Complex Analysis

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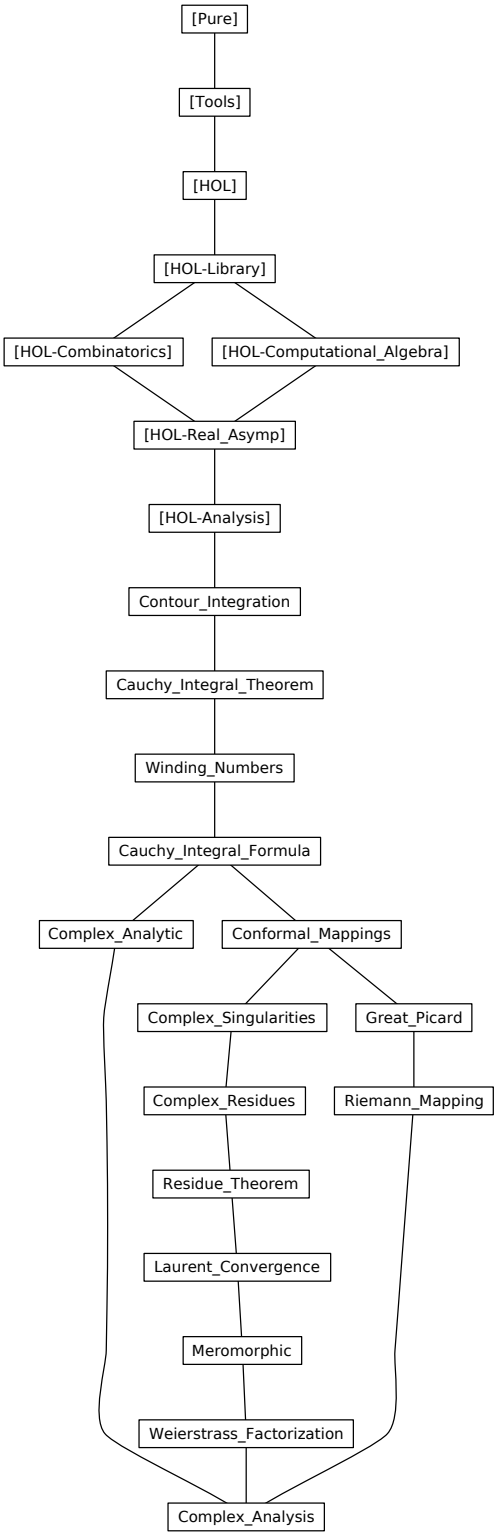
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1 Contour integration

theory *Contour_Integration*
imports *HOL−Analysis.Analysis*
begin

1.1 Definition

definition *has_contour_integral* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ (*real* ⇒ *complex*) ⇒ *bool*
(**infixr** ‹*has'_contour'_integral*› 50)
where (*f has_contour_integral i*) *g* ≡
((λ*x*. *f*(*g x*) ∗ *vector_derivative g* (*at x within* {*0..1*}))
has_integral i) {*0..1*}

definition *contour_integrable_on*
(**infixr** ‹*contour'_integrable'_on*› 50)
where *f contour_integrable_on g* ≡ ∃ *i*. (*f has_contour_integral i*) *g*

definition *contour_integral*
where *contour_integral g f* ≡ *SOME i*. (*f has_contour_integral i*) *g* ∨ ¬ *f contour_integrable_on g* ∧ *i=0*

1.2 Relation to subpath construction

1.3 Cauchy's theorem where there's a primitive

corollary *Cauchy_theorem_primitive*:
assumes ⋀*x*. *x* ∈ *S* ⟹ (*f has_field_derivative f' x*) (*at x within S*)
and *valid_path g path_image g* ⊆ *S pathfinish g* = *pathstart g*
shows (*f' has_contour_integral 0*) *g*

1.4 Reversing the order in a double path integral

proposition *contour_integral_swap*:
assumes *fcon*: *continuous_on* (*path_image g* × *path_image h*) (λ(*y1*,*y2*). *f y1 y2*)
and *vp*: *valid_path g valid_path h*
and *gvcon*: *continuous_on* {*0..1*} (λ*t*. *vector_derivative g* (*at t*))
and *hvcon*: *continuous_on* {*0..1*} (λ*t*. *vector_derivative h* (*at t*))

shows *contour_integral g* (λ*w*. *contour_integral h* (*f w*)) =
contour_integral h (λ*z*. *contour_integral g* (λ*w*. *f w z*))

1.5 Partial circle path

definition *part_circlepath* :: [*complex*, *real*, *real*, *real*, *real*] ⇒ *complex*
where *part_circlepath z r s t* ≡ λ*x*. *z* + *of_real r* ∗ *exp* (i ∗ *of_real* (*linepath s t x*))

proposition *path_image_part_circlepath*:
assumes *s* ≤ *t*
shows *path_image* (*part_circlepath z r s t*) = {*z* + *r* ∗ *exp*(i ∗ *of_real x*) | *x*. *s* ≤ *x* ∧ *x* ≤ *t*}

corollary *contour_integral_bound_part_circlepath_strong*:
assumes *f contour_integrable_on part_circlepath z r s t*
and *finite k* **and** *0* ≤ *B 0* < *r s* ≤ *t*
and ⋀*x*. *x* ∈ *path_image*(*part_circlepath z r s t*) − *k* ⟹ *norm*(*f x*) ≤ *B*
shows *cmod* (*contour_integral* (*part_circlepath z r s t*) *f*) ≤ *B* ∗ *r* ∗ (*t* − *s*)

1.6 Special case of one complete circle

definition *circlepath* :: [*complex*, *real*, *real*] ⇒ *complex*
where *circlepath z r* ≡ *part_circlepath z r 0* (*2*∗*pi*)

1.7 Uniform convergence of path integral

proposition *contour_integral_uniform_limit*:
assumes *ev_fint*: *eventually* (λ*n*::′*a*. (*f n*) *contour_integrable_on* γ) *F*
and *ul_f*: *uniform_limit* (*path_image* γ) *f l F*
and *noleB*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *norm* (*vector_derivative* γ (*at t*)) ≤ *B*
and γ: *valid_path* γ
and [*simp*]: ¬ *trivial_limit F*
shows *l contour_integrable_on* γ ((λ*n*. *contour_integral* γ (*f n*)) ⟶ *contour_integral* γ *l*) *F*

end

2 Complex Path Integrals and Cauchy's Integral Theorem

theory *Cauchy_Integral_Theorem*
imports
HOL−Analysis.Analysis

Contour_Integration
begin

proposition *Cauchy_theorem_triangle_interior*:
assumes *contf*: *continuous_on* (*convex hull* {*a*,*b*,*c*}) *f*
and *holf*: *f holomorphic_on interior* (*convex hull* {*a*,*b*,*c*})
shows (*f has_contour_integral 0*) (*linepath a b +++ linepath b c +++ linepath c a*)

2.1 Cauchy's theorem for a convex set

corollary *Cauchy_theorem_convex_simple*:
assumes *holf*: *f holomorphic_on S*
and *convex S valid_path g path_image g* ⊆ *S pathfinish g* = *pathstart g*
shows (*f has_contour_integral 0*) *g*

2.2 Homotopy forms of Cauchy's theorem

proposition *Cauchy_theorem_homotopic_paths*:
assumes *hom*: *homotopic_paths S g h*
and *open S* **and** *f*: *f holomorphic_on S*
and *vpg*: *valid_path g* **and** *vph*: *valid_path h*
shows *contour_integral g f* = *contour_integral h f*

proposition *Cauchy_theorem_homotopic_loops*:
assumes *hom*: *homotopic_loops S g h*
and *open S* **and** *f*: *f holomorphic_on S*
and *vpg*: *valid_path g* **and** *vph*: *valid_path h*
shows *contour_integral g f* = *contour_integral h f*

end

3 Winding numbers

theory *Winding_Numbers*
imports *Cauchy_Integral_Theorem*
begin

3.1 Definition

definition *winding_number_prop* :: [*real* ⇒ *complex*, *complex*, *real*, *real* ⇒ *complex*, *complex*] ⇒ *bool* **where**
winding_number_prop γ *z e p n* ≡
valid_path p ∧ *z* ∉ *path_image p* ∧
pathstart p = *pathstart* γ ∧
pathfinish p = *pathfinish* γ ∧
(∀ *t* ∈ {*0..1*}. *norm*(γ *t* − *p t*) < *e*) ∧
contour_integral p (λ*w*. *1*/(*w* − *z*)) = *2* ∗ *pi* ∗ i ∗ *n*

definition *winding_number*:: [*real* ⇒ *complex*, *complex*] ⇒ *complex* **where**
winding_number γ *z* ≡ *SOME n*. ∀ *e* > *0*. ∃ *p*. *winding_number_prop* γ *z e p n*

proposition *winding_number_valid_path*:
assumes *valid_path* γ *z* ∉ *path_image* γ
shows *winding_number* γ *z* = *1*/(*2*∗*pi*∗i) ∗ *contour_integral* γ (λ*w*. *1*/(*w* − *z*))

proposition *has_contour_integral_winding_number*:
assumes γ: *valid_path* γ *z* ∉ *path_image* γ
shows ((λ*w*. *1*/(*w* − *z*)) *has_contour_integral* (*2*∗*pi*∗i∗*winding_number* γ *z*)) γ

3.2 The winding number is an integer

theorem *integer_winding_number*:
⟦*path* γ; *pathfinish* γ = *pathstart* γ; *z* ∉ *path_image* γ⟧ ⟹ *winding_number* γ *z* ∈ ℤ

3.3 Continuity of winding number and invariance on connected sets

theorem *continuous_at_winding_number*:
fixes *z*::*complex*
assumes γ: *path* γ **and** *z*: *z* ∉ *path_image* γ
shows *continuous* (*at z*) (*winding_number* γ)

corollary *continuous_on_winding_number*:
path γ ⟹ *continuous_on* (− *path_image* γ) (λ*w*. *winding_number* γ *w*)

3.4 Winding number is zero "outside" a curve

proposition *winding_number_zero_in_outside*:

assumes γ: *path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ **and** z: $z \in$ *outside* (*path_image* γ)
shows *winding_number* γ z = *0*

proposition *winding_number_part_circlepath_pos_less*:
assumes $s < t$ **and** *no*: *norm*$(w - z) < r$
shows *0* < *Re* (*winding_number*(*part_circlepath z r s t*) *w*)

proposition *winding_number_circlepath*:
assumes *norm*$(w - z) < r$ **shows** *winding_number*(*circlepath z r*) *w* = *1*

3.5 Winding number for a triangle

proposition *winding_number_triangle*:
assumes z: $z \in$ *interior*(*convex hull* {*a*,*b*,*c*})
shows *winding_number*(*linepath a b +++ linepath b c +++ linepath c a*) z =
(*if 0* < *Im*((*b* − *a*) ∗ *cnj* (*b* − *z*)) *then 1 else* −*1*)

3.6 Winding numbers for simple closed paths

proposition *simple_closed_path_winding_number_inside*:
assumes *simple_path* γ
obtains $\bigwedge z.\ z \in$ *inside*(*path_image* γ) $\Longrightarrow$ *winding_number* γ z = *1*
| $\bigwedge z.\ z \in$ *inside*(*path_image* γ) $\Longrightarrow$ *winding_number* γ z = −*1*

3.7 Winding number for rectangular paths

proposition *winding_number_rectpath*:
assumes $z \in$ *box a1 a3*
shows *winding_number* (*rectpath a1 a3*) z = *1*

proposition *winding_number_rectpath_outside*:
assumes *Re a1* ≤ *Re a3 Im a1* ≤ *Im a3*
assumes $z \notin$ *cbox a1 a3*
shows *winding_number* (*rectpath a1 a3*) z = *0*

end

4 Cauchy's Integral Formula

theory *Cauchy_Integral_Formula*
imports *Winding_Numbers*
begin

4.1 Proof

theorem *Cauchy_integral_formula_convex_simple*:
assumes *convex S* **and** *holf*: *f holomorphic_on S* **and** $z \in$ *interior S valid_path* γ *path_image* $\gamma \subseteq S - \{z\}$
pathfinish γ = *pathstart* γ
shows $((\lambda w.\ f\ w\ /\ (w-z))$ *has_contour_integral* $(2*pi * \mathrm{i} *$ *winding_number* $\gamma\ z * f\ z))\ \gamma$
theorem *Cauchy_integral_circlepath*:
assumes *contf*: *continuous_on* (*cball z r*) *f* **and** *holf*: *f holomorphic_on* (*ball z r*) **and** *wz*: $norm(w-z) < r$
shows $((\lambda u.\ f\ u/(u-w))$ *has_contour_integral* $(2 *$ *of_real pi* $* \mathrm{i} * f\ w))$
(*circlepath z r*)

4.2 Existence of all higher derivatives

proposition *derivative_is_holomorphic*:
assumes *open S*
and *fder*: $\bigwedge z.\ z \in S \Longrightarrow (f$ *has_field_derivative* $f'\ z)$ (*at z*)
shows *f' holomorphic_on S*

corollary *contour_integral_affine*:
assumes *valid_path* γ $c \neq 0$
shows *contour_integral* $((\lambda x.\ c * x + b) \circ \gamma)\ f =$ *contour_integral* $\gamma\ (\lambda w.\ c * f\ (c * w + b))$

4.3 Morera's theorem

proposition *Morera_triangle*:
⟦*continuous_on S f*; *open S*;
$\bigwedge a\ b\ c.$ *convex hull* {*a*,*b*,*c*} $\subseteq S$
$\longrightarrow$ *contour_integral* (*linepath a b*) *f* +
contour_integral (*linepath b c*) *f* +
contour_integral (*linepath c a*) *f* = 0⟧
$\Longrightarrow$ *f analytic_on S*

4.4 Combining theorems for higher derivatives including Leibniz rule

proposition *no_isolated_singularity*:
fixes *z*::*complex*
assumes *f*: *continuous_on S f* **and** *holf*: *f holomorphic_on* $(S-K)$ **and** *S*: *open S* **and** *K*: *finite K*
shows *f holomorphic_on S*

proposition *Cauchy_integral_formula_convex*:
assumes *S*: *convex S* **and** *K*: *finite K* **and** *contf*: *continuous_on S f*
and *fcd*: $(\bigwedge x.\ x \in \mathit{interior}\ S - K \Longrightarrow f\ \mathit{field_differentiable\ at}\ x)$
and *z*: $z \in \mathit{interior}\ S$ **and** *vpg*: *valid_path* γ
and *pasz*: *path_image* $\gamma \subseteq S - \{z\}$ **and** *loop*: *pathfinish* γ = *pathstart* γ
shows $((\lambda w.\ f\ w\ /\ (w-z))\ \mathit{has_contour_integral}\ (2*\mathit{pi} * \mathrm{i} * \mathit{winding_number}\ \gamma\ z * f\ z))\ \gamma$

corollary *Cauchy_contour_integral_circlepath*:
assumes *continuous_on* (*cball z r*) *f f holomorphic_on ball z r w* $\in$ *ball z r*
shows *contour_integral*(*circlepath z r*) $(\lambda u.\ f\ u/(u-w)\widehat{\ }(\mathit{Suc}\ k)) = (2 * \mathit{pi} * \mathrm{i}) * (\mathit{deriv}\ \widehat{\ \ }\ k)\ f\ w\ /\ (\mathit{fact}\ k)$

4.5 A holomorphic function is analytic, i.e. has local power series

theorem *holomorphic_power_series*:
assumes *holf*: *f holomorphic_on ball z r*
and *w*: $w \in \mathit{ball}\ z\ r$
shows $((\lambda n.\ (\mathit{deriv}\ \widehat{\ \ }\ n)\ f\ z\ /\ (\mathit{fact}\ n) * (w-z)\widehat{\ }n)\ \mathit{sums}\ f\ w)$

4.6 The Liouville theorem and the Fundamental Theorem of Algebra

proposition *Liouville_weak*:
assumes *f holomorphic_on UNIV* **and** $(f \longrightarrow l)$ *at_infinity*
shows $f\ z = l$

proposition *Liouville_weak_inverse*:
assumes *f holomorphic_on UNIV* **and** *unbounded*: $\bigwedge B.$ *eventually* $(\lambda x.\ \mathit{norm}\ (f\ x) \geq B)$ *at_infinity*
obtains *z* **where** $f\ z = 0$
theorem *fundamental_theorem_of_algebra*:
fixes *a* :: *nat* $\Rightarrow$ *complex*
assumes $a\ 0 = 0 \vee (\exists i \in \{1..n\}.\ a\ i \neq 0)$
obtains *z* **where** $(\sum i{\leq}n.\ a\ i * z\widehat{\ }i) = 0$

4.7 Weierstrass convergence theorem

proposition *has_complex_derivative_uniform_limit*:
fixes *z*::*complex*
assumes *cont*: *eventually* $(\lambda n.$ *continuous_on* (*cball z r*) $(f\ n) \wedge$
$(\forall w \in \mathit{ball}\ z\ r.\ ((f\ n)\ \mathit{has_field_derivative}\ (f'\ n\ w))\ (\mathit{at}\ w)))\ F$
and *ulim*: *uniform_limit* (*cball z r*) *f g F*

and F: $\neg$ *trivial_limit* F **and** $0 < r$
obtains g' **where**
continuous_on (*cball* z r) g
$\bigwedge w.\ w \in$ *ball* z $r \Longrightarrow$ (g *has_field_derivative* (g' w)) (*at* w) $\wedge$ (($\lambda n.\ f'\ n\ w$) $\longrightarrow$ g' w) F

4.8 More on winding numbers

4.9 On analytic functions defined by a series

corollary *holomorphic_iff_power_series*:
f *holomorphic_on* *ball* z $r \longleftrightarrow$
($\forall w \in$ *ball* z r. ($\lambda n.$ (*deriv* ^^ n) f z / (*fact* n) $*$ ($w-z$)^n) *sums* f w)

4.10 General, homology form of Cauchy's theorem

theorem *Cauchy_integral_formula_global*:
assumes S: *open* S **and** *holf*: f *holomorphic_on* S
and z: $z \in S$ **and** *vpg*: *valid_path* γ
and *pasz*: *path_image* $\gamma \subseteq S - \{z\}$ **and** *loop*: *pathfinish* γ = *pathstart* γ
and *zero*: $\bigwedge w.\ w \notin S \Longrightarrow$ *winding_number* γ $w = 0$
shows (($\lambda w.\ f\ w$ / ($w-z$)) *has_contour_integral* (*2*pi* $*$ i $*$ *winding_number* γ z $*$ f z)) γ

theorem *Cauchy_theorem_global*:
assumes S: *open* S **and** *holf*: f *holomorphic_on* S
and *vpg*: *valid_path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ
and *pas*: *path_image* $\gamma \subseteq S$
and *zero*: $\bigwedge w.\ w \notin S \Longrightarrow$ *winding_number* γ $w = 0$
shows (f *has_contour_integral* 0) γ

corollary *Cauchy_theorem_global_outside*:
assumes *open* S f *holomorphic_on* S *valid_path* γ *pathfinish* γ = *pathstart* γ *path_image* $\gamma \subseteq S$
$\bigwedge w.\ w \notin S \Longrightarrow w \in$ *outside*(*path_image* γ)
shows (f *has_contour_integral* 0) γ

4.11 Cauchy's inequality and more versions of Liouville

theorem *Liouville_theorem*:
assumes *holf*: f *holomorphic_on* *UNIV*
and *bf*: *bounded* (*range* f)
shows f *constant_on* *UNIV*

4.12 The uniform Weierstrass convergence theorem

theorem *logderiv_prod_uniform_limit*:
fixes $f :: 'a \Rightarrow complex \Rightarrow complex$ **and** $idx :: 'n \Rightarrow 'a\ set$ **and** $F :: 'n\ filter$
assumes *ulim*: $\bigwedge L.\ compact\ L \Longrightarrow L \subseteq A \Longrightarrow$
$uniform_limit\ L\ (\lambda n\ x.\ \prod k{\in}idx\ n.\ f\ k\ x)\ P\ F$
and *holf*: $\bigwedge k.\ k \in I \Longrightarrow f\ k\ holomorphic_on\ A$
and *A*: $open\ A$
and *nzP*: $\bigwedge z.\ z \in A \Longrightarrow P\ z \neq 0$
and *fin*: $\forall_F\ n\ in\ F.\ finite\ (idx\ n) \wedge idx\ n \subseteq I$
and *cover*: $\bigwedge k.\ k \in I \Longrightarrow \forall_F\ n\ in\ F.\ k \in idx\ n$
and *F*: $F \neq bot$
and *K*: $compact\ K\ K \subseteq A$
shows $uniform_limit\ K\ (\lambda n\ z.\ \sum k{\in}idx\ n.\ deriv\ (f\ k)\ z\ /\ f\ k\ z)$
$(\lambda z.\ deriv\ P\ z\ /\ P\ z)\ F$
corollary *logderiv_prodinf_uniform_limit*:
fixes $f :: nat \Rightarrow complex \Rightarrow complex$
assumes *ulim*: $\bigwedge L.\ compact\ L \Longrightarrow L \subseteq A \Longrightarrow$
$uniform_limit\ L\ (\lambda n\ x.\ \prod k{<}n.\ f\ k\ x)\ P\ sequentially$
and *holf*: $\bigwedge k.\ f\ k\ holomorphic_on\ A$
and *A*: $open\ A$
and *nzP*: $\bigwedge z.\ z \in A \Longrightarrow P\ z \neq 0$
and *K*: $compact\ K\ K \subseteq A$
shows $uniform_limit\ K\ (\lambda n\ z.\ \sum k{<}n.\ deriv\ (f\ k)\ z\ /\ f\ k\ z)$
$(\lambda z.\ deriv\ P\ z\ /\ P\ z)\ sequentially$
corollary *logderiv_infprod_uniform_limit*:
fixes $f :: 'a \Rightarrow complex \Rightarrow complex$
assumes *ulim*: $\bigwedge L.\ compact\ L \Longrightarrow L \subseteq A \Longrightarrow$
$uniform_limit\ L\ (\lambda X\ x.\ \prod k{\in}X.\ f\ k\ x)\ P\ (finite_subsets_at_top\ I)$
and *holf*: $\bigwedge k.\ k \in I \Longrightarrow f\ k\ holomorphic_on\ A$
and *A*: $open\ A$
and *nzP*: $\bigwedge z.\ z \in A \Longrightarrow P\ z \neq 0$
and *K*: $compact\ K\ K \subseteq A$
shows $uniform_limit\ K\ (\lambda X\ z.\ \sum k{\in}X.\ deriv\ (f\ k)\ z\ /\ f\ k\ z)$
$(\lambda z.\ deriv\ P\ z\ /\ P\ z)\ (finite_subsets_at_top\ I)$
corollary *has_sum_logderiv_infprod*:
fixes $f :: 'a \Rightarrow complex \Rightarrow complex$
assumes *ulim*: $\bigwedge L.\ compact\ L \Longrightarrow L \subseteq A \Longrightarrow$
$uniform_limit\ L\ (\lambda X\ x.\ \prod k{\in}X.\ f\ k\ x)\ P\ (finite_subsets_at_top\ I)$
and *holf*: $\bigwedge k.\ k \in I \Longrightarrow f\ k\ holomorphic_on\ A$
and *A*: $open\ A$
and *nzP*: $\bigwedge z.\ z \in A \Longrightarrow P\ z \neq 0$
and *z*: $z \in A$
shows $((\lambda k.\ deriv\ (f\ k)\ z\ /\ f\ k\ z)\ has_sum\ deriv\ P\ z\ /\ P\ z)\ I$

4.13 Complex functions and power series

definition *fps_expansion* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *complex fps*
where
fps_expansion f z0 = *Abs_fps* (λ*n*. (*deriv* ^^ *n*) *f z0* / *fact n*)

end

5 Conformal Mappings and Consequences of Cauchy's Integral Theorem

theory *Conformal_Mappings*
imports *Cauchy_Integral_Formula*

begin

5.1 Analytic continuation

proposition *isolated_zeros*:
assumes *holf*: *f holomorphic_on S*
and *open S connected S* ξ ∈ *S f* ξ = *0* β ∈ *S f* β ≠ *0*
obtains *r* **where** *0* < *r* **and** *ball* ξ *r* ⊆ *S* **and**
⋀*z*. *z* ∈ *ball* ξ *r* − {ξ} ⟹ *f z* ≠ *0*

proposition *analytic_continuation*:
assumes *holf*: *f holomorphic_on S*
and *open S* **and** *connected S*
and *U* ⊆ *S* **and** ξ ∈ *S*
and ξ *islimpt U*
and *fU0* [*simp*]: ⋀*z*. *z* ∈ *U* ⟹ *f z* = *0*
and *w* ∈ *S*
shows *f w* = *0*

corollary *analytic_continuation_open*:
assumes *open s* **and** *open s′* **and** *s* ≠ {} **and** *connected s′*
and *s* ⊆ *s′*
assumes *f holomorphic_on s′* **and** *g holomorphic_on s′*
and ⋀*z*. *z* ∈ *s* ⟹ *f z* = *g z*
assumes *z* ∈ *s′*
shows *f z* = *g z*

corollary *analytic_continuation′*:
assumes *f holomorphic_on S open S connected S*
and *U* ⊆ *S* ξ ∈ *S* ξ *islimpt U*
and *f constant_on U*
shows *f constant_on S*

5.2 Open mapping theorem

theorem *open_mapping_thm*:
assumes *holf*: *f holomorphic_on S*
and *S*: *open S* **and** *connected S*
and *open U* **and** $U \subseteq S$
and *fne*: $\neg$ *f constant_on S*
shows *open* (*f* ‘ *U*)

5.3 Maximum modulus principle

proposition *maximum_modulus_principle*:
assumes *holf*: *f holomorphic_on S*
and *S*: *open S* **and** *connected S*
and *open U* **and** $U \subseteq S$ **and** $\xi \in U$
and *no*: $\bigwedge z.\ z \in U \Longrightarrow norm(f\ z) \leq norm(f\ \xi)$
shows *f constant_on S*

proposition *maximum_modulus_frontier*:
assumes *holf*: *f holomorphic_on* (*interior S*)
and *contf*: *continuous_on* (*closure S*) *f*
and *bos*: *bounded S*
and *leB*: $\bigwedge z.\ z \in frontier\ S \Longrightarrow norm(f\ z) \leq B$
and $\xi \in S$
shows $norm(f\ \xi) \leq B$

5.4 Relating invertibility and nonvanishing of derivative

proposition *holomorphic_has_inverse*:
assumes *holf*: *f holomorphic_on S*
and *open S* **and** *injf*: *inj_on f S*
obtains *g* **where** *g holomorphic_on* (*f* ‘ *S*)
$\bigwedge z.\ z \in S \Longrightarrow deriv\ f\ z * deriv\ g\ (f\ z) = 1$
$\bigwedge z.\ z \in S \Longrightarrow g(f\ z) = z$

5.5 The Schwarz Lemma

proposition *Schwarz_Lemma*:
assumes *holf*: *f holomorphic_on* (*ball 0 1*) **and** [*simp*]: *f 0 = 0*
and *no*: $\bigwedge z.\ norm\ z < 1 \Longrightarrow norm\ (f\ z) < 1$
and ξ: $norm\ \xi < 1$

shows $norm\ (f\ \xi) \leq norm\ \xi$ **and** $norm(deriv\ f\ 0) \leq 1$
and $((\exists z.\ norm\ z < 1 \wedge z \neq 0 \wedge norm(f\ z) = norm\ z)$
$\vee\ norm(deriv\ f\ 0) = 1)$
$\Longrightarrow \exists \alpha.\ (\forall z.\ norm\ z < 1 \longrightarrow f\ z = \alpha * z) \wedge norm\ \alpha = 1$
(**is** $?P \Longrightarrow ?Q$)

corollary *Schwarz_Lemma'*:
assumes *holf*: f *holomorphic_on* $(ball\ 0\ 1)$ **and** [*simp*]: $f\ 0 = 0$
and *no*: $\bigwedge z.\ norm\ z < 1 \Longrightarrow norm\ (f\ z) < 1$
shows $((\forall \xi.\ norm\ \xi < 1 \longrightarrow norm\ (f\ \xi) \leq norm\ \xi)$
$\wedge\ norm(deriv\ f\ 0) \leq 1)$
$\wedge\ (((\exists z.\ norm\ z < 1 \wedge z \neq 0 \wedge norm(f\ z) = norm\ z)$
$\vee\ norm(deriv\ f\ 0) = 1)$
$\longrightarrow (\exists \alpha.\ (\forall z.\ norm\ z < 1 \longrightarrow f\ z = \alpha * z) \wedge norm\ \alpha = 1))$

5.6 The Schwarz reflection principle

proposition *Schwarz_reflection*:
assumes *open* S **and** *cnjs*: $cnj\ `\ S \subseteq S$
and *holf*: f *holomorphic_on* $(S \cap \{z.\ 0 < Im\ z\})$
and *contf*: *continuous_on* $(S \cap \{z.\ 0 \leq Im\ z\})\ f$
and f: $\bigwedge z.\ [\![z \in S;\ z \in \mathbb{R}]\!] \Longrightarrow (f\ z) \in \mathbb{R}$
shows $(\lambda z.\ if\ 0 \leq Im\ z\ then\ f\ z\ else\ cnj(f(cnj\ z)))$ *holomorphic_on* S

5.7 Bloch's theorem

proposition *Bloch_unit*:
assumes *holf*: f *holomorphic_on* $ball\ a\ 1$ **and** [*simp*]: $deriv\ f\ a = 1$
obtains $b\ r$ **where** $1/12 < r$ **and** $ball\ b\ r \subseteq f\ `\ (ball\ a\ 1)$

theorem *Bloch*:
assumes *holf*: f *holomorphic_on* $ball\ a\ r$ **and** $0 < r$
and r': $r' \leq r * norm\ (deriv\ f\ a)\ /\ 12$
obtains b **where** $ball\ b\ r' \subseteq f\ `\ (ball\ a\ r)$

corollary *Bloch_general*:
assumes *holf*: f *holomorphic_on* S **and** $a \in S$
and *tle*: $\bigwedge z.\ z \in frontier\ S \Longrightarrow t \leq dist\ a\ z$
and *rle*: $r \leq t * norm(deriv\ f\ a)\ /\ 12$
obtains b **where** $ball\ b\ r \subseteq f\ `\ S$

end

6 The Great Picard Theorem and its Applications

theory *Great_Picard*
imports *Conformal_Mappings*
begin

6.1 Schottky's theorem

theorem *Schottky*:
assumes *holf*: *f holomorphic_on cball 0 1*
and *nof0*: *norm(f 0) ≤ r*
and *not01*: *⋀z. z ∈ cball 0 1 ⟹ ¬(f z = 0 ∨ f z = 1)*
and *0 < t t < 1 norm z ≤ t*
shows *norm(f z) ≤ exp(pi * exp(pi * (2 + 2 * r + 12 * t / (1 − t))))*

6.2 The Little Picard Theorem

theorem *Landau_Picard*:
obtains *R*
where *⋀z. 0 < R z*
⋀f. ⟦f holomorphic_on cball 0 (R(f 0)); ⋀z. norm z ≤ R(f 0) ⟹ f z ≠ 0 ∧ f z ≠ 1⟧ ⟹ norm(deriv f 0) < 1

theorem *little_Picard*:
assumes *holf*: *f holomorphic_on UNIV*
and *a ≠ b range f ∩ {a,b} = {}*
obtains *c* **where** *f = (λx. c)*

6.3 The Arzelà–Ascoli theorem

theorem *Arzela_Ascoli*:
fixes *ℱ :: [nat,'a::euclidean_space] ⇒ 'b::{real_normed_vector,heine_borel}*
assumes *compact S*
and *M*: *⋀n x. x ∈ S ⟹ norm(ℱ n x) ≤ M*
and *equicont*:
⋀x e. ⟦x ∈ S; 0 < e⟧ ⟹ ∃d. 0 < d ∧ (∀n y. y ∈ S ∧ norm(x − y) < d ⟶ norm(ℱ n x − ℱ n y) < e)

obtains g k **where** *continuous_on* S g *strict_mono* $(k :: nat \Rightarrow nat)$
$\bigwedge e.\ 0 < e \Longrightarrow \exists N.\ \forall n\ x.\ n \geq N \wedge x \in S \longrightarrow norm(\mathcal{F}(k\ n)\ x - g\ x) < e$

6.3.1 Montel's theorem

theorem *Montel*:
fixes $\mathcal{F} :: [nat, complex] \Rightarrow complex$
assumes *open* S
and $\mathcal{H}$: $\bigwedge h.\ h \in \mathcal{H} \Longrightarrow h$ *holomorphic_on* S
and *bounded*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow \exists B.\ \forall h \in \mathcal{H}.\ \forall\ z \in K.\ norm(h\ z) \leq B$
and *rng_f*: *range* $\mathcal{F} \subseteq \mathcal{H}$
obtains g r
where g *holomorphic_on* S *strict_mono* $(r :: nat \Rightarrow nat)$
$\bigwedge x.\ x \in S \Longrightarrow ((\lambda n.\ \mathcal{F}\ (r\ n)\ x) \longrightarrow g\ x)$ *sequentially*
$\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow$ *uniform_limit* K $(\mathcal{F} \circ r)$ g *sequentially*

6.4 Some simple but useful cases of Hurwitz's theorem

proposition *Hurwitz_no_zeros*:
assumes S: *open* S *connected* S
and *holf*: $\bigwedge n{::}nat.\ \mathcal{F}\ n$ *holomorphic_on* S
and *holg*: g *holomorphic_on* S
and *ul_g*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow$ *uniform_limit* $K\ \mathcal{F}\ g$ *sequentially*
and *nonconst*: $\neg\ g$ *constant_on* S
and *nz*: $\bigwedge n\ z.\ z \in S \Longrightarrow \mathcal{F}\ n\ z \neq 0$
and $z0 \in S$
shows $g\ z0 \neq 0$

corollary *Hurwitz_injective*:
assumes S: *open* S *connected* S
and *holf*: $\bigwedge n{::}nat.\ \mathcal{F}\ n$ *holomorphic_on* S
and *holg*: g *holomorphic_on* S
and *ul_g*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow$ *uniform_limit* $K\ \mathcal{F}\ g$ *sequentially*
and *nonconst*: $\neg\ g$ *constant_on* S
and *inj*: $\bigwedge n.$ *inj_on* $(\mathcal{F}\ n)\ S$
shows *inj_on* $g\ S$

6.5 The Great Picard theorem

theorem *great_Picard*:
assumes *open M z* ∈ *M a* ≠ *b* **and** *holf*: *f holomorphic_on* (*M* − {*z*})
and *fab*: ⋀*w*. *w* ∈ *M* − {*z*} ⟹ *f w* ≠ *a* ∧ *f w* ≠ *b*
obtains *l* **where** (*f* ⟶ *l*) (*at z*) ∨ ((*inverse* ∘ *f*) ⟶ *l*) (*at z*)

corollary *great_Picard_alt*:
assumes *M*: *open M z* ∈ *M* **and** *holf*: *f holomorphic_on* (*M* − {*z*})
and *non*: ⋀*l*. ¬ (*f* ⟶ *l*) (*at z*) ⋀*l*. ¬ ((*inverse* ∘ *f*) ⟶ *l*) (*at z*)
obtains *a* **where** − {*a*} ⊆ *f* ‘ (*M* − {*z*})

corollary *great_Picard_infinite*:
assumes *M*: *open M z* ∈ *M* **and** *holf*: *f holomorphic_on* (*M* − {*z*})
and *non*: ⋀*l*. ¬ (*f* ⟶ *l*) (*at z*) ⋀*l*. ¬ ((*inverse* ∘ *f*) ⟶ *l*) (*at z*)
obtains *a* **where** ⋀*w*. *w* ≠ *a* ⟹ *infinite* {*x*. *x* ∈ *M* − {*z*} ∧ *f x* = *w*}

theorem *Casorati_Weierstrass*:
assumes *open M z* ∈ *M f holomorphic_on* (*M* − {*z*})
and ⋀*l*. ¬ (*f* ⟶ *l*) (*at z*) ⋀*l*. ¬ ((*inverse* ∘ *f*) ⟶ *l*) (*at z*)
shows *closure*(*f* ‘ (*M* − {*z*})) = *UNIV*

end

7 Moebius functions, Equivalents of Simply Connected Sets, Riemann Mapping Theorem

theory *Riemann_Mapping*
imports *Great_Picard*
begin

7.1 Moebius functions are biholomorphisms of the unit disc

definition *Moebius_function* :: [*real*,*complex*,*complex*] ⇒ *complex* **where**
Moebius_function ≡ λ*t w z*. *exp*(i ∗ *of_real t*) ∗ (*z* − *w*) / (*1* − *cnj w* ∗ *z*)

7.2 A big chain of equivalents of simple connectedness for an open set

proposition
assumes *open S*
shows *simply_connected_eq_winding_number_zero*:
simply_connected S ⟷
connected S ∧

$(\forall\, g\ z.\ path\ g \wedge path_image\ g \subseteq S \wedge$
$\quad pathfinish\ g = pathstart\ g \wedge (z \notin S)$
$\quad \longrightarrow winding_number\ g\ z = 0)$ (**is** *?wn0*)
and *simply_connected_eq_contour_integral_zero*:
$simply_connected\ S \longleftrightarrow$
$connected\ S \wedge$
$(\forall\, g\ f.\ valid_path\ g \wedge path_image\ g \subseteq S \wedge$
$\quad pathfinish\ g = pathstart\ g \wedge f\ holomorphic_on\ S$
$\longrightarrow (f\ has_contour_integral\ 0)\ g)$ (**is** *?ci0*)
and *simply_connected_eq_global_primitive*:
$simply_connected\ S \longleftrightarrow$
$connected\ S \wedge$
$(\forall f.\ f\ holomorphic_on\ S \longrightarrow$
$\quad (\exists\, h.\ \forall z.\ z \in S \longrightarrow (h\ has_field_derivative\ f\ z)\ (at\ z)))$ (**is** *?gp*)
and *simply_connected_eq_holomorphic_log*:
$simply_connected\ S \longleftrightarrow$
$connected\ S \wedge$
$(\forall f.\ f\ holomorphic_on\ S \wedge (\forall z \in S.\ f\ z \neq 0)$
$\quad \longrightarrow (\exists\, g.\ g\ holomorphic_on\ S \wedge (\forall z \in S.\ f\ z = exp(g\ z))))$ (**is** *?log*)
and *simply_connected_eq_holomorphic_sqrt*:
$simply_connected\ S \longleftrightarrow$
$connected\ S \wedge$
$(\forall f.\ f\ holomorphic_on\ S \wedge (\forall z \in S.\ f\ z \neq 0)$
$\quad \longrightarrow (\exists\, g.\ g\ holomorphic_on\ S \wedge (\forall z \in S.\ \ f\ z = (g\ z)^2)))$ (**is** *?sqrt*)
and *simply_connected_eq_biholomorphic_to_disc*:
$simply_connected\ S \longleftrightarrow$
$S = \{\} \vee S = UNIV \vee$
$(\exists f\ g.\ f\ holomorphic_on\ S \wedge g\ holomorphic_on\ ball\ 0\ 1 \wedge$
$\quad (\forall z \in S.\ f\ z \in ball\ 0\ 1 \wedge g(f\ z) = z) \wedge$
$\quad (\forall z \in ball\ 0\ 1.\ g\ z \in S \wedge f(g\ z) = z))$ (**is** *?bih*)
and *simply_connected_eq_homeomorphic_to_disc*:
$simply_connected\ S \longleftrightarrow S = \{\} \vee S\ homeomorphic\ ball\ (0{::}complex)\ 1$
(**is** *?disc*)

corollary *contractible_eq_simply_connected_2d*:
fixes S :: *complex set*
assumes *open S*
shows *contractible S* $\longleftrightarrow$ *simply_connected S*

7.3 A further chain of equivalences about components of the complement of a simply connected set

proposition
fixes S :: *complex set*
assumes *open S*
shows *simply_connected_eq_frontier_properties*:

simply_connected S ⟷
connected S ∧
(*if bounded S then connected*(*frontier S*)
else (∀ *C* ∈ *components*(*frontier S*). ¬*bounded C*)) (**is** *?fp*)
and *simply_connected_eq_unbounded_complement_components*:
simply_connected S ⟷
connected S ∧ (∀ *C* ∈ *components*(− *S*). ¬*bounded C*) (**is** *?ucc*)
and *simply_connected_eq_empty_inside*:
simply_connected S ⟷
connected S ∧ *inside S* = {} (**is** *?ei*)

7.4 Further equivalences based on continuous logs and sqrts

proposition
fixes *S* :: *complex set*
assumes *open S*
shows *simply_connected_eq_continuous_log*:
simply_connected S ⟷
connected S ∧
(∀ *f*::*complex*⇒*complex*. *continuous_on S f* ∧ (∀ *z* ∈ *S*. *f z* ≠ *0*)
⟶ (∃ *g*. *continuous_on S g* ∧ (∀ *z* ∈ *S*. *f z* = *exp* (*g z*)))) (**is** *?log*)
and *simply_connected_eq_continuous_sqrt*:
simply_connected S ⟷
connected S ∧
(∀ *f*::*complex*⇒*complex*. *continuous_on S f* ∧ (∀ *z* ∈ *S*. *f z* ≠ *0*)
⟶ (∃ *g*. *continuous_on S g* ∧ (∀ *z* ∈ *S*. *f z* = (*g z*)^2))) (**is** *?sqrt*)

7.5 Finally, the Riemann Mapping Theorem

theorem *Riemann_mapping_theorem*:
open S ∧ *simply_connected S* ⟷
S = {} ∨ *S* = *UNIV* ∨
(∃ *f g*. *f holomorphic_on S* ∧ *g holomorphic_on ball 0 1* ∧
(∀ *z* ∈ *S*. *f z* ∈ *ball 0 1* ∧ *g*(*f z*) = *z*) ∧
(∀ *z* ∈ *ball 0 1*. *g z* ∈ *S* ∧ *f*(*g z*) = *z*))
(**is** _ = *?rhs*)

7.6 Applications to Winding Numbers

7.7 Winding number equality is the same as path/loop homotopy in C - 0

proposition *winding_number_homotopic_paths_eq*:
assumes *path p* **and** ζ*p*: ζ ∉ *path_image p*
and *path q* **and** ζ*q*: ζ ∉ *path_image q*

and *qp*: *pathstart q* = *pathstart p pathfinish q* = *pathfinish p*
shows *winding_number p ζ* = *winding_number q ζ* ⟷ *homotopic_paths* (−{*ζ*}) *p q*
(**is** *?lhs* = *?rhs*)

end
theory *Complex_Singularities*
imports *Conformal_Mappings*
begin

7.8 Non-essential singular points

definition
is_pole :: (*'a*::*topological_space* ⇒ *'b*::*real_normed_vector*) ⇒ *'a* ⇒ *bool*
where *is_pole f a* = (*LIM x* (*at a*). *f x* :> *at_infinity*)

7.9 Isolated singularities

7.10 The order of non-essential singularities (i.e. removable singularities or poles)

definition *zorder* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *int* **where**
zorder f z = (*THE n*. (∃ *h r*. *r*>*0* ∧ *h holomorphic_on cball z r* ∧ *h z*≠*0*
∧ (∀ *w*∈*cball z r* − {*z*}. *f w* = *h w* ∗ (*w*−*z*) *powi n*
∧ *h w* ≠*0*)))

definition *zor_poly*
:: [*complex* ⇒ *complex*, *complex*] ⇒ *complex* ⇒ *complex* **where**
zor_poly f z = (*SOME h*. ∃ *r*. *r* > *0* ∧ *h holomorphic_on cball z r* ∧ *h z* ≠ *0*
∧ (∀ *w*∈*cball z r* − {*z*}. *f w* = *h w* ∗ (*w*−*z*) *powi* (*zorder f z*)
∧ *h w* ≠*0*))

7.11 Isolated points

7.12 Isolated zeros

end
theory *Complex_Residues*
imports *Complex_Singularities*
begin

7.13 Definition of residues

definition *residue* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *complex* **where**
residue f z = (*SOME int*. ∃ *e*>*0*. ∀ *ε*>*0*. *ε*<*e*
⟶ (*f has_contour_integral 2*∗*pi*∗ i ∗*int*) (*circlepath z ε*))

theorem *residue_fps_expansion_over_power_at_0*:
assumes *f has_fps_expansion F*
shows *residue* (λ*z*. *f z* / *z* ^ *Suc n*) *0* = *fps_nth F n*

7.14 Poles and residues of some well-known functions

end

8 The Residue Theorem, the Argument Principle and Rouché's Theorem

theory *Residue_Theorem*
imports *Complex_Residues HOL−Library.Landau_Symbols*
begin

8.1 Cauchy's residue theorem

theorem *Residue_theorem*:
fixes *S pts*::*complex set* **and** *f*::*complex* ⇒ *complex*
and *g*::*real* ⇒ *complex*
assumes *open S connected S finite pts* **and**
holo:*f holomorphic_on S−pts* **and**
valid_path g **and**
loop:*pathfinish g* = *pathstart g* **and**
path_image g ⊆ *S−pts* **and**
homo:∀ *z*. (*z* ∉ *S*) ⟶ *winding_number g z* = *0*
shows *contour_integral g f* = *2* ∗ *pi* ∗ i ∗(∑ *p*∈*pts*. *winding_number g p* ∗ *residue f p*)

8.2 The argument principle

theorem *argument_principle*:
fixes *f*::*complex* ⇒ *complex* **and** *poles S*:: *complex set*
defines *pz* ≡ {*w*∈*S*. *f w* = *0* ∨ *w* ∈ *poles*} — *pz* is the set of poles and zeros
assumes *open S connected S* **and**
f_holo:*f holomorphic_on S−poles* **and**
h_holo:*h holomorphic_on S* **and**
valid_path g **and**
loop:*pathfinish g* = *pathstart g* **and**

path_img:*path_image* $g \subseteq S - pz$ **and**
homo:$\forall z.\ (z \notin S) \longrightarrow$ *winding_number* $g\ z = 0$ **and**
finite:*finite* pz **and**
poles:$\forall p{\in}S{\cap}$*poles. is_pole* $f\ p$
shows *contour_integral* $g\ (\lambda x.\ deriv\ f\ x * h\ x\ /\ f\ x) = 2 * pi * \mathrm{i} *$
$(\sum p{\in}pz.\ winding_number\ g\ p * h\ p * zorder\ f\ p)$
(**is** *?L*=*?R*)

8.3 Coefficient asymptotics for generating functions

theorem
fixes f :: *complex* ⇒ *complex* **and** n :: *nat* **and** r :: *real*
defines $g \equiv (\lambda w.\ f\ w\ /\ w\ \hat{}\ Suc\ n)$ **and** $\gamma \equiv$ *circlepath* 0 r
assumes *open* A *connected* A *cball* 0 $r \subseteq A$ $r > 0$
assumes f *holomorphic_on* $A - S$ $S \subseteq$ *ball* 0 r *finite* S 0 $\notin S$
shows *fps_coeff_conv_residues*:
$(deriv\ \hat{}\hat{}\ n)\ f\ 0\ /\ fact\ n =$
contour_integral $\gamma\ g\ /\ (2 * pi * \mathrm{i}) - (\sum z{\in}S.\ residue\ g\ z)$ (**is** *?thesis1*)
and *fps_coeff_residues_bound*:
$(\bigwedge z.\ norm\ z = r \Longrightarrow z \notin k \Longrightarrow norm\ (f\ z) \leq C) \Longrightarrow C \geq 0 \Longrightarrow$ *finite* $k \Longrightarrow$
$norm\ ((deriv\ \hat{}\hat{}\ n)\ f\ 0\ /\ fact\ n + (\sum z{\in}S.\ residue\ g\ z)) \leq C\ /\ r\ \hat{}\ n$
corollary *fps_coeff_residues_bigo*:
fixes f :: *complex* ⇒ *complex* **and** r :: *real*
assumes *open* A *connected* A *cball* 0 $r \subseteq A$ $r > 0$
assumes f *holomorphic_on* $A - S$ $S \subseteq$ *ball* 0 r *finite* S 0 $\notin S$
assumes g: *eventually* $(\lambda n.\ g\ n = -(\sum z{\in}S.\ residue\ (\lambda z.\ f\ z\ /\ z\ \hat{}\ Suc\ n)\ z))$ *sequentially*
(**is** *eventually* $(\lambda n.\ _ = -?g'\ n)$ _)
shows $(\lambda n.\ (deriv\ \hat{}\hat{}\ n)\ f\ 0\ /\ fact\ n - g\ n) \in O(\lambda n.\ 1\ /\ r\ \hat{}\ n)$ (**is** $(\lambda n.\ ?c\ n - _) \in O(_)$)

corollary *fps_coeff_residues_bigo′*:
fixes f :: *complex* ⇒ *complex* **and** r :: *real*
assumes *exp*: f *has_fps_expansion* F
assumes *open* A *connected* A *cball* 0 $r \subseteq A$ $r > 0$
assumes f *holomorphic_on* $A - S$ $S \subseteq$ *ball* 0 r *finite* S 0 $\notin S$
assumes *eventually* $(\lambda n.\ g\ n = -(\sum z{\in}S.\ residue\ (\lambda z.\ f\ z\ /\ z\ \hat{}\ Suc\ n)\ z))$ *sequentially*
(**is** *eventually* $(\lambda n.\ _ = -?g'\ n)$ _)
shows $(\lambda n.\ fps_nth\ F\ n - g\ n) \in O(\lambda n.\ 1\ /\ r\ \hat{}\ n)$ (**is** $(\lambda n.\ ?c\ n - _) \in O(_)$)

8.4 Rouche's theorem

theorem *Rouche_theorem*:
fixes $f\ g$::*complex* ⇒ *complex* **and** s:: *complex set*
defines $fg{\equiv}(\lambda p.\ f\ p + g\ p)$

defines $zeros_fg \equiv \{p \in s.\ fg\ p = 0\}$ **and** $zeros_f \equiv \{p \in s.\ f\ p = 0\}$
assumes
$open\ s$ **and** $connected\ s$ **and**
$finite\ zeros_fg$ **and**
$finite\ zeros_f$ **and**
$f_holo{:}f\ holomorphic_on\ s$ **and**
$g_holo{:}g\ holomorphic_on\ s$ **and**
$valid_path\ \gamma$ **and**
$loop{:}pathfinish\ \gamma = pathstart\ \gamma$ **and**
$path_img{:}path_image\ \gamma \subseteq s$ **and**
$path_less{:}\forall\, z \in path_image\ \gamma.\ cmod(f\ z) > cmod(g\ z)$ **and**
$homo{:}\forall\, z.\ (z \notin s) \longrightarrow winding_number\ \gamma\ z = 0$
shows $(\sum p \in zeros_fg.\ winding_number\ \gamma\ p * zorder\ fg\ p)$
$= (\sum p \in zeros_f.\ winding_number\ \gamma\ p * zorder\ f\ p)$

end
theory *Laurent_Convergence*
imports *HOL−Computational_Algebra.Formal_Laurent_Series HOL−Library.Landau_Symbols*
Residue_Theorem

begin

definition $fls_conv_radius :: complex\ fls \Rightarrow ereal$ **where**
$fls_conv_radius\ f = fps_conv_radius\ (fls_regpart\ f)$

definition $eval_fls :: complex\ fls \Rightarrow complex \Rightarrow complex$ **where**
$eval_fls\ F\ z = eval_fps\ (fls_base_factor_to_fps\ F)\ z * z\ powi\ fls_subdegree\ F$

definition
$has_laurent_expansion :: (complex \Rightarrow complex) \Rightarrow complex\ fls \Rightarrow bool$
(**infixl** ‹*has'_laurent'_expansion*› 60)
where $(f\ has_laurent_expansion\ F) \longleftrightarrow$
$fls_conv_radius\ F > 0 \wedge eventually\ (\lambda z.\ eval_fls\ F\ z = f\ z)\ (at\ 0)$

theorem *sums_eval_fls*:
fixes f
defines $n \equiv fls_subdegree\ f$
assumes $norm\ z < fls_conv_radius\ f$ **and** $z \neq 0 \vee n \geq 0$
shows $(\lambda k.\ fls_nth\ f\ (int\ k + n) * z\ powi\ (int\ k + n))\ sums\ eval_fls\ f\ z$

theorem *not_essential_has_laurent_expansion_0*:
assumes $isolated_singularity_at\ f\ 0\ not_essential\ f\ 0$
shows $f\ has_laurent_expansion\ laurent_expansion\ f\ 0$

8.5 More Laurent expansions

8.6 Formal convergence versus analytic convergence

proposition *uniform_limit_imp_fps_expansion_eq*:
fixes $f :: 'a \Rightarrow$ *complex fps*
assumes *lim1*: $(f \longrightarrow h)\ F$
assumes *lim2*: *uniform_limit* $A\ (\lambda x\ z.\ f'\ x\ z)\ g'\ F$
assumes *expansions*: *eventually* $(\lambda x.\ f'\ x$ *has_fps_expansion* $f\ x)\ F\ g'$ *has_fps_expansion* g
assumes *holo*: *eventually* $(\lambda x.\ f'\ x$ *holomorphic_on* $A)\ F$
assumes A: *open* $A\ 0 \in A$
assumes *nontriv* [*simp*]: $F \neq$ *bot*
shows $g = h$

end

theory *Meromorphic* **imports**
Laurent_Convergence
Cauchy_Integral_Formula
HOL−Analysis.Sparse_In
begin

8.7 Remove singular points

definition *remove_sings* :: $(complex \Rightarrow complex) \Rightarrow complex \Rightarrow complex$ **where**
remove_sings $f\ z = (if\ \exists c.\ f\ -z\rightarrow\ c\ then\ Lim\ (at\ z)\ f\ else\ 0)$

8.8 Meromorphicity

definition *meromorphic_on* :: $(complex \Rightarrow complex) \Rightarrow complex\ set \Rightarrow bool$
(**infixl** ‹*meromorphic'_on*› *50*) **where**
f *meromorphic_on* $A \longleftrightarrow (\forall z{\in}A.\ \exists F.\ (\lambda w.\ f\ (z + w))$ *has_laurent_expansion* $F)$

8.9 Nice meromorphicity

8.10 Closure properties and proofs for individual functions

8.11 Meromorphic functions and zero order

8.12 More on poles and zeros

8.13 A relation for the zero order of a function

end

9 The Weierstraß Factorisation Theorem

theory *Weierstrass_Factorization*
  imports *Meromorphic*
begin

9.1 The elementary factors

9.2 Infinite products of elementary factors

theorem *has_zorder*: *has_zorder f z* (*card* (*a* −‘ {*z*} ∩ *I*))

theorem *logderiv_eq*:
  assumes *z* ∉ *a* ‘ *I*
  shows *deriv f z* / *f z* = (∑∞*x*∈*I*. (*z* / *a x*) ^ *p x* / (*z* − *a x*))

9.3 Writing a quotient as an exponential

9.4 Constructing the sequence of zeros

9.5 The factorisation theorem for holomorphic functions

theorem *weierstrass_factorization*:
  assumes *g holomorphic_on A open A connected A*
  assumes ⋀*z*. *z* ∈ *frontier A* ⟹ ¬*z islimpt* {*w*∈*A*. *g w* = *0*}
  obtains *h f* **where**

h holomorphic_on A f holomorphic_on UNIV
$\forall z.\ f\ z = 0 \longleftrightarrow (\forall z{\in}A.\ g\ z = 0) \vee (z \in A \wedge g\ z = 0)$
$\forall z{\in}A.$ *zorder f z = zorder g z*
$\forall z{\in}A.\ h\ z \neq 0$
$\forall z{\in}A.\ g\ z = h\ z * f\ z$
theorem *weierstrass_factorization_UNIV*:
assumes *g holomorphic_on UNIV*
obtains *h f* **where**
h holomorphic_on UNIV f holomorphic_on UNIV
$\forall z.\ f\ z = 0 \longleftrightarrow g\ z = 0$
$\forall z.$ *zorder f z = zorder g z*
$\forall z.\ h\ z \neq 0$
$\forall z.\ g\ z = h\ z * f\ z$

9.6 The factorisation theorem for meromorphic functions

theorem *weierstrass_factorization_meromorphic*:
assumes *mero*: *g nicely_meromorphic_on A* **and** *A*: *open A connected A*
assumes *no_limpt*: $\bigwedge z.\ z \in$ *frontier A* $\Longrightarrow \neg z$ *islimpt* $\{w{\in}A.\ g\ w = 0 \vee$ *is_pole g w*$\}$
obtains *h f1 f2* **where**
h holomorphic_on A f1 holomorphic_on UNIV f2 holomorphic_on UNIV
$\forall z{\in}A.$ *f1 z = 0* $\longleftrightarrow \neg$*is_pole g z* $\wedge$ *g z = 0*
$\forall z{\in}A.$ *f2 z = 0* $\longleftrightarrow$ *is_pole g z*
$\forall z{\in}A.\ \neg$*is_pole g z* $\longrightarrow$ *zorder f1 z = zorder g z*
$\forall z{\in}A.$ *is_pole g z* $\longrightarrow$ *zorder f2 z* $= -$*zorder g z*
$\forall z{\in}A.\ h\ z \neq 0$
$\forall z{\in}A.\ g\ z = h\ z *$ *f1 z / f2 z*
theorem *weierstrass_factorization_meromorphic_UNIV*:
assumes *g nicely_meromorphic_on UNIV*
obtains *h f1 f2* **where**
h holomorphic_on UNIV f1 holomorphic_on UNIV f2 holomorphic_on UNIV
$\forall z.$ *f1 z = 0* $\longleftrightarrow \neg$*is_pole g z* $\wedge$ *g z = 0*
$\forall z.$ *f2 z = 0* $\longleftrightarrow$ *is_pole g z*
$\forall z.\ \neg$*is_pole g z* $\longrightarrow$ *zorder f1 z = zorder g z*
$\forall z.$ *is_pole g z* $\longrightarrow$ *zorder f2 z* $= -$*zorder g z*
$\forall z.\ h\ z \neq 0$
$\forall z.\ g\ z = h\ z *$ *f1 z / f2 z*

end

10 Complex Analyticity and the Complexification of Real-Analytic Functions

theory *Complex_Analytic*
imports *HOL−Analysis.Real_Analytic Cauchy_Integral_Formula*
begin

10.1 A CR-linear self-map of the plane is multiplication by a scalar

10.2 Several-variable Cauchy–Riemann criterion

theorem *CauchyRiemann_imp_holomorphic*:
fixes $f :: complex \Rightarrow complex$
assumes *Sopen*: *open S*
and *diff*: $\bigwedge x.\ x \in S \Longrightarrow (f\ has_derivative\ (L\ x))\ (at\ x)$
and *CR*: $\bigwedge x.\ x \in S \Longrightarrow CR_linear\ (L\ x)$
shows *f holomorphic_on S*

theorem *holomorphic_imp_Cinfinity_on*:
assumes *f holomorphic_on U* **and** *open U*
shows *Cinfinity_on f U*

10.3 Real analyticity via holomorphic extension

theorem *real_analytic_at_1d_iff_holo_extension*:
fixes $f :: real \Rightarrow real$
shows $real_analytic_at_1d\ f\ c \longleftrightarrow has_holo_extension_at\ f\ c$

end
theory *Complex_Analysis*
imports
Riemann_Mapping
Residue_Theorem
Weierstrass_Factorization
Complex_Analytic
begin

end

References

[1]