈ *sets M* ⟹ *emeasure M A* < ∞ ⟹ *P* (λ*x*. *indicator A x* *$_R$ *y*)
assumes *add*: ⋀*f g*. *simple_function M f* ⟹ *emeasure M* {*y* ∈ *space M*. *f y* ≠

$0\} \neq \infty \Longrightarrow$
$simple_function\ M\ g \Longrightarrow emeasure\ M\ \{y \in space\ M.\ g\ y \neq 0\}$
$\neq \infty \Longrightarrow$
$(\bigwedge z.\ z \in space\ M \Longrightarrow norm\ (f\ z + g\ z) = norm\ (f\ z) + norm$
$(g\ z)) \Longrightarrow$
$P\ f \Longrightarrow P\ g \Longrightarrow P\ (\lambda x.\ f\ x + g\ x)$
shows $P\ f$
lemma *integrable_simple_function_induct_nn*[*consumes 3*, *case_names cong indicator add*, *induct set*: *simple_function*]:
fixes $f :: 'a \Rightarrow 'b :: \{second_countable_topology, banach, linorder_topology, ordered_real_vector\}$
assumes *f*: *simple_function M f emeasure M* $\{y \in space\ M.\ f\ y \neq 0\} \neq \infty \bigwedge x.$
$x \in space\ M \longrightarrow f\ x \geq 0$
assumes *cong*: $\bigwedge f\ g.\ simple_function\ M\ f \Longrightarrow emeasure\ M\ \{y \in space\ M.\ f\ y$
$\neq 0\} \neq \infty \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow f\ x \geq 0) \Longrightarrow simple_function\ M\ g \Longrightarrow$
$emeasure\ M\ \{y \in space\ M.\ g\ y \neq 0\} \neq \infty \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow g\ x \geq 0)$
$\Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow f\ x = g\ x) \Longrightarrow P\ f \Longrightarrow P\ g$
assumes *indicator*: $\bigwedge A\ y.\ y \geq 0 \Longrightarrow A \in sets\ M \Longrightarrow emeasure\ M\ A < \infty \Longrightarrow$
$P\ (\lambda x.\ indicator\ A\ x *_R y)$
assumes *add*: $\bigwedge f\ g.\ (\bigwedge x.\ x \in space\ M \Longrightarrow f\ x \geq 0) \Longrightarrow simple_function\ M\ f$
$\Longrightarrow emeasure\ M\ \{y \in space\ M.\ f\ y \neq 0\} \neq \infty \Longrightarrow$
$(\bigwedge x.\ x \in space\ M \Longrightarrow g\ x \geq 0) \Longrightarrow simple_function\ M\ g \Longrightarrow$
$emeasure\ M\ \{y \in space\ M.\ g\ y \neq 0\} \neq \infty \Longrightarrow$
$(\bigwedge z.\ z \in space\ M \Longrightarrow norm\ (f\ z + g\ z) = norm\ (f\ z) + norm$
$(g\ z)) \Longrightarrow$
$P\ f \Longrightarrow P\ g \Longrightarrow P\ (\lambda x.\ f\ x + g\ x)$
shows $P\ f$

10.7.1 Totally Ordered Banach Spaces

10.7.2 Auxiliary Lemmas for Set Integrals

10.7.3 Integrability and Measurability of the Diameter

10.7.4 Averaging Theorem

corollary *integral_nonneg_eq_0_iff_AE_banach*:
fixes $f :: 'a \Rightarrow 'b :: \{second_countable_topology, banach, linorder_topology, ordered_real_vector\}$
assumes *f*[*measurable*]: *integrable M f* **and** *nonneg*: *AE x in M.* $0 \leq f\ x$
shows $integral^L\ M\ f = 0 \longleftrightarrow$ *(AE x in M.* $f\ x = 0$*)*

corollary *integral_eq_mono_AE_eq_AE*:
fixes $f\ g :: 'a \Rightarrow 'b :: \{second_countable_topology, banach, linorder_topology, ordered_real_vector\}$
assumes *integrable M f integrable M g* $integral^L\ M\ f = integral^L\ M\ g$ *AE x in M.* $f\ x \leq g\ x$
shows *AE x in M.* $f\ x = g\ x$

end

10.8 Homeomorphism Theorems

theory *Homeomorphism*
imports *Homotopy*
begin

10.8.1 Homeomorphism of all convex compact sets with nonempty interior

proposition
fixes S :: *'a*::*euclidean_space set*
assumes *compact S* **and** *0*: $0 \in$ *rel_interior S*
and *star*: $\bigwedge x.\ x \in S \Longrightarrow$ *open_segment 0 x* $\subseteq$ *rel_interior S*
shows *starlike_compact_projective1_0*:
$S -$ *rel_interior S homeomorphic sphere 0 1* $\cap$ *affine hull S*
(**is** *?SMINUS homeomorphic ?SPHER*)
and *starlike_compact_projective2_0*:
S homeomorphic cball 0 1 $\cap$ *affine hull S*
(**is** *S homeomorphic ?CBALL*)

corollary
fixes S :: *'a*::*euclidean_space set*
assumes *compact S* **and** *a*: $a \in$ *rel_interior S*
and *star*: $\bigwedge x.\ x \in S \Longrightarrow$ *open_segment a x* $\subseteq$ *rel_interior S*
shows *starlike_compact_projective1*:
$S -$ *rel_interior S homeomorphic sphere a 1* $\cap$ *affine hull S*
and *starlike_compact_projective2*:
S homeomorphic cball a 1 $\cap$ *affine hull S*

corollary *starlike_compact_projective_special*:
assumes *compact S*
and *cb01*: *cball* (*0*::*'a*::*euclidean_space*) *1* $\subseteq S$
and *scale*: $\bigwedge x\ u.\ [\![x \in S;\ 0 \le u;\ u < 1]\!] \Longrightarrow u *_R x \in S -$ *frontier S*
shows *S homeomorphic* (*cball* (*0*::*'a*::*euclidean_space*) *1*)

10.8.2 Homeomorphisms between punctured spheres and affine sets

theorem *homeomorphic_punctured_affine_sphere_affine*:
fixes a :: *'a* :: *euclidean_space*
assumes $0 < r$ $b \in$ *sphere a r affine T* $a \in T$ $b \in T$ *affine p*
and *aff*: *aff_dim T* = *aff_dim p* + *1*
shows (*sphere a r* $\cap$ *T*) $-$ {*b*} *homeomorphic p*

corollary *homeomorphic_punctured_sphere_affine*:
fixes $a :: 'a :: euclidean_space$
assumes $0 < r$ **and** b: $b \in sphere\ a\ r$
and *affine* T **and** *affS*: $aff_dim\ T + 1 = DIM('a)$
shows $(sphere\ a\ r - \{b\})$ *homeomorphic* T

corollary *homeomorphic_punctured_sphere_hyperplane*:
fixes $a :: 'a :: euclidean_space$
assumes $0 < r$ **and** b: $b \in sphere\ a\ r$
and $c \neq 0$
shows $(sphere\ a\ r - \{b\})$ *homeomorphic* $\{x::'a.\ c \cdot x = d\}$

proposition *homeomorphic_punctured_sphere_affine_gen*:
fixes $a :: 'a :: euclidean_space$
assumes *convex* S *bounded* S **and** a: $a \in rel_frontier\ S$
and *affine* T **and** *affS*: $aff_dim\ S = aff_dim\ T + 1$
shows $rel_frontier\ S - \{a\}$ *homeomorphic* T
proposition *homeomorphic_closedin_convex*:
fixes $S :: 'm::euclidean_space\ set$
assumes $aff_dim\ S < DIM('n)$
obtains U **and** $T :: 'n::euclidean_space\ set$
where *convex* U $U \neq \{\}$ *closedin* $(top_of_set\ U)$ T
S *homeomorphic* T

10.8.3 Locally compact sets in an open set

proposition *locally_compact_homeomorphic_closed*:
fixes $S :: 'a::euclidean_space\ set$
assumes *locally compact* S **and** *dimlt*: $DIM('a) < DIM('b)$
obtains $T :: 'b::euclidean_space\ set$ **where** *closed* T S *homeomorphic* T

proposition *homeomorphic_convex_compact_cball*:
fixes $e :: real$
and $S :: 'a::euclidean_space\ set$
assumes S: *convex* S *compact* S *interior* $S \neq \{\}$ **and** $e > 0$
shows S *homeomorphic* $(cball\ (b::'a)\ e)$

corollary *homeomorphic_convex_compact*:
fixes $S :: 'a::euclidean_space\ set$
and $T :: 'a\ set$
assumes *convex* S *compact* S *interior* $S \neq \{\}$
and *convex* T *compact* T *interior* $T \neq \{\}$
shows S *homeomorphic* T

10.8.4 Covering spaces and lifting results for them

definition *covering_space*
:: *'a::topological_space set* ⇒ (*'a* ⇒ *'b*) ⇒ *'b::topological_space set* ⇒ *bool*
where
covering_space c p S ≡
continuous_on c p ∧ *p ' c* = *S* ∧
(∀ *x* ∈ *S*. ∃ *T*. *x* ∈ *T* ∧ *openin* (*top_of_set S*) *T* ∧
(∃ *v*. ⋃ *v* = *c* ∩ *p* −' *T* ∧
(∀ *u* ∈ *v*. *openin* (*top_of_set c*) *u*) ∧
pairwise disjnt v ∧
(∀ *u* ∈ *v*. ∃ *q*. *homeomorphism u T p q*)))

proposition *covering_space_open_map*:
fixes *S* :: *'a* :: *metric_space set* **and** *T* :: *'b* :: *metric_space set*
assumes *p*: *covering_space c p S* **and** *T*: *openin* (*top_of_set c*) *T*
shows *openin* (*top_of_set S*) (*p ' T*)

proposition *covering_space_lift_unique*:
fixes *f* :: *'a::topological_space* ⇒ *'b::topological_space*
fixes *g1* :: *'a* ⇒ *'c::real_normed_vector*
assumes *covering_space c p S*
g1 a = *g2 a*
continuous_on T f f ∈ *T* → *S*
continuous_on T g1 g1 ∈ *T* → *c* ⋀*x*. *x* ∈ *T* ⟹ *f x* = *p*(*g1 x*)
continuous_on T g2 g2 ∈ *T* → *c* ⋀*x*. *x* ∈ *T* ⟹ *f x* = *p*(*g2 x*)
connected T a ∈ *T x* ∈ *T*
shows *g1 x* = *g2 x*

proposition *covering_space_locally_eq*:
fixes *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
assumes *cov*: *covering_space C p S*
and *pim*: ⋀*T*. ⟦*T* ⊆ *C*; *φ T*⟧ ⟹ *ψ*(*p ' T*)
and *qim*: ⋀*q U*. ⟦*U* ⊆ *S*; *continuous_on U q*; *ψ U*⟧ ⟹ *φ*(*q ' U*)
shows *locally ψ S* ⟷ *locally φ C*
(**is** *?lhs* = *?rhs*)

proposition *covering_space_lift_homotopy*:
fixes *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
and *h* :: *real* × *'c::real_normed_vector* ⇒ *'b*
assumes *cov*: *covering_space C p S*
and *conth*: *continuous_on* ({*0..1*} × *U*) *h*
and *him*: *h* ∈ ({*0..1*} × *U*) → *S*

and *heq*: $\bigwedge y.\ y \in U \Longrightarrow h\ (0,y) = p(f\ y)$
and *contf*: *continuous_on* $U\ f$ **and** *fim*: $f \in U \to C$
obtains k **where** *continuous_on* $(\{0..1\} \times U)\ k$
$k \in (\{0..1\} \times U) \to C$
$\bigwedge y.\ y \in U \Longrightarrow k(0,\ y) = f\ y$
$\bigwedge z.\ z \in \{0..1\} \times U \Longrightarrow h\ z = p(k\ z)$

corollary *covering_space_lift_homotopy_alt*:
fixes p :: $'a$::*real_normed_vector* $\Rightarrow$ $'b$::*real_normed_vector*
and h :: $'c$::*real_normed_vector* $\times$ *real* $\Rightarrow$ $'b$
assumes *cov*: *covering_space* $C\ p\ S$
and *conth*: *continuous_on* $(U \times \{0..1\})\ h$
and *him*: $h \in (U \times \{0..1\}) \to S$
and *heq*: $\bigwedge y.\ y \in U \Longrightarrow h\ (y,0) = p(f\ y)$
and *contf*: *continuous_on* $U\ f$ **and** *fim*: $f \in U \to C$
obtains k **where** *continuous_on* $(U \times \{0..1\})\ k$
$k \in (U \times \{0..1\}) \to C$
$\bigwedge y.\ y \in U \Longrightarrow k(y,\ 0) = f\ y$
$\bigwedge z.\ z \in U \times \{0..1\} \Longrightarrow h\ z = p(k\ z)$

corollary *covering_space_lift_homotopic_function*:
fixes p :: $'a$::*real_normed_vector* $\Rightarrow$ $'b$::*real_normed_vector* **and** g:: $'c$::*real_normed_vector* $\Rightarrow$ $'a$
assumes *cov*: *covering_space* $C\ p\ S$
and *contg*: *continuous_on* $U\ g$
and *gim*: $g \in U \to C$
and *pgeq*: $\bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$
and *hom*: *homotopic_with_canon* $(\lambda x.\ True)\ U\ S\ f\ f'$
obtains g' **where** *continuous_on* $U\ g'$ *image* $g'\ U \subseteq C\ \bigwedge y.\ y \in U \Longrightarrow p(g'\ y) = f'\ y$

corollary *covering_space_lift_inessential_function*:
fixes p :: $'a$::*real_normed_vector* $\Rightarrow$ $'b$::*real_normed_vector* **and** U :: $'c$::*real_normed_vector set*
assumes *cov*: *covering_space* $C\ p\ S$
and *hom*: *homotopic_with_canon* $(\lambda x.\ True)\ U\ S\ f\ (\lambda x.\ a)$
obtains g **where** *continuous_on* $U\ g\ g\ `\ U \subseteq C\ \bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$

10.8.5 Lifting of general functions to covering space

proposition *covering_space_lift_path_strong*:
fixes p :: $'a$::*real_normed_vector* $\Rightarrow$ $'b$::*real_normed_vector*
and f :: $'c$::*real_normed_vector* $\Rightarrow$ $'b$
assumes *cov*: *covering_space* $C\ p\ S$ **and** $a \in C$
and *path* g **and** *pag*: *path_image* $g \subseteq S$ **and** *pas*: *pathstart* $g = p\ a$
obtains h **where** *path* h *path_image* $h \subseteq C$ *pathstart* $h = a$
and $\bigwedge t.\ t \in \{0..1\} \Longrightarrow p(h\ t) = g\ t$

corollary *covering_space_lift_path*:
fixes $p :: 'a::real_normed_vector \Rightarrow 'b::real_normed_vector$
assumes *cov*: *covering_space C p S* **and** *path g* **and** *pig*: *path_image g* $\subseteq S$
obtains *h* **where** *path h path_image h* $\subseteq C \bigwedge t.\ t \in \{0..1\} \Longrightarrow p(h\ t) = g\ t$

proposition *covering_space_lift_homotopic_paths*:
fixes $p :: 'a::real_normed_vector \Rightarrow 'b::real_normed_vector$
assumes *cov*: *covering_space C p S*
and *path g1* **and** *pig1*: *path_image g1* $\subseteq S$
and *path g2* **and** *pig2*: *path_image g2* $\subseteq S$
and *hom*: *homotopic_paths S g1 g2*
and *path h1* **and** *pih1*: *path_image h1* $\subseteq C$ **and** *ph1*: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow p(h1\ t) = g1\ t$
and *path h2* **and** *pih2*: *path_image h2* $\subseteq C$ **and** *ph2*: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow p(h2\ t) = g2\ t$
and *h1h2*: *pathstart h1* $=$ *pathstart h2*
shows *homotopic_paths C h1 h2*

corollary *covering_space_monodromy*:
fixes $p :: 'a::real_normed_vector \Rightarrow 'b::real_normed_vector$
assumes *cov*: *covering_space C p S*
and *path g1* **and** *pig1*: *path_image g1* $\subseteq S$
and *path g2* **and** *pig2*: *path_image g2* $\subseteq S$
and *hom*: *homotopic_paths S g1 g2*
and *path h1* **and** *pih1*: *path_image h1* $\subseteq C$ **and** *ph1*: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow p(h1\ t) = g1\ t$
and *path h2* **and** *pih2*: *path_image h2* $\subseteq C$ **and** *ph2*: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow p(h2\ t) = g2\ t$
and *h1h2*: *pathstart h1* $=$ *pathstart h2*
shows *pathfinish h1* $=$ *pathfinish h2*

corollary *covering_space_lift_homotopic_path*:
fixes $p :: 'a::real_normed_vector \Rightarrow 'b::real_normed_vector$
assumes *cov*: *covering_space C p S*
and *hom*: *homotopic_paths S f f'*
and *path g* **and** *pig*: *path_image g* $\subseteq C$
and *a*: *pathstart g* $= a$ **and** *b*: *pathfinish g* $= b$
and *pgeq*: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow p(g\ t) = f\ t$
obtains g' **where** *path g' path_image g'* $\subseteq C$
pathstart g' $= a$ *pathfinish g'* $= b$ $\bigwedge t.\ t \in \{0..1\} \Longrightarrow p(g'\ t) = f'\ t$

proposition *covering_space_lift_general*:
fixes $p :: 'a::real_normed_vector \Rightarrow 'b::real_normed_vector$
and $f :: 'c::real_normed_vector \Rightarrow 'b$
assumes *cov*: *covering_space C p S* **and** $a \in C\ z \in U$

and *U*: *path_connected U locally path_connected U*
and *contf*: *continuous_on U f* **and** *fim*: $f \in U \to S$
and *feq*: $f\ z = p\ a$
and *hom*: $\bigwedge r.\ [\![path\ r;\ path_image\ r \subseteq U;\ pathstart\ r = z;\ pathfinish\ r = z]\!]$
$\Longrightarrow \exists q.\ path\ q \wedge path_image\ q \subseteq C \wedge$
$pathstart\ q = a \wedge pathfinish\ q = a \wedge$
$homotopic_paths\ S\ (f \circ r)\ (p \circ q)$
obtains *g* **where** *continuous_on U g* $g \in U \to C$ $g\ z = a$ $\bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$

corollary *covering_space_lift_stronger*:
fixes $p :: 'a$::*real_normed_vector* $\Rightarrow$ $'b$::*real_normed_vector*
and $f :: 'c$::*real_normed_vector* $\Rightarrow$ $'b$
assumes *cov*: *covering_space C p S* $a \in C$ $z \in U$
and *U*: *path_connected U locally path_connected U*
and *contf*: *continuous_on U f* **and** *fim*: $f \in U \to S$
and *feq*: $f\ z = p\ a$
and *hom*: $\bigwedge r.\ [\![path\ r;\ path_image\ r \subseteq U;\ pathstart\ r = z;\ pathfinish\ r = z]\!]$
$\Longrightarrow \exists b.\ homotopic_paths\ S\ (f \circ r)\ (linepath\ b\ b)$
obtains *g* **where** *continuous_on U g* $g \in U \to C$ $g\ z = a$ $\bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$

corollary *covering_space_lift_strong*:
fixes $p :: 'a$::*real_normed_vector* $\Rightarrow$ $'b$::*real_normed_vector*
and $f :: 'c$::*real_normed_vector* $\Rightarrow$ $'b$
assumes *cov*: *covering_space C p S* $a \in C$ $z \in U$
and *scU*: *simply_connected U* **and** *lpcU*: *locally path_connected U*
and *contf*: *continuous_on U f* **and** *fim*: $f \in U \to S$
and *feq*: $f\ z = p\ a$
obtains *g* **where** *continuous_on U g* $g \in U \to C$ $g\ z = a$ $\bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$

corollary *covering_space_lift*:
fixes $p :: 'a$::*real_normed_vector* $\Rightarrow$ $'b$::*real_normed_vector*
and $f :: 'c$::*real_normed_vector* $\Rightarrow$ $'b$
assumes *cov*: *covering_space C p S*
and *U*: *simply_connected U locally path_connected U*
and *contf*: *continuous_on U f* **and** *fim*: $f \in U \to S$
obtains *g* **where** *continuous_on U g* $g \in U \to C$ $\bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$

end

theory *Equivalence_Lebesgue_Henstock_Integration*
imports
Lebesgue_Measure
Henstock_Kurzweil_Integration
Complete_Measure
Set_Integral

Homeomorphism
Cartesian_Euclidean_Space
begin

10.8.6 Equivalence Lebesgue integral on *lborel* and HK-integral

10.8.7 Absolute integrability (this is the same as Lebesgue integrability)

corollary *absolutely_integrable_spike_set*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes *f*: f *absolutely_integrable_on* S **and** *neg*: *negligible* $\{x \in S - T.\ f\ x \neq 0\}$ *negligible* $\{x \in T - S.\ f\ x \neq 0\}$
shows f *absolutely_integrable_on* T

10.8.8 Applications to Negligibility

corollary *eventually_ae_filter_negligible*:
eventually P (*ae_filter lebesgue*) $\longleftrightarrow$ $(\exists N.$ *negligible* $N \wedge \{x.\ \neg\ P\ x\} \subseteq N)$

proposition *negligible_convex_frontier*:
fixes $S :: 'N :: euclidean_space\ set$
assumes *convex* S
shows *negligible*(*frontier* S)

corollary *negligible_sphere*: *negligible* (*sphere* a e)

proposition *open_not_negligible*:
assumes *open* S $S \neq \{\}$
shows $\neg$ *negligible* S

10.8.9 Negligibility of image under non-injective linear map

10.8.10 Negligibility of a Lipschitz image of a negligible set

proposition *negligible_locally_Lipschitz_image*:
fixes $f :: 'M::euclidean_space \Rightarrow 'N::euclidean_space$
assumes *MleN*: $DIM('M) \leq DIM('N)$ *negligible* S
and *lips*: $\bigwedge x.\ x \in S \Longrightarrow \exists T\ B.\ open\ T \wedge x \in T \wedge (\forall y \in S \cap T.\ norm(f\ y - f\ x) \leq B * norm(y - x))$
shows *negligible* $(f\ `\ S)$

corollary *negligible_differentiable_image_negligible*:
fixes $f :: 'M::euclidean_space \Rightarrow 'N::euclidean_space$
assumes *MleN*: $DIM('M) \leq DIM('N)$ *negligible S*
and *diff_f*: *f differentiable_on S*
shows *negligible* $(f \text{ ` } S)$

corollary *negligible_differentiable_image_lowdim*:
fixes $f :: 'M::euclidean_space \Rightarrow 'N::euclidean_space$
assumes *MlessN*: $DIM('M) < DIM('N)$ **and** *diff_f*: *f differentiable_on S*
shows *negligible* $(f \text{ ` } S)$

10.8.11 Measurability of countable unions and intersections of various kinds.

10.8.12 Negligibility is a local property

10.8.13 Integral bounds

proposition *bounded_variation_absolutely_integrable_interval*:
fixes $f :: 'n::euclidean_space \Rightarrow 'm::euclidean_space$
assumes *f*: *f integrable_on cbox a b*
and $*$: $\bigwedge d.\ d\ division_of\ (cbox\ a\ b) \Longrightarrow sum\ (\lambda K.\ norm(integral\ K\ f))\ d \leq B$
shows *f absolutely_integrable_on cbox a b*

10.8.14 Outer and inner approximation of measurable sets by well-behaved sets.

proposition *measurable_outer_intervals_bounded*:
assumes $S \in lmeasurable\ S \subseteq cbox\ a\ b\ e > 0$
obtains $\mathcal{D}$
where *countable* $\mathcal{D}$
$\bigwedge K.\ K \in \mathcal{D} \Longrightarrow K \subseteq cbox\ a\ b \wedge K \neq \{\} \wedge (\exists\, c\ d.\ K = cbox\ c\ d)$
$pairwise\ (\lambda A\ B.\ interior\ A \cap interior\ B = \{\})\ \mathcal{D}$
$\bigwedge u\ v.\ cbox\ u\ v \in \mathcal{D} \Longrightarrow \exists\, n.\ \forall\, i \in Basis.\ v \cdot i - u \cdot i = (b \cdot i - a \cdot i)/2\hat{}n$
$\bigwedge K.\ [\![K \in \mathcal{D};\ box\ a\ b \neq \{\}]\!] \Longrightarrow interior\ K \neq \{\}$
$S \subseteq \bigcup \mathcal{D}\ \bigcup \mathcal{D} \in lmeasurable\ measure\ lebesgue\ (\bigcup \mathcal{D}) \leq measure\ lebesgue\ S + e$

10.8.15 Transformation of measure by linear maps

proposition *measure_linear_sufficient*:
fixes $f :: 'n::euclidean_space \Rightarrow 'n$
assumes *linear f* **and** *S*: $S \in lmeasurable$

and *im*: $\bigwedge a\ b.\ measure\ lebesgue\ (f\ `\ (cbox\ a\ b)) = m * measure\ lebesgue\ (cbox\ a\ b)$
shows $f\ `\ S \in lmeasurable \wedge m * measure\ lebesgue\ S = measure\ lebesgue\ (f\ `\ S)$

10.8.16 Lemmas about absolute integrability

corollary *absolutely_integrable_on_const* [*simp*]:
fixes $c :: 'a::euclidean_space$
assumes $S \in lmeasurable$
shows $(\lambda x.\ c)\ absolutely_integrable_on\ S$

10.8.17 Componentwise

proposition *absolutely_integrable_componentwise_iff*:
shows $f\ absolutely_integrable_on\ A \longleftrightarrow (\forall\, b{\in}Basis.\ (\lambda x.\ f\ x \cdot b)\ absolutely_integrable_on\ A)$

corollary *absolutely_integrable_max_1*:
fixes $f :: 'n::euclidean_space \Rightarrow real$
assumes $f\ absolutely_integrable_on\ S\ g\ absolutely_integrable_on\ S$
shows $(\lambda x.\ max\ (f\ x)\ (g\ x))\ absolutely_integrable_on\ S$

corollary *absolutely_integrable_min_1*:
fixes $f :: 'n::euclidean_space \Rightarrow real$
assumes $f\ absolutely_integrable_on\ S\ g\ absolutely_integrable_on\ S$
shows $(\lambda x.\ min\ (f\ x)\ (g\ x))\ absolutely_integrable_on\ S$

10.8.18 Dominated convergence

proposition *integral_countable_UN*:
fixes $f :: real\hat{}'m \Rightarrow real\hat{}'n$
assumes *f*: $f\ absolutely_integrable_on\ (\bigcup (range\ s))$
and *s*: $\bigwedge m.\ s\ m \in sets\ lebesgue$
shows $\bigwedge n.\ f\ absolutely_integrable_on\ (\bigcup m{\le}n.\ s\ m)$
and $(\lambda n.\ integral\ (\bigcup m{\le}n.\ s\ m)\ f) \longrightarrow integral\ (\bigcup (s\ `\ UNIV))\ f$ (**is** *?F* $\longrightarrow$ *?I*)

10.8.19 Fundamental Theorem of Calculus for the Lebesgue integral

10.8.20 Integration by parts

10.8.21 A non-negative continuous function whose integral is zero must be zero

corollary *integral_cbox_eq_0_iff*:
fixes $f :: 'a::euclidean_space \Rightarrow real$
assumes *continuous_on* $(cbox\ a\ b)\ f$ **and** $box\ a\ b \neq \{\}$
and $\bigwedge x.\ x \in cbox\ a\ b \Longrightarrow f\ x \geq 0$
shows $integral\ (cbox\ a\ b)\ f = 0 \longleftrightarrow (\forall\, x \in cbox\ a\ b.\ f\ x = 0)$ (**is** *?lhs = ?rhs*)

10.8.22 Various common equivalent forms of function measurability

10.8.23 Lebesgue sets and continuous images

proposition *lebesgue_regular_inner*:
assumes $S \in sets\ lebesgue$
obtains $K\ C$ **where** $negligible\ K\ \bigwedge n::nat.\ compact(C\ n)\ S = (\bigcup n.\ C\ n) \cup K$

10.8.24 Affine lemmas

lemma *lebesgue_integral_real_affine*:
fixes $f :: real \Rightarrow 'a :: euclidean_space$ **and** $c :: real$
assumes *c*: $c \neq 0$ **shows** $(\int x.\ f\ x\ \partial\ lebesgue) = |c| *_R (\int x.\ f(t + c * x)\ \partial lebesgue)$

10.8.25 More results on integrability

proposition *measurable_bounded_by_integrable_imp_integrable*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes *f*: $f \in borel_measurable\ (lebesgue_on\ S)$ **and** *g*: $g\ integrable_on\ S$
and *normf*: $\bigwedge x.\ x \in S \Longrightarrow norm(f\ x) \leq g\ x$ **and** *S*: $S \in sets\ lebesgue$
shows $f\ integrable_on\ S$

corollary *measurable_bounded_by_integrable_imp_lebesgue_integrable*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes *f*: $f \in borel_measurable\ (lebesgue_on\ S)$ **and** *g*: $integrable\ (lebesgue_on\ S)\ g$

and *normf*: $\bigwedge x.\ x \in S \Longrightarrow norm(f\ x) \leq g\ x$ **and** *S*: $S \in sets\ lebesgue$
shows *integrable* (*lebesgue_on S*) *f*

corollary *measurable_bounded_by_integrable_imp_integrable_real*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *real*
assumes $f \in borel_measurable\ (lebesgue_on\ S)$ *g integrable_on S* $\bigwedge x.\ x \in S \Longrightarrow abs(f\ x) \leq g\ x\ S \in sets\ lebesgue$
shows *f integrable_on S*

10.8.26 Relation between Borel measurability and integrability.

proposition *negligible_differentiable_vimage*:
fixes *f* :: *'a* ⇒ *'a*::*euclidean_space*
assumes *negligible T*
and *f'*: $\bigwedge x.\ x \in S \Longrightarrow inj(f'\ x)$
and *derf*: $\bigwedge x.\ x \in S \Longrightarrow (f\ has_derivative\ f'\ x)\ (at\ x\ within\ S)$
shows *negligible* $\{x \in S.\ f\ x \in T\}$
proposition *has_derivative_inverse_within*:
fixes *f* :: *'a*::*real_normed_vector* ⇒ *'b*::*euclidean_space*
assumes *der_f*: $(f\ has_derivative\ f')\ (at\ a\ within\ S)$
and *cont_g*: *continuous* $(at\ (f\ a)\ within\ f\ `\ S)\ g$
and $a \in S$ *linear g'* **and** *id*: $g' \circ f' = id$
and *gf*: $\bigwedge x.\ x \in S \Longrightarrow g(f\ x) = x$
shows $(g\ has_derivative\ g')\ (at\ (f\ a)\ within\ f\ `\ S)$

end

10.9 Continuity of the indefinite integral; improper integral theorem

theory *Improper_Integral*
imports *Equivalence_Lebesgue_Henstock_Integration*
begin

10.9.1 Equiintegrability

definition *equiintegrable_on* (**infixr** ‹*equiintegrable'_on*› *46*)
where *F equiintegrable_on I* ≡
$(\forall f \in F.\ f\ integrable_on\ I) \wedge$
$(\forall e > 0.\ \exists \gamma.\ gauge\ \gamma \wedge$
$(\forall f\ \mathcal{D}.\ f \in F \wedge \mathcal{D}\ tagged_division_of\ I \wedge \gamma\ fine\ \mathcal{D}$
$\longrightarrow norm\ ((\sum (x,K) \in \mathcal{D}.\ content\ K *_R f\ x) - integral\ I\ f) < e))$

corollary *equiintegrable_sum_real*:
fixes $F :: (real \Rightarrow 'b::euclidean_space)\ set$
assumes F *equiintegrable_on* $\{a..b\}$
shows $(\bigcup I \in Collect\ finite.\ \bigcup c \in \{c.\ (\forall i \in I.\ c\ i \geq 0) \wedge sum\ c\ I = 1\}.$
$\bigcup f \in I \to F.\ \{(\lambda x.\ sum\ (\lambda i.\ c\ i *_R f\ i\ x)\ I)\})$
equiintegrable_on $\{a..b\}$
theorem *equiintegrable_limit*:
fixes $g :: 'a :: euclidean_space \Rightarrow 'b :: banach$
assumes *feq*: *range f equiintegrable_on cbox a b*
and *to_g*: $\bigwedge x.\ x \in cbox\ a\ b \Longrightarrow (\lambda n.\ f\ n\ x) \longrightarrow g\ x$
shows *g integrable_on cbox a b* $\wedge$ $(\lambda n.\ integral\ (cbox\ a\ b)\ (f\ n)) \longrightarrow integral\ (cbox\ a\ b)\ g$

10.9.2 Subinterval restrictions for equiintegrable families

proposition *sum_content_area_over_thin_division*:
assumes *div*: $\mathcal{D}$ *division_of S* **and** *S*: $S \subseteq cbox\ a\ b$ **and** *i*: $i \in Basis$
and $a \cdot i \leq c\ c \leq b \cdot i$
and *nonmt*: $\bigwedge K.\ K \in \mathcal{D} \Longrightarrow K \cap \{x.\ x \cdot i = c\} \neq \{\}$
shows $(b \cdot i - a \cdot i) * (\sum K \in \mathcal{D}.\ content\ K\ /\ (interval_upperbound\ K \cdot i - interval_lowerbound\ K \cdot i))$
$\leq 2 * content(cbox\ a\ b)$

proposition *bounded_equiintegral_over_thin_tagged_partial_division*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes *F*: *F equiintegrable_on cbox a b* **and** *f*: $f \in F$ **and** $0 < \varepsilon$
and *norm_f*: $\bigwedge h\ x.\ [\![h \in F;\ x \in cbox\ a\ b]\!] \Longrightarrow norm(h\ x) \leq norm(f\ x)$
obtains γ **where** *gauge* γ
$\bigwedge c\ i\ S\ h.\ [\![c \in cbox\ a\ b;\ i \in Basis;\ S\ tagged_partial_division_of\ cbox\ a\ b;$
$\gamma\ fine\ S;\ h \in F;\ \bigwedge x\ K.\ (x,K) \in S \Longrightarrow (K \cap \{x.\ x \cdot i = c \cdot i\} \neq \{\})]\!]$
$\Longrightarrow (\sum (x,K) \in S.\ norm\ (integral\ K\ h)) < \varepsilon$

proposition *equiintegrable_halfspace_restrictions_le*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes *F*: *F equiintegrable_on cbox a b* **and** *f*: $f \in F$
and *norm_f*: $\bigwedge h\ x.\ [\![h \in F;\ x \in cbox\ a\ b]\!] \Longrightarrow norm(h\ x) \leq norm(f\ x)$

shows $(\bigcup i \in Basis.\ \bigcup c.\ \bigcup h \in F.\ \{(\lambda x.\ if\ x \cdot i \le c\ then\ h\ x\ else\ 0)\})$
equiintegrable_on cbox a b

corollary *equiintegrable_halfspace_restrictions_ge*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes F: F *equiintegrable_on cbox a b* **and** f: $f \in F$
and *norm_f*: $\bigwedge h\ x.\ [\![h \in F;\ x \in cbox\ a\ b]\!] \Longrightarrow norm(h\ x) \le norm(f\ x)$
shows $(\bigcup i \in Basis.\ \bigcup c.\ \bigcup h \in F.\ \{(\lambda x.\ if\ x \cdot i \ge c\ then\ h\ x\ else\ 0)\})$
equiintegrable_on cbox a b

corollary *equiintegrable_halfspace_restrictions_lt*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes F: F *equiintegrable_on cbox a b* **and** f: $f \in F$
and *norm_f*: $\bigwedge h\ x.\ [\![h \in F;\ x \in cbox\ a\ b]\!] \Longrightarrow norm(h\ x) \le norm(f\ x)$
shows $(\bigcup i \in Basis.\ \bigcup c.\ \bigcup h \in F.\ \{(\lambda x.\ if\ x \cdot i < c\ then\ h\ x\ else\ 0)\})$ *equiintegrable_on cbox a b*
(**is** *?G equiintegrable_on cbox a b*)

corollary *equiintegrable_halfspace_restrictions_gt*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes F: F *equiintegrable_on cbox a b* **and** f: $f \in F$
and *norm_f*: $\bigwedge h\ x.\ [\![h \in F;\ x \in cbox\ a\ b]\!] \Longrightarrow norm(h\ x) \le norm(f\ x)$
shows $(\bigcup i \in Basis.\ \bigcup c.\ \bigcup h \in F.\ \{(\lambda x.\ if\ x \cdot i > c\ then\ h\ x\ else\ 0)\})$ *equiintegrable_on cbox a b*
(**is** *?G equiintegrable_on cbox a b*)

proposition *equiintegrable_closed_interval_restrictions*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes f: *f integrable_on cbox a b*
shows $(\bigcup c\ d.\ \{(\lambda x.\ if\ x \in cbox\ c\ d\ then\ f\ x\ else\ 0)\})$ *equiintegrable_on cbox a b*

10.9.3 Continuity of the indefinite integral

proposition *indefinite_integral_continuous*:
fixes $f :: 'a :: euclidean_space \Rightarrow 'b :: euclidean_space$
assumes *int_f*: *f integrable_on cbox a b*
and c: $c \in cbox\ a\ b$ **and** d: $d \in cbox\ a\ b\ 0 < \varepsilon$
obtains δ **where** $0 < \delta$
$\bigwedge c'\ d'.\ [\![c' \in cbox\ a\ b;\ d' \in cbox\ a\ b;\ norm(c' - c) \le \delta;\ norm(d' - d) \le \delta]\!]$
$\Longrightarrow norm(integral(cbox\ c'\ d')\ f - integral(cbox\ c\ d)\ f) < \varepsilon$

corollary *indefinite_integral_uniformly_continuous*:
fixes $f :: 'a :: euclidean_space \Rightarrow 'b :: euclidean_space$
assumes *f integrable_on cbox a b*
shows *uniformly_continuous_on* $(cbox\ (Pair\ a\ a)\ (Pair\ b\ b))\ (\lambda y.\ integral\ (cbox\ (fst\ y)\ (snd\ y))\ f)$

corollary *bounded_integrals_over_subintervals*:
fixes $f :: 'a :: euclidean_space \Rightarrow 'b :: euclidean_space$
assumes *f integrable_on cbox a b*
shows *bounded* $\{integral\ (cbox\ c\ d)\ f\ |c\ d.\ cbox\ c\ d \subseteq cbox\ a\ b\}$
theorem *absolutely_integrable_improper*:
fixes $f :: 'M::euclidean_space \Rightarrow 'N::euclidean_space$
assumes *int_f*: $\bigwedge c\ d.\ cbox\ c\ d \subseteq box\ a\ b \Longrightarrow f\ integrable_on\ cbox\ c\ d$
and *bo*: *bounded* $\{integral\ (cbox\ c\ d)\ f\ |c\ d.\ cbox\ c\ d \subseteq box\ a\ b\}$
and *absi*: $\bigwedge i.\ i \in Basis$
$\Longrightarrow \exists g.\ g\ absolutely_integrable_on\ cbox\ a\ b \wedge$
$((\forall x \in cbox\ a\ b.\ f\ x \cdot i \leq g\ x) \vee (\forall x \in cbox\ a\ b.\ f\ x \cdot i \geq g\ x))$
shows *f absolutely_integrable_on cbox a b*

10.9.4 Second mean value theorem and corollaries

theorem *second_mean_value_theorem_full*:
fixes $f :: real \Rightarrow real$
assumes *f*: *f integrable_on* $\{a..b\}$ **and** $a \leq b$
and *g*: $\bigwedge x\ y.\ [\![a \leq x; x \leq y; y \leq b]\!] \Longrightarrow g\ x \leq g\ y$
obtains *c* **where** $c \in \{a..b\}$
and $((\lambda x.\ g\ x * f\ x)\ has_integral\ (g\ a * integral\ \{a..c\}\ f + g\ b * integral\ \{c..b\}\ f))\ \{a..b\}$

corollary *second_mean_value_theorem*:
fixes $f :: real \Rightarrow real$
assumes *f*: *f integrable_on* $\{a..b\}$ **and** $a \leq b$
and *g*: $\bigwedge x\ y.\ [\![a \leq x; x \leq y; y \leq b]\!] \Longrightarrow g\ x \leq g\ y$
obtains *c* **where** $c \in \{a..b\}$
$integral\ \{a..b\}\ (\lambda x.\ g\ x * f\ x) = g\ a * integral\ \{a..c\}\ f + g\ b * integral\ \{c..b\}\ f$

end

10.10 Continuous Extensions of Functions

theory *Continuous_Extension*
imports *Starlike*
begin

10.10.1 Partitions of unity subordinate to locally finite open coverings

proposition *subordinate_partition_of_unity*:
fixes $S :: 'a::metric_space\ set$
assumes $S \subseteq \bigcup \mathcal{C}$ **and** *opC*: $\bigwedge T.\ T \in \mathcal{C} \Longrightarrow open\ T$
and *fin*: $\bigwedge x.\ x \in S \Longrightarrow \exists V.\ open\ V \wedge x \in V \wedge finite\ \{U \in \mathcal{C}.\ U \cap V \neq \{\}\}$
obtains $F :: ['a\ set, 'a] \Rightarrow real$
where $\bigwedge U.\ U \in \mathcal{C} \Longrightarrow continuous_on\ S\ (F\ U) \wedge (\forall x \in S.\ 0 \leq F\ U\ x)$
and $\bigwedge x\ U.\ [\![U \in \mathcal{C}; x \in S; x \notin U]\!] \Longrightarrow F\ U\ x = 0$
and $\bigwedge x.\ x \in S \Longrightarrow supp_sum\ (\lambda W.\ F\ W\ x)\ \mathcal{C} = 1$
and $\bigwedge x.\ x \in S \Longrightarrow \exists V.\ open\ V \wedge x \in V \wedge finite\ \{U \in \mathcal{C}.\ \exists x \in V.\ F\ U\ x \neq 0\}$

10.10.2 Urysohn's Lemma for Euclidean Spaces

proposition *Urysohn_local_strong*:
assumes *US*: $closedin\ (top_of_set\ U)\ S$
and *UT*: $closedin\ (top_of_set\ U)\ T$
and $S \cap T = \{\}\ a \neq b$
obtains $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
where $continuous_on\ U\ f$
$\bigwedge x.\ x \in U \Longrightarrow f\ x \in closed_segment\ a\ b$
$\bigwedge x.\ x \in U \Longrightarrow (f\ x = a \longleftrightarrow x \in S)$
$\bigwedge x.\ x \in U \Longrightarrow (f\ x = b \longleftrightarrow x \in T)$

proposition *Urysohn*:
assumes *US*: $closed\ S$
and *UT*: $closed\ T$
and $S \cap T = \{\}$
obtains $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
where $continuous_on\ UNIV\ f$
$\bigwedge x.\ f\ x \in closed_segment\ a\ b$
$\bigwedge x.\ x \in S \Longrightarrow f\ x = a$
$\bigwedge x.\ x \in T \Longrightarrow f\ x = b$

10.10.3 Dugundji's Extension Theorem and Tietze Variants

theorem *Dugundji*:
fixes $f :: 'a::\{metric_space, second_countable_topology\} \Rightarrow 'b::real_inner$
assumes $convex\ C\ C \neq \{\}$
and *cloin*: $closedin\ (top_of_set\ U)\ S$
and *contf*: $continuous_on\ S\ f$ **and** $f\ `\ S \subseteq C$
obtains g **where** $continuous_on\ U\ g\ g\ `\ U \subseteq C$
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

corollary *Tietze*:
fixes f :: $'a$::{*metric_space*,*second_countable_topology*} ⇒ $'b$::*real_inner*
assumes *continuous_on* S f
and *closedin* (*top_of_set* U) S
and $0 \leq B$
and $\bigwedge x.\ x \in S \Longrightarrow norm(f\ x) \leq B$
obtains g **where** *continuous_on* U g $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
$\bigwedge x.\ x \in U \Longrightarrow norm(g\ x) \leq B$

end

10.11 Equivalence Between Classical Borel Measurability and HOL Light's

theory *Equivalence_Measurable_On_Borel*
imports *Equivalence_Lebesgue_Henstock_Integration Improper_Integral Continuous_Extension*
begin

10.11.1 Austin's Lemma

10.11.2 A differentiability-like property of the indefinite integral.

proposition *integrable_ccontinuous_explicit*:
fixes f :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
assumes $\bigwedge a\ b$::$'a$. f *integrable_on cbox* a b
obtains N **where**
negligible N
$\bigwedge x\ e.\ [\![x \notin N;\ 0 < e]\!] \Longrightarrow$
$\exists\, d{>}0.\ \forall h.\ 0 < h \wedge h < d \longrightarrow$
$norm(integral\ (cbox\ x\ (x + h *_R One))\ f\ /_R\ h\ \hat{}\ DIM('a) - f\ x) < e$

10.11.3 HOL Light measurability

proposition *integrable_subintervals_imp_measurable*:
fixes f :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
assumes $\bigwedge a\ b.\ f$ *integrable_on cbox* a b
shows f *measurable_on UNIV*

10.11.4 Composing continuous and measurable functions; a few variants

proposition *indicator_measurable_on*:
assumes $S \in sets\ lebesgue$
shows *indicat_real S measurable_on UNIV*

lemma *simple_function_induct_real*
[*consumes 1*, *case_names cong set mult add*, *induct set*: *simple_function*]:
fixes $u :: 'a \Rightarrow real$
assumes *u*: *simple_function M u*
assumes *cong*: $\bigwedge f\ g.$ *simple_function M f* $\Longrightarrow$ *simple_function M g* $\Longrightarrow$ (*AE x in M*. *f x* = *g x*) $\Longrightarrow$ *P f* $\Longrightarrow$ *P g*
assumes *set*: $\bigwedge A.\ A \in sets\ M \Longrightarrow P\ (indicator\ A)$
assumes *mult*: $\bigwedge u\ c.\ P\ u \Longrightarrow P\ (\lambda x.\ c * u\ x)$
assumes *add*: $\bigwedge u\ v.\ P\ u \Longrightarrow P\ v \Longrightarrow P\ (\lambda x.\ u\ x + v\ x)$
and *nn*: $\bigwedge x.\ u\ x \geq 0$
shows *P u*

proposition *simple_function_measurable_on_UNIV*:
fixes *f* :: *'a*::*euclidean_space* $\Rightarrow$ *real*
assumes *f*: *simple_function lebesgue f* **and** *nn*: $\bigwedge x.\ f\ x \geq 0$
shows *f measurable_on UNIV*

corollary *simple_function_measurable_on*:
fixes *f* :: *'a*::*euclidean_space* $\Rightarrow$ *real*
assumes *f*: *simple_function lebesgue f* **and** *nn*: $\bigwedge x.\ f\ x \geq 0$ **and** *S*: $S \in sets\ lebesgue$
shows *f measurable_on S*

proposition *measurable_on_componentwise_UNIV*:
f measurable_on UNIV $\longleftrightarrow$ ($\forall\, i{\in}Basis.$ ($\lambda x.$ (*f x* $\cdot$ *i*) $*_R$ *i*) *measurable_on UNIV*)
(**is** *?lhs* = *?rhs*)

corollary *measurable_on_componentwise*:
f measurable_on S $\longleftrightarrow$ ($\forall\, i{\in}Basis.$ ($\lambda x.$ (*f x* $\cdot$ *i*) $*_R$ *i*) *measurable_on S*)

lemma *borel_measurable_implies_simple_function_sequence_real*:
fixes *u* :: *'a* $\Rightarrow$ *real*
assumes *u*[*measurable*]: *u* $\in$ *borel_measurable M* **and** *nn*: $\bigwedge x.\ u\ x \geq 0$
shows $\exists f.$ *incseq f* $\wedge$ ($\forall\, i.$ *simple_function M* (*f i*)) $\wedge$ ($\forall\, x.$ *bdd_above* (*range* ($\lambda i.$ *f i x*))) $\wedge$
($\forall\, i\ x.\ 0 \leq f\ i\ x$) $\wedge$ *u* = (*SUP i*. *f i*)

proposition *homeomorphic_box_UNIV*:

fixes *a b*:: *'a*::*euclidean_space*
assumes *box a b* ≠ {}
shows *box a b homeomorphic* (*UNIV*::*'a set*)

proposition *measurable_on_imp_borel_measurable_lebesgue_UNIV*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *f measurable_on UNIV*
shows *f* ∈ *borel_measurable lebesgue*

corollary *measurable_on_imp_borel_measurable_lebesgue*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *f measurable_on S* **and** *S*: *S* ∈ *sets lebesgue*
shows *f* ∈ *borel_measurable* (*lebesgue_on S*)

proposition *measurable_on_limit*:
fixes *f* :: *nat* ⇒ *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *f*: ⋀*n*. *f n measurable_on S* **and** *N*: *negligible N*
and *lim*: ⋀*x*. *x* ∈ *S* − *N* ⟹ (λ*n*. *f n x*) ⟶ *g x*
shows *g measurable_on S*

proposition *lebesgue_measurable_imp_measurable_on*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *f*: *f* ∈ *borel_measurable lebesgue* **and** *S*: *S* ∈ *sets lebesgue*
shows *f measurable_on S*

proposition *measurable_on_iff_borel_measurable*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *S* ∈ *sets lebesgue*
shows *f measurable_on S* ⟷ *f* ∈ *borel_measurable* (*lebesgue_on S*) (**is** *?lhs* = *?rhs*)

10.11.5 Monotonic functions are Lebesgue integrable

10.11.6 Measurability on generalisations of the binary product

end
theory *Rectifiable_Path*
imports *Bounded_Variation Path_Connected Equivalence_Lebesgue_Henstock_Integration*
begin

10.12 Rectifiable paths and arc-length reparametrization

corollary *bounded_variation_absolutely_integrable_real_interval*:
fixes f :: *real* $\Rightarrow$ *'m*::*euclidean_space*
assumes f *integrable_on* $\{a..b\}$
and $\bigwedge d.$ d *division_of* $\{a..b\} \Longrightarrow$ *sum* $(\lambda K.$ *norm*(*integral* K $f))$ $d \leq B$
shows f *absolutely_integrable_on* $\{a..b\}$

10.13 Basic properties

10.14 Invariance under transformations

10.15 Shiftpath

10.16 Lipschitz and distance bounds

10.17 Linepath

10.18 Rectifiable path image bound

end

10.19 Brouwer's Fixed Point Theorem

theory *Brouwer_Fixpoint*
imports *Homeomorphism Derivative*
begin

10.19.1 Retractions

10.19.2 Kuhn Simplices

10.19.3 Brouwer's fixed point theorem

theorem *brouwer*:
fixes $f :: 'a::euclidean_space \Rightarrow 'a$
assumes S: *compact S convex S* $S \neq \{\}$
and *contf*: *continuous_on S f*
and *fim*: $f \in S \to S$
obtains x **where** $x \in S$ **and** $f\ x = x$

10.19.4 Applications

corollary *no_retraction_cball*:
fixes $a :: 'a::euclidean_space$
assumes $e > 0$
shows $\neg$ (*frontier* (*cball a e*) *retract_of* (*cball a e*))

corollary *contractible_sphere*:
fixes $a :: 'a::euclidean_space$
shows *contractible*(*sphere a r*) $\longleftrightarrow$ $r \leq 0$

corollary *connected_sphere_eq*:
fixes $a :: 'a :: euclidean_space$
shows *connected*(*sphere a r*) $\longleftrightarrow$ $2 \leq DIM('a) \vee r \leq 0$
(**is** *?lhs* = *?rhs*)

corollary *path_connected_sphere_eq*:
fixes $a :: 'a :: euclidean_space$
shows *path_connected*(*sphere a r*) $\longleftrightarrow$ $2 \leq DIM('a) \vee r \leq 0$
(**is** *?lhs* = *?rhs*)

proposition *frontier_subset_retraction*:
fixes $S :: 'a::euclidean_space\ set$
assumes *bounded S* **and** *fros*: *frontier S* $\subseteq T$
and *contf*: *continuous_on* (*closure S*) *f*
and *fim*: $f \in S \to T$
and *fid*: $\bigwedge x.\ x \in T \Longrightarrow f\ x = x$
shows $S \subseteq T$

corollary *rel_frontier_retract_of_punctured_affine_hull*:
fixes $S :: 'a::euclidean_space\ set$
assumes *bounded S convex S a* $\in$ *rel_interior S*
shows *rel_frontier S retract_of* (*affine hull S* $-$ $\{a\}$)

corollary *rel_boundary_retract_of_punctured_affine_hull*:
fixes $S :: 'a::euclidean_space\ set$
assumes *compact S convex S a* $\in$ *rel_interior S*
shows (*S* $-$ *rel_interior S*) *retract_of* (*affine hull S* $-$ $\{a\}$)

theorem *has_derivative_inverse_on*:
fixes $f :: 'n::euclidean_space \Rightarrow 'n$
assumes *open S*
and $\bigwedge x.\ x \in S \Longrightarrow (f\ has_derivative\ f'(x))\ (at\ x)$
and $\bigwedge x.\ x \in S \Longrightarrow g\ (f\ x) = x$
and $f'\ x \circ g'\ x = id$
and $x \in S$
shows $(g\ has_derivative\ g'(x))\ (at\ (f\ x))$

end

10.20 Lipschitz Continuity

theory *Lipschitz*
imports
Derivative Abstract_Metric_Spaces Brouwer_Fixpoint
begin

definition *lipschitz_on*
where $lipschitz_on\ C\ U\ f \longleftrightarrow (0 \le C \wedge (\forall x \in U.\ \forall y \in U.\ dist\ (f\ x)\ (f\ y) \le C * dist\ x\ y))$
notation
$lipschitz_on\ (\langle(\langle open_block\ notation=\langle postfix\ lipschitz_on\rangle\rangle_-lipschitz'_on)\rangle [1000])$
proposition *lipschitz_on_uniformly_continuous*:
assumes $L-lipschitz_on\ X\ f$
shows *uniformly_continuous_on X f*

proposition *lipschitz_on_continuous_on*:
continuous_on X f **if** $L-lipschitz_on\ X\ f$
proposition *bounded_derivative_imp_lipschitz*:
assumes $\bigwedge x.\ x \in X \Longrightarrow (f\ has_derivative\ f'\ x)\ (at\ x\ within\ X)$
assumes *convex*: *convex X*
assumes $\bigwedge x.\ x \in X \Longrightarrow onorm\ (f'\ x) \le C\ 0 \le C$
shows $C-lipschitz_on\ X\ f$

10.20.1 Local Lipschitz continuity

proposition *lipschitz_on_closed_Union*:
assumes $\bigwedge i.\ i \in I \Longrightarrow lipschitz_on\ M\ (U\ i)\ f$
$\bigwedge i.\ i \in I \Longrightarrow closed\ (U\ i)$
finite I
$M \ge 0$
$\{u..(v::real)\} \subseteq (\bigcup i \in I.\ U\ i)$
shows $lipschitz_on\ M\ \{u..v\}\ f$

10.20.2 Local Lipschitz continuity (uniform for a family of functions)

definition *local_lipschitz*::
'a::*metric_space set* ⇒ *'b*::*metric_space set* ⇒ (*'a* ⇒ *'b* ⇒ *'c*::*metric_space*) ⇒ *bool*
where
local_lipschitz T X f ≡ ∀ *x* ∈ *X*. ∀ *t* ∈ *T*.
∃ *u*>*0*. ∃ *L*. ∀ *t* ∈ *cball t u* ∩ *T*. *L*−*lipschitz_on* (*cball x u* ∩ *X*) (*f t*)

proposition *c1_implies_local_lipschitz*:
fixes *T*::*real set* **and** *X*::*'a*::{*banach*,*heine_borel*} *set*
and *f*::*real* ⇒ *'a* ⇒ *'a*
assumes *f'*: ⋀*t x*. *t* ∈ *T* ⟹ *x* ∈ *X* ⟹ (*f t has_derivative blinfun_apply* (*f'* (*t*, *x*))) (*at x*)
assumes *cont_f'*: *continuous_on* (*T* × *X*) *f'*
assumes *open T*
assumes *open X*
shows *local_lipschitz T X f*

10.20.3 bilipschitz homeomorphism results

proposition *bilipschitz_homeomorphism_spherical_projection*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *convex S bounded S 0* ∈ *rel_interior S*
shows ∃ *g*. *homeomorphism* (*rel_frontier S*) (*sphere 0 1* ∩ *affine hull S*)
(λ*x*. *x* /*R* *norm x*) *g* ∧
(∃ *B*. ∀ *x y*. *x* ∈ *rel_frontier S* ⟶ *y* ∈ *rel_frontier S* ⟶
norm (*x* /*R* *norm x* − *y* /*R* *norm y*) ≤ *B* ∗ *norm* (*x* − *y*)) ∧
(∃ *B*. ∀ *x y*. *x* ∈ *sphere 0 1* ∩ *affine hull S* ⟶
y ∈ *sphere 0 1* ∩ *affine hull S* ⟶
norm (*g x* − *g y*) ≤ *B* ∗ *norm* (*x* − *y*))

end
theory *Absolute_Continuity*
imports *Bounded_Variation Equivalence_Measurable_On_Borel Rectifiable_Path Lipschitz*

begin

10.21 Absolute set-continuity

10.22 Absolute continuity for functions

10.22.1 Basic properties

10.22.2 Relationship to bounded variation and continuity

10.22.3 Combining intervals

10.22.4 Bilinear and product

10.22.5 Fundamental theorem of calculus

theorem *fundamental_theorem_of_calculus_Bartle*:
fixes $f :: \langle real \Rightarrow 'a::real_normed_vector\rangle$ **and** $f' :: \langle real \Rightarrow 'a\rangle$
assumes *neg*: $\langle negligible\ S\rangle$
and $\langle a \le b\rangle$
and *deriv*: $\langle \bigwedge x.\ x \in \{a..b\} - S \Longrightarrow (f\ has_vector_derivative\ f'\ x)\ (at\ x\ within\ \{a..b\})\rangle$
and *HS*: $\langle \bigwedge \varepsilon.\ \varepsilon > 0 \Longrightarrow$
$\exists g.\ gauge\ g \wedge$
$(\forall p.\ p\ tagged_partial_division_of\ cbox\ a\ b \wedge g\ fine\ p \wedge fst\ `\ p \subseteq S \longrightarrow$
$norm\ (\sum (x,k)\in p.\ f\ (Sup\ k) - f\ (Inf\ k)) < \varepsilon)\rangle$
shows $\langle (f'\ has_integral\ (f\ b - f\ a))\ \{a..b\}\rangle$

theorem *fundamental_theorem_of_calculus_absolutely_continuous*:
fixes $f :: real \Rightarrow 'a::euclidean_space$
assumes *negligible* S $a \le b$
absolutely_continuous_on $\{a..b\}$ f
and *fvd*: $\bigwedge x.\ x \in \{a..b\} - S \Longrightarrow (f\ has_vector_derivative\ f'\ x)\ (at\ x\ within\ \{a..b\})$
shows $(f'\ has_integral\ (f\ b - f\ a))\ \{a..b\}$

theorem *fundamental_theorem_of_calculus_strong*:
fixes $f :: real \Rightarrow 'a::banach$ **and** $f' :: real \Rightarrow 'a$
assumes *countable* S
and $a \le b$
and $\bigwedge x.\ x \in \{a..b\} - S \Longrightarrow (f\ has_vector_derivative\ f'\ x)\ (at\ x\ within\ \{a..b\})$
and *contf*: *continuous_on* $\{a..b\}$ f
shows $(f'\ has_integral\ (f\ b - f\ a))\ \{a..b\}$
proposition *has_integral_substitution_ac*:
fixes $\varphi :: real \Rightarrow real$ **and** $\varphi' :: real \Rightarrow real$ **and** $f :: real \Rightarrow real$
assumes $a \le b$ $\varphi\ a \le \varphi\ b$
and φ: *absolutely_continuous_on* $\{a..b\}$ φ
and *negligible* S
and *vec*: $\bigwedge t.\ t \in \{a..b\} - S \Longrightarrow (\varphi\ has_vector_derivative\ \varphi'\ t)\ (at\ t)$
and *contf*: *continuous_on* $\{\varphi\ a..\varphi\ b\}$ f
and *mono*: $\bigwedge x\ y.\ x \in \{a..b\} \Longrightarrow y \in \{a..b\} \Longrightarrow x \le y \Longrightarrow \varphi\ x \le \varphi\ y$
shows $((\lambda t.\ \varphi'\ t * f\ (\varphi\ t))\ has_integral\ (integral\ \{\varphi\ a..\varphi\ b\}\ f))\ \{a..b\}$

10.22.6 Integration by parts

10.22.7 Integration by substitution

10.22.8 This doesn't directly involve integration, but that gives an easy proof

10.22.9 Generalize a bit to any convex set

10.22.10 Closure and interior

10.23 Bounded variation and absolutely integrable derivatives

theorem *absolutely_integrable_approximate_continuous*:
fixes f :: ‹$'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*›
and S :: ‹$'a$ *set*›
assumes *S_meas*: ‹$S \in$ *sets lebesgue*›
and *f_int*: ‹f *absolutely_integrable_on* S›
and *e_pos*: ‹$e > 0$›
obtains g **where** ‹g *absolutely_integrable_on* S› ‹*continuous_on UNIV* g›
‹*bounded* (g ‘ *UNIV*)›
‹*norm* (*integral* S (λx. *norm* ($f\ x - g\ x$))) $< e$›

theorem *absolute_integration_by_parts*:
fixes *bop* :: ‹$'a$::*euclidean_space* ⇒ $'b$::*euclidean_space* ⇒ $'c$::*euclidean_space*›
and f :: ‹*real* ⇒ $'a$› **and** g :: ‹*real* ⇒ $'b$›
and f' :: ‹*real* ⇒ $'a$› **and** g' :: ‹*real* ⇒ $'b$›
assumes ‹*bilinear bop*›
and *ab*: ‹$a \le b$›
and *f'abs*: ‹f' *absolutely_integrable_on* $\{a..b\}$›
and *g'abs*: ‹g' *absolutely_integrable_on* $\{a..b\}$›
and *f'int*: ‹$\bigwedge x.\ x \in \{a..b\} \Longrightarrow (f'$ *has_integral* $f\ x)\ \{a..x\}$›
and *g'int*: ‹$\bigwedge x.\ x \in \{a..b\} \Longrightarrow (g'$ *has_integral* $g\ x)\ \{a..x\}$›
shows ‹(λx. *bop* ($f\ x$) ($g'\ x$)) *absolutely_integrable_on* $\{a..b\}$›
and ‹(λx. *bop* ($f'\ x$) ($g\ x$)) *absolutely_integrable_on* $\{a..b\}$›
and ‹*integral* $\{a..b\}$ (λx. *bop* ($f\ x$) ($g'\ x$)) + *integral* $\{a..b\}$ (λx. *bop* ($f'\ x$) ($g\ x$))
= *bop* ($f\ b$) ($g\ b$) − *bop* ($f\ a$) ($g\ a$)›

theorem *has_vector_derivative_indefinite_integral*:
fixes f :: ‹*real* ⇒ *'a*::*euclidean_space*›
assumes ‹f *integrable_on* $\{a..b\}$›
obtains k **where** ‹*negligible* k›
‹$\bigwedge x.\ x \in \{a..b\} - k \Longrightarrow$
$((\lambda u.\ \textit{integral}\ \{a..u\}\ f)\ \textit{has_vector_derivative}\ f\ x)\ (\textit{at}\ x\ \textit{within}\ \{a..b\})$›

theorem *absolute_integration_by_parts_sum*:
fixes *bop* :: ‹*'a*::*euclidean_space* ⇒ *'b*::*euclidean_space* ⇒ *'c*::*euclidean_space*›
and f :: ‹*real* ⇒ *'a*› **and** g :: ‹*real* ⇒ *'b*›
and f' :: ‹*real* ⇒ *'a*› **and** g' :: ‹*real* ⇒ *'b*›
and $a\ b$ :: *real*
assumes *bop*: ‹*bilinear bop*›
and *ab*: ‹$a \le b$›
and *f'abs*: ‹f' *absolutely_integrable_on* $\{a..b\}$›
and *g'abs*: ‹g' *absolutely_integrable_on* $\{a..b\}$›
and *f'int*: ‹$\bigwedge x.\ x \in \{a..b\} \Longrightarrow (f'\ \textit{has_integral}\ (f\ x - f\ a))\ \{a..x\}$›
and *g'int*: ‹$\bigwedge x.\ x \in \{a..b\} \Longrightarrow (g'\ \textit{has_integral}\ (g\ x - g\ a))\ \{a..x\}$›
shows ‹$(\lambda x.\ \textit{bop}\ (f\ x)\ (g'\ x) + \textit{bop}\ (f'\ x)\ (g\ x))$ *absolutely_integrable_on* $\{a..b\}$›
and ‹$\bigwedge x.\ x \in \{a..b\} \Longrightarrow$
$((\lambda x.\ \textit{bop}\ (f\ x)\ (g'\ x) + \textit{bop}\ (f'\ x)\ (g\ x))$
$\textit{has_integral}\ (\textit{bop}\ (f\ x)\ (g\ x) - \textit{bop}\ (f\ a)\ (g\ a)))\ \{a..x\}$›

10.24 Lipschitz and distance bounds

10.25 Linepath

10.26 Rectifiable path image bound

10.27 Absolutely continuous paths

end
theory *Lebesgue_Differentiation*
imports *Equivalence_Measurable_On_Borel Bounded_Variation Weierstrass_Theorems*
begin

theorem *Lebesgue_differentiation_thm_compact*:
fixes f :: *real* ⇒ *'a*::*euclidean_space*
assumes *has_bounded_variation_on* f (*cbox* a b)
shows *negligible* $\{x \in \textit{cbox}\ a\ b.\ \neg\ f\ \textit{differentiable}\ (\textit{at}\ x)\}$

corollary *Lebesgue_differentiation_thm*:
fixes $f :: real \Rightarrow 'a::euclidean_space$
assumes *is_interval S has_bounded_variation_on f S*
shows *negligible* $\{x \in S.\ \neg\ f\ differentiable\ (at\ x)\}$

corollary *Lebesgue_differentiation_thm_alt*:
fixes $f :: real \Rightarrow 'a::euclidean_space$
assumes *is_interval S has_bounded_variation_on f S*
shows $\exists t.\ t \subseteq S \wedge negligible\ t \wedge (\forall x \in S - t.\ f\ differentiable\ (at\ x))$

corollary *Lebesgue_differentiation_thm_gen*:
fixes $f :: real \Rightarrow 'a::euclidean_space$
assumes *countable* (*components S*) *has_bounded_variation_on f S*
shows *negligible* $\{x \in S.\ \neg\ f\ differentiable\ (at\ x)\}$
corollary *Lebesgue_differentiation_thm_increasing*:
fixes $f :: real \Rightarrow real$
assumes *is_interval S mono_on S f*
shows *negligible* $\{x \in S.\ \neg\ f\ differentiable\ (at\ x)\}$

corollary *Lebesgue_differentiation_thm_decreasing*:
fixes $f :: real \Rightarrow real$
assumes *is_interval S antimono_on S f*
shows *negligible* $\{x \in S.\ \neg\ f\ differentiable\ (at\ x)\}$

end

10.28 Harmonic Numbers

theory *Harmonic_Numbers*
imports
Complex_Transcendental
Summation_Tests
begin

10.28.1 The Harmonic numbers

definition $harm :: nat \Rightarrow 'a :: real_normed_field$ **where**
$harm\ n = (\sum k{=}1..n.\ inverse\ (of_nat\ k))$

theorem *not_convergent_harm*: $\neg convergent\ (harm :: nat \Rightarrow 'a :: real_normed_field)$

10.28.2 The Euler-Mascheroni constant

lemma *euler_mascheroni_LIMSEQ*:
$(\lambda n.\ harm\ n - ln\ (of_nat\ n) :: real) \longrightarrow euler_mascheroni$

theorem *alternating_harmonic_series_sums*: $(\lambda k.\ (-1)\hat{}k\ /\ real_of_nat\ (Suc\ k))\ sums\ ln\ 2$

end

10.29 The Gamma Function

theory *Gamma_Function*
imports
Absolute_Continuity
Summation_Tests
Harmonic_Numbers
HOL−Library.Nonpos_Ints
HOL−Library.Periodic_Fun
begin

10.29.1 The Euler form and the logarithmic Gamma function

definition *Gamma_series* :: $('a :: \{banach, real_normed_field\}) \Rightarrow nat \Rightarrow 'a$ **where**
Gamma_series z n = *fact n* ∗ *exp* (*z* ∗ *of_real* (*ln* (*of_nat n*))) / *pochhammer z* (*n*+1)
definition *ln_Gamma_series* :: $('a :: \{banach, real_normed_field, ln\}) \Rightarrow nat \Rightarrow 'a$ **where**
ln_Gamma_series z n = *z* ∗ *ln* (*of_nat n*) − *ln z* − (∑ *k*=1..*n*. *ln* (*z* / *of_nat k* + 1))

theorem *ln_Gamma_complex_LIMSEQ*: $(z :: complex) \notin \mathbb{Z}_{\leq 0} \Longrightarrow ln_Gamma_series\ z \longrightarrow ln_Gamma\ z$

10.29.2 The Polygamma functions

definition *Polygamma* :: $nat \Rightarrow ('a :: \{real_normed_field, banach\}) \Rightarrow 'a$ **where**
Polygamma n z = (*if n* = 0 *then*
(∑ *k*. *inverse* (*of_nat* (*Suc k*)) − *inverse* (*z* + *of_nat k*)) − *euler_mascheroni*
else
(−1)^*Suc n* ∗ *fact n* ∗ (∑ *k*. *inverse* ((*z* + *of_nat k*)^*Suc n*)))

abbreviation *Digamma* :: $('a :: \{real_normed_field, banach\}) \Rightarrow 'a$ **where**
Digamma ≡ *Polygamma* 0

theorem *Digamma_LIMSEQ*:
fixes *z* :: $'a :: \{banach, real_normed_field\}$
assumes *z*: $z \neq 0$

shows $(\lambda m.\ of_real\ (ln\ (real\ m)) - (\sum n{<}m.\ inverse\ (z + of_nat\ n))) \longrightarrow Digamma\ z$

theorem *Polygamma_LIMSEQ*:
fixes $z :: 'a :: \{banach, real_normed_field\}$
assumes $z \neq 0$ **and** $n > 0$
shows $(\lambda k.\ inverse\ ((z + of_nat\ k) \hat{}\ Suc\ n))\ sums\ ((-1) \hat{}\ Suc\ n * Polygamma\ n\ z\ /\ fact\ n)$

theorem *has_field_derivative_ln_Gamma_complex* [*derivative_intros*]:
fixes $z :: complex$
assumes $z\text{: } z \notin \mathbb{R}_{\leq 0}$
shows $(ln_Gamma\ has_field_derivative\ Digamma\ z)\ (at\ z)$

theorem *Polygamma_plus1*:
assumes $z \neq 0$
shows $Polygamma\ n\ (z + 1) = Polygamma\ n\ z + (-1) \hat{}\ n * fact\ n\ /\ (z \hat{}\ Suc\ n)$

theorem *Digamma_of_nat*:
$Digamma\ (of_nat\ (Suc\ n) :: 'a :: \{real_normed_field, banach\}) = harm\ n - euler_mascheroni$

theorem *has_field_derivative_Polygamma* [*derivative_intros*]:
fixes $z :: 'a :: \{real_normed_field, euclidean_space\}$
assumes $z\text{: } z \notin \mathbb{Z}_{\leq 0}$
shows $(Polygamma\ n\ has_field_derivative\ Polygamma\ (Suc\ n)\ z)\ (at\ z\ within\ A)$

10.29.3 Basic properties

theorem *Gamma_series_LIMSEQ* [*tendsto_intros*]:
$Gamma_series\ z \longrightarrow Gamma\ z$

theorem *Gamma_plus1*: $z \notin \mathbb{Z}_{\leq 0} \Longrightarrow Gamma\ (z + 1) = z * Gamma\ z$

theorem *pochhammer_Gamma*: $z \notin \mathbb{Z}_{\leq 0} \Longrightarrow pochhammer\ z\ n = Gamma\ (z + of_nat\ n)\ /\ Gamma\ z$

theorem *Gamma_fact*: $Gamma\ (1 + of_nat\ n) = fact\ n$

10.29.4 Differentiability

theorem *has_field_derivative_Gamma* [*derivative_intros*]:
$z \notin \mathbb{Z}_{\leq 0} \Longrightarrow (Gamma\ has_field_derivative\ Gamma\ z * Digamma\ z)\ (at\ z\ within\ A)$

theorem *log_convex_Gamma_real*: $convex_on\ \{0<..\}\ (ln \circ Gamma :: real \Rightarrow real)$

10.29.5 The uniqueness of the real Gamma function

theorem *Gamma_pos_real_unique*:
assumes x: $x > 0$
shows $G\ x = Gamma\ x$

10.29.6 The Beta function

theorem *Beta_plus1_plus1*:
assumes $x \notin \mathbb{Z}_{\leq 0}$ $y \notin \mathbb{Z}_{\leq 0}$
shows $Beta\ (x + 1)\ y + Beta\ x\ (y + 1) = Beta\ x\ y$

theorem *Beta_plus1_left*:
assumes $x \notin \mathbb{Z}_{\leq 0}$
shows $(x + y) * Beta\ (x + 1)\ y = x * Beta\ x\ y$

theorem *Beta_plus1_right*:
assumes $y \notin \mathbb{Z}_{\leq 0}$
shows $(x + y) * Beta\ x\ (y + 1) = y * Beta\ x\ y$

10.29.7 Legendre duplication theorem

theorem *Gamma_legendre_duplication*:
fixes z :: *complex*
assumes $z \notin \mathbb{Z}_{\leq 0}$ $z + 1/2 \notin \mathbb{Z}_{\leq 0}$
shows $Gamma\ z * Gamma\ (z + 1/2) =$
$exp\ ((1 - 2*z) * of_real\ (ln\ 2)) * of_real\ (sqrt\ pi) * Gamma\ (2*z)$

10.29.8 Alternative definitions

theorem *Gamma_series_euler'*:
assumes *z*: $(z :: 'a :: Gamma) \notin \mathbb{Z}_{\leq 0}$
shows $(\lambda n.\ Gamma_series_euler'\ z\ n) \longrightarrow Gamma\ z$

theorem *Gamma_Weierstrass_complex*: $Gamma_series_Weierstrass\ z \longrightarrow Gamma\ (z :: complex)$

theorem *gbinomial_Gamma*:
assumes $z + 1 \notin \mathbb{Z}_{\leq 0}$
shows $(z\ gchoose\ n) = Gamma\ (z + 1)\ /\ (fact\ n * Gamma\ (z - of_nat\ n + 1))$

theorem *Gamma_integral_complex*:
assumes *z*: $Re\ z > 0$
shows $((\lambda t.\ of_real\ t\ powr\ (z - 1)\ /\ of_real\ (exp\ t))\ has_integral\ Gamma\ z)\ \{0..\}$

theorem *has_integral_Beta_real*:
assumes *a*: $a > 0$ **and** *b*: $b > (0 :: real)$
shows $((\lambda t.\ t\ powr\ (a - 1) * (1 - t)\ powr\ (b - 1))\ has_integral\ Beta\ a\ b)\ \{0..1\}$

10.29.9 The Weierstraß product formula for the sine

theorem *sin_product_formula_complex*:
fixes $z :: complex$
shows $(\lambda n.\ of_real\ pi * z * (\prod k{=}1..n.\ 1 - z\hat{}2\ /\ of_nat\ k\hat{}2)) \longrightarrow sin\ (of_real\ pi * z)$

theorem *wallis*: $(\lambda n.\ \prod k{=}1..n.\ (4 * real\ k\hat{}2)\ /\ (4 * real\ k\hat{}2 - 1)) \longrightarrow pi\ /\ 2$

10.29.10 The Solution to the Basel problem

theorem *inverse_squares_sums*: $(\lambda n.\ 1\ /\ (n + 1)^2)\ sums\ (pi^2\ /\ 6)$

end

theory *Interval_Integral*
imports *Equivalence_Lebesgue_Henstock_Integration*
begin

10.29.11 Approximating a (possibly infinite) interval

proposition *einterval_Icc_approximation*:
fixes $a\ b :: ereal$
assumes $a < b$
obtains $u\ l :: nat \Rightarrow real$ **where**
$einterval\ a\ b = (\bigcup i.\ \{l\ i\ ..\ u\ i\})$
$incseq\ u\ decseq\ l\ \bigwedge i.\ l\ i < u\ i\ \bigwedge i.\ a < l\ i\ \bigwedge i.\ u\ i < b$
$l \longrightarrow a\ u \longrightarrow b$

definition *interval_lebesgue_integral* :: *real measure* ⇒ *ereal* ⇒ *ereal* ⇒ (*real* ⇒ *'a*) ⇒ *'a*::{*banach*, *second_countable_topology*} **where**
interval_lebesgue_integral M a b f =
(*if a* ≤ *b then* (*LINT x*:*einterval a b*|*M*. *f x*) *else* − (*LINT x*:*einterval b a*|*M*. *f x*))

definition *interval_lebesgue_integrable* :: *real measure* ⇒ *ereal* ⇒ *ereal* ⇒ (*real* ⇒ *'a*::{*banach*, *second_countable_topology*}) ⇒ *bool* **where**
interval_lebesgue_integrable M a b f =
(*if a* ≤ *b then set_integrable M* (*einterval a b*) *f else set_integrable M* (*einterval b a*) *f*)

10.29.12 Basic properties of integration over an interval

proposition *interval_integrable_to_infinity_eq*: (*interval_lebesgue_integrable M a* ∞ *f*) =
(*set_integrable M* {*a*<..} *f*)

10.29.13 Basic properties of integration over an interval wrt lebesgue measure

10.29.14 General limit approximation arguments

proposition *interval_integral_Icc_approx_nonneg*:
fixes $a\ b :: ereal$
assumes $a < b$
fixes $u\ l :: nat \Rightarrow real$
assumes *approx*: $einterval\ a\ b = (\bigcup i.\ \{l\ i\ ..\ u\ i\})$
$incseq\ u\ decseq\ l\ \bigwedge i.\ l\ i < u\ i\ \bigwedge i.\ a < l\ i\ \bigwedge i.\ u\ i < b$
$l \longrightarrow a\ u \longrightarrow b$
fixes $f :: real \Rightarrow real$

assumes *f_integrable*: $\bigwedge i.$ *set_integrable lborel* $\{l\ i..u\ i\}$ f
assumes *f_nonneg*: *AE x in lborel.* $a <$ *ereal* $x \longrightarrow$ *ereal* $x < b \longrightarrow 0 \leq f\ x$
assumes *f_measurable*: *set_borel_measurable lborel* (*einterval a b*) f
assumes *lbint_lim*: $(\lambda i.$ *LBINT* $x{=}l\ i..\ u\ i.\ f\ x) \longrightarrow C$
shows
set_integrable lborel (*einterval a b*) f
(*LBINT* $x{=}a..b.\ f\ x) = C$

proposition *interval_integral_Icc_approx_integrable*:
fixes $u\ l ::$ *nat* $\Rightarrow$ *real* **and** $a\ b ::$ *ereal*
fixes $f ::$ *real* $\Rightarrow$ $'a::\{$*banach*, *second_countable_topology*$\}$
assumes $a < b$
assumes *approx*: *einterval a b* $= (\bigcup i.\ \{l\ i\ ..\ u\ i\})$
incseq u decseq l $\bigwedge i.\ l\ i < u\ i\ \bigwedge i.\ a < l\ i\ \bigwedge i.\ u\ i < b$
$l \longrightarrow a\ u \longrightarrow b$
assumes *f_integrable*: *set_integrable lborel* (*einterval a b*) f
shows $(\lambda i.$ *LBINT* $x{=}l\ i..\ u\ i.\ f\ x) \longrightarrow$ (*LBINT* $x{=}a..b.\ f\ x)$

10.29.15 A slightly stronger Fundamental Theorem of Calculus

theorem *interval_integral_FTC_integrable*:
fixes $f\ F ::$ *real* $\Rightarrow$ $'a::$*euclidean_space* **and** $a\ b ::$ *ereal*
assumes $a < b$
assumes F: $\bigwedge x.\ a <$ *ereal* $x \Longrightarrow$ *ereal* $x < b \Longrightarrow (F$ *has_vector_derivative* $f\ x)$ $(at\ x)$
assumes f: $\bigwedge x.\ a <$ *ereal* $x \Longrightarrow$ *ereal* $x < b \Longrightarrow$ *isCont* $f\ x$
assumes *f_integrable*: *set_integrable lborel* (*einterval a b*) f
assumes A: $((F \circ$ *real_of_ereal*$) \longrightarrow A)$ (*at_right a*)
assumes B: $((F \circ$ *real_of_ereal*$) \longrightarrow B)$ (*at_left b*)
shows (*LBINT* $x{=}a..b.\ f\ x) = B - A$

theorem *interval_integral_FTC2*:
fixes $a\ b\ c ::$ *real* **and** $f ::$ *real* $\Rightarrow$ $'a::$*euclidean_space*
assumes $a \leq c\ c \leq b$
and *contf*: *continuous_on* $\{a..b\}$ f
fixes $x ::$ *real*
assumes $a \leq x$ **and** $x \leq b$
shows $((\lambda u.$ *LBINT* $y{=}c..u.\ f\ y)$ *has_vector_derivative* $(f\ x))$ $(at\ x$ *within* $\{a..b\})$

proposition *einterval_antiderivative*:
fixes $a\ b ::$ *ereal* **and** $f ::$ *real* $\Rightarrow$ $'a::$*euclidean_space*
assumes $a < b$ **and** *contf*: $\bigwedge x ::$ *real.* $a < x \Longrightarrow x < b \Longrightarrow$ *isCont* $f\ x$
shows $\exists F.\ \forall x ::$ *real.* $a < x \longrightarrow x < b \longrightarrow (F$ *has_vector_derivative* $f\ x)$ $(at\ x)$

10.29.16 The substitution theorem

theorem *interval_integral_substitution_finite*:
fixes $a\ b :: real$ **and** $f :: real \Rightarrow 'a::euclidean_space$
assumes $a \leq b$
and *derivg*: $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow (g\ has_real_derivative\ (g'\ x))\ (at\ x\ within\ \{a..b\})$
and *contf* : $continuous_on\ (g\ `\ \{a..b\})\ f$
and *contg'*: $continuous_on\ \{a..b\}\ g'$
shows $(LBINT\ x{=}a..b.\ g'\ x *_R f\ (g\ x)) = (LBINT\ y{=}g\ a..g\ b.\ f\ y)$

theorem *interval_integral_substitution_integrable*:
fixes $f :: real \Rightarrow 'a::euclidean_space$ **and** $a\ b\ u\ v :: ereal$
assumes $a < b$
and *deriv_g*: $\bigwedge x.\ a < ereal\ x \Longrightarrow ereal\ x < b \Longrightarrow DERIV\ g\ x :> g'\ x$
and *contf*: $\bigwedge x.\ a < ereal\ x \Longrightarrow ereal\ x < b \Longrightarrow isCont\ f\ (g\ x)$
and *contg'*: $\bigwedge x.\ a < ereal\ x \Longrightarrow ereal\ x < b \Longrightarrow isCont\ g'\ x$
and *g'_nonneg*: $\bigwedge x.\ a \leq ereal\ x \Longrightarrow ereal\ x \leq b \Longrightarrow 0 \leq g'\ x$
and *A*: $((ereal \circ g \circ real_of_ereal) \longrightarrow A)\ (at_right\ a)$
and *B*: $((ereal \circ g \circ real_of_ereal) \longrightarrow B)\ (at_left\ b)$
and *integrable*: $set_integrable\ lborel\ (einterval\ a\ b)\ (\lambda x.\ g'\ x *_R f\ (g\ x))$
and *integrable2*: $set_integrable\ lborel\ (einterval\ A\ B)\ (\lambda x.\ f\ x)$
shows $(LBINT\ x{=}A..B.\ f\ x) = (LBINT\ x{=}a..b.\ g'\ x *_R f\ (g\ x))$

theorem *interval_integral_substitution_nonneg*:
fixes $f\ g\ g' :: real \Rightarrow real$ **and** $a\ b\ u\ v :: ereal$
assumes $a < b$
and *deriv_g*: $\bigwedge x.\ a < ereal\ x \Longrightarrow ereal\ x < b \Longrightarrow DERIV\ g\ x :> g'\ x$
and *contf*: $\bigwedge x.\ a < ereal\ x \Longrightarrow ereal\ x < b \Longrightarrow isCont\ f\ (g\ x)$
and *contg'*: $\bigwedge x.\ a < ereal\ x \Longrightarrow ereal\ x < b \Longrightarrow isCont\ g'\ x$
and *f_nonneg*: $\bigwedge x.\ a < ereal\ x \Longrightarrow ereal\ x < b \Longrightarrow 0 \leq f\ (g\ x)$
and *g'_nonneg*: $\bigwedge x.\ a \leq ereal\ x \Longrightarrow ereal\ x \leq b \Longrightarrow 0 \leq g'\ x$
and *A*: $((ereal \circ g \circ real_of_ereal) \longrightarrow A)\ (at_right\ a)$
and *B*: $((ereal \circ g \circ real_of_ereal) \longrightarrow B)\ (at_left\ b)$
and *integrable_fg*: $set_integrable\ lborel\ (einterval\ a\ b)\ (\lambda x.\ f\ (g\ x) * g'\ x)$
shows
$set_integrable\ lborel\ (einterval\ A\ B)\ f$
$(LBINT\ x{=}A..B.\ f\ x) = (LBINT\ x{=}a..b.\ (f\ (g\ x) * g'\ x))$

proposition *interval_integral_norm*:
fixes $f :: real \Rightarrow 'a :: \{banach, second_countable_topology\}$
shows $interval_lebesgue_integrable\ lborel\ a\ b\ f \Longrightarrow a \leq b \Longrightarrow$

$norm\ (LBINT\ t{=}a..b.\ f\ t) \leq LBINT\ t{=}a..b.\ norm\ (f\ t)$

proposition *interval_integral_norm2*:
interval_lebesgue_integrable lborel a b f $\Longrightarrow$
$norm\ (LBINT\ t{=}a..b.\ f\ t) \leq |LBINT\ t{=}a..b.\ norm\ (f\ t)|$

10.29.17 Fubini theorems

end

10.30 Integration by Substition for the Lebesgue Integral

theory *Lebesgue_Integral_Substitution*
imports *Interval_Integral*
begin

theorem *nn_integral_substitution*:
fixes $f :: real \Rightarrow real$
assumes *Mf*[*measurable*]: *set_borel_measurable borel* $\{g\ a..g\ b\}\ f$
assumes *derivg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow (g\ has_real_derivative\ g'\ x)\ (at\ x)$
assumes *contg'*: *continuous_on* $\{a..b\}\ g'$
assumes *derivg_nonneg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow g'\ x \geq 0$
assumes $a \leq b$
shows $(\int^+ x.\ f\ x * indicator\ \{g\ a..g\ b\}\ x\ \partial lborel) =$
$(\int^+ x.\ f\ (g\ x) * g'\ x * indicator\ \{a..b\}\ x\ \partial lborel)$

theorem *integral_substitution*:
assumes *integrable*: *set_integrable lborel* $\{g\ a..g\ b\}\ f$
assumes *derivg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow (g\ has_real_derivative\ g'\ x)\ (at\ x)$
assumes *contg'*: *continuous_on* $\{a..b\}\ g'$
assumes *derivg_nonneg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow g'\ x \geq 0$
assumes $a \leq b$
shows *set_integrable lborel* $\{a..b\}\ (\lambda x.\ f\ (g\ x) * g'\ x)$
and $(LBINT\ x.\ f\ x * indicator\ \{g\ a..g\ b\}\ x) = (LBINT\ x.\ f\ (g\ x) * g'\ x *$
$indicator\ \{a..b\}\ x)$

theorem *interval_integral_substitution*:
assumes *integrable*: *set_integrable lborel* $\{g\ a..g\ b\}\ f$
assumes *derivg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow (g\ has_real_derivative\ g'\ x)\ (at\ x)$
assumes *contg'*: *continuous_on* $\{a..b\}\ g'$
assumes *derivg_nonneg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow g'\ x \geq 0$
assumes $a \leq b$

shows *set_integrable lborel* {*a*..*b*} (λ*x*. *f* (*g x*) ∗ *g*′ *x*)
and (*LBINT x*=*g a*..*g b*. *f x*) = (*LBINT x*=*a*..*b*. *f* (*g x*) ∗ *g*′ *x*)

end

10.31 The Volume of an n-Dimensional Ball

theory *Ball_Volume*
imports *Gamma_Function Lebesgue_Integral_Substitution*
begindefinition *unit_ball_vol* :: *real* ⇒ *real* **where**
unit_ball_vol n = *pi powr* (*n* / *2*) / *Gamma* (*n* / *2* + *1*)

corollary *content_ball*:
content (*ball c r*) = *unit_ball_vol* (*DIM*(′*a*)) ∗ *r* ^ *DIM*(′*a*)

end

10.32 Integral Test for Summability

theory *Integral_Test*
imports *Henstock_Kurzweil_Integration*
beginlocale *antimono_fun_sum_integral_diff* =
fixes *f* :: *real* ⇒ *real*
assumes *dec*: ⋀*x y*. *x* ≥ *0* ⟹ *x* ≤ *y* ⟹ *f x* ≥ *f y*
assumes *nonneg*: ⋀*x*. *x* ≥ *0* ⟹ *f x* ≥ *0*
assumes *cont*: *continuous_on* {*0*..} *f*
begin

theorem *integral_test*:
summable (λ*n*. *f* (*of_nat n*)) ⟷ *convergent* (λ*n*. *integral* {*0*..*of_nat n*} *f*)

end

10.33 Embedding Measure Spaces with a Function

theory *Embed_Measure*
imports *Binary_Product_Measure*
begindefinition *embed_measure* :: ′*a measure* ⇒ (′*a* ⇒ ′*b*) ⇒ ′*b measure* **where**
embed_measure M f = *measure_of* (*f* ‘ *space M*) {*f* ‘ *A* |*A*. *A* ∈ *sets M*}
(λ*A*. *emeasure M* (*f* −‘ *A* ∩ *space M*))

end

10.34 Fashoda Meet Theorem

theory *Fashoda_Theorem*

imports *Brouwer_Fixpoint Path_Connected Cartesian_Euclidean_Space*
begin

10.34.1 Bijections between intervals

definition *interval_bij* :: $'a \times 'a \Rightarrow 'a \times 'a \Rightarrow 'a \Rightarrow 'a$::*euclidean_space*
where *interval_bij* =
$(\lambda(a,\ b)\ (u,\ v)\ x.\ (\sum i{\in}Basis.\ (u{\cdot}i + (x{\cdot}i - a{\cdot}i)\ /\ (b{\cdot}i - a{\cdot}i) * (v{\cdot}i - u{\cdot}i))\ *_R\ i))$

10.34.2 Fashoda meet theorem

proposition *fashoda_unit*:
fixes $f\ g$ :: $real \Rightarrow real\hat{}2$
assumes $f\ `\ \{-1\ ..\ 1\} \subseteq cbox\ (-1)\ 1$
and $g\ `\ \{-1\ ..\ 1\} \subseteq cbox\ (-1)\ 1$
and *continuous_on* $\{-1\ ..\ 1\}\ f$
and *continuous_on* $\{-1\ ..\ 1\}\ g$
and $f\ (-1)\$1 = -1$
and $f\ 1\$1 = 1\ g\ (-1)\ \$2 = -1$
and $g\ 1\ \$2 = 1$
shows $\exists s{\in}\{-1\ ..\ 1\}.\ \exists t{\in}\{-1\ ..\ 1\}.\ f\ s = g\ t$

proposition *fashoda_unit_path*:
fixes $f\ g$ :: $real \Rightarrow real\hat{}2$
assumes *path* f
and *path* g
and *path_image* $f \subseteq cbox\ (-1)\ 1$
and *path_image* $g \subseteq cbox\ (-1)\ 1$
and $(pathstart\ f)\$1 = -1$
and $(pathfinish\ f)\$1 = 1$
and $(pathstart\ g)\$2 = -1$
and $(pathfinish\ g)\$2 = 1$
obtains z **where** $z \in path_image\ f$ **and** $z \in path_image\ g$

theorem *fashoda*:
fixes b :: $real\hat{}2$
assumes *path* f
and *path* g
and *path_image* $f \subseteq cbox\ a\ b$
and *path_image* $g \subseteq cbox\ a\ b$
and $(pathstart\ f)\$1 = a\$1$
and $(pathfinish\ f)\$1 = b\$1$
and $(pathstart\ g)\$2 = a\$2$
and $(pathfinish\ g)\$2 = b\$2$
obtains z **where** $z \in path_image\ f$ **and** $z \in path_image\ g$

10.34.3 Useful Fashoda corollary pointed out to me by Tom Hales

corollary *fashoda_interlace*:
fixes $a :: real\hat{}2$
assumes *path f*
and *path g*
and *paf*: $path_image\ f \subseteq cbox\ a\ b$
and *pag*: $path_image\ g \subseteq cbox\ a\ b$
and $(pathstart\ f)\$2 = a\$2$
and $(pathfinish\ f)\$2 = a\$2$
and $(pathstart\ g)\$2 = a\$2$
and $(pathfinish\ g)\$2 = a\$2$
and $(pathstart\ f)\$1 < (pathstart\ g)\$1$
and $(pathstart\ g)\$1 < (pathfinish\ f)\$1$
and $(pathfinish\ f)\$1 < (pathfinish\ g)\$1$
obtains *z* **where** $z \in path_image\ f$ **and** $z \in path_image\ g$

end

10.35 Vector Cross Products in 3 Dimensions

theory *Cross3*
imports *Determinants Cartesian_Euclidean_Space*
begin

definition $cross3 :: [real\hat{}3, real\hat{}3] \Rightarrow real\hat{}3$ (**infixr** ‹×› *80*)
where $a \times b \equiv$
$vector\ [a\$2 * b\$3 - a\$3 * b\$2,$
$a\$3 * b\$1 - a\$1 * b\$3,$
$a\$1 * b\$2 - a\$2 * b\$1]$

10.35.1 Basic lemmas

proposition *Jacobi*: $x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0$ **for** *x*::$real\hat{}3$

proposition *Lagrange*: $x \times (y \times z) = (x \cdot z) *_R y - (x \cdot y) *_R z$

proposition *cross_triple*: $(x \times y) \cdot z = (y \times z) \cdot x$

proposition *dot_cross*: $(w \times x) \cdot (y \times z) = (w \cdot y) * (x \cdot z) - (w \cdot z) * (x \cdot y)$

proposition *norm_cross*: $(norm\ (x \times y))^2 = (norm\ x)^2 * (norm\ y)^2 - (x \cdot y)^2$

10.35.2 Preservation by rotation, or other orthogonal transformation up to sign

10.35.3 Continuity

end

10.36 Bounded Continuous Functions

theory *Bounded_Continuous_Function*
imports
Topology_Euclidean_Space
Uniform_Limit
begin

10.36.1 Definition

definition *bcontfun* = $\{f.$ *continuous_on UNIV* $f \wedge$ *bounded* (*range* $f)\}$

instantiation *bcontfun* :: (*topological_space*, *metric_space*) *metric_space*
begin

lift_definition *dist_bcontfun* :: $'a \Rightarrow_C 'b \Rightarrow 'a \Rightarrow_C 'b \Rightarrow$ *real*
is $\lambda f\ g.$ (*SUP* $x.$ *dist* ($f\ x$) ($g\ x$))

10.36.2 Complete Space

instance *bcontfun* :: (*metric_space*, *complete_space*) *complete_space*

end

10.37 Infinite Products

theory *Infinite_Products*
imports *Topology_Euclidean_Space Complex_Transcendental HOL−Real_Asymp.Real_Asymp*
begin

10.37.1 Definitions and basic properties

definition *raw_has_prod* :: [*nat* $\Rightarrow$ $'a$::{*t2_space*, *comm_semiring_1*}, *nat*, $'a$]
$\Rightarrow$ *bool*
where *raw_has_prod f M p* $\equiv (\lambda n.\ \prod i{\le}n.\ f\ (i{+}M)) \longrightarrow p \wedge p \neq 0$

definition
has_prod :: (*nat* $\Rightarrow$ $'a$::{*t2_space*, *comm_semiring_1*}) $\Rightarrow$ $'a$ $\Rightarrow$ *bool* (**infixr** ‹*has'_prod*› *80*)

where f *has_prod* $p \equiv$ *raw_has_prod* f 0 $p \vee (\exists i\ q.\ p = 0 \wedge f\ i = 0 \wedge$ *raw_has_prod* f (*Suc* i) q)

definition *convergent_prod* :: (*nat* ⇒ $'a$ :: {*t2_space*,*comm_semiring_1*}) ⇒ *bool*
where
convergent_prod $f \equiv \exists M\ p.$ *raw_has_prod* f M p

definition *prodinf* :: (*nat* ⇒ $'a$::{*t2_space*, *comm_semiring_1*}) ⇒ $'a$
(**binder** ‹$\prod$› *10*)
where *prodinf* f = (*THE* $p.$ f *has_prod* p)

10.37.2 Absolutely convergent products

definition *abs_convergent_prod* :: (*nat* ⇒ _) ⇒ *bool* **where**
abs_convergent_prod $f \longleftrightarrow$ *convergent_prod* ($\lambda i.$ *1* + *norm* ($f\ i$ − *1*))

lemma *convergent_prod_iff_convergent*:
fixes f :: *nat* ⇒ $'a$ :: {*topological_semigroup_mult*,*t2_space*,*idom*}
assumes $\bigwedge i.\ f\ i \neq 0$
shows *convergent_prod* $f \longleftrightarrow$ *convergent* ($\lambda n.$ $\prod i{\leq}n.$ $f\ i$) ∧ *lim* ($\lambda n.$ $\prod i{\leq}n.$ f i) ≠ *0*

theorem *abs_convergent_prod_conv_summable*:
fixes f :: *nat* ⇒ $'a$ :: *real_normed_div_algebra*
shows *abs_convergent_prod* $f \longleftrightarrow$ *summable* ($\lambda i.$ *norm* ($f\ i$ − *1*))

10.37.3 More elementary properties

theorem *abs_convergent_prod_imp_convergent_prod*:
fixes f :: *nat* ⇒ $'a$ :: {*real_normed_div_algebra*,*complete_space*,*comm_ring_1*}
assumes *abs_convergent_prod* f
shows *convergent_prod* f

corollary *convergent_prod_offset_0*:
fixes f :: *nat* ⇒ $'a$ :: {*idom*,*topological_semigroup_mult*,*t2_space*}
assumes *convergent_prod* f $\bigwedge i.\ f\ i \neq 0$
shows $\exists p.$ *raw_has_prod* f *0* p

theorem *has_prod_iff*: f *has_prod* $x \longleftrightarrow$ *convergent_prod* $f \wedge$ *prodinf* $f = x$

10.37.4 Exponentials and logarithms

theorem *convergent_prod_iff_summable_real*:
fixes a :: *nat* ⇒ *real*

assumes $\bigwedge n.\ a\ n > 0$
shows *convergent_prod* $(\lambda k.\ 1 + a\ k) \longleftrightarrow$ *summable* a (**is** *?lhs* = *?rhs*)

theorem *Ln_prodinf_complex*:
fixes z :: *nat* ⇒ *complex*
assumes z: $\bigwedge j.\ z\ j \neq 0$ **and** ξ: $\xi \neq 0$
shows $((\lambda n.\ \prod j{\leq}n.\ z\ j) \longrightarrow \xi) \longleftrightarrow (\exists k.\ (\lambda n.\ (\sum j{\leq}n.\ Ln\ (z\ j))) \longrightarrow Ln\ \xi + of_int\ k * (of_real(2{*}pi) * \mathrm{i}))$ (**is** *?lhs* = *?rhs*)
proposition *convergent_prod_iff_summable_complex*:
fixes z :: *nat* ⇒ *complex*
assumes $\bigwedge k.\ z\ k \neq 0$
shows *convergent_prod* $(\lambda k.\ z\ k) \longleftrightarrow$ *summable* $(\lambda k.\ Ln\ (z\ k))$ (**is** *?lhs* = *?rhs*)
proposition *summable_imp_convergent_prod_complex*:
fixes z :: *nat* ⇒ *complex*
assumes z: *summable* $(\lambda k.\ norm\ (z\ k))$ **and** *non0*: $\bigwedge k.\ z\ k \neq -1$
shows *convergent_prod* $(\lambda k.\ 1 + z\ k)$

corollary *summable_imp_convergent_prod_real*:
fixes z :: *nat* ⇒ *real*
assumes z: *summable* $(\lambda k.\ |z\ k|)$ **and** *non0*: $\bigwedge k.\ z\ k \neq -1$
shows *convergent_prod* $(\lambda k.\ 1 + z\ k)$

10.37.5 Convergence criteria: especially uniform convergence of infinite products

end
theory *Infinite_Product*
imports *Infinite_Products*

begin

10.38 Unordered infinite products

10.38.1 General properties

10.38.2 Subsets

corollary *multipliable_on_subset_nonzero*:

fixes $f :: 'a \Rightarrow 'b :: \{real_normed_field, complete_space\}$
assumes $(f\ has_setprod\ L)\ A$ **and** $L \neq 0$ **and** $B \subseteq A$
shows $f\ multipliable_on\ B$

10.38.3 Strong multipliability

corollary *infprod_eq_0_iff*:
assumes $f\ strongly_multipliable_on\ A$
shows $infprod\ f\ A = 0 \longleftrightarrow (\exists x{\in}A.\ f\ x = 0)$

10.38.4 Absolute convergence

corollary *multipliable_on_norm*:
fixes $f :: 'a \Rightarrow 'b :: \{real_normed_div_algebra, semidom\}$
assumes $f\ multipliable_on\ A$
shows $(\lambda x.\ norm\ (f\ x))\ multipliable_on\ A$

corollary *abs_multipliable_on_SigmaI*:
fixes $f :: 'a \times 'b \Rightarrow 'c :: \{banach, real_normed_algebra_1\}$
assumes $\bigwedge x.\ x \in A \Longrightarrow (\lambda y.\ f\ (x, y))\ abs_multipliable_on\ B\ x$
and $(\lambda x.\ \sum_\infty y{\in}B\ x.\ norm\ (f\ (x, y) - 1))\ summable_on\ A$
shows $f\ abs_multipliable_on\ Sigma\ A\ B$

corollary *multipliable_on_SigmaI*:
fixes $f :: 'a \times 'b \Rightarrow 'c :: \{banach, real_normed_field\}$
assumes $\bigwedge x.\ x \in A \Longrightarrow (\lambda y.\ f\ (x, y))\ abs_multipliable_on\ B\ x$
and $(\lambda x.\ \sum_\infty y{\in}B\ x.\ norm\ (f\ (x, y) - 1))\ summable_on\ A$
shows $f\ multipliable_on\ Sigma\ A\ B$

corollary *abs_multipliable_on_UnionI*:
fixes $f :: 'b \Rightarrow 'c :: \{banach, real_normed_algebra_1\}$
assumes *f*: $\bigwedge x.\ x \in A \Longrightarrow f\ abs_multipliable_on\ B\ x$
and *sum*: $(\lambda x.\ \sum_\infty y{\in}B\ x.\ norm\ (f\ y - 1))\ summable_on\ A$
and *disj*: $disjoint_family_on\ B\ A$
shows $f\ abs_multipliable_on\ (\bigcup x{\in}A.\ B\ x)$

corollary *multipliable_on_UnionI*:
fixes $f :: 'b \Rightarrow 'c :: \{banach, real_normed_field\}$
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ abs_multipliable_on\ B\ x$
and $(\lambda x.\ \sum_\infty y \in B\ x.\ norm\ (f\ y - 1))\ summable_on\ A$
and *disjoint_family_on B A*
shows $f\ multipliable_on\ (\bigcup x \in A.\ B\ x)$

corollary *uniform_limit_infprod_M_test_strong*:
fixes $f :: 'k \Rightarrow 'a :: topological_space \Rightarrow 'b :: \{real_normed_div_algebra, comm_ring_1, banach, semidom, topological_semigroup_mult, t2_space\}$
assumes *le*: $\bigwedge k\ y.\ k \in A' \Longrightarrow y \in B \Longrightarrow norm\ (f\ k\ y - 1) \leq M\ k$
assumes *bounded*: $\bigwedge k.\ k \in A - A' \Longrightarrow bounded\ ((\lambda y.\ f\ k\ y - 1)\ `\ B)$
and *M*: *M summable_on A'*
and *finite* $(A - A')$
shows *uniform_limit B* $(\lambda X\ y.\ \prod k \in X.\ f\ k\ y)\ (\lambda y.\ \prod_\infty k \in A.\ f\ k\ y)$ (*finite_subsets_at_top A*)

corollary *uniform_limit_infprod_M_test*:
fixes $f :: 'k \Rightarrow 'a :: topological_space \Rightarrow 'b :: \{real_normed_div_algebra, comm_ring_1, banach, semidom, topological_semigroup_mult, t2_space\}$
assumes *le*: $\bigwedge k\ y.\ k \in A \Longrightarrow y \in B \Longrightarrow norm\ (f\ k\ y - 1) \leq M\ k$
and *M*: *M summable_on A*
shows *uniform_limit B* $(\lambda X\ y.\ \prod k \in X.\ f\ k\ y)\ (\lambda y.\ \prod_\infty k \in A.\ f\ k\ y)$ (*finite_subsets_at_top A*)

corollary *uniform_limit_prod_lessThan*:
fixes $f :: nat \Rightarrow 'a \Rightarrow 'b :: \{metric_space, comm_monoid_mult\}$
assumes *uniform_limit B* $(\lambda X\ y.\ \prod x \in X.\ f\ x\ y)\ P$ (*finite_subsets_at_top UNIV*)
shows *uniform_limit B* $(\lambda n\ y.\ \prod k < n.\ f\ k\ y)\ P\ sequentially$

10.38.5 Real numbers

10.38.6 Complex numbers

corollary *has_sum_Ln_of_has_setprod_complex*:
fixes $f :: 'a \Rightarrow complex$
assumes *P*: $(f\ has_setprod\ P)\ A$ **and** *nz*: $P \neq 0$
shows $\exists F.\ finite\ F \wedge F \subseteq A \wedge$
$((\lambda x.\ Ln\ (f\ x))\ has_sum\ Ln\ (\prod_\infty x \in A - F.\ f\ x))\ (A - F)$
corollary *multipliable_on_imp_abs_multipliable_on_complex*:
fixes $f :: 'a \Rightarrow complex$
assumes *mult*: *f multipliable_on A* **and** *nz*: *infprod f A* $\neq 0$
shows *f abs_multipliable_on A*

corollary *multipliable_on_imp_abs_multipliable_on_real*:
fixes $f :: 'a \Rightarrow real$
assumes *mult*: *f multipliable_on A* **and** *nz*: *infprod f A* $\neq 0$

shows *f abs_multipliable_on A*

end

10.39 Sums over Infinite Sets

theory *Infinite_Set_Sum*
imports *Set_Integral Infinite_Sum*
begin

definition *abs_summable_on* ::
(*'a* ⇒ *'b* :: {*banach*, *second_countable_topology*}) ⇒ *'a set* ⇒ *bool*
(**infix** ‹*abs'_summable'_on*› *50*)
where
f abs_summable_on A ⟷ *integrable* (*count_space A*) *f*

definition *infsetsum* ::
(*'a* ⇒ *'b* :: {*banach*, *second_countable_topology*}) ⇒ *'a set* ⇒ *'b*
where
infsetsum f A = *lebesgue_integral* (*count_space A*) *f*

theorem *infsetsum_reindex*:
assumes *inj_on g A*
shows *infsetsum f* (*g* ‘ *A*) = *infsetsum* (λ*x*. *f* (*g x*)) *A*

theorem *infsetsum_Sigma*:
fixes *A* :: *'a set* **and** *B* :: *'a* ⇒ *'b set*
assumes [*simp*]: *countable A* **and** ⋀*i*. *countable* (*B i*)
assumes *summable*: *f abs_summable_on* (*Sigma A B*)
shows *infsetsum f* (*Sigma A B*) = *infsetsum* (λ*x*. *infsetsum* (λ*y*. *f* (*x*, *y*)) (*B x*)) *A*

theorem *abs_summable_on_Sigma_iff*:
assumes [*simp*]: *countable A* **and** ⋀*x*. *x* ∈ *A* ⟹ *countable* (*B x*)
shows *f abs_summable_on Sigma A B* ⟷
(∀ *x*∈*A*. (λ*y*. *f* (*x*, *y*)) *abs_summable_on B x*) ∧
((λ*x*. *infsetsum* (λ*y*. *norm* (*f* (*x*, *y*))) (*B x*)) *abs_summable_on A*)

theorem *infsetsum_prod_PiE*:
fixes *f* :: *'a* ⇒ *'b* ⇒ *'c* :: {*real_normed_field*,*banach*,*second_countable_topology*}
assumes *finite*: *finite A* **and** *countable*: ⋀*x*. *x* ∈ *A* ⟹ *countable* (*B x*)
assumes *summable*: ⋀*x*. *x* ∈ *A* ⟹ *f x abs_summable_on B x*
shows *infsetsum* (λ*g*. ∏ *x*∈*A*. *f x* (*g x*)) (*PiE A B*) = (∏ *x*∈*A*. *infsetsum* (*f x*) (*B x*))

end

10.40 Faces, Extreme Points, Polytopes, Polyhedra etc

theory *Polytope*
imports *Cartesian_Euclidean_Space Path_Connected*
begin

10.40.1 Faces of a (usually convex) set

definition *face_of* :: [*'a*::*real_vector set*, *'a set*] ⇒ *bool* (**infixr** ‹(*face'_of*)› *50*)
where
T face_of S ⟷
T ⊆ *S* ∧ *convex T* ∧
(∀ *a* ∈ *S*. ∀ *b* ∈ *S*. ∀ *x* ∈ *T*. *x* ∈ *open_segment a b* ⟶ *a* ∈ *T* ∧ *b* ∈ *T*)

proposition *face_of_imp_eq_affine_Int*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *S*: *convex S* **and** *T*: *T face_of S*
shows *T* = (*affine hull T*) ∩ *S*

proposition *face_of_conic*:
assumes *conic S f face_of S*
shows *conic f*

proposition *face_of_convex_hulls*:
assumes *S*: *finite S T* ⊆ *S* **and** *disj*: *affine hull T* ∩ *convex hull* (*S* − *T*) = {}
shows (*convex hull T*) *face_of* (*convex hull S*)

proposition *face_of_convex_hull_insert*:
assumes *finite S a* ∉ *affine hull S* **and** *T*: *T face_of convex hull S*
shows *T face_of convex hull insert a S*

proposition *face_of_affine_trivial*:
assumes *affine S T face_of S*
shows *T* = {} ∨ *T* = *S*

proposition *Inter_faces_finite_altbound*:
fixes *T* :: *'a*::*euclidean_space set set*
assumes *cfaI*: ⋀*c*. *c* ∈ *T* ⟹ *c face_of S*

shows $\exists F'.\ \mathit{finite}\ F' \wedge F' \subseteq T \wedge \mathit{card}\ F' \leq \mathit{DIM}('a) + 2 \wedge \bigcap F' = \bigcap T$

proposition *face_of_Times*:
assumes F *face_of* S **and** F' *face_of* S'
shows $(F \times F')$ *face_of* $(S \times S')$

corollary *face_of_Times_decomp*:
fixes $S :: 'a{::}\mathit{euclidean_space}\ \mathit{set}$ **and** $S' :: 'b{::}\mathit{euclidean_space}\ \mathit{set}$
shows C *face_of* $(S \times S') \longleftrightarrow (\exists F\ F'.\ F$ *face_of* $S \wedge F'$ *face_of* $S' \wedge C = F \times F')$
(**is** *?lhs* = *?rhs*)

10.40.2 Exposed faces

definition *exposed_face_of* :: $['a{::}\mathit{euclidean_space}\ \mathit{set}, 'a\ \mathit{set}] \Rightarrow \mathit{bool}$
(**infixr** ‹*exposed'_face'_of*› 50)
where T *exposed_face_of* $S \longleftrightarrow$
T *face_of* $S \wedge (\exists a\ b.\ S \subseteq \{x.\ a \cdot x \leq b\} \wedge T = S \cap \{x.\ a \cdot x = b\})$

proposition *exposed_face_of_Int*:
assumes T *exposed_face_of* S
and U *exposed_face_of* S
shows $(T \cap U)$ *exposed_face_of* S

proposition *exposed_face_of_Inter*:
fixes $P :: 'a{::}\mathit{euclidean_space}\ \mathit{set}\ \mathit{set}$
assumes $P \neq \{\}$
and $\bigwedge T.\ T \in P \Longrightarrow T$ *exposed_face_of* S
shows $\bigcap P$ *exposed_face_of* S

proposition *exposed_face_of_sums*:
assumes *convex* S **and** *convex* T
and F *exposed_face_of* $\{x + y \mid x\ y.\ x \in S \wedge y \in T\}$
(**is** F *exposed_face_of ?ST*)
obtains $k\ l$
where k *exposed_face_of* $S\ l$ *exposed_face_of* T
$F = \{x + y \mid x\ y.\ x \in k \wedge y \in l\}$

proposition *exposed_face_of_parallel*:
T *exposed_face_of* $S \longleftrightarrow$
T *face_of* $S \wedge$
$(\exists a\ b.\ S \subseteq \{x.\ a \cdot x \leq b\} \wedge T = S \cap \{x.\ a \cdot x = b\} \wedge$
$(T \neq \{\} \longrightarrow T \neq S \longrightarrow a \neq 0) \wedge$
$(T \neq S \longrightarrow (\forall w \in \mathit{affine\ hull}\ S.\ (w + a) \in \mathit{affine\ hull}\ S)))$
(**is** *?lhs* = *?rhs*)

10.40.3 Extreme points of a set: its singleton faces

definition *extreme_point_of* :: [*'a*::*real_vector*, *'a set*] ⇒ *bool*
(**infixr** ‹(*extreme'_point'_of*)› *50*)
where *x extreme_point_of S* ⟷
x ∈ *S* ∧ (∀ *a* ∈ *S*. ∀ *b* ∈ *S*. *x* ∉ *open_segment a b*)

proposition *extreme_points_of_convex_hull*:
{*x*. *x extreme_point_of* (*convex hull S*)} ⊆ *S*

10.40.4 Facets

definition *facet_of* :: [*'a*::*euclidean_space set*, *'a set*] ⇒ *bool*
(**infixr** ‹(*facet'_of*)› *50*)
where *F facet_of S* ⟷ *F face_of S* ∧ *F* ≠ {} ∧ *aff_dim F* = *aff_dim S* − *1*

10.40.5 Edges: faces of affine dimension 1

definition *edge_of* :: [*'a*::*euclidean_space set*, *'a set*] ⇒ *bool* (**infixr** ‹(*edge'_of*)› *50*)
where *e edge_of S* ⟷ *e face_of S* ∧ *aff_dim e* = *1*

10.40.6 Existence of extreme points

proposition *different_norm_3_collinear_points*:
fixes *a* :: *'a*::*euclidean_space*
assumes *x* ∈ *open_segment a b norm*(*a*) = *norm*(*b*) *norm*(*x*) = *norm*(*b*)
shows *False*

proposition *extreme_point_exists_convex*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *compact S convex S S* ≠ {}
obtains *x* **where** *x extreme_point_of S*

10.40.7 Krein-Milman, the weaker form

proposition *Krein_Milman*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *compact S convex S*
shows *S* = *closure*(*convex hull* {*x*. *x extreme_point_of S*})

theorem *Krein_Milman_Minkowski*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *compact S convex S*
shows *S* = *convex hull* {*x*. *x extreme_point_of S*}

10.40.8 Applying it to convex hulls of explicitly indicated finite sets

corollary *Krein_Milman_polytope*:
fixes $S :: 'a::euclidean_space\ set$
shows
finite S
$\Longrightarrow$ *convex hull S* =
convex hull $\{x.\ x\ extreme_point_of\ (convex\ hull\ S)\}$

proposition *face_of_convex_hull_insert_eq*:
fixes $a :: 'a :: euclidean_space$
assumes *finite S* **and** *a*: $a \notin affine\ hull\ S$
shows (*F face_of* (*convex hull* (*insert a S*)) $\longleftrightarrow$
F face_of (*convex hull S*) $\vee$
$(\exists F'.\ F'\ face_of\ (convex\ hull\ S) \wedge F = convex\ hull\ (insert\ a\ F'))$)
(**is** *F face_of ?CAS* $\longleftrightarrow$ _)

proposition *face_of_convex_hull_affine_independent*:
fixes $S :: 'a::euclidean_space\ set$
assumes $\neg$ *affine_dependent S*
shows (*T face_of* (*convex hull S*) $\longleftrightarrow$ $(\exists c.\ c \subseteq S \wedge T = convex\ hull\ c)$)
(**is** *?lhs* = *?rhs*)

proposition *Krein_Milman_frontier*:
fixes $S :: 'a::euclidean_space\ set$
assumes *convex S compact S*
shows *S* = *convex hull* (*frontier S*)
(**is** *?lhs* = *?rhs*)

10.40.9 Polytopes

definition *polytope* **where**
polytope S $\equiv \exists v.$ *finite v* $\wedge$ *S* = *convex hull v*

proposition *face_of_polytope_insert2*:
fixes $a :: 'a :: euclidean_space$
assumes *polytope S* $a \notin$ *affine hull S F face_of S*
shows *convex hull* (*insert a F*) *face_of convex hull* (*insert a S*)

10.40.10 Polyhedra

definition *polyhedron* **where**

polyhedron $S \equiv$
$\exists F.$ *finite* $F \wedge$
$S = \bigcap F \wedge$
$(\forall h \in F.\ \exists a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\})$

10.40.11 Canonical polyhedron representation making facial structure explicit

proposition *polyhedron_Int_affine*:
fixes $S :: 'a :: euclidean_space\ set$
shows *polyhedron* $S \longleftrightarrow$
$(\exists F.$ *finite* $F \wedge S =$ (*affine hull* S) $\cap \bigcap F \wedge$
$(\forall h \in F.\ \exists a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\}))$

proposition *rel_interior_polyhedron_explicit*:
assumes *finite* F
and *seq*: $S =$ *affine hull* $S \cap \bigcap F$
and *faceq*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
and *psub*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset$ *affine hull* $S \cap \bigcap F'$
shows *rel_interior* $S = \{x \in S.\ \forall h \in F.\ a\ h \cdot x < b\ h\}$

proposition *polyhedron_Int_affine_parallel_minimal*:
fixes $S :: 'a :: euclidean_space\ set$
shows *polyhedron* $S \longleftrightarrow$
$(\exists F.$ *finite* $F \wedge$
$S =$ (*affine hull* S) $\cap (\bigcap F) \wedge$
$(\forall h \in F.\ \exists a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\} \wedge$
$(\forall x \in$ *affine hull* $S.\ (x + a) \in$ *affine hull* $S)) \wedge$
$(\forall F'.\ F' \subset F \longrightarrow S \subset$ (*affine hull* S) $\cap (\bigcap F')))$
(**is** *?lhs* = *?rhs*)

proposition *facet_of_polyhedron_explicit*:
assumes *finite* F
and *seq*: $S =$ *affine hull* $S \cap \bigcap F$
and *faceq*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
and *psub*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset$ *affine hull* $S \cap \bigcap F'$
shows C *facet_of* $S \longleftrightarrow (\exists h.\ h \in F \wedge C = S \cap \{x.\ a\ h \cdot x = b\ h\})$

proposition *face_of_polyhedron_explicit*:
fixes $S :: 'a :: euclidean_space\ set$
assumes *finite* F
and *seq*: $S =$ *affine hull* $S \cap \bigcap F$
and *faceq*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
and *psub*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset$ *affine hull* $S \cap \bigcap F'$

and C: C *face_of* S **and** $C \neq \{\}$ $C \neq S$
shows $C = \bigcap \{S \cap \{x.\ a\ h \cdot x = b\ h\} \mid h.\ h \in F \wedge C \subseteq S \cap \{x.\ a\ h \cdot x = b\ h\}\}$

10.40.12 More general corollaries from the explicit representation

corollary *facet_of_polyhedron*:
assumes *polyhedron* S **and** C *facet_of* S
obtains $a\ b$ **where** $a \neq 0$ $S \subseteq \{x.\ a \cdot x \leq b\}$ $C = S \cap \{x.\ a \cdot x = b\}$

corollary *face_of_polyhedron*:
assumes *polyhedron* S **and** C *face_of* S **and** $C \neq \{\}$ **and** $C \neq S$
shows $C = \bigcap \{F.\ F$ *facet_of* $S \wedge C \subseteq F\}$

proposition *rel_interior_of_polyhedron*:
fixes $S :: 'a :: euclidean_space\ set$
assumes *polyhedron* S
shows *rel_interior* $S = S - \bigcup \{F.\ F$ *facet_of* $S\}$
proposition *polyhedron_eq_finite_exposed_faces*:
fixes $S :: 'a :: euclidean_space\ set$
shows *polyhedron* $S \longleftrightarrow$ *closed* $S \wedge$ *convex* $S \wedge$ *finite* $\{F.\ F$ *exposed_face_of* $S\}$
(**is** *?lhs* = *?rhs*)

corollary *polyhedron_eq_finite_faces*:
fixes $S :: 'a :: euclidean_space\ set$
shows *polyhedron* $S \longleftrightarrow$ *closed* $S \wedge$ *convex* $S \wedge$ *finite* $\{F.\ F$ *face_of* $S\}$
(**is** *?lhs* = *?rhs*)

10.40.13 Relation between polytopes and polyhedra

proposition *polytope_eq_bounded_polyhedron*:
fixes $S :: 'a :: euclidean_space\ set$
shows *polytope* $S \longleftrightarrow$ *polyhedron* $S \wedge$ *bounded* S
(**is** *?lhs* = *?rhs*)

10.40.14 Relative and absolute frontier of a polytope

proposition *frontier_of_convex_hull*:
fixes $S :: 'a::euclidean_space\ set$
assumes *card* $S = Suc\ (DIM('a))$
shows *frontier*(*convex hull* $S) = \bigcup \{$*convex hull* $(S - \{a\}) \mid a.\ a \in S\}$

10.40.15 Special case of a triangle

proposition *frontier_of_triangle*:
fixes $a :: 'a::euclidean_space$
assumes $DIM('a) = 2$
shows $frontier(convex\ hull\ \{a,b,c\}) = closed_segment\ a\ b \cup closed_segment\ b\ c \cup closed_segment\ c\ a$
(**is** *?lhs = ?rhs*)

corollary *inside_of_triangle*:
fixes $a :: 'a::euclidean_space$
assumes $DIM('a) = 2$
shows $inside\ (closed_segment\ a\ b \cup closed_segment\ b\ c \cup closed_segment\ c\ a) = interior(convex\ hull\ \{a,b,c\})$

corollary *interior_of_triangle*:
fixes $a :: 'a::euclidean_space$
assumes $DIM('a) = 2$
shows $interior(convex\ hull\ \{a,b,c\}) =$
$convex\ hull\ \{a,b,c\} - (closed_segment\ a\ b \cup closed_segment\ b\ c \cup closed_segment\ c\ a)$

10.40.16 Subdividing a cell complex

proposition *cell_complex_subdivision_exists*:
fixes $\mathcal{F} :: 'a::euclidean_space\ set\ set$
assumes $0 < e$ *finite* $\mathcal{F}$
and *poly*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow polytope\ X$
and *aff*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow aff_dim\ X \leq d$
and *face*: $\bigwedge X\ Y.\ [\![X \in \mathcal{F};\ Y \in \mathcal{F}]\!] \Longrightarrow X \cap Y\ face_of\ X$
obtains $\mathcal{F}'$ **where** *finite* $\mathcal{F}'$ $\bigcup \mathcal{F}' = \bigcup \mathcal{F}$ $\bigwedge X.\ X \in \mathcal{F}' \Longrightarrow diameter\ X < e$
$\bigwedge X.\ X \in \mathcal{F}' \Longrightarrow polytope\ X$ $\bigwedge X.\ X \in \mathcal{F}' \Longrightarrow aff_dim\ X \leq d$
$\bigwedge X\ Y.\ [\![X \in \mathcal{F}';\ Y \in \mathcal{F}']\!] \Longrightarrow X \cap Y\ face_of\ X$
$\bigwedge C.\ C \in \mathcal{F}' \Longrightarrow \exists D.\ D \in \mathcal{F} \wedge C \subseteq D$
$\bigwedge C\ x.\ C \in \mathcal{F} \wedge x \in C \Longrightarrow \exists D.\ D \in \mathcal{F}' \wedge x \in D \wedge D \subseteq C$

10.40.17 Simplexes

definition $simplex :: int \Rightarrow 'a::euclidean_space\ set \Rightarrow bool$ (**infix** ‹*simplex*› *50*)
where $n\ simplex\ S \equiv \exists C.\ \neg\ affine_dependent\ C \wedge int(card\ C) = n + 1 \wedge S = convex\ hull\ C$

10.40.18 Simplicial complexes and triangulations

definition *simplicial_complex* **where**
simplicial_complex $\mathcal{C} \equiv$
$\quad$ *finite* $\mathcal{C} \wedge$
$\quad$ $(\forall\, S \in \mathcal{C}.\ \exists\, n.\ n$ *simplex* $S) \wedge$
$\quad$ $(\forall\, F\ S.\ S \in \mathcal{C} \wedge F$ *face_of* $S \longrightarrow F \in \mathcal{C}) \wedge$
$\quad$ $(\forall\, S\ S'.\ S \in \mathcal{C} \wedge S' \in \mathcal{C} \longrightarrow (S \cap S')$ *face_of* $S)$

definition *triangulation* **where**
triangulation $\mathcal{T} \equiv$
$\quad$ *finite* $\mathcal{T} \wedge$
$\quad$ $(\forall\, T \in \mathcal{T}.\ \exists\, n.\ n$ *simplex* $T) \wedge$
$\quad$ $(\forall\, T\ T'.\ T \in \mathcal{T} \wedge T' \in \mathcal{T} \longrightarrow (T \cap T')$ *face_of* $T)$

10.40.19 Refining a cell complex to a simplicial complex

proposition *convex_hull_insert_Int_eq*:
fixes $z :: {'a} ::$ *euclidean_space*
assumes *z*: $z \in$ *rel_interior* S
and *T*: $T \subseteq$ *rel_frontier* S
and *U*: $U \subseteq$ *rel_frontier* S
and *convex* S *convex* T *convex* U
shows *convex hull* (*insert* z T) $\cap$ *convex hull* (*insert* z U) $=$ *convex hull* (*insert* z ($T \cap U$))
(**is** *?lhs* $=$ *?rhs*)

proposition *simplicial_subdivision_of_cell_complex*:
assumes *finite* $\mathcal{M}$
and *poly*: $\bigwedge C.\ C \in \mathcal{M} \Longrightarrow$ *polytope* C
and *face*: $\bigwedge C1\ C2.\ [\![C1 \in \mathcal{M};\ C2 \in \mathcal{M}]\!] \Longrightarrow C1 \cap C2$ *face_of* $C1$
obtains $\mathcal{T}$ **where** *simplicial_complex* $\mathcal{T}$
$\bigcup \mathcal{T} = \bigcup \mathcal{M}$
$\bigwedge C.\ C \in \mathcal{M} \Longrightarrow \exists\, F.$ *finite* $F \wedge F \subseteq \mathcal{T} \wedge C = \bigcup F$
$\bigwedge K.\ K \in \mathcal{T} \Longrightarrow \exists\, C.\ C \in \mathcal{M} \wedge K \subseteq C$

corollary *fine_simplicial_subdivision_of_cell_complex*:
assumes $0 < e$ *finite* $\mathcal{M}$
and *poly*: $\bigwedge C.\ C \in \mathcal{M} \Longrightarrow$ *polytope* C
and *face*: $\bigwedge C1\ C2.\ [\![C1 \in \mathcal{M};\ C2 \in \mathcal{M}]\!] \Longrightarrow C1 \cap C2$ *face_of* $C1$
obtains $\mathcal{T}$ **where** *simplicial_complex* $\mathcal{T}$
$\bigwedge K.\ K \in \mathcal{T} \Longrightarrow$ *diameter* $K < e$
$\bigcup \mathcal{T} = \bigcup \mathcal{M}$
$\bigwedge C.\ C \in \mathcal{M} \Longrightarrow \exists\, F.$ *finite* $F \wedge F \subseteq \mathcal{T} \wedge C = \bigcup F$
$\bigwedge K.\ K \in \mathcal{T} \Longrightarrow \exists\, C.\ C \in \mathcal{M} \wedge K \subseteq C$

10.40.20 Some results on cell division with full-dimensional cells only

proposition *fine_triangular_subdivision_of_cell_complex*:
assumes $0 < e$ *finite* $\mathcal{M}$
and *poly*: $\bigwedge C.\ C \in \mathcal{M} \Longrightarrow$ *polytope* C
and *aff*: $\bigwedge C.\ C \in \mathcal{M} \Longrightarrow$ *aff_dim* $C = d$
and *face*: $\bigwedge C1\ C2.\ [\![C1 \in \mathcal{M};\ C2 \in \mathcal{M}]\!] \Longrightarrow C1 \cap C2$ *face_of* $C1$
obtains $\mathcal{T}$ **where** *triangulation* $\mathcal{T}$ $\bigwedge k.\ k \in \mathcal{T} \Longrightarrow$ *diameter* $k < e$
$\bigwedge k.\ k \in \mathcal{T} \Longrightarrow$ *aff_dim* $k = d$ $\bigcup \mathcal{T} = \bigcup \mathcal{M}$
$\bigwedge C.\ C \in \mathcal{M} \Longrightarrow \exists f.$ *finite* $f \wedge f \subseteq \mathcal{T} \wedge C = \bigcup f$
$\bigwedge k.\ k \in \mathcal{T} \Longrightarrow \exists C.\ C \in \mathcal{M} \wedge k \subseteq C$

10.41 Finitely generated cone is polyhedral, and hence closed

proposition *polyhedron_convex_cone_hull*:
fixes S :: *'a*::*euclidean_space set*
assumes *finite* S
shows *polyhedron*(*convex_cone hull* S)

end

10.42 Absolute Retracts, Absolute Neighbourhood Retracts and Euclidean Neighbourhood Retracts

theory *Retracts*
imports
Brouwer_Fixpoint
Continuous_Extension
begindefinition *AR* :: *'a*::*topological_space set* ⇒ *bool* **where**
AR $S \equiv \forall U.\ \forall S'$::(*'a* ∗ *real*) *set*.
S *homeomorphic* $S' \wedge$ *closedin* (*top_of_set* U) $S' \longrightarrow S'$ *retract_of* U

definition *ANR* :: *'a*::*topological_space set* ⇒ *bool* **where**
ANR $S \equiv \forall U.\ \forall S'$::(*'a* ∗ *real*) *set*.
S *homeomorphic* $S' \wedge$ *closedin* (*top_of_set* U) S'
$\longrightarrow (\exists T.$ *openin* (*top_of_set* U) $T \wedge S'$ *retract_of* T)

definition *ENR* :: *'a*::*topological_space set* ⇒ *bool* **where**
ENR $S \equiv \exists U.$ *open* $U \wedge S$ *retract_of* U

corollary *ANR_imp_absolute_neighbourhood_retract*:
fixes $S :: 'a::euclidean_space\ set$ **and** $S' :: 'b::euclidean_space\ set$
assumes *ANR S S homeomorphic S'*
and *clo*: *closedin* (*top_of_set U*) *S'*
obtains *V* **where** *openin* (*top_of_set U*) *V S' retract_of V*

corollary *ANR_imp_absolute_neighbourhood_retract_UNIV*:
fixes $S :: 'a::euclidean_space\ set$ **and** $S' :: 'b::euclidean_space\ set$
assumes *ANR S* **and** *hom*: *S homeomorphic S'* **and** *clo*: *closed S'*
obtains *V* **where** *open V S' retract_of V*

corollary *neighbourhood_extension_into_ANR*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes *contf*: *continuous_on S f* **and** *fim*: $f \in S \to T$ **and** *ANR T closed S*
obtains *V g* **where** $S \subseteq V$ *open V continuous_on V g*
$g \in V \to T \bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

10.42.1 Analogous properties of ENRs

corollary *ENR_imp_absolute_neighbourhood_retract_UNIV*:
fixes $S :: 'a::euclidean_space\ set$ **and** $S' :: 'b::euclidean_space\ set$
assumes *ENR S S homeomorphic S'*
obtains *T'* **where** *open T' S' retract_of T'*

corollary *AR_closed_Un*:
fixes $S :: 'a::euclidean_space\ set$
shows $[\![closed\ S;\ closed\ T;\ AR\ S;\ AR\ T;\ AR\ (S \cap T)]\!] \Longrightarrow AR\ (S \cup T)$

corollary *ANR_closed_Un*:
fixes $S :: 'a::euclidean_space\ set$
shows $[\![closed\ S;\ closed\ T;\ ANR\ S;\ ANR\ T;\ ANR\ (S \cap T)]\!] \Longrightarrow ANR\ (S \cup T)$

10.42.2 More advanced properties of ANRs and ENRs

10.42.3 Original ANR material, now for ENRs

10.42.4 Finally, spheres are ANRs and ENRs

10.42.5 Spheres are connected, etc

10.42.6 Borsuk homotopy extension theorem

theorem *Borsuk_homotopy_extension_homotopic*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes *cloTS*: *closedin* (*top_of_set T*) *S*
and *anr*: $(ANR\ S \wedge ANR\ T) \vee ANR\ U$
and *contf*: *continuous_on T f*
and $f \in T \to U$
and *homotopic_with_canon* $(\lambda x.\ True)\ S\ U\ f\ g$
obtains g' **where** *homotopic_with_canon* $(\lambda x.\ True)\ T\ U\ f\ g'$
continuous_on T g' image g' T $\subseteq U$
$\bigwedge x.\ x \in S \Longrightarrow g'\ x = g\ x$

10.42.7 More extension theorems

10.42.8 The complement of a set and path-connectedness

theorem *connected_complement_homeomorphic_convex_compact*:
fixes $S :: 'a::euclidean_space\ set$ **and** $T :: 'b::euclidean_space\ set$
assumes *hom*: *S homeomorphic T* **and** *T*: *convex T compact T* **and** *2*: $2 \leq DIM('a)$
shows *connected*$(-\ S)$

corollary *path_connected_complement_homeomorphic_convex_compact*:
fixes $S :: 'a::euclidean_space\ set$ **and** $T :: 'b::euclidean_space\ set$
assumes *hom*: *S homeomorphic T convex T compact T* $2 \leq DIM('a)$
shows *path_connected*$(-\ S)$

end

10.43 Extending Continous Maps, Invariance of Domain, etc

theory *Further_Topology*
imports *Weierstrass_Theorems Polytope Complex_Transcendental Equivalence_Lebesgue_Henstock_I*
Retracts
begin

10.43.1 A map from a sphere to a higher dimensional sphere is nullhomotopic

proposition *inessential_spheremap_lowdim_gen*:
fixes f :: *'M::euclidean_space* ⇒ *'a::euclidean_space*
assumes *convex S bounded S convex T bounded T*
and *affST*: *aff_dim S* < *aff_dim T*
and *contf*: *continuous_on* (*rel_frontier S*) *f*
and *fim*: $f \in$ (*rel_frontier S*) → *rel_frontier T*
obtains *c* **where** *homotopic_with_canon* (λ*z*. *True*) (*rel_frontier S*) (*rel_frontier T*) *f* (λ*x*. *c*)

10.43.2 Some technical lemmas about extending maps from cell complexes

theorem *extend_map_cell_complex_to_sphere*:
assumes *finite* $\mathcal{F}$ **and** *S*: $S \subseteq \bigcup \mathcal{F}$ *closed S* **and** *T*: *convex T bounded T*
and *poly*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *polytope X*
and *aff*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *aff_dim X* < *aff_dim T*
and *face*: $\bigwedge X\ Y.\ [\![X \in \mathcal{F};\ Y \in \mathcal{F}]\!] \Longrightarrow (X \cap Y)$ *face_of X*
and *contf*: *continuous_on S f* **and** *fim*: $f \in S \to$ *rel_frontier T*
obtains *g* **where** *continuous_on* $(\bigcup \mathcal{F})$ *g*
$g \in (\bigcup \mathcal{F}) \to$ *rel_frontier T* $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

theorem *extend_map_cell_complex_to_sphere_cofinite*:
assumes *finite* $\mathcal{F}$ **and** *S*: $S \subseteq \bigcup \mathcal{F}$ *closed S* **and** *T*: *convex T bounded T*
and *poly*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *polytope X*
and *aff*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *aff_dim X* ≤ *aff_dim T*
and *face*: $\bigwedge X\ Y.\ [\![X \in \mathcal{F};\ Y \in \mathcal{F}]\!] \Longrightarrow (X \cap Y)$ *face_of X*
and *contf*: *continuous_on S f* **and** *fim*: $f \in S \to$ *rel_frontier T*
obtains *C g* **where** *finite C disjnt C S continuous_on* $(\bigcup \mathcal{F} - C)$ *g*
$g \in (\bigcup \mathcal{F} - C) \to$ *rel_frontier T* $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

10.43.3 Special cases and corollaries involving spheres

proposition *extend_map_affine_to_sphere_cofinite_simple*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *compact S convex U bounded U*
and *aff*: *aff_dim T* ≤ *aff_dim U*
and $S \subseteq T$ **and** *contf*: *continuous_on S f*
and *fim*: $f \in S \to$ *rel_frontier U*

obtains K g **where** *finite* K $K \subseteq T$ *disjnt* K S *continuous_on* $(T - K)$ g
$g \in (T - K) \to$ *rel_frontier* U
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

10.43.4 Extending maps to spheres

proposition *extend_map_affine_to_sphere_cofinite_gen*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *SUT*: *compact* S *convex* U *bounded* U *affine* T $S \subseteq T$
and *aff*: *aff_dim* $T \leq$ *aff_dim* U
and *contf*: *continuous_on* S f
and *fim*: $f \in S \to$ *rel_frontier* U
and *dis*: $\bigwedge C.\ [\![C \in components(T - S);\ bounded\ C]\!] \Longrightarrow C \cap L \neq \{\}$
obtains K g **where** *finite* K $K \subseteq L$ $K \subseteq T$ *disjnt* K S *continuous_on* $(T - K)$ g
$g \in (T - K) \to$ *rel_frontier* U
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

corollary *extend_map_affine_to_sphere_cofinite*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *SUT*: *compact* S *affine* T $S \subseteq T$
and *aff*: *aff_dim* $T \leq DIM('b)$ **and** $0 \leq r$
and *contf*: *continuous_on* S f
and *fim*: $f \in S \to$ *sphere* a r
and *dis*: $\bigwedge C.\ [\![C \in components(T - S);\ bounded\ C]\!] \Longrightarrow C \cap L \neq \{\}$
obtains K g **where** *finite* K $K \subseteq L$ $K \subseteq T$ *disjnt* K S *continuous_on* $(T - K)$ g
$g \in (T - K) \to$ *sphere* a r $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

corollary *extend_map_UNIV_to_sphere_cofinite*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes $DIM('a) \leq DIM('b)$ **and** $0 \leq r$
and *compact* S
and *continuous_on* S f
and $f \in S \to$ *sphere* a r
and $\bigwedge C.\ [\![C \in components(- S);\ bounded\ C]\!] \Longrightarrow C \cap L \neq \{\}$
obtains K g **where** *finite* K $K \subseteq L$ *disjnt* K S *continuous_on* $(- K)$ g
$g \in (- K) \to$ *sphere* a r $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

corollary *extend_map_UNIV_to_sphere_no_bounded_component*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *aff*: $DIM('a) \leq DIM('b)$ **and** $0 \leq r$
and *SUT*: *compact* S
and *contf*: *continuous_on* S f

and *fim*: $f \in S \to$ *sphere a r*
and *dis*: $\bigwedge C.\ C \in$ *components*$(-\ S) \Longrightarrow \neg$ *bounded C*
obtains *g* **where** *continuous_on UNIV g g* $\in$ *UNIV* $\to$ *sphere a r* $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

theorem *Borsuk_separation_theorem_gen*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *compact S*
shows $(\forall\, c \in$ *components*$(-\ S).\ \neg$*bounded c*$) \longleftrightarrow$
$(\forall f.$ *continuous_on S f* $\wedge\ f \in S \to$ *sphere* (*0*::*'a*) *1*
$\longrightarrow (\exists\, c.$ *homotopic_with_canon* $(\lambda x.$ *True*$)$ *S* (*sphere 0 1*) *f* $(\lambda x.$ *c*)))
(**is** *?lhs* = *?rhs*)

corollary *Borsuk_separation_theorem*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *compact S* **and** *2*: *2* $\le$ *DIM*(*'a*)
shows *connected*$(-\ S) \longleftrightarrow$
$(\forall f.$ *continuous_on S f* $\wedge\ f \in S \to$ *sphere* (*0*::*'a*) *1*
$\longrightarrow (\exists\, c.$ *homotopic_with_canon* $(\lambda x.$ *True*$)$ *S* (*sphere 0 1*) *f* $(\lambda x.$ *c*)))
(**is** *?lhs* = *?rhs*)

proposition *Jordan_Brouwer_separation*:
fixes *S* :: *'a*::*euclidean_space set* **and** *a*::*'a*
assumes *hom*: *S homeomorphic sphere a r* **and** *0* $<$ *r*
shows $\neg$ *connected*$(-\ S)$

proposition *Jordan_Brouwer_frontier*:
fixes *S* :: *'a*::*euclidean_space set* **and** *a*::*'a*
assumes *S*: *S homeomorphic sphere a r* **and** *T*: *T* $\in$ *components*$(-\ S)$ **and** *2*: *2* $\le$ *DIM*(*'a*)
shows *frontier T* = *S*

proposition *Jordan_Brouwer_nonseparation*:
fixes *S* :: *'a*::*euclidean_space set* **and** *a*::*'a*
assumes *S*: *S homeomorphic sphere a r* **and** *T* $\subset$ *S* **and** *2*: *2* $\le$ *DIM*(*'a*)
shows *connected*$(-\ T)$

10.43.5 Invariance of domain and corollaries

theorem *invariance_of_domain*:
fixes *f* :: *'a* $\Rightarrow$ *'a*::*euclidean_space*
assumes *continuous_on S f open S inj_on f S*

shows *open*(*f* ‘ *S*)

corollary *invariance_of_domain_subspaces*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *ope*: *openin* (*top_of_set U*) *S*
and *subspace U subspace V* **and** *VU*: *dim V* ≤ *dim U*
and *contf*: *continuous_on S f* **and** *fim*: *f* ∈ *S* → *V*
and *injf*: *inj_on f S*
shows *openin* (*top_of_set V*) (*f* ‘ *S*)

corollary *invariance_of_dimension_subspaces*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *ope*: *openin* (*top_of_set U*) *S*
and *subspace U subspace V*
and *contf*: *continuous_on S f* **and** *fim*: *f* ∈ *S* → *V*
and *injf*: *inj_on f S* **and** *S* ≠ {}
shows *dim U* ≤ *dim V*

corollary *invariance_of_domain_affine_sets*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *ope*: *openin* (*top_of_set U*) *S*
and *aff*: *affine U affine V aff_dim V* ≤ *aff_dim U*
and *contf*: *continuous_on S f* **and** *fim*: *f* ∈ *S* → *V*
and *injf*: *inj_on f S*
shows *openin* (*top_of_set V*) (*f* ‘ *S*)

corollary *invariance_of_dimension_affine_sets*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *ope*: *openin* (*top_of_set U*) *S*
and *aff*: *affine U affine V*
and *contf*: *continuous_on S f* **and** *fim*: *f* ∈ *S* → *V*
and *injf*: *inj_on f S* **and** *S* ≠ {}
shows *aff_dim U* ≤ *aff_dim V*

corollary *invariance_of_dimension*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *contf*: *continuous_on S f* **and** *open S*
and *injf*: *inj_on f S* **and** *S* ≠ {}
shows *DIM*(*'a*) ≤ *DIM*(*'b*)

corollary *continuous_injective_image_subspace_dim_le*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *subspace S subspace T*
and *contf*: *continuous_on S f* **and** *fim*: *f* ∈ *S* → *T*
and *injf*: *inj_on f S*
shows *dim S* ≤ *dim T*

corollary *invariance_of_domain_homeomorphic*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes *open S continuous_on S f DIM('b)* $\leq$ *DIM('a) inj_on f S*
shows *S homeomorphic (f ' S)*

proposition *homeomorphic_interiors*:
fixes $S :: 'a::euclidean_space\ set$ **and** $T :: 'b::euclidean_space\ set$
assumes *S homeomorphic T interior S* $= \{\} \longleftrightarrow$ *interior T* $= \{\}$
shows *(interior S) homeomorphic (interior T)*

proposition *uniformly_continuous_homeomorphism_UNIV_trivial*:
fixes $f :: 'a::euclidean_space \Rightarrow 'a$
assumes *contf*: *uniformly_continuous_on S f* **and** *hom*: *homeomorphism S UNIV f g*
shows *S* $=$ *UNIV*

10.43.6 Formulation of loop homotopy in terms of maps out of type complex

proposition *simply_connected_eq_homotopic_circlemaps*:
fixes $S :: 'a::real_normed_vector\ set$
shows *simply_connected S* $\longleftrightarrow$
$(\forall f\ g::complex \Rightarrow 'a.$
continuous_on (sphere 0 1) f $\wedge$ *f* $\in$ *(sphere 0 1)* $\to$ *S* $\wedge$
continuous_on (sphere 0 1) g $\wedge$ *g* $\in$ *(sphere 0 1)* $\to$ *S*
$\longrightarrow$ *homotopic_with_canon* $(\lambda h.\ True)$ *(sphere 0 1) S f g)*

proposition *simply_connected_eq_contractible_circlemap*:
fixes $S :: 'a::real_normed_vector\ set$
shows *simply_connected S* $\longleftrightarrow$
path_connected S $\wedge$
$(\forall f::complex \Rightarrow 'a.$
continuous_on (sphere 0 1) f $\wedge$ *f '(sphere 0 1)* $\subseteq$ *S*
$\longrightarrow (\exists a.$ *homotopic_with_canon* $(\lambda h.\ True)$ *(sphere 0 1) S f* $(\lambda x.\ a)))$

corollary *homotopy_eqv_simple_connectedness*:
fixes $S :: 'a::real_normed_vector\ set$ **and** $T :: 'b::real_normed_vector\ set$
shows *S homotopy_eqv T* $\Longrightarrow$ *simply_connected S* $\longleftrightarrow$ *simply_connected T*

10.43.7 Homeomorphism of simple closed curves to circles

proposition *homeomorphic_simple_path_image_circle*:
fixes a :: *complex* **and** γ :: *real* $\Rightarrow$ *'a*::*t2_space*
assumes *simple_path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ **and** $0 < r$
shows (*path_image* γ) *homeomorphic sphere a r*

10.43.8 Dimension-based conditions for various homeomorphisms

10.43.9 more invariance of domain

proposition *invariance_of_domain_sphere_affine_set_gen*:
fixes f :: *'a*::*euclidean_space* $\Rightarrow$ *'b*::*euclidean_space*
assumes *contf*: *continuous_on S f* **and** *injf*: *inj_on f S* **and** *fim*: $f \in S \to T$
and *U*: *bounded U convex U*
and *affine T* **and** *affTU*: *aff_dim T* < *aff_dim U*
and *ope*: *openin* (*top_of_set* (*rel_frontier U*)) *S*
shows *openin* (*top_of_set T*) (*f* ' *S*)

proposition *simply_connected_punctured_convex*:
fixes a :: *'a*::*euclidean_space*
assumes *convex S* **and** *3*: $3 \leq$ *aff_dim S*
shows *simply_connected*$(S - \{a\})$

corollary *simply_connected_punctured_universe*:
fixes a :: *'a*::*euclidean_space*
assumes $3 \leq$ *DIM*(*'a*)
shows *simply_connected*$(- \{a\})$

10.43.10 The power, squaring and exponential functions as covering maps

proposition *covering_space_power_punctured_plane*:
assumes $0 < n$
shows *covering_space* $(- \{0\})$ $(\lambda z$::*complex*. $z\hat{}n)$ $(- \{0\})$

corollary *covering_space_square_punctured_plane*:
covering_space $(- \{0\})$ $(\lambda z$::*complex*. $z\hat{}2)$ $(- \{0\})$

proposition *covering_space_exp_punctured_plane*:
covering_space UNIV $(\lambda z$::*complex*. *exp z*$)$ $(- \{0\})$

10.43.11 Hence the Borsukian results about mappings into circles

corollary *inessential_imp_continuous_logarithm_circle*:
fixes $f :: 'a::real_normed_vector \Rightarrow complex$
assumes $homotopic_with_canon\ (\lambda h.\ True)\ S\ (sphere\ 0\ 1)\ f\ (\lambda t.\ a)$
obtains g **where** $continuous_on\ S\ g$ **and** $\bigwedge x.\ x \in S \Longrightarrow f\ x = exp(g\ x)$

proposition *homotopic_with_sphere_times*:
fixes $f :: 'a::real_normed_vector \Rightarrow complex$
assumes *hom*: $homotopic_with_canon\ (\lambda x.\ True)\ S\ (sphere\ 0\ 1)\ f\ g$ **and** *conth*: $continuous_on\ S\ h$
and *hin*: $\bigwedge x.\ x \in S \Longrightarrow h\ x \in sphere\ 0\ 1$
shows $homotopic_with_canon\ (\lambda x.\ True)\ S\ (sphere\ 0\ 1)\ (\lambda x.\ f\ x * h\ x)\ (\lambda x.\ g\ x * h\ x)$

proposition *homotopic_circlemaps_divide*:
fixes $f :: 'a::real_normed_vector \Rightarrow complex$
shows $homotopic_with_canon\ (\lambda x.\ True)\ S\ (sphere\ 0\ 1)\ f\ g \longleftrightarrow$
$continuous_on\ S\ f \wedge f \in S \to sphere\ 0\ 1 \wedge$
$continuous_on\ S\ g \wedge g \in S \to sphere\ 0\ 1 \wedge$
$(\exists c.\ homotopic_with_canon\ (\lambda x.\ True)\ S\ (sphere\ 0\ 1)\ (\lambda x.\ f\ x\ /\ g\ x)\ (\lambda x.\ c))$

10.43.12 Upper and lower hemicontinuous functions

proposition *upper_lower_hemicontinuous_explicit*:
fixes $T :: ('b::\{real_normed_vector, heine_borel\})\ set$
assumes *fST*: $\bigwedge x.\ x \in S \Longrightarrow f\ x \subseteq T$
and *ope*: $\bigwedge U.\ openin\ (top_of_set\ T)\ U \Longrightarrow openin\ (top_of_set\ S)\ \{x \in S.\ f\ x \subseteq U\}$
and *clo*: $\bigwedge U.\ closedin\ (top_of_set\ T)\ U \Longrightarrow closedin\ (top_of_set\ S)\ \{x \in S.\ f\ x \subseteq U\}$
and $x \in S\ 0 < e$ **and** *bofx*: $bounded(f\ x)$ **and** *fx_ne*: $f\ x \neq \{\}$
obtains d **where** $0 < d$
$\bigwedge x'.\ [\![x' \in S;\ dist\ x\ x' < d]\!] \Longrightarrow (\forall y \in f\ x.\ \exists y'.\ y' \in f\ x' \wedge dist\ y\ y' < e) \wedge (\forall y' \in f\ x'.\ \exists y.\ y \in f\ x \wedge dist\ y'\ y < e)$

10.43.13 Complex logs exist on various "well-behaved" sets

10.43.14 Another simple case where sphere maps are null-homotopic

10.43.15 Holomorphic logarithms and square roots

10.43.16 The "Borsukian" property of sets

definition *Borsukian* **where**
  Borsukian S ≡
    ∀*f*. *continuous_on S f* ∧ *f* ∈ *S* → (− {*0*::*complex*})
      ⟶ (∃ *a*. *homotopic_with_canon* (λ*h*. *True*) *S* (− {*0*}) *f* (λ*x*. *a*))

proposition *Borsukian_sphere*:
  fixes *a* :: *'a*::*euclidean_space*
  shows *3* ≤ *DIM*(*'a*) ⟹ *Borsukian* (*sphere a r*)

proposition *Borsukian_open_Un*:
  fixes *S* :: *'a*::*real_normed_vector set*
  assumes *opeS*: *openin* (*top_of_set* (*S* ∪ *T*)) *S*
    and *opeT*: *openin* (*top_of_set* (*S* ∪ *T*)) *T*
    and *BS*: *Borsukian S* **and** *BT*: *Borsukian T* **and** *ST*: *connected*(*S* ∩ *T*)
  shows *Borsukian*(*S* ∪ *T*)

proposition *closed_irreducible_separator*:
  fixes *a* :: *'a*::*real_normed_vector*
  assumes *closed S* **and** *ab*: ¬ *connected_component* (− *S*) *a b*
  obtains *T* **where** *T* ⊆ *S closed T T* ≠ {} ¬ *connected_component* (− *T*) *a b*
        ⋀*U*. *U* ⊂ *T* ⟹ *connected_component* (− *U*) *a b*

10.43.17 Unicoherence (closed)

definition *unicoherent* **where**
  unicoherent U ≡
  ∀*S T*. *connected S* ∧ *connected T* ∧ *S* ∪ *T* = *U* ∧
      closedin (*top_of_set U*) *S* ∧ *closedin* (*top_of_set U*) *T*
      ⟶ *connected* (*S* ∩ *T*)

proposition *homeomorphic_unicoherent*:
  assumes *ST*: *S homeomorphic T* **and** *S*: *unicoherent S*
  shows *unicoherent T*

corollary *contractible_imp_unicoherent*:
fixes U :: *'a::euclidean_space set*
assumes *contractible* U **shows** *unicoherent* U

corollary *convex_imp_unicoherent*:
fixes U :: *'a::euclidean_space set*
assumes *convex* U **shows** *unicoherent* U
corollary *unicoherent_UNIV*: *unicoherent* (*UNIV* :: *'a* :: *euclidean_space set*)

10.43.18 Several common variants of unicoherence

10.43.19 Some separation results

proposition *separation_by_component_open*:
fixes S :: *'a* :: *euclidean_space set*
assumes *open* S **and** *non*: ¬ *connected*(− S)
obtains C **where** $C \in$ *components* S ¬ *connected*(− C)

proposition *inessential_eq_extensible*:
fixes f :: *'a::euclidean_space* ⇒ *complex*
assumes *closed* S
shows (∃ *a*. *homotopic_with_canon* (λ*h*. *True*) S (−{*0*}) f (λ*t*. *a*)) ⟷
(∃ *g*. *continuous_on UNIV g* ∧ (∀ $x \in S$. $g\ x = f\ x$) ∧ (∀ *x*. $g\ x \neq 0$))
(**is** *?lhs* = *?rhs*)

proposition *Janiszewski_dual*:
fixes S :: *complex set*
assumes *compact* S *compact* T *connected* S *connected* T *connected*(− ($S \cup T$))
shows *connected*($S \cap T$)

end

10.44 The Jordan Curve Theorem and Applications

theory *Jordan_Curve*
imports *Arcwise_Connected Further_Topology*
begin

10.44.1 Janiszewski's theorem

theorem *Janiszewski*:
fixes *a b* :: *complex*
assumes *compact S closed T* **and** *conST*: *connected* (*S* ∩ *T*)
and *ccS*: *connected_component* (− *S*) *a b* **and** *ccT*: *connected_component* (− *T*) *a b*
shows *connected_component* (− (*S* ∪ *T*)) *a b*

10.44.2 The Jordan Curve theorem

corollary *Jordan_inside_outside*:
fixes *c* :: *real* ⇒ *complex*
assumes *simple_path c pathfinish c* = *pathstart c*
shows *inside*(*path_image c*) ≠ {} ∧
open(*inside*(*path_image c*)) ∧
connected(*inside*(*path_image c*)) ∧
outside(*path_image c*) ≠ {} ∧
open(*outside*(*path_image c*)) ∧
connected(*outside*(*path_image c*)) ∧
bounded(*inside*(*path_image c*)) ∧
¬ *bounded*(*outside*(*path_image c*)) ∧
inside(*path_image c*) ∩ *outside*(*path_image c*) = {} ∧
inside(*path_image c*) ∪ *outside*(*path_image c*) =
− *path_image c* ∧
frontier(*inside*(*path_image c*)) = *path_image c* ∧
frontier(*outside*(*path_image c*)) = *path_image c*
theorem *split_inside_simple_closed_curve*:
fixes *c* :: *real* ⇒ *complex*
assumes *simple_path c1* **and** *c1*: *pathstart c1* = *a pathfinish c1* = *b*
and *simple_path c2* **and** *c2*: *pathstart c2* = *a pathfinish c2* = *b*
and *simple_path c* **and** *c*: *pathstart c* = *a pathfinish c* = *b*
and *a* ≠ *b*
and *c1c2*: *path_image c1* ∩ *path_image c2* = {*a*,*b*}
and *c1c*: *path_image c1* ∩ *path_image c* = {*a*,*b*}
and *c2c*: *path_image c2* ∩ *path_image c* = {*a*,*b*}
and *ne_12*: *path_image c* ∩ *inside*(*path_image c1* ∪ *path_image c2*) ≠ {}
obtains *inside*(*path_image c1* ∪ *path_image c*) ∩ *inside*(*path_image c2* ∪ *path_image c*) = {}
inside(*path_image c1* ∪ *path_image c*) ∪ *inside*(*path_image c2* ∪ *path_image c*) ∪
(*path_image c* − {*a*,*b*}) = *inside*(*path_image c1* ∪ *path_image c2*)

end

10.45 Polynomial Functions: Extremal Behaviour and Root Counts

theory *Poly_Roots*
imports *Complex_Main*
begin

10.45.1 Basics about polynomial functions: extremal behaviour and root counts

proposition *polyfun_extremal_lemma*:
fixes $c :: nat \Rightarrow 'a$::*real_normed_div_algebra*
assumes $e > 0$
shows $\exists M.\ \forall z.\ M \leq norm\ z \longrightarrow norm(\sum i{\leq}n.\ c\ i * z\hat{}i) \leq e * norm(z)\ \hat{}$ *Suc n*

proposition *polyfun_extremal*:
fixes $c :: nat \Rightarrow 'a$::*real_normed_div_algebra*
assumes $\exists k.\ k \neq 0 \wedge k \leq n \wedge c\ k \neq 0$
shows *eventually* $(\lambda z.\ norm(\sum i{\leq}n.\ c\ i * z\hat{}i) \geq B)$ *at_infinity*

proposition *polyfun_rootbound*:
fixes $c :: nat \Rightarrow 'a$::{*comm_ring*,*real_normed_div_algebra*}
assumes $\exists k.\ k \leq n \wedge c\ k \neq 0$
shows *finite* $\{z.\ (\sum i{\leq}n.\ c\ i * z\hat{}i) = 0\} \wedge$ *card* $\{z.\ (\sum i{\leq}n.\ c\ i * z\hat{}i) = 0\} \leq n$

corollary
fixes $c :: nat \Rightarrow 'a$::{*comm_ring*,*real_normed_div_algebra*}
assumes $\exists k.\ k \leq n \wedge c\ k \neq 0$
shows *polyfun_rootbound_finite*: *finite* $\{z.\ (\sum i{\leq}n.\ c\ i * z\hat{}i) = 0\}$
and *polyfun_rootbound_card*: *card* $\{z.\ (\sum i{\leq}n.\ c\ i * z\hat{}i) = 0\} \leq n$

proposition *polyfun_finite_roots*:
fixes $c :: nat \Rightarrow 'a$::{*comm_ring*,*real_normed_div_algebra*}
shows *finite* $\{z.\ (\sum i{\leq}n.\ c\ i * z\hat{}i) = 0\} \longleftrightarrow (\exists k.\ k \leq n \wedge c\ k \neq 0)$

theorem *polyfun_eq_const*:
fixes $c :: nat \Rightarrow 'a$::{*comm_ring*,*real_normed_div_algebra*}
shows $(\forall z.\ (\sum i{\leq}n.\ c\ i * z\hat{}i) = k) \longleftrightarrow c\ 0 = k \wedge (\forall k.\ k \neq 0 \wedge k \leq n \longrightarrow c\ k = 0)$

end

10.46 Generalised Binomial Theorem

theory *Generalised_Binomial_Theorem*

imports
Complex_Main
Complex_Transcendental
Summation_Tests
begin

theorem *gen_binomial_complex*:
fixes z :: *complex*
assumes *norm* $z < 1$
shows $(\lambda n.\ (a\ gchoose\ n) * z\hat{}n)\ sums\ (1 + z)\ powr\ a$

end

10.47 Vitali Covering Theorem and an Application to Negligibility

theory *Vitali_Covering_Theorem*
imports
HOL−Combinatorics.Permutations
Equivalence_Lebesgue_Henstock_Integration
begin

10.47.1 Vitali covering theorem

theorem *Vitali_covering_theorem_cballs*:
fixes $a :: 'a \Rightarrow 'n::euclidean_space$
assumes r: $\bigwedge i.\ i \in K \Longrightarrow 0 < r\ i$
and S: $\bigwedge x\ d.\ [\![x \in S;\ 0 < d]\!] \Longrightarrow \exists i.\ i \in K \wedge x \in cball\ (a\ i)\ (r\ i) \wedge r\ i < d$
obtains C **where** *countable* C $C \subseteq K$
pairwise $(\lambda i\ j.\ disjnt\ (cball\ (a\ i)\ (r\ i))\ (cball\ (a\ j)\ (r\ j)))\ C$
negligible$(S - (\bigcup i \in C.\ cball\ (a\ i)\ (r\ i)))$

theorem *Vitali_covering_theorem_balls*:
fixes $a :: 'a \Rightarrow 'b::euclidean_space$
assumes S: $\bigwedge x\ d.\ [\![x \in S;\ 0 < d]\!] \Longrightarrow \exists i.\ i \in K \wedge x \in ball\ (a\ i)\ (r\ i) \wedge r\ i < d$
obtains C **where** *countable* C $C \subseteq K$
pairwise $(\lambda i\ j.\ disjnt\ (ball\ (a\ i)\ (r\ i))\ (ball\ (a\ j)\ (r\ j)))\ C$
negligible$(S - (\bigcup i \in C.\ ball\ (a\ i)\ (r\ i)))$

proposition *negligible_eq_zero_density*:
negligible $S \longleftrightarrow$

$(\forall x{\in}S.\ \forall r{>}0.\ \forall e{>}0.\ \exists d.\ 0 < d \wedge d \leq r \wedge$
$(\exists U.\ S \cap ball\ x\ d \subseteq U \wedge U \in lmeasurable \wedge measure\ lebesgue\ U$
$< e * measure\ lebesgue\ (ball\ x\ d)))$

end
theory *Determinant_Linear_Function*

imports *Determinants*

begin

10.47.2 Determinants of linear functions between Euclidean spaces

theorem *det_compose*:
assumes *f*: *Vector_Spaces.linear scale scale f* **and** *g*: *Vector_Spaces.linear scale scale g*
shows *local.det* $(f \circ g)$ = *local.det f* $*$ *local.det g*

corollary *det_inv*:
assumes *Vector_Spaces.linear scale scale f inj f*
shows *local.det* $(inv\ f)$ = *inverse* $(local.det\ f)$
theorem *det_eq_0_iff*:
assumes *Vector_Spaces.linear scale scale f*
shows *local.det f* = 0 $\longleftrightarrow$ $\neg$*inj f*
theorem (**in** *finite_dimensional_vector_space_pair*) *det_basis_change*:
assumes *Vector_Spaces.linear* $(*a)$ $(*b)$ *h bij_betw h B1 B2*
shows *vs1.det f* = *vs2.det* $(h \circ f \circ inv\ h)$
corollary (**in** *finite_dimensional_vector_space*) *det_eq*:
assumes *f*: *Vector_Spaces.linear scale scale f* **and** *B*: *independent B span B* = *UNIV*
shows *local.det f* =
$(\sum p \mid p\ permutes\ B.\ of_int\ (sign\ p) * (\prod b{\in}B.\ representation\ B\ (f\ (p\ b))\ b))$

10.47.3 Absolute determinant of a multiset

end

10.48 Transfer from $\mathbb{R}^n$ to euclidean spaces

theory *Euclidean_Space_Transfer*
imports *Determinant_Linear_Function Lebesgue_Measure Path_Connected*

begin

10.48.1 General missing parametricity rules

10.48.2 Parametricity of measures

10.48.3 A type for the basis of a Euclidean space

instantiation *basis* :: (*euclidean_space*) *linorder*
begin

10.48.4 Transfer for Euclidean spaces

end

10.49 Change of Variables Theorems

theory *Change_Of_Vars*
imports *Vitali_Covering_Theorem Determinants*
Determinant_Linear_Function Euclidean_Space_Transfer

begin

10.49.1 Measurable Shear and Stretch

proposition
fixes *a* :: *real^'n*
assumes *m* ≠ *n* **and** *ab_ne*: *cbox a b* ≠ {} **and** *an*: *0* ≤ *a$n*
shows *measurable_shear_interval*: (λ*x*. χ *i*. *if i* = *m then x$m* + *x$n else x$i*) ‘ (*cbox a b*) ∈ *lmeasurable*
(**is** *?f* ‘ _ ∈ _)
and *measure_shear_interval*:
measure lebesgue ((λ*x*. χ *i*. *if i* = *m then x$m* + *x$n else x$i*) ‘ *cbox a b*)
= *measure lebesgue* (*cbox a b*) (**is** *?Q*)

proposition
fixes *S* :: (*real^'n*) *set*
assumes *S* ∈ *lmeasurable*
shows *measurable_stretch*: ((λ*x*. χ *k*. *m k* ∗ *x$k*) ‘ *S*) ∈ *lmeasurable* (**is** *?f* ‘ *S* ∈ _)
and *measure_stretch*: *measure lebesgue* ((λ*x*. χ *k*. *m k* ∗ *x$k*) ‘ *S*) = |*prod m UNIV*| ∗ *measure lebesgue S*
(**is** *?MEQ*)

proposition
fixes f :: *real^'n::{finite,wellorder}* ⇒ *real^'n::_*
assumes *linear f S* ∈ *lmeasurable*
shows *measurable_linear_image_cart*: (*f* ‘ *S*) ∈ *lmeasurable*
and *measure_linear_image_cart*: *measure lebesgue* (*f* ‘ *S*) = |*matrix_det* (*matrix f*)| ∗ *measure lebesgue S* (**is** *?Q f S*)

proposition
fixes f :: *'a :: euclidean_space* ⇒ *'a*
assumes *linear f S* ∈ *lmeasurable*
shows *measurable_linear_image*: (*f* ‘ *S*) ∈ *lmeasurable*
and *measure_linear_image*: *measure lebesgue* (*f* ‘ *S*) = |*eucl.det f*| ∗ *measure lebesgue S*

proposition *measure_semicontinuous_with_hausdist_explicit*:
assumes *bounded S* **and** *neg*: *negligible*(*frontier S*) **and** *e* > *0*
obtains *d* **where** *d* > *0*
⋀*T*. ⟦*T* ∈ *lmeasurable*; ⋀*y*. *y* ∈ *T* ⟹ ∃*x*. *x* ∈ *S* ∧ *dist x y* < *d*⟧
⟹ *measure lebesgue T* < *measure lebesgue S* + *e*

proposition
fixes f :: *'a::euclidean_space* ⇒ *'a*
assumes *S*: *S* ∈ *lmeasurable*
and *deriv*: ⋀*x*. *x* ∈ *S* ⟹ (*f has_derivative f' x*) (*at x within S*)
and *int*: (λ*x*. |*eucl.det* (*f' x*)|) *integrable_on S*
and *bounded*: ⋀*x*. *x* ∈ *S* ⟹ |*eucl.det* (*f' x*)| ≤ *B*
shows *measurable_bounded_differentiable_image*:
f ‘ *S* ∈ *lmeasurable*
and *measure_bounded_differentiable_image*:
measure lebesgue (*f* ‘ *S*) ≤ *B* ∗ *measure lebesgue S* (**is** *?M*)

theorem
fixes f :: *'a::euclidean_space* ⇒ *'a*
assumes *S*: *S* ∈ *sets lebesgue*
and *deriv*: ⋀*x*. *x* ∈ *S* ⟹ (*f has_derivative f' x*) (*at x within S*)
and *int*: (λ*x*. |*eucl.det* (*f' x*)|) *integrable_on S*
shows *measurable_differentiable_image*: *f* ‘ *S* ∈ *lmeasurable*
and *measure_differentiable_image*:
measure lebesgue (*f* ‘ *S*) ≤ *integral S* (λ*x*. |*eucl.det* (*f' x*)|) (**is** *?M*)

10.49.2 Borel measurable Jacobian determinant

proposition *linear_rational_approximation*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*

assumes *linear f e* > *0*
obtains *g* **where** *linear g* $\bigwedge i\ j.\ i \in Basis \Longrightarrow j \in Basis \Longrightarrow g\ i \cdot j \in \mathbb{Q}$
onorm $(f - g) < e$

proposition *orthogonal_transformation_exists*:
fixes *a b* :: *'a*::*euclidean_space*
assumes *norm a* = *norm b*
obtains *T* **where** *orthogonal_transformation T T a* = *b*

proposition *borel_measurable_partial_derivatives*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *S*: *S* ∈ *sets lebesgue*
and *f*: $\bigwedge x.\ x \in S \Longrightarrow$ (*f has_derivative f' x*) (*at x within S*)
and *uB*: *u* ∈ *Basis* **and** *vB*: *v* ∈ *Basis*
shows $(\lambda x.\ f'\ x\ u \cdot v) \in$ *borel_measurable* (*lebesgue_on S*)

theorem *borel_measurable_det_Jacobian*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'a*
assumes *S*: *S* ∈ *sets lebesgue* **and** *f*: $\bigwedge x.\ x \in S \Longrightarrow$ (*f has_derivative f' x*) (*at x within S*)
shows $(\lambda x.\ eucl.det\ (f'\ x)) \in$ *borel_measurable* (*lebesgue_on S*)
theorem *borel_measurable_lebesgue_on_preimage_borel*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *S* ∈ *sets lebesgue*
shows *f* ∈ *borel_measurable* (*lebesgue_on S*) ⟷
$(\forall T.\ T \in sets\ borel \longrightarrow \{x \in S.\ f\ x \in T\} \in sets\ lebesgue)$

10.49.3 Simplest case of Sard's theorem (we don't need continuity of derivative)

theorem *baby_Sard*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *mlen*: *DIM*(*'a*) ≤ *DIM*(*'b*)
and *der*: $\bigwedge x.\ x \in S \Longrightarrow$ (*f has_derivative f' x*) (*at x within S*)
and *rank*: $\bigwedge x.\ x \in S \Longrightarrow dim\ (f'\ x\ `\ UNIV) < DIM('b)$
shows *negligible*(*f* ` *S*)

10.49.4 A one-way version of change-of-variables not assuming injectivity.

proposition *absolutely_integrable_on_image*:

fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$ **and** $g :: 'a \Rightarrow 'a$
assumes *der_g*: $\bigwedge x.\ x \in S \Longrightarrow (g\ has_derivative\ g'\ x)\ (at\ x\ within\ S)$
and *intS*: $(\lambda x.\ |eucl.det\ (g'\ x)| *_R f(g\ x))\ absolutely_integrable_on\ S$
shows $f\ absolutely_integrable_on\ (g\ `\ S)$

proposition *integral_on_image_ubound*:
fixes $f :: 'a::euclidean_space \Rightarrow real$ **and** $g :: 'a \Rightarrow 'a$
assumes $\bigwedge x.\ x \in S \Longrightarrow 0 \le f(g\ x)$
and $\bigwedge x.\ x \in S \Longrightarrow (g\ has_derivative\ g'\ x)\ (at\ x\ within\ S)$
and $(\lambda x.\ |eucl.det\ (g'\ x)| * f(g\ x))\ integrable_on\ S$
shows $integral\ (g\ `\ S)\ f \le integral\ S\ (\lambda x.\ |eucl.det\ (g'\ x)| * f(g\ x))$

10.49.5 Change-of-variables theorem

theorem *has_absolute_integral_change_of_variables_invertible*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$ **and** $g :: 'a \Rightarrow 'a$
assumes *der_g*: $\bigwedge x.\ x \in S \Longrightarrow (g\ has_derivative\ g'\ x)\ (at\ x\ within\ S)$
and *hg*: $\bigwedge x.\ x \in S \Longrightarrow h(g\ x) = x$
and *conth*: $continuous_on\ (g\ `\ S)\ h$
shows $(\lambda x.\ |eucl.det(g'\ x)| *_R f(g\ x))\ absolutely_integrable_on\ S \wedge integral\ S$
$(\lambda x.\ |eucl.det(g'\ x)| *_R f(g\ x)) = b \longleftrightarrow$
$f\ absolutely_integrable_on\ (g\ `\ S) \wedge integral\ (g\ `\ S)\ f = b$
(**is** *?lhs = ?rhs*)

theorem *has_absolute_integral_change_of_variables_compact*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$ **and** $g :: 'a \Rightarrow 'a$
assumes $compact\ S$
and *der_g*: $\bigwedge x.\ x \in S \Longrightarrow (g\ has_derivative\ g'\ x)\ (at\ x\ within\ S)$
and *inj*: $inj_on\ g\ S$
shows $((\lambda x.\ |eucl.det(g'\ x)| *_R f(g\ x))\ absolutely_integrable_on\ S \wedge$
$integral\ S\ (\lambda x.\ |eucl.det(g'\ x)| *_R f(g\ x)) = b$
$\longleftrightarrow f\ absolutely_integrable_on\ (g\ `\ S) \wedge integral\ (g\ `\ S)\ f = b)$

theorem *has_absolute_integral_change_of_variables*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$ **and** $g :: 'a \Rightarrow 'a$
assumes *S*: $S \in sets\ lebesgue$
and *der_g*: $\bigwedge x.\ x \in S \Longrightarrow (g\ has_derivative\ g'\ x)\ (at\ x\ within\ S)$
and *inj*: $inj_on\ g\ S$
shows $(\lambda x.\ |eucl.det(g'\ x)| *_R f(g\ x))\ absolutely_integrable_on\ S \wedge$
$integral\ S\ (\lambda x.\ |eucl.det(g'\ x)| *_R f(g\ x)) = b$
$\longleftrightarrow f\ absolutely_integrable_on\ (g\ `\ S) \wedge integral\ (g\ `\ S)\ f = b$

corollary *absolutely_integrable_change_of_variables*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$ **and** $g :: 'a \Rightarrow 'a$
assumes $S \in sets\ lebesgue$
and $\bigwedge x.\ x \in S \Longrightarrow (g\ has_derivative\ g'\ x)\ (at\ x\ within\ S)$
and $inj_on\ g\ S$
shows $f\ absolutely_integrable_on\ (g\ `\ S)$
$\longleftrightarrow (\lambda x.\ |eucl.det(g'\ x)| *_R f(g\ x))\ absolutely_integrable_on\ S$

corollary *integral_change_of_variables*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$ **and** $g :: 'a \Rightarrow 'a$
assumes S: $S \in sets\ lebesgue$
and *der_g*: $\bigwedge x.\ x \in S \Longrightarrow (g\ has_derivative\ g'\ x)\ (at\ x\ within\ S)$
and *inj*: $inj_on\ g\ S$
and *disj*: $(f\ absolutely_integrable_on\ (g\ `\ S) \vee$
$(\lambda x.\ |eucl.det(g'\ x)| *_R f(g\ x))\ absolutely_integrable_on\ S)$
shows $integral\ (g\ `\ S)\ f = integral\ S\ (\lambda x.\ |eucl.det(g'\ x)| *_R f(g\ x))$

corollary *absolutely_integrable_change_of_variables_1*:
fixes $f :: real \Rightarrow 'b::euclidean_space$ **and** $g :: real \Rightarrow real$
assumes S: $S \in sets\ lebesgue$
and *der_g*: $\bigwedge x.\ x \in S \Longrightarrow (g\ has_vector_derivative\ g'\ x)\ (at\ x\ within\ S)$
and *inj*: $inj_on\ g\ S$
shows $(f\ absolutely_integrable_on\ g\ `\ S \longleftrightarrow$
$(\lambda x.\ |g'\ x| *_R f(g\ x))\ absolutely_integrable_on\ S)$

10.49.6 Change of variables for integrals: special case of linear function

10.49.7 Change of variable for measure

proposition *measure_lebesgue_linear_transformation*:
fixes $A :: 'a::euclidean_space\ set$
fixes $f :: _ \Rightarrow 'a$
assumes $bounded\ A\ A \in sets\ lebesgue\ linear\ f$
shows $measure\ lebesgue\ (f\ `\ A) = |det\ f| * measure\ lebesgue\ A$

end
theory
Multivariate_Analysis
imports
Ordered_Euclidean_Space
Determinants
Cross3
Lipschitz
Starlike
beginend

10.50 Volume of a Simplex

theory *Simplex_Content*
imports *Change_Of_Vars*
begin

theorem *content_std_simplex*:
measure lborel (convex hull (insert 0 Basis :: 'a :: euclidean_space set)) =
1 / fact DIM('a)

theorem *content_simplex*:
fixes *X :: (real ^ 'n :: {finite, wellorder}) set* **and** *f :: 'n :: _ ⇒ real ^ ('n :: _)*
assumes *finite X card X = Suc CARD('n)* **and** *x0*: *x0 ∈ X* **and** *bij*: *bij_betw f UNIV (X − {x0})*
defines *M ≡ (χ i. χ j. f j $ i − x0 $ i)*
shows *content (convex hull X) = |matrix_det M| / fact (CARD('n))*

theorem *content_triangle*:
fixes *A B C :: real ^ 2*
shows *content (convex hull {A, B, C}) =*
|(C $ 1 − A $ 1) ∗ (B $ 2 − A $ 2) − (B $ 1 − A $ 1) ∗ (C $ 2 − A $ 2)| / 2

theorem *heron*:
fixes *A B C :: real ^ 2*
defines *a ≡ dist B C* **and** *b ≡ dist A C* **and** *c ≡ dist A B*
defines *s ≡ (a + b + c) / 2*
shows *content (convex hull {A, B, C}) = sqrt (s ∗ (s − a) ∗ (s − b) ∗ (s − c))*

end

10.51 Convergence of Formal Power Series

theory *FPS_Convergence*
imports
Generalised_Binomial_Theorem
HOL−Computational_Algebra.Formal_Power_Series
HOL−Computational_Algebra.Polynomial_FPS

begin

10.51.1 Basic properties of convergent power series

definition *fps_conv_radius :: 'a :: {banach, real_normed_div_algebra} fps ⇒ ereal* **where**
fps_conv_radius f = conv_radius (fps_nth f)

definition *eval_fps :: 'a :: {banach, real_normed_div_algebra} fps ⇒ 'a ⇒ 'a*

where
eval_fps f z = (∑ *n*. *fps_nth f n* ∗ *z* ^ *n*)

theorem *sums_eval_fps*:
fixes *f* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*
assumes *norm z* < *fps_conv_radius f*
shows (λ*n*. *fps_nth f n* ∗ *z* ^ *n*) *sums eval_fps f z*

10.51.2 Evaluating power series

theorem *eval_fps_deriv*:
assumes *norm z* < *fps_conv_radius f*
shows *eval_fps* (*fps_deriv f*) *z* = *deriv* (*eval_fps f*) *z*

theorem *fps_nth_conv_deriv*:
fixes *f* :: *complex fps*
assumes *fps_conv_radius f* > *0*
shows *fps_nth f n* = (*deriv* ^^ *n*) (*eval_fps f*) *0* / *fact n*

theorem *eval_fps_eqD*:
fixes *f g* :: *complex fps*
assumes *fps_conv_radius f* > *0 fps_conv_radius g* > *0*
assumes *eventually* (λ*z*. *eval_fps f z* = *eval_fps g z*) (*nhds 0*)
shows *f* = *g*

10.51.3 FPS of a polynomial

10.51.4 Power series expansions of analytic functions

definition
has_fps_expansion :: (*'a* :: {*banach*,*real_normed_div_algebra*} ⇒ *'a*) ⇒ *'a fps*
⇒ *bool*
(**infixl** ‹*has'_fps'_expansion*› *60*)
where (*f has_fps_expansion F*) ⟷
fps_conv_radius F > *0* ∧ *eventually* (λ*z*. *eval_fps F z* = *f z*) (*nhds 0*)

end
theory *Smooth_Paths*
imports *Retracts*
begin

10.51.5 Piecewise differentiability of paths

10.51.6 Valid paths, and their start and finish

definition *valid_path* :: (*real* ⇒ *'a* :: *real_normed_vector*) ⇒ *bool*

where *valid_path f* ≡ *f piecewise_C1_differentiable_on* {*0..1*::*real*}

end
theory *Analysis*
imports

Convex
Determinants

FSigma
Sum_Topology
Abstract_Topological_Spaces
Abstract_Metric_Spaces
Urysohn
Connected
Abstract_Limits
Isolated
Sparse_In

Elementary_Normed_Spaces
Norm_Arith

Convex_Euclidean_Space
Operator_Norm

Line_Segment
Derivative
Cartesian_Euclidean_Space
Kronecker_Approximation_Theorem
Weierstrass_Theorems
Ck_Implicit_Function
Analytic_Inverse_Function

Absolute_Continuity
Rectifiable_Path
Lebesgue_Differentiation
Ball_Volume
Integral_Test
Improper_Integral
Equivalence_Measurable_On_Borel
Lebesgue_Integral_Substitution
Embed_Measure
Complete_Measure
Radon_Nikodym
Fashoda_Theorem
Cross3
Homeomorphism
Bounded_Continuous_Function
Abstract_Topology

Product_Topology
Lindelof_Spaces
Infinite_Product
Infinite_Sum
Infinite_Set_Sum
Polytope
Jordan_Curve
Poly_Roots
Generalised_Binomial_Theorem
Gamma_Function
Change_Of_Vars
Multivariate_Analysis
Simplex_Content
FPS_Convergence
Smooth_Paths
Abstract_Euclidean_Space
Function_Metric
begin

end

10.52 Poly Mappings as a Real Normed Vector

theory *Finite_Function_Topology*
imports *Function_Topology HOL−Library.Poly_Mapping*

begin

instantiation *poly_mapping* :: (*type*, *real_vector*) *real_vector*
begin

instantiation *poly_mapping* :: (*type*, *real_normed_vector*) *metric_space*
begin

instantiation *poly_mapping* :: (*type*, *real_normed_vector*) *real_normed_vector*
begin

end

Bibliography

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