Measure and Probability Theory

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Contents

1 Probability measure **3**
- 1.1 Introduce binder for probability . . . 6
- 1.2 Distributions . . . 9

2 Distribution Functions **16**
- 2.1 Properties of cdf's . . . 17
- 2.2 Uniqueness . . . 18

3 Weak Convergence of Functions and Distributions **20**

4 Weak Convergence of Functions **20**

5 Weak Convergence of Distributions **20**

6 Skorohod's theorem **20**

7 The Giry monad **23**
- 7.1 Sub-probability spaces . . . 23
- 7.2 Properties of "return" . . . 27
- 7.3 Join . . . 30
- 7.4 Giry monad on probability spaces . . . 36

8 Projective Family **39**

9 Infinite Product Measure **44**
- 9.1 Sequence space . . . 46

10 Independent families of events, event sets, and random variables **48**

11 Convolution Measure **55**

12 Information theory **58**
12.1 Information theory . . . 58
12.2 Kullback−Leibler divergence . . . 58
12.3 Finite Entropy . . . 60
12.4 Mutual Information . . . 61
12.5 Entropy . . . 63
12.6 Conditional Mutual Information . . . 64
12.7 Conditional Entropy . . . 65
12.8 Equalities . . . 66

13 Properties of Various Distributions **68**
13.1 Erlang . . . 69
13.2 Exponential distribution . . . 71
13.3 Uniform distribution . . . 73
13.4 Normal distribution . . . 75

14 Characteristic Functions **80**
14.1 Application of the FTC: integrating e^ix . . . 80
14.2 The Characteristic Function of a Real Measure. . . . 80
14.3 Independence . . . 81
14.4 Approximations to e^{ix} . . . 81
14.5 Calculation of the Characteristic Function of the Standard Distribution . . . 83

15 Helly's selection theorem **84**

16 Integral of sinc **85**
16.1 Various preparatory integrals . . . 85

17 The sinc function, and the sine integral (Si) **86**
17.1 The final theorems: boundedness and scalability . . . 87

18 The Levy inversion theorem, and the Levy continuity theorem. **88**
18.1 The Levy inversion theorem . . . 88
18.2 The Levy continuity theorem . . . 89

19 The Central Limit Theorem **89**

20 Probability mass function **91**
20.1 PMF as measure . . . 91
20.2 Monad Interpretation . . . 95
20.3 PMFs as function . . . 100
20.4 Conditional Probabilities . . . 103
20.5 Relator . . . 104

20.6 Distributions . . . 108
20.6.1 Bernoulli Distribution . . . 108
20.6.2 Geometric Distribution . . . 109
20.6.3 Uniform Multiset Distribution . . . 110
20.6.4 Uniform Distribution . . . 110
20.6.5 Poisson Distribution . . . 112
20.6.6 Binomial Distribution . . . 112
20.7 Negative Binomial distribution . . . 114
20.8 PMFs from association lists . . . 116

21 Code generation for PMFs **118**
21.1 General code generation setup . . . 118
21.2 Code abbreviations for integrals and probabilities . . . 125

22 Finite Maps **127**
22.1 Domain and Application . . . 127
22.2 Constructor of Finite Maps . . . 128
22.3 Product set of Finite Maps . . . 128
22.3.1 Basic Properties of Pi' . . . 129
22.4 Topological Space of Finite Maps . . . 129
22.5 Metric Space of Finite Maps . . . 130
22.6 Complete Space of Finite Maps . . . 131
22.7 Second Countable Space of Finite Maps . . . 131
22.8 Polish Space of Finite Maps . . . 132
22.9 Product Measurable Space of Finite Maps . . . 132
22.10 Isomorphism between Functions and Finite Maps . . . 137

23 Projective Limit **139**
23.1 Sequences of Finite Maps in Compact Sets . . . 139
23.2 Daniell-Kolmogorov Theorem . . . 140

24 Random Permutations **141**

25 Discrete subprobability distribution **143**
25.1 Auxiliary material . . . 143
25.1.1 More about extended reals . . . 143
25.1.2 More about *'a option* . . . 144
25.1.3 A relator for sets that treats sets like predicates . . . 146
25.1.4 Monotonicity rules . . . 147
25.1.5 Bijections . . . 147
25.2 Subprobability mass function . . . 147
25.3 Support . . . 150
25.4 Functorial structure . . . 150
25.5 Monad operations . . . 152

25.5.1 Return . . . 152
25.5.2 Bind . . . 152
25.6 Relator . . . 154
25.7 From *'a pmf* to *'a spmf* . . . 156
25.8 Weight of a subprobability . . . 157
25.9 From density to spmfs . . . 159
25.10 Ordering on spmfs . . . 159
25.11 CCPO structure for the flat ccpo *ord-option* (=) . . . 161
25.11.1 Admissibility of *rel-spmf* . . . 166
25.12 Restrictions on spmfs . . . 167
25.13 Subprobability distributions of sets . . . 169
25.14 Losslessness . . . 171
25.15 Scaling . . . 172
25.16 Conditional spmfs . . . 175
25.17 Product spmf . . . 176
25.18 Assertions . . . 177
25.19 Try . . . 178
25.20 Miscellaneous . . . 180

26 Indexed products of PMFs **180**
26.1 Preliminaries . . . 180
26.2 Definition . . . 180
26.3 Dependent product sets with a default . . . 181
26.4 Common PMF operations on products . . . 183
26.5 Merging and splitting PMF products . . . 183
26.6 Additional properties . . . 185
26.7 Applications . . . 186

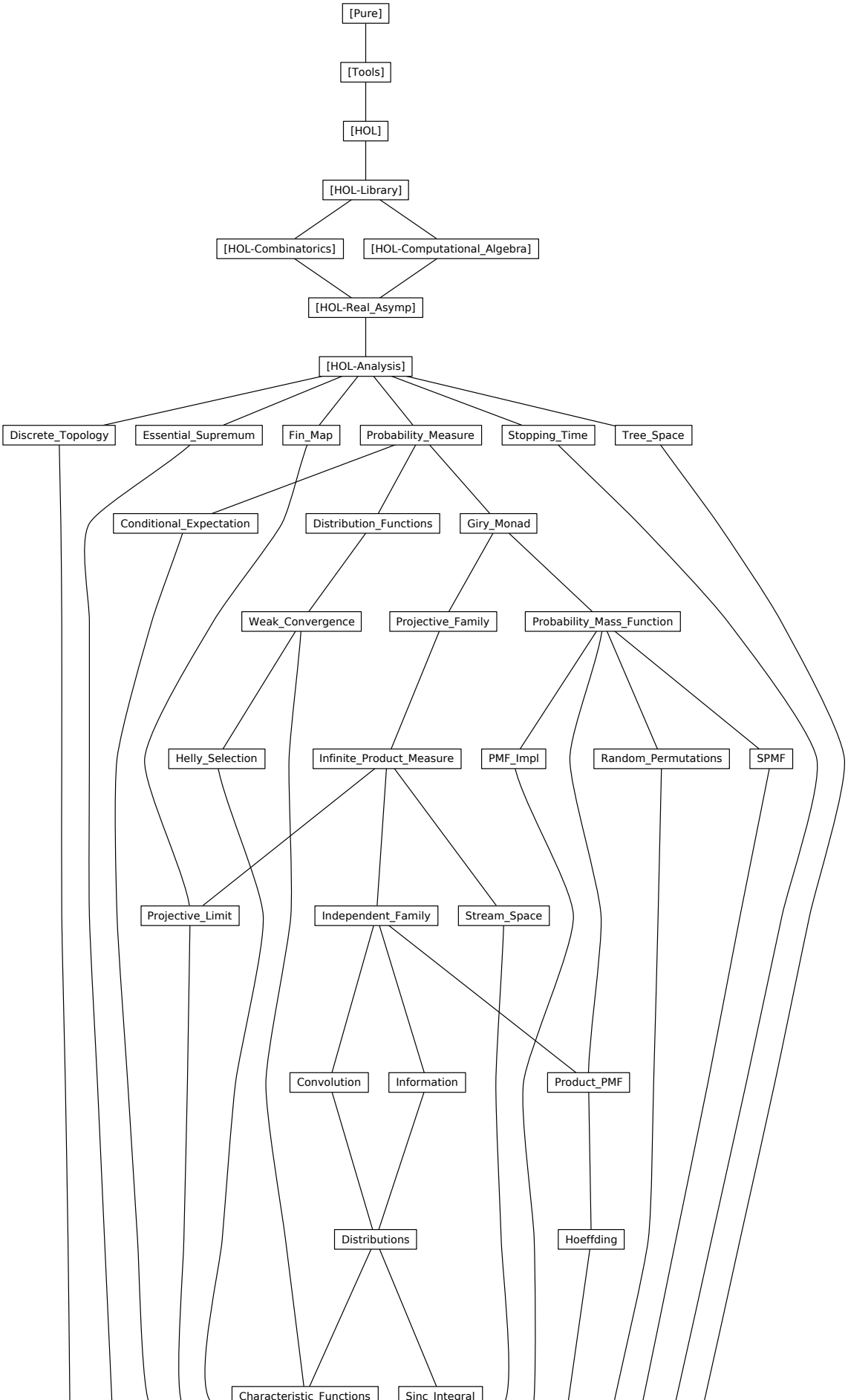
27 Hoeffding's Lemma and Hoeffding's Inequality **186**
27.1 Hoeffding's Lemma . . . 186
27.2 Hoeffding's Inequality . . . 188
27.3 Hoeffding's inequality for i.i.d. bounded random variables . . . 190
27.4 Hoeffding's Inequality for the Binomial distribution . . . 191
27.5 Tail bounds for the negative binomial distribution . . . 193

28 Conditional Expectation **202**
28.1 Restricting a measure to a sub-sigma-algebra . . . 202
28.2 Nonnegative conditional expectation . . . 205
28.3 Real conditional expectation . . . 207

29 The essential supremum **212**

30 Stopping times **214**
30.1 Stopping Time . . . 214

31 Filtration **215**
31.1 σ-algebra of a Stopping Time 215


[Pure]
[Tools]
[HOL]
[HOL-Library]
[HOL-Combinatorics]
[HOL-Computational_Algebra]
[HOL-Real_Asymp]
[HOL-Analysis]
Discrete_Topology
Essential_Supremum
Fin_Map
Probability_Measure
Stopping_Time
Tree_Space
Conditional_Expectation
Distribution_Functions
Giry_Monad
Weak_Convergence
Projective_Family
Probability_Mass_Function
Helly_Selection
Infinite_Product_Measure
PMF_Impl
Random_Permutations
SPMF
Projective_Limit
Independent_Family
Stream_Space
Convolution
Information
Product_PMF
Distributions
Hoeffding
Characteristic_Functions
Sinc_Integral

1 Probability measure

theory *Probability-Measure*
imports *HOL−Analysis.Analysis*
begin

locale *prob-space* = *finite-measure* +
assumes *emeasure-space-1*: *emeasure M* (*space M*) = *1*

lemma *prob-spaceI*[*Pure.intro*!]:
assumes ∗: *emeasure M* (*space M*) = *1*
shows *prob-space M*
⟨*proof*⟩

lemma *prob-space-imp-sigma-finite*: *prob-space M* ⟹ *sigma-finite-measure M*
⟨*proof*⟩

abbreviation (**in** *prob-space*) *events* ≡ *sets M*
abbreviation (**in** *prob-space*) *prob* ≡ *measure M*
abbreviation (**in** *prob-space*) *random-variable M′ X* ≡ *X* ∈ *measurable M M′*
abbreviation (**in** *prob-space*) *expectation* ≡ *integral*^{L} *M*
abbreviation (**in** *prob-space*) *variance X* ≡ *integral*^{L} *M* (λ*x*. (*X x* − *expectation X*)^2)

lemma (**in** *prob-space*) *finite-measure* [*simp*]: *finite-measure M*
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-space-distr*:
assumes *f*: *f* ∈ *measurable M M′* **shows** *prob-space* (*distr M M′ f*)
⟨*proof*⟩

lemma *prob-space-distrD*:
assumes *f*: *f* ∈ *measurable M N* **and** *M*: *prob-space* (*distr M N f*) **shows** *prob-space M*
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-space*: *prob* (*space M*) = *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-le-1*[*simp*, *intro*]: *prob A* ≤ *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *not-empty*: *space M* ≠ {}
⟨*proof*⟩

lemma (**in** *prob-space*) *emeasure-eq-1-AE*:
S ∈ *sets M* ⟹ *AE x in M*. *x* ∈ *S* ⟹ *emeasure M S* = *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *emeasure-le-1*: $emeasure\ M\ S \leq 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *emeasure-ge-1-iff*: $emeasure\ M\ A \geq 1 \longleftrightarrow emeasure\ M\ A = 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-iff-emeasure-eq-1*:
assumes [*measurable*]: *Measurable.pred M P*
shows $(AE\ x\ in\ M.\ P\ x) \longleftrightarrow emeasure\ M\ \{x \in space\ M.\ P\ x\} = 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *measure-le-1*: $emeasure\ M\ X \leq 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *measure-ge-1-iff*: $measure\ M\ A \geq 1 \longleftrightarrow measure\ M\ A = 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-I-eq-1*:
assumes $emeasure\ M\ \{x \in space\ M.\ P\ x\} = 1$ $\{x \in space\ M.\ P\ x\} \in sets\ M$
shows $AE\ x\ in\ M.\ P\ x$
⟨*proof*⟩

lemma *prob-space-restrict-space*:
$S \in sets\ M \Longrightarrow emeasure\ M\ S = 1 \Longrightarrow prob\text{-}space\ (restrict\text{-}space\ M\ S)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-compl*:
assumes *A*: $A \in events$
shows $prob\ (space\ M - A) = 1 - prob\ A$
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-in-set-eq-1*:
assumes *A*[*measurable*]: $A \in events$ **shows** $(AE\ x\ in\ M.\ x \in A) \longleftrightarrow prob\ A = 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-False*: $(AE\ x\ in\ M.\ False) \longleftrightarrow False$
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-prob-1*:
assumes $prob\ A = 1$ **shows** $AE\ x\ in\ M.\ x \in A$
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-const*[*simp*]: $(AE\ x\ in\ M.\ P) \longleftrightarrow P$
⟨*proof*⟩

lemma (**in** *prob-space*) *ae-filter-bot*: $ae\text{-}filter\ M \neq bot$

⟨*proof*⟩

lemma (**in** *prob-space*) *AE-contr*:
assumes *ae*: *AE* ω *in* *M*. *P* ω *AE* ω *in* *M*. ¬ *P* ω
shows *False*
⟨*proof*⟩

lemma (**in** *prob-space*) *integral-ge-const*:
fixes *c* :: *real*
shows *integrable* *M* *f* ⟹ (*AE* *x* *in* *M*. *c* ≤ *f* *x*) ⟹ *c* ≤ (∫ *x*. *f* *x* ∂*M*)
⟨*proof*⟩

lemma (**in** *prob-space*) *integral-le-const*:
fixes *c* :: *real*
shows *integrable* *M* *f* ⟹ (*AE* *x* *in* *M*. *f* *x* ≤ *c*) ⟹ (∫ *x*. *f* *x* ∂*M*) ≤ *c*
⟨*proof*⟩

lemma (**in** *prob-space*) *nn-integral-ge-const*:
(*AE* *x* *in* *M*. *c* ≤ *f* *x*) ⟹ *c* ≤ (∫⁺*x*. *f* *x* ∂*M*)
⟨*proof*⟩

lemma (**in** *prob-space*) *expectation-less*:
fixes *X* :: - ⇒ *real*
assumes [*simp*]: *integrable* *M* *X*
assumes *gt*: *AE* *x* *in* *M*. *X* *x* < *b*
shows *expectation* *X* < *b*
⟨*proof*⟩

lemma (**in** *prob-space*) *expectation-greater*:
fixes *X* :: - ⇒ *real*
assumes [*simp*]: *integrable* *M* *X*
assumes *gt*: *AE* *x* *in* *M*. *a* < *X* *x*
shows *a* < *expectation* *X*
⟨*proof*⟩

lemma (**in** *prob-space*) *jensens-inequality*:
fixes *q* :: *real* ⇒ *real*
assumes *X*: *integrable* *M* *X* *AE* *x* *in* *M*. *X* *x* ∈ *I*
assumes *I*: *I* = {*a* <..< *b*} ∨ *I* = {*a* <..} ∨ *I* = {..< *b*} ∨ *I* = *UNIV*
assumes *q*: *integrable* *M* (λ*x*. *q* (*X* *x*)) *convex-on* *I* *q*
shows *q* (*expectation* *X*) ≤ *expectation* (λ*x*. *q* (*X* *x*))
⟨*proof*⟩

lemma (**in** *prob-space*) *finite-borel-measurable-integrable*:
assumes *f*∈ *borel-measurable* *M*
and *finite* (*f*‘(*space* *M*))
shows *integrable* *M* *f*
⟨*proof*⟩

1.1 Introduce binder for probability

syntax
-prob :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* (‹('𝒫'((/- *in* -./ -)'))›)
syntax-consts
-prob == *measure*
translations
$\mathcal{P}(x\ in\ M.\ P)$ => *CONST measure M* $\{x \in$ *CONST space M*. $P\}$

⟨*ML*⟩

definition
cond-prob M P Q = $\mathcal{P}(\omega\ in\ M.\ P\ \omega \wedge Q\ \omega)$ / $\mathcal{P}(\omega\ in\ M.\ Q\ \omega)$

syntax
-conditional-prob :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* (‹('𝒫'(- *in* -. - |/ -'))›)
syntax-consts
-conditional-prob == *cond-prob*
translations
$\mathcal{P}(x\ in\ M.\ P \mid Q)$ => *CONST cond-prob M* ($\lambda x.\ P$) ($\lambda x.\ Q$)

lemma (**in** *prob-space*) *AE-E-prob*:
assumes *ae*: *AE x in M*. *P x*
obtains *S* **where** $S \subseteq \{x \in$ *space M*. *P x*$\}$ *S* ∈ *events prob S* = *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-neg*: $\{x \in$ *space M*. *P x*$\}$ ∈ *events* ⟹ $\mathcal{P}(x\ in\ M.\ \neg\ P\ x) = 1 - \mathcal{P}(x\ in\ M.\ P\ x)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-eq-AE*:
(*AE x in M*. *P x* ⟷ *Q x*) ⟹ $\{x \in$ *space M*. *P x*$\}$ ∈ *events* ⟹ $\{x \in$ *space M*. *Q x*$\}$ ∈ *events* ⟹ $\mathcal{P}(x\ in\ M.\ P\ x) = \mathcal{P}(x\ in\ M.\ Q\ x)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-eq-0-AE*:
assumes *not*: *AE x in M*. ¬ *P x* **shows** $\mathcal{P}(x\ in\ M.\ P\ x) = 0$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-Collect-eq-0*:
$\{x \in$ *space M*. *P x*$\}$ ∈ *sets M* ⟹ $\mathcal{P}(x\ in\ M.\ P\ x) = 0$ ⟷ (*AE x in M*. ¬ *P x*)
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-Collect-eq-1*:
$\{x \in$ *space M*. *P x*$\}$ ∈ *sets M* ⟹ $\mathcal{P}(x\ in\ M.\ P\ x) = 1$ ⟷ (*AE x in M*. *P x*)
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-eq-0*:
A ∈ *sets M* ⟹ *prob A* = *0* ⟷ (*AE x in M*. *x* ∉ *A*)
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-eq-1*:
A ∈ *sets M* ⟹ *prob A* = *1* ⟷ (*AE x in M*. *x* ∈ *A*)
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-sums*:
assumes *P*: ⋀*n*. {*x*∈*space M*. *P n x*} ∈ *events*
assumes *Q*: {*x*∈*space M*. *Q x*} ∈ *events*
assumes *ae*: *AE x in M*. (∀ *n*. *P n x* ⟶ *Q x*) ∧ (*Q x* ⟶ (∃!*n*. *P n x*))
shows (λ*n*. *P*(*x in M*. *P n x*)) *sums P*(*x in M*. *Q x*)
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-sum*:
assumes [*simp*, *intro*]: *finite I*
assumes *P*: ⋀*n*. *n* ∈ *I* ⟹ {*x*∈*space M*. *P n x*} ∈ *events*
assumes *Q*: {*x*∈*space M*. *Q x*} ∈ *events*
assumes *ae*: *AE x in M*. (∀ *n*∈*I*. *P n x* ⟶ *Q x*) ∧ (*Q x* ⟶ (∃!*n*∈*I*. *P n x*))
shows *P*(*x in M*. *Q x*) = (∑ *n*∈*I*. *P*(*x in M*. *P n x*))
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-EX-countable*:
assumes *sets*: ⋀*i*. *i* ∈ *I* ⟹ {*x*∈*space M*. *P i x*} ∈ *sets M* **and** *I*: *countable I*
assumes *disj*: *AE x in M*. ∀ *i*∈*I*. ∀ *j*∈*I*. *P i x* ⟶ *P j x* ⟶ *i* = *j*
shows *P*(*x in M*. ∃ *i*∈*I*. *P i x*) = (∫ ⁺*i*. *P*(*x in M*. *P i x*) ∂*count-space I*)
⟨*proof*⟩

lemma (**in** *prob-space*) *cond-prob-eq-AE*:
assumes *P*: *AE x in M*. *Q x* ⟶ *P x* ⟷ *P*′ *x* {*x*∈*space M*. *P x*} ∈ *events* {*x*∈*space M*. *P*′ *x*} ∈ *events*
assumes *Q*: *AE x in M*. *Q x* ⟷ *Q*′ *x* {*x*∈*space M*. *Q x*} ∈ *events* {*x*∈*space M*. *Q*′ *x*} ∈ *events*
shows *cond-prob M P Q* = *cond-prob M P*′ *Q*′
⟨*proof*⟩

lemma (**in** *prob-space*) *joint-distribution-Times-le-fst*:
random-variable MX X ⟹ *random-variable MY Y* ⟹ *A* ∈ *sets MX* ⟹ *B* ∈ *sets MY*
⟹ *emeasure* (*distr M* (*MX* ⨂$_M$ *MY*) (λ*x*. (*X x*, *Y x*))) (*A* × *B*) ≤ *emeasure* (*distr M MX X*) *A*
⟨*proof*⟩

lemma (**in** *prob-space*) *joint-distribution-Times-le-snd*:
random-variable MX X ⟹ *random-variable MY Y* ⟹ *A* ∈ *sets MX* ⟹ *B* ∈ *sets MY*
⟹ *emeasure* (*distr M* (*MX* ⨂$_M$ *MY*) (λ*x*. (*X x*, *Y x*))) (*A* × *B*) ≤ *emeasure* (*distr M MY Y*) *B*
⟨*proof*⟩

lemma (**in** *prob-space*) *variance-eq*:
fixes $X :: 'a \Rightarrow real$
assumes [*simp*]: *integrable M X*
assumes [*simp*]: *integrable M* $(\lambda x.\ (X\ x)^2)$
shows *variance X* = *expectation* $(\lambda x.\ (X\ x)^2)$ − (*expectation X*)^2
⟨*proof*⟩

lemma (**in** *prob-space*) *variance-positive*: *0* ≤ *variance* $(X::'a \Rightarrow real)$
⟨*proof*⟩

lemma (**in** *prob-space*) *variance-mean-zero*:
expectation X = *0* ⟹ *variance X* = *expectation* $(\lambda x.\ (X\ x)\hat{}2)$
⟨*proof*⟩

theorem (**in** *prob-space*) *Chebyshev-inequality*:
assumes [*measurable*]: *random-variable borel f*
assumes *integrable M* $(\lambda x.\ f\ x\ \hat{}\ 2)$
defines $\mu \equiv$ *expectation f*
assumes *a* > *0*
shows *prob* $\{x \in$ *space M*. $|f\ x - \mu| \geq a\} \leq$ *variance f* / a^2
unfolding μ-*def*
proof (*rule second-moment-method*)
have *integrable*: *integrable M f*
using *assms* **by** (*blast dest*: *square-integrable-imp-integrable*)
show *integrable M* $(\lambda x.\ (f\ x - expectation\ f)^2)$
using *assms integrable* **unfolding** *power2-eq-square ring-distribs*
by (*intro Bochner-Integration.integrable-diff*) *auto*
qed (*use assms* **in** *auto*)

locale *pair-prob-space* = *pair-sigma-finite M1 M2* + *M1*: *prob-space M1* + *M2*: *prob-space M2* **for** *M1 M2*

sublocale *pair-prob-space* ⊆ *P?*: *prob-space M1* $\bigotimes_M$ *M2*
⟨*proof*⟩

locale *product-prob-space* = *product-sigma-finite M* **for** $M :: 'i \Rightarrow 'a$ *measure* +
fixes $I :: 'i$ *set*
assumes *prob-space*: $\bigwedge i.$ *prob-space* $(M\ i)$

sublocale *product-prob-space* ⊆ *M?*: *prob-space M i* **for** *i*
⟨*proof*⟩

locale *finite-product-prob-space* = *finite-product-sigma-finite M I* + *product-prob-space M I* **for** *M I*

sublocale *finite-product-prob-space* ⊆ *prob-space* $\Pi_M\ i \in I.\ M\ i$
⟨*proof*⟩

lemma (**in** *finite-product-prob-space*) *prob-times*:

assumes X: $\bigwedge i.\ i \in I \Longrightarrow X\ i \in sets\ (M\ i)$
shows $prob\ (\Pi_E\ i{\in}I.\ X\ i) = (\prod i{\in}I.\ M.prob\ i\ (X\ i))$
⟨*proof*⟩

lemma *product-prob-spaceI*:
assumes $\bigwedge i.$ *prob-space* $(M\ i)$
shows *product-prob-space M*
⟨*proof*⟩

1.2 Distributions

definition *distributed* :: *'a measure* ⇒ *'b measure* ⇒ (*'a* ⇒ *'b*) ⇒ (*'b* ⇒ *ennreal*) ⇒ *bool*
where
distributed M N X f ⟷
distr M N X = *density N f* ∧ *f* ∈ *borel-measurable N* ∧ *X* ∈ *measurable M N*

lemma
assumes *distributed M N X f*
shows *distributed-distr-eq-density*: *distr M N X* = *density N f*
and *distributed-measurable*: *X* ∈ *measurable M N*
and *distributed-borel-measurable*: *f* ∈ *borel-measurable N*
⟨*proof*⟩

lemma
assumes *D*: *distributed M N X f*
shows *distributed-measurable'*[*measurable-dest*]:
g ∈ *measurable L M* ⟹ (λ*x*. *X* (*g x*)) ∈ *measurable L N*
and *distributed-borel-measurable'*[*measurable-dest*]:
h ∈ *measurable L N* ⟹ (λ*x*. *f* (*h x*)) ∈ *borel-measurable L*
⟨*proof*⟩

lemma *distributed-real-measurable*:
(⋀*x*. *x* ∈ *space N* ⟹ 0 ≤ *f x*) ⟹ *distributed M N X* (λ*x*. *ennreal* (*f x*)) ⟹ *f* ∈ *borel-measurable N*
⟨*proof*⟩

lemma *distributed-real-measurable'*:
(⋀*x*. *x* ∈ *space N* ⟹ 0 ≤ *f x*) ⟹ *distributed M N X* (λ*x*. *ennreal* (*f x*)) ⟹
h ∈ *measurable L N* ⟹ (λ*x*. *f* (*h x*)) ∈ *borel-measurable L*
⟨*proof*⟩

lemma *joint-distributed-measurable1*:
distributed M (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*)) *f* ⟹ *h1* ∈ *measurable N M* ⟹ (λ*x*. *X* (*h1 x*)) ∈ *measurable N S*
⟨*proof*⟩

lemma *joint-distributed-measurable2*:
distributed M (*S* ⨂$_M$ *T*) (λ*x*. (*X x*, *Y x*)) *f* ⟹ *h2* ∈ *measurable N M* ⟹ (λ*x*.

$Y\ (h2\ x)) \in$ *measurable* $N\ T$
⟨*proof*⟩

lemma *distributed-count-space*:
assumes X: *distributed* M (*count-space* A) $X\ P$ **and** a: $a \in A$ **and** A: *finite* A
shows $P\ a =$ *emeasure* $M\ (X -` \{a\} \cap$ *space* $M)$
⟨*proof*⟩

lemma *distributed-cong-density*:
$(AE\ x$ *in* $N.\ f\ x = g\ x) \Longrightarrow g \in$ *borel-measurable* $N \Longrightarrow f \in$ *borel-measurable* N
$\Longrightarrow$
distributed $M\ N\ X\ f \longleftrightarrow$ *distributed* $M\ N\ X\ g$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-imp-emeasure-nonzero*:
assumes X: *distributed* $M\ MX\ X\ Px$
shows *emeasure* $MX\ \{x \in$ *space* $MX.\ Px\ x \neq 0\} \neq 0$
⟨*proof*⟩

lemma *subdensity*:
assumes T: $T \in$ *measurable* $P\ Q$
assumes f: *distributed* $M\ P\ X\ f$
assumes g: *distributed* $M\ Q\ Y\ g$
assumes Y: $Y = T \circ X$
shows $AE\ x$ *in* $P.\ g\ (T\ x) = 0 \longrightarrow f\ x = 0$
⟨*proof*⟩

lemma *subdensity-real*:
fixes $g :: 'a \Rightarrow$ *real* **and** $f :: 'b \Rightarrow$ *real*
assumes T: $T \in$ *measurable* $P\ Q$
assumes f: *distributed* $M\ P\ X\ f$
assumes g: *distributed* $M\ Q\ Y\ g$
assumes Y: $Y = T \circ X$
shows $(AE\ x$ *in* $P.\ 0 \leq g\ (T\ x)) \Longrightarrow (AE\ x$ *in* $P.\ 0 \leq f\ x) \Longrightarrow AE\ x$ *in* $P.\ g\ (T\ x) = 0 \longrightarrow f\ x = 0$
⟨*proof*⟩

lemma *distributed-emeasure*:
distributed $M\ N\ X\ f \Longrightarrow A \in$ *sets* $N \Longrightarrow$ *emeasure* $M\ (X -` A \cap$ *space* $M) = (\int^+ x.\ f\ x * indicator\ A\ x\ \partial N)$
⟨*proof*⟩

lemma *distributed-nn-integral*:
distributed $M\ N\ X\ f \Longrightarrow g \in$ *borel-measurable* $N \Longrightarrow (\int^+ x.\ f\ x * g\ x\ \partial N) = (\int^+ x.\ g\ (X\ x)\ \partial M)$
⟨*proof*⟩

lemma *distributed-integral*:
distributed $M\ N\ X\ f \Longrightarrow g \in$ *borel-measurable* $N \Longrightarrow (\bigwedge x.\ x \in$ *space* $N \Longrightarrow 0 \leq$

$f\ x) \Longrightarrow$
$(\int x.\ f\ x * g\ x\ \partial N) = (\int x.\ g\ (X\ x)\ \partial M)$
⟨*proof*⟩

lemma *distributed-transform-integral*:
assumes *Px*: *distributed M N X Px* $\bigwedge x.\ x \in$ *space N* $\Longrightarrow 0 \leq Px\ x$
assumes *distributed M P Y Py* $\bigwedge x.\ x \in$ *space P* $\Longrightarrow 0 \leq Py\ x$
assumes *Y*: $Y = T \circ X$ **and** *T*: $T \in$ *measurable N P* **and** *f*: $f \in$ *borel-measurable P*
shows $(\int x.\ Py\ x * f\ x\ \partial P) = (\int x.\ Px\ x * f\ (T\ x)\ \partial N)$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-unique*:
assumes *Px*: *distributed M S X Px*
assumes *Py*: *distributed M S X Py*
shows *AE x in S. Px x = Py x*
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-jointI*:
assumes *sigma-finite-measure S sigma-finite-measure T*
assumes *X*[*measurable*]: $X \in$ *measurable M S* **and** *Y*[*measurable*]: $Y \in$ *measurable M T*
assumes [*measurable*]: $f \in$ *borel-measurable* $(S \bigotimes_M T)$ **and** *f*: *AE x in* $S \bigotimes_M T$. $0 \leq f\ x$
assumes *eq*: $\bigwedge A\ B.\ A \in$ *sets S* $\Longrightarrow B \in$ *sets T* $\Longrightarrow$
emeasure M $\{x \in$ *space M*. $X\ x \in A \wedge Y\ x \in B\} = (\int^+ x.\ (\int^+ y.\ f\ (x,\ y) *$ *indicator B y* $\partial T) *$ *indicator A x* $\partial S)$
shows *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))\ f$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-swap*:
assumes *sigma-finite-measure S sigma-finite-measure T*
assumes *Pxy*: *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
shows *distributed M* $(T \bigotimes_M S)$ $(\lambda x.\ (Y\ x,\ X\ x))$ $(\lambda(x,\ y).\ Pxy\ (y,\ x))$
⟨*proof*⟩

lemma (**in** *prob-space*) *distr-marginal1*:
assumes *sigma-finite-measure S sigma-finite-measure T*
assumes *Pxy*: *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
defines $Px \equiv \lambda x.\ (\int^+ z.\ Pxy\ (x,\ z)\ \partial T)$
shows *distributed M S X Px*
⟨*proof*⟩

lemma (**in** *prob-space*) *distr-marginal2*:
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *Pxy*: *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
shows *distributed M T Y* $(\lambda y.\ (\int^+ x.\ Pxy\ (x,\ y)\ \partial S))$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-marginal-eq-joint1*:
assumes T: *sigma-finite-measure T*
assumes S: *sigma-finite-measure S*
assumes Px: *distributed M S X Px*
assumes Pxy: *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
shows *AE x in S. Px x* $= (\int^+ y.\ Pxy\ (x,\ y)\ \partial T)$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-marginal-eq-joint2*:
assumes T: *sigma-finite-measure T*
assumes S: *sigma-finite-measure S*
assumes Py: *distributed M T Y Py*
assumes Pxy: *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
shows *AE y in T. Py y* $= (\int^+ x.\ Pxy\ (x,\ y)\ \partial S)$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-joint-indep'*:
assumes S: *sigma-finite-measure S* **and** T: *sigma-finite-measure T*
assumes X[*measurable*]: *distributed M S X Px* **and** Y[*measurable*]: *distributed M T Y Py*
assumes *indep*: *distr M S X* $\bigotimes_M$ *distr M T Y* $=$ *distr M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$
shows *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ $(\lambda(x,\ y).\ Px\ x * Py\ y)$
⟨*proof*⟩

lemma *distributed-integrable*:
distributed M N X f $\Longrightarrow$ *g* $\in$ *borel-measurable N* $\Longrightarrow$ $(\bigwedge x.\ x \in space\ N \Longrightarrow 0 \leq f\ x) \Longrightarrow$
integrable N $(\lambda x.\ f\ x * g\ x) \longleftrightarrow$ *integrable M* $(\lambda x.\ g\ (X\ x))$
⟨*proof*⟩

lemma *distributed-transform-integrable*:
assumes Px: *distributed M N X Px* $\bigwedge x.\ x \in space\ N \Longrightarrow 0 \leq Px\ x$
assumes *distributed M P Y Py* $\bigwedge x.\ x \in space\ P \Longrightarrow 0 \leq Py\ x$
assumes Y: $Y = (\lambda x.\ T\ (X\ x))$ **and** T: $T \in measurable\ N\ P$ **and** f: $f \in$ *borel-measurable P*
shows *integrable P* $(\lambda x.\ Py\ x * f\ x) \longleftrightarrow$ *integrable N* $(\lambda x.\ Px\ x * f\ (T\ x))$
⟨*proof*⟩

lemma *distributed-integrable-var*:
fixes $X :: 'a \Rightarrow real$
shows *distributed M lborel X* $(\lambda x.\ ennreal\ (f\ x)) \Longrightarrow (\bigwedge x.\ 0 \leq f\ x) \Longrightarrow$
integrable lborel $(\lambda x.\ f\ x * x) \Longrightarrow$ *integrable M X*
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-variance*:
fixes f::*real* $\Rightarrow$ *real*
assumes D: *distributed M lborel X f* **and** [*simp*]: $\bigwedge x.\ 0 \leq f\ x$
shows *variance X* $= (\int x.\ x^2 * f\ (x + expectation\ X)\ \partial lborel)$

⟨*proof*⟩

lemma (**in** *prob-space*) *variance-affine*:
fixes f::*real* ⇒ *real*
assumes [*arith*]: $b \neq 0$
assumes D[*intro*]: *distributed M lborel X f*
assumes [*simp*]: *prob-space* (*density lborel f*)
assumes I[*simp*]: *integrable M X*
assumes *I2*[*simp*]: *integrable M* ($\lambda x.$ $(X\ x)^2$)
shows *variance* ($\lambda x.$ $a + b * X\ x$) = b^2 * *variance X*
⟨*proof*⟩

definition
simple-distributed M X f ⟷
($\forall x.$ $0 \leq f\ x$) ∧
distributed M (*count-space* (*X*‘*space M*)) *X* ($\lambda x.$ *ennreal* (*f x*)) ∧
finite (*X*‘*space M*)

lemma *simple-distributed-nonneg*[*dest*]: *simple-distributed M X f* ⟹ $0 \leq f\ x$
⟨*proof*⟩

lemma *simple-distributed*:
simple-distributed M X Px ⟹ *distributed M* (*count-space* (*X*‘*space M*)) *X Px*
⟨*proof*⟩

lemma *simple-distributed-finite*[*dest*]: *simple-distributed M X P* ⟹ *finite* (*X*‘*space M*)
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-simple-function-superset*:
assumes *X*: *simple-function M X* ⋀$x.$ $x \in X$ ‘ *space M* ⟹ *P x* = *measure M* (X −‘ {x} ∩ *space M*)
assumes *A*: *X*‘*space M* ⊆ *A finite A*
defines *S* ≡ *count-space A* **and** *P′* ≡ ($\lambda x.$ *if x* ∈ *X*‘*space M then P x else* 0)
shows *distributed M S X P′*
⟨*proof*⟩

lemma (**in** *prob-space*) *simple-distributedI*:
assumes *X*: *simple-function M X*
⋀$x.$ $0 \leq P\ x$
⋀$x.$ $x \in X$ ‘ *space M* ⟹ *P x* = *measure M* (X −‘ {x} ∩ *space M*)
shows *simple-distributed M X P*
⟨*proof*⟩

lemma *simple-distributed-joint-finite*:
assumes *X*: *simple-distributed M* ($\lambda x.$ ($X\ x$, $Y\ x$)) *Px*
shows *finite* (*X* ‘ *space M*) *finite* (*Y* ‘ *space M*)
⟨*proof*⟩

lemma *simple-distributed-joint2-finite*:
assumes *X*: *simple-distributed M* (λ*x*. (*X x*, *Y x*, *Z x*)) *Px*
shows *finite* (*X* ' *space M*) *finite* (*Y* ' *space M*) *finite* (*Z* ' *space M*)
⟨*proof*⟩

lemma *simple-distributed-simple-function*:
simple-distributed M X Px ⟹ *simple-function M X*
⟨*proof*⟩

lemma *simple-distributed-measure*:
simple-distributed M X P ⟹ *a* ∈ *X*'*space M* ⟹ *P a* = *measure M* (*X* −' {*a*} ∩ *space M*)
⟨*proof*⟩

lemma (**in** *prob-space*) *simple-distributed-joint*:
assumes *X*: *simple-distributed M* (λ*x*. (*X x*, *Y x*)) *Px*
defines *S* ≡ *count-space* (*X*'*space M*) ⨂$_M$ *count-space* (*Y*'*space M*)
defines *P* ≡ (λ*x*. *if x* ∈ (λ*x*. (*X x*, *Y x*))'*space M then Px x else 0*)
shows *distributed M S* (λ*x*. (*X x*, *Y x*)) *P*
⟨*proof*⟩

lemma (**in** *prob-space*) *simple-distributed-joint2*:
assumes *X*: *simple-distributed M* (λ*x*. (*X x*, *Y x*, *Z x*)) *Px*
defines *S* ≡ *count-space* (*X*'*space M*) ⨂$_M$ *count-space* (*Y*'*space M*) ⨂$_M$ *count-space* (*Z*'*space M*)
defines *P* ≡ (λ*x*. *if x* ∈ (λ*x*. (*X x*, *Y x*, *Z x*))'*space M then Px x else 0*)
shows *distributed M S* (λ*x*. (*X x*, *Y x*, *Z x*)) *P*
⟨*proof*⟩

lemma (**in** *prob-space*) *simple-distributed-sum-space*:
assumes *X*: *simple-distributed M X f*
shows *sum f* (*X*'*space M*) = *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-marginal-eq-joint-simple*:
assumes *Px*: *simple-function M X*
assumes *Py*: *simple-distributed M Y Py*
assumes *Pxy*: *simple-distributed M* (λ*x*. (*X x*, *Y x*)) *Pxy*
assumes *y*: *y* ∈ *Y*'*space M*
shows *Py y* = (∑ *x*∈*X*'*space M*. *if* (*x*, *y*) ∈ (λ*x*. (*X x*, *Y x*)) ' *space M then Pxy* (*x*, *y*) *else 0*)
⟨*proof*⟩

lemma *distributedI-real*:
fixes *f* :: '*a* ⇒ *real*
assumes *gen*: *sets M1* = *sigma-sets* (*space M1*) *E* **and** *Int-stable E*
and *A*: *range A* ⊆ *E* (⋃ *i*::*nat*. *A i*) = *space M1* ⋀*i*. *emeasure* (*distr M M1 X*) (*A i*) ≠ ∞
and *X*: *X* ∈ *measurable M M1*

and *f*: $f \in$ *borel-measurable M1 AE x in M1*. $0 \leq f\ x$
and *eq*: $\bigwedge A$. $A \in E \Longrightarrow$ *emeasure M* $(X -` A \cap$ *space M*$) = (\int^+ x.\ f\ x * indicator\ A\ x\ \partial M1)$
shows *distributed M M1 X f*
⟨*proof*⟩

lemma *distributedI-borel-atMost*:
fixes $f :: real \Rightarrow real$
assumes [*measurable*]: $X \in$ *borel-measurable M*
and [*measurable*]: $f \in$ *borel-measurable borel* **and** *f*[*simp*]: *AE x in lborel*. $0 \leq f\ x$
and *g-eq*: $\bigwedge a$. $(\int^+ x.\ f\ x * indicator\ \{..a\}\ x\ \partial lborel) = ennreal\ (g\ a)$
and *M-eq*: $\bigwedge a$. *emeasure M* $\{x \in$ *space M*. $X\ x \leq a\} = ennreal\ (g\ a)$
shows *distributed M lborel X f*
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-params*:
assumes *X*: *distributed M MX X* $(\lambda x.$ *indicator A x / measure MX A*$)$
shows $A \in$ *sets MX measure MX A* $\neq 0$
⟨*proof*⟩

lemma *prob-space-uniform-measure*:
assumes *A*: *emeasure M A* $\neq 0$ *emeasure M A* $\neq \infty$
shows *prob-space* (*uniform-measure M A*)
⟨*proof*⟩

lemma *prob-space-uniform-count-measure*: *finite* $A \Longrightarrow A \neq \{\} \Longrightarrow$ *prob-space* (*uniform-count-measure A*)
⟨*proof*⟩

lemma (**in** *prob-space*) *measure-uniform-measure-eq-cond-prob*:
assumes [*measurable*]: *Measurable.pred M P Measurable.pred M Q*
shows $\mathcal{P}(x$ *in uniform-measure M* $\{x \in$ *space M*. *Q x*$\}$. *P x*$) = \mathcal{P}(x$ *in M*. *P x* | *Q x*$)$
⟨*proof*⟩

lemma *prob-space-point-measure*:
finite $S \Longrightarrow (\bigwedge s.\ s \in S \Longrightarrow 0 \leq p\ s) \Longrightarrow (\sum s \in S.\ p\ s) = 1 \Longrightarrow$ *prob-space* (*point-measure S p*)
⟨*proof*⟩

lemma (**in** *prob-space*) *distr-pair-fst*: *distr* $(N \bigotimes_M M)\ N\ fst = N$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *distr-reorder*:
assumes *inj-on t J* $t \in J \to K$ *finite K*
shows *distr* $(PiM\ K\ M)\ (Pi_M\ J\ (\lambda x.\ M\ (t\ x)))\ (\lambda \omega.\ \lambda n \in J.\ \omega\ (t\ n)) = PiM\ J\ (\lambda x.\ M\ (t\ x))$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *distr-restrict*:
$J \subseteq K \Longrightarrow$ *finite* $K \Longrightarrow (\Pi_M\ i{\in}J.\ M\ i) = distr\ (\Pi_M\ i{\in}K.\ M\ i)\ (\Pi_M\ i{\in}J.\ M\ i)\ (\lambda f.\ restrict\ f\ J)$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *emeasure-prod-emb*[*simp*]:
assumes *L*: $J \subseteq L$ *finite* L **and** *X*: $X \in sets\ (Pi_M\ J\ M)$
shows $emeasure\ (Pi_M\ L\ M)\ (prod\text{-}emb\ L\ M\ J\ X) = emeasure\ (Pi_M\ J\ M)\ X$
⟨*proof*⟩

lemma *emeasure-distr-restrict*:
assumes $I \subseteq K$ **and** Q[*measurable-cong*]: $sets\ Q = sets\ (PiM\ K\ M)$ **and** A[*measurable*]: $A \in sets\ (PiM\ I\ M)$
shows $emeasure\ (distr\ Q\ (PiM\ I\ M)\ (\lambda\omega.\ restrict\ \omega\ I))\ A = emeasure\ Q\ (prod\text{-}emb\ K\ M\ I\ A)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-space-completion*: *prob-space* (*completion* M)
⟨*proof*⟩

lemma *distr-PiM-finite-prob-space*:
assumes *fin*: *finite* I
assumes *product-prob-space* M
assumes *product-prob-space* M'
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow f \in measurable\ (M\ i)\ (M'\ i)$
shows $distr\ (PiM\ I\ M)\ (PiM\ I\ M')\ (compose\ I\ f) = PiM\ I\ (\lambda i.\ distr\ (M\ i)\ (M'\ i)\ f)$
⟨*proof*⟩

end

2 Distribution Functions

Shows that the cumulative distribution function (cdf) of a distribution (a measure on the reals) is nondecreasing and right continuous, which tends to 0 and 1 in either direction.

Conversely, every such function is the cdf of a unique distribution. This direction defines the measure in the obvious way on half-open intervals, and then applies the Caratheodory extension theorem.

theory *Distribution-Functions*
imports *Probability-Measure*
begin

lemma *UN-Ioc-eq-UNIV*: $(\bigcup n.\ \{-real\ n <..\ real\ n\}) = UNIV$
⟨*proof*⟩

2.1 Properties of cdf's

definition
cdf :: *real measure* ⇒ *real* ⇒ *real*
where
cdf M ≡ λ*x*. *measure M* {..*x*}

lemma *cdf-def2*: *cdf M x* = *measure M* {..*x*}
⟨*proof*⟩

locale *finite-borel-measure* = *finite-measure M* **for** *M* :: *real measure* +
assumes *M-is-borel*: *sets M* = *sets borel*
begin

lemma *sets-M*[*intro*]: *a* ∈ *sets borel* ⟹ *a* ∈ *sets M*
⟨*proof*⟩

lemma *cdf-diff-eq*:
assumes *x* < *y*
shows *cdf M y* − *cdf M x* = *measure M* {*x*<..*y*}
⟨*proof*⟩

lemma *cdf-nondecreasing*: *x* ≤ *y* ⟹ *cdf M x* ≤ *cdf M y*
⟨*proof*⟩

lemma *borel-UNIV*: *space M* = *UNIV*
⟨*proof*⟩

lemma *cdf-nonneg*: *cdf M x* ≥ *0*
⟨*proof*⟩

lemma *cdf-bounded*: *cdf M x* ≤ *measure M* (*space M*)
⟨*proof*⟩

lemma *cdf-lim-infty*:
(λ*i*. *cdf M* (*real i*)) ⟶ *measure M* (*space M*))
⟨*proof*⟩

lemma *cdf-lim-at-top*: (*cdf M* ⟶ *measure M* (*space M*)) *at-top*
⟨*proof*⟩

lemma *cdf-lim-neg-infty*: ((λ*i*. *cdf M* (− *real i*)) ⟶ *0*)
⟨*proof*⟩

lemma *cdf-lim-at-bot*: (*cdf M* ⟶ *0*) *at-bot*
⟨*proof*⟩

lemma *cdf-is-right-cont*: *continuous* (*at-right a*) (*cdf M*)
⟨*proof*⟩

lemma *cdf-at-left*: $(cdf\ M \longrightarrow measure\ M\ \{..<a\})\ (at\text{-}left\ a)$
⟨*proof*⟩

lemma *isCont-cdf*: $isCont\ (cdf\ M)\ x \longleftrightarrow measure\ M\ \{x\} = 0$
⟨*proof*⟩

lemma *countable-atoms*: $countable\ \{x.\ measure\ M\ \{x\} > 0\}$
⟨*proof*⟩

end

locale *real-distribution* = *prob-space M* **for** *M* :: *real measure* +
assumes *events-eq-borel* [*simp*, *measurable-cong*]: *sets M* = *sets borel*
begin

lemma *finite-borel-measure-M*: *finite-borel-measure M*
⟨*proof*⟩

sublocale *finite-borel-measure M*
⟨*proof*⟩

lemma *space-eq-univ* [*simp*]: *space M* = *UNIV*
⟨*proof*⟩

lemma *cdf-bounded-prob*: $\bigwedge x.\ cdf\ M\ x \leq 1$
⟨*proof*⟩

lemma *cdf-lim-infty-prob*: $(\lambda i.\ cdf\ M\ (real\ i)) \longrightarrow 1$
⟨*proof*⟩

lemma *cdf-lim-at-top-prob*: $(cdf\ M \longrightarrow 1)\ at\text{-}top$
⟨*proof*⟩

lemma *measurable-finite-borel* [*simp*]:
$f \in borel\text{-}measurable\ borel \Longrightarrow f \in borel\text{-}measurable\ M$
⟨*proof*⟩

end

lemma (**in** *prob-space*) *real-distribution-distr* [*intro*, *simp*]:
$random\text{-}variable\ borel\ X \Longrightarrow real\text{-}distribution\ (distr\ M\ borel\ X)$
⟨*proof*⟩

2.2 Uniqueness

lemma (**in** *finite-borel-measure*) *emeasure-Ioc*:
assumes $a \leq b$ **shows** $emeasure\ M\ \{a <..\ b\} = cdf\ M\ b - cdf\ M\ a$
⟨*proof*⟩

lemma *cdf-unique'*:
fixes *M1 M2*
assumes *finite-borel-measure M1* **and** *finite-borel-measure M2*
assumes *cdf M1 = cdf M2*
shows *M1 = M2*
⟨*proof*⟩

lemma *cdf-unique*:
real-distribution M1 ⟹ real-distribution M2 ⟹ cdf M1 = cdf M2 ⟹ M1 = M2
⟨*proof*⟩

lemma
fixes *F* :: *real ⇒ real*
assumes *nondecF* : ⋀ *x y*. *x ≤ y ⟹ F x ≤ F y*
and *right-cont-F* : ⋀*a*. *continuous* (*at-right a*) *F*
and *lim-F-at-bot* : (*F* ⟶ *0*) *at-bot*
and *lim-F-at-top* : (*F* ⟶ *m*) *at-top*
and *m*: *0 ≤ m*
shows *interval-measure-UNIV*: *emeasure* (*interval-measure F*) *UNIV = m*
and *finite-borel-measure-interval-measure*: *finite-borel-measure* (*interval-measure F*)
⟨*proof*⟩

lemma *real-distribution-interval-measure*:
fixes *F* :: *real ⇒ real*
assumes *nondecF* : ⋀ *x y*. *x ≤ y ⟹ F x ≤ F y* **and**
right-cont-F : ⋀*a*. *continuous* (*at-right a*) *F* **and**
lim-F-at-bot : (*F* ⟶ *0*) *at-bot* **and**
lim-F-at-top : (*F* ⟶ *1*) *at-top*
shows *real-distribution* (*interval-measure F*)
⟨*proof*⟩

lemma
fixes *F* :: *real ⇒ real*
assumes *nondecF* : ⋀ *x y*. *x ≤ y ⟹ F x ≤ F y* **and**
right-cont-F : ⋀*a*. *continuous* (*at-right a*) *F* **and**
lim-F-at-bot : (*F* ⟶ *0*) *at-bot*
shows *emeasure-interval-measure-Iic*: *emeasure* (*interval-measure F*) {.. *x*} = *F x*
and *measure-interval-measure-Iic*: *measure* (*interval-measure F*) {.. *x*} = *F x*
⟨*proof*⟩

lemma *cdf-interval-measure*:
(⋀ *x y*. *x ≤ y ⟹ F x ≤ F y*) ⟹ (⋀*a*. *continuous* (*at-right a*) *F*) ⟹ (*F* ⟶ *0*) *at-bot ⟹ cdf* (*interval-measure F*) = *F*
⟨*proof*⟩

end

3 Weak Convergence of Functions and Distributions

Properties of weak convergence of functions and measures, including the portmanteau theorem.

theory *Weak-Convergence*
imports *Distribution-Functions*
begin

4 Weak Convergence of Functions

definition
weak-conv :: $(nat \Rightarrow (real \Rightarrow real)) \Rightarrow (real \Rightarrow real) \Rightarrow bool$
where
$weak\text{-}conv\ F\text{-}seq\ F \equiv \forall x.\ isCont\ F\ x \longrightarrow (\lambda n.\ F\text{-}seq\ n\ x) \longrightarrow F\ x$

5 Weak Convergence of Distributions

definition
weak-conv-m :: $(nat \Rightarrow real\ measure) \Rightarrow real\ measure \Rightarrow bool$
where
$weak\text{-}conv\text{-}m\ M\text{-}seq\ M \equiv weak\text{-}conv\ (\lambda n.\ cdf\ (M\text{-}seq\ n))\ (cdf\ M)$

6 Skorohod's theorem

locale *right-continuous-mono* =
fixes $f :: real \Rightarrow real$ **and** $a\ b :: real$
assumes *cont*: $\bigwedge x.\ continuous\ (at\text{-}right\ x)\ f$
assumes *mono*: $mono\ f$
assumes *bot*: $(f \longrightarrow a)\ at\text{-}bot$
assumes *top*: $(f \longrightarrow b)\ at\text{-}top$
begin

abbreviation $I :: real \Rightarrow real$ **where**
$I\ \omega \equiv Inf\ \{x.\ \omega \leq f\ x\}$

lemma *pseudoinverse*: **assumes** $a < \omega\ \omega < b$ **shows** $\omega \leq f\ x \longleftrightarrow I\ \omega \leq x$
⟨*proof*⟩

lemma *pseudoinverse'*: $\forall \omega \in \{a<..<b\}.\ \forall x.\ \omega \leq f\ x \longleftrightarrow I\ \omega \leq x$
⟨*proof*⟩

lemma *mono-I*: $mono\text{-}on\ \{a <..< b\}\ I$
⟨*proof*⟩

end

locale *cdf-distribution = real-distribution*
begin

abbreviation $C \equiv cdf\ M$

sublocale *right-continuous-mono C 0 1*
⟨*proof*⟩

lemma *measurable-C*[*measurable*]: $C \in borel\text{-}measurable\ borel$
⟨*proof*⟩

lemma *measurable-CI*[*measurable*]: $I \in borel\text{-}measurable\ (restrict\text{-}space\ borel\ \{0<..<1\})$
⟨*proof*⟩

lemma *emeasure-distr-I*: $emeasure\ (distr\ (restrict\text{-}space\ lborel\ \{0<..<1::real\})\ borel\ I)\ UNIV = 1$
⟨*proof*⟩

lemma *distr-I-eq-M*: $distr\ (restrict\text{-}space\ lborel\ \{0<..<1::real\})\ borel\ I = M$ (**is** *?I = -*)
⟨*proof*⟩

end

context
fixes $\mu :: nat \Rightarrow real\ measure$
and $M :: real\ measure$
assumes μ: $\bigwedge n.\ real\text{-}distribution\ (\mu\ n)$
assumes M: $real\text{-}distribution\ M$
assumes $\mu\text{-}to\text{-}M$: $weak\text{-}conv\text{-}m\ \mu\ M$
begin

theorem *Skorohod*:
$\exists\ (\Omega :: real\ measure)\ (Y\text{-}seq :: nat \Rightarrow real \Rightarrow real)\ (Y :: real \Rightarrow real).$
$prob\text{-}space\ \Omega \wedge$
$(\forall n.\ Y\text{-}seq\ n \in measurable\ \Omega\ borel) \wedge$
$(\forall n.\ distr\ \Omega\ borel\ (Y\text{-}seq\ n) = \mu\ n) \wedge$
$Y \in measurable\ \Omega\ lborel \wedge$
$distr\ \Omega\ borel\ Y = M \wedge$
$(\forall x \in space\ \Omega.\ (\lambda n.\ Y\text{-}seq\ n\ x) \longrightarrow Y\ x)$
⟨*proof*⟩

The Portmanteau theorem, that is, the equivalence of various definitions of weak convergence.

theorem *weak-conv-imp-bdd-ae-continuous-conv*:
fixes
$f :: real \Rightarrow 'a::\{banach, second\text{-}countable\text{-}topology\}$

assumes
discont-null: $M\ (\{x.\ \neg\ isCont\ f\ x\}) = 0$ **and**
f-bdd: $\bigwedge x.\ norm\ (f\ x) \leq B$ **and**
[*measurable*]: $f \in borel\text{-}measurable\ borel$
shows
$(\lambda\ n.\ integral^L\ (\mu\ n)\ f) \longrightarrow integral^L\ M\ f$
⟨*proof*⟩

theorem *weak-conv-imp-integral-bdd-continuous-conv*:
fixes f :: *real* ⇒ $'a$::{*banach*, *second-countable-topology*}
assumes
$\bigwedge x.\ isCont\ f\ x$ **and**
$\bigwedge x.\ norm\ (f\ x) \leq B$
shows
$(\lambda\ n.\ integral^L\ (\mu\ n)\ f) \longrightarrow integral^L\ M\ f$
⟨*proof*⟩

theorem *weak-conv-imp-continuity-set-conv*:
fixes f :: *real* ⇒ *real*
assumes [*measurable*]: $A \in sets\ borel$ **and** $M\ (frontier\ A) = 0$
shows $(\lambda n.\ measure\ (\mu\ n)\ A) \longrightarrow measure\ M\ A$
⟨*proof*⟩

end

definition
cts-step :: *real* ⇒ *real* ⇒ *real* ⇒ *real*
where
$cts\text{-}step\ a\ b\ x \equiv if\ x \leq a\ then\ 1\ else\ if\ x \geq b\ then\ 0\ else\ (b - x)\ /\ (b - a)$

lemma *cts-step-uniformly-continuous*:
assumes [*arith*]: $a < b$
shows *uniformly-continuous-on UNIV* (*cts-step a b*)
⟨*proof*⟩

lemma (**in** *real-distribution*) *integrable-cts-step*: $a < b \Longrightarrow integrable\ M\ (cts\text{-}step\ a\ b)$
⟨*proof*⟩

lemma (**in** *real-distribution*) *cdf-cts-step*:
assumes [*arith*]: $x < y$
shows $cdf\ M\ x \leq integral^L\ M\ (cts\text{-}step\ x\ y)$ **and** $integral^L\ M\ (cts\text{-}step\ x\ y) \leq cdf\ M\ y$
⟨*proof*⟩

context
fixes *M-seq* :: *nat* ⇒ *real measure*
and *M* :: *real measure*
assumes *distr-M-seq* [*simp*]: $\bigwedge n.\ real\text{-}distribution\ (M\text{-}seq\ n)$

assumes *distr-M* [*simp*]: *real-distribution M*
begin

theorem *continuity-set-conv-imp-weak-conv*:
fixes *f* :: *real* ⇒ *real*
assumes ∗: ⋀*A*. *A* ∈ *sets borel* ⟹ *M* (*frontier A*) = *0* ⟹ (λ *n*. (*measure* (*M-seq n*) *A*)) ⟶ *measure M A*
shows *weak-conv-m M-seq M*
⟨*proof*⟩

theorem *integral-cts-step-conv-imp-weak-conv*:
assumes *integral-conv*: ⋀*x y*. *x* < *y* ⟹ (λ*n*. *integral*^{L} (*M-seq n*) (*cts-step x y*)) ⟶ *integral*^{L} *M* (*cts-step x y*)
shows *weak-conv-m M-seq M*
⟨*proof*⟩

theorem *integral-bdd-continuous-conv-imp-weak-conv*:
assumes
⋀*f*. (⋀*x*. *isCont f x*) ⟹ (⋀*x*. *abs* (*f x*) ≤ *1*) ⟹ (λ*n*. *integral*^{L} (*M-seq n*) *f*::*real*) ⟶ *integral*^{L} *M f*
shows
weak-conv-m M-seq M
⟨*proof*⟩

end

end

7 The Giry monad

theory *Giry-Monad*
imports *Probability-Measure HOL−Library.Monad-Syntax*
begin

7.1 Sub-probability spaces

locale *subprob-space* = *finite-measure* +
assumes *emeasure-space-le-1*: *emeasure M* (*space M*) ≤ *1*
assumes *subprob-not-empty*: *space M* ≠ {}

lemma *subprob-spaceI*[*Pure.intro*!]:
assumes ∗: *emeasure M* (*space M*) ≤ *1*
assumes *space M* ≠ {}
shows *subprob-space M*
⟨*proof*⟩

lemma (**in** *subprob-space*) *emeasure-subprob-space-less-top*: *emeasure M A* ≠ *top*
⟨*proof*⟩

lemma *prob-space-imp-subprob-space*:
prob-space M $\Longrightarrow$ *subprob-space M*
⟨*proof*⟩

lemma *subprob-space-imp-sigma-finite*: *subprob-space M* $\Longrightarrow$ *sigma-finite-measure M*
⟨*proof*⟩

sublocale *prob-space* $\subseteq$ *subprob-space*
⟨*proof*⟩

lemma *subprob-space-sigma* [*simp*]: $\Omega \neq \{\}$ $\Longrightarrow$ *subprob-space* (*sigma* Ω *X*)
⟨*proof*⟩

lemma *subprob-space-null-measure*: *space M* $\neq \{\}$ $\Longrightarrow$ *subprob-space* (*null-measure M*)
⟨*proof*⟩

lemma (**in** *subprob-space*) *subprob-space-distr*:
assumes *f*: *f* $\in$ *measurable M M'* **and** *space M'* $\neq \{\}$ **shows** *subprob-space* (*distr M M' f*)
⟨*proof*⟩

lemma (**in** *subprob-space*) *subprob-emeasure-le-1*: *emeasure M X* $\leq$ *1*
⟨*proof*⟩

lemma (**in** *subprob-space*) *subprob-measure-le-1*: *measure M X* $\leq$ *1*
⟨*proof*⟩

lemma (**in** *subprob-space*) *nn-integral-le-const*:
assumes *0* $\leq$ *c AE x in M*. *f x* $\leq$ *c*
shows $(\int^+ x.\ f\ x\ \partial M) \leq c$
⟨*proof*⟩

lemma *emeasure-density-distr-interval*:
fixes *h* :: *real* $\Rightarrow$ *real* **and** *g* :: *real* $\Rightarrow$ *real* **and** *g'* :: *real* $\Rightarrow$ *real*
assumes [*simp*]: *a* $\leq$ *b*
assumes *Mf*[*measurable*]: *f* $\in$ *borel-measurable borel*
assumes *Mg*[*measurable*]: *g* $\in$ *borel-measurable borel*
assumes *Mg'*[*measurable*]: *g'* $\in$ *borel-measurable borel*
assumes *Mh*[*measurable*]: *h* $\in$ *borel-measurable borel*
assumes *prob*: *subprob-space* (*density lborel f*)
assumes *nonnegf*: $\bigwedge x.\ f\ x \geq 0$
assumes *derivg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow$ (*g has-real-derivative g' x*) (*at x*)
assumes *contg'*: *continuous-on* $\{a..b\}$ *g'*
assumes *mono*: *strict-mono-on* $\{a..b\}$ *g* **and** *inv*: $\bigwedge x.\ h\ x \in \{a..b\} \Longrightarrow g\ (h\ x) = x$
assumes *range*: $\{a..b\} \subseteq$ *range h*
shows *emeasure* (*distr* (*density lborel f*) *lborel h*) $\{a..b\}$ =

emeasure (*density lborel* (λ*x*. *f* (*g x*) ∗ *g*′ *x*)) {*a*..*b*}
⟨*proof*⟩

locale *pair-subprob-space* =
pair-sigma-finite M1 M2 + *M1*: *subprob-space M1* + *M2*: *subprob-space M2* **for** *M1 M2*

sublocale *pair-subprob-space* ⊆ *P?*: *subprob-space M1* ⨂$_M$ *M2*
⟨*proof*⟩

lemma *subprob-space-null-measure-iff*:
subprob-space (*null-measure M*) ⟷ *space M* ≠ {}
⟨*proof*⟩

lemma *subprob-space-restrict-space*:
assumes *M*: *subprob-space M*
and *A*: *A* ∩ *space M* ∈ *sets M A* ∩ *space M* ≠ {}
shows *subprob-space* (*restrict-space M A*)
⟨*proof*⟩

definition *subprob-algebra* :: *'a measure* ⇒ *'a measure measure* **where**
subprob-algebra K =
(*SUP A* ∈ *sets K*. *vimage-algebra* {*M*. *subprob-space M* ∧ *sets M* = *sets K*}
(λ*M*. *emeasure M A*) *borel*)

lemma *space-subprob-algebra*: *space* (*subprob-algebra A*) = {*M*. *subprob-space M* ∧ *sets M* = *sets A*}
⟨*proof*⟩

lemma *subprob-algebra-cong*: *sets M* = *sets N* ⟹ *subprob-algebra M* = *subprob-algebra N*
⟨*proof*⟩

lemma *measurable-emeasure-subprob-algebra*[*measurable*]:
a ∈ *sets A* ⟹ (λ*M*. *emeasure M a*) ∈ *borel-measurable* (*subprob-algebra A*)
⟨*proof*⟩

lemma *measurable-measure-subprob-algebra*[*measurable*]:
a ∈ *sets A* ⟹ (λ*M*. *measure M a*) ∈ *borel-measurable* (*subprob-algebra A*)
⟨*proof*⟩

lemma *subprob-measurableD*:
assumes *N*: *N* ∈ *measurable M* (*subprob-algebra S*) **and** *x*: *x* ∈ *space M*
shows *space* (*N x*) = *space S*
and *sets* (*N x*) = *sets S*
and *measurable* (*N x*) *K* = *measurable S K*
and *measurable K* (*N x*) = *measurable K S*
⟨*proof*⟩

⟨*ML*⟩

context
fixes *K M N* **assumes** *K*: $K \in$ *measurable M* (*subprob-algebra N*)
begin

lemma *subprob-space-kernel*: $a \in$ *space M* $\Longrightarrow$ *subprob-space* (*K a*)
⟨*proof*⟩

lemma *sets-kernel*: $a \in$ *space M* $\Longrightarrow$ *sets* (*K a*) = *sets N*
⟨*proof*⟩

lemma *measurable-emeasure-kernel*[*measurable*]:
$A \in$ *sets N* $\Longrightarrow$ (λa. *emeasure* (*K a*) *A*) $\in$ *borel-measurable M*
⟨*proof*⟩

end

lemma *measurable-subprob-algebra*:
($\bigwedge a$. $a \in$ *space M* $\Longrightarrow$ *subprob-space* (*K a*)) $\Longrightarrow$
($\bigwedge a$. $a \in$ *space M* $\Longrightarrow$ *sets* (*K a*) = *sets N*) $\Longrightarrow$
($\bigwedge A$. $A \in$ *sets N* $\Longrightarrow$ (λa. *emeasure* (*K a*) *A*) $\in$ *borel-measurable M*) $\Longrightarrow$
K $\in$ *measurable M* (*subprob-algebra N*)
⟨*proof*⟩

lemma *measurable-submarkov*:
K $\in$ *measurable M* (*subprob-algebra M*) $\longleftrightarrow$
($\forall x \in$*space M*. *subprob-space* (*K x*) $\wedge$ *sets* (*K x*) = *sets M*) $\wedge$
($\forall A \in$*sets M*. (λx. *emeasure* (*K x*) *A*) $\in$ *measurable M borel*)
⟨*proof*⟩

lemma *measurable-subprob-algebra-generated*:
assumes *eq*: *sets N* = *sigma-sets* Ω *G* **and** *Int-stable G G* $\subseteq$ *Pow* Ω
assumes *subsp*: $\bigwedge a$. $a \in$ *space M* $\Longrightarrow$ *subprob-space* (*K a*)
assumes *sets*: $\bigwedge a$. $a \in$ *space M* $\Longrightarrow$ *sets* (*K a*) = *sets N*
assumes $\bigwedge A$. $A \in G \Longrightarrow$ (λa. *emeasure* (*K a*) *A*) $\in$ *borel-measurable M*
assumes Ω: (λa. *emeasure* (*K a*) Ω) $\in$ *borel-measurable M*
shows *K* $\in$ *measurable M* (*subprob-algebra N*)
⟨*proof*⟩

lemma *space-subprob-algebra-empty-iff*:
space (*subprob-algebra N*) = {} $\longleftrightarrow$ *space N* = {}
⟨*proof*⟩

lemma *nn-integral-measurable-subprob-algebra*[*measurable*]:
assumes *f*: *f* $\in$ *borel-measurable N*
shows (λM. *integral*^{N} *M f*) $\in$ *borel-measurable* (*subprob-algebra N*) (**is** - $\in$ *?B*)
⟨*proof*⟩

lemma *measurable-distr*:
assumes [*measurable*]: $f \in$ *measurable* M N
shows $(\lambda M'.$ *distr* M' N $f) \in$ *measurable* (*subprob-algebra* M) (*subprob-algebra* N)
⟨*proof*⟩

lemma *emeasure-space-subprob-algebra*[*measurable*]:
$(\lambda a.$ *emeasure* a (*space* a)) $\in$ *borel-measurable* (*subprob-algebra* N)
⟨*proof*⟩

lemma *integrable-measurable-subprob-algebra*[*measurable*]:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes [*measurable*]: $f \in$ *borel-measurable* N
shows *Measurable.pred* (*subprob-algebra* N) $(\lambda M.$ *integrable* M $f)$
⟨*proof*⟩

lemma *integral-measurable-subprob-algebra*[*measurable*]:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes f [*measurable*]: $f \in$ *borel-measurable* N
shows $(\lambda M.$ *integral*^{L} M $f) \in$ *subprob-algebra* $N \rightarrow_M$ *borel*
⟨*proof*⟩

lemma *measurable-pair-measure*:
assumes f: $f \in$ *measurable* M (*subprob-algebra* N)
assumes g: $g \in$ *measurable* M (*subprob-algebra* L)
shows $(\lambda x.$ f $x \bigotimes_M$ g $x) \in$ *measurable* M (*subprob-algebra* $(N \bigotimes_M L)$)
⟨*proof*⟩

lemma *restrict-space-measurable*:
assumes X: $X \neq \{\}$ $X \in$ *sets* K
assumes N: $N \in$ *measurable* M (*subprob-algebra* K)
shows $(\lambda x.$ *restrict-space* $(N\,x)$ $X) \in$ *measurable* M (*subprob-algebra* (*restrict-space* K X))
⟨*proof*⟩

7.2 Properties of "return"

definition *return* :: $'a$ *measure* $\Rightarrow$ $'a \Rightarrow$ $'a$ *measure* **where**
return R x = *measure-of* (*space* R) (*sets* R) $(\lambda A.$ *indicator* A $x)$

lemma *space-return*[*simp*]: *space* (*return* M x) = *space* M
⟨*proof*⟩

lemma *sets-return*[*simp*]: *sets* (*return* M x) = *sets* M
⟨*proof*⟩

lemma *measurable-return1*[*simp*]: *measurable* (*return* N x) L = *measurable* N L
⟨*proof*⟩

lemma *measurable-return2*[*simp*]: *measurable L* (*return N x*) = *measurable L N*
⟨*proof*⟩

lemma *return-sets-cong*: *sets M* = *sets N* ⟹ *return M* = *return N*
⟨*proof*⟩

lemma *return-cong*: *sets A* = *sets B* ⟹ *return A x* = *return B x*
⟨*proof*⟩

lemma *emeasure-return*[*simp*]:
assumes *A* ∈ *sets M*
shows *emeasure* (*return M x*) *A* = *indicator A x*
⟨*proof*⟩

lemma *prob-space-return*: *x* ∈ *space M* ⟹ *prob-space* (*return M x*)
⟨*proof*⟩

lemma *subprob-space-return*: *x* ∈ *space M* ⟹ *subprob-space* (*return M x*)
⟨*proof*⟩

lemma *subprob-space-return-ne*:
assumes *space M* ≠ {} **shows** *subprob-space* (*return M x*)
⟨*proof*⟩

lemma *measure-return*: **assumes** *X*: *X* ∈ *sets M* **shows** *measure* (*return M x*) *X* = *indicator X x*
⟨*proof*⟩

lemma *AE-return*:
assumes [*simp*]: *x* ∈ *space M* **and** [*measurable*]: *Measurable.pred M P*
shows (*AE y in return M x*. *P y*) ⟷ *P x*
⟨*proof*⟩

lemma *nn-integral-return*:
assumes *x* ∈ *space M g* ∈ *borel-measurable M*
shows (∫⁺ *a*. *g a* ∂*return M x*) = *g x*
⟨*proof*⟩

lemma *integral-return*:
fixes *g* :: - ⇒ *'a* :: {*banach*, *second-countable-topology*}
assumes *x* ∈ *space M g* ∈ *borel-measurable M*
shows (∫ *a*. *g a* ∂*return M x*) = *g x*
⟨*proof*⟩

lemma *return-measurable*[*measurable*]: *return N* ∈ *measurable N* (*subprob-algebra N*)
⟨*proof*⟩

lemma *distr-return*:
assumes $f \in$ *measurable* M N **and** $x \in$ *space* M
shows *distr* (*return* M x) N f = *return* N (f x)
⟨*proof*⟩

lemma *return-restrict-space*:
$\Omega \in$ *sets* $M \Longrightarrow$ *return* (*restrict-space* M Ω) x = *restrict-space* (*return* M x) Ω
⟨*proof*⟩

lemma *measurable-distr2*:
assumes f[*measurable*]: *case-prod* $f \in$ *measurable* ($L \bigotimes_M M$) N
assumes g[*measurable*]: $g \in$ *measurable* L (*subprob-algebra* M)
shows (λx. *distr* (g x) N (f x)) $\in$ *measurable* L (*subprob-algebra* N)
⟨*proof*⟩

lemma *nn-integral-measurable-subprob-algebra2*:
assumes f[*measurable*]: ($\lambda(x, y)$. f x y) $\in$ *borel-measurable* ($M \bigotimes_M N$)
assumes N[*measurable*]: $L \in$ *measurable* M (*subprob-algebra* N)
shows (λx. *integral*^{N} (L x) (f x)) $\in$ *borel-measurable* M
⟨*proof*⟩

lemma *emeasure-measurable-subprob-algebra2*:
assumes A[*measurable*]: (*SIGMA* x:*space* M. A x) $\in$ *sets* ($M \bigotimes_M N$)
assumes L[*measurable*]: $L \in$ *measurable* M (*subprob-algebra* N)
shows (λx. *emeasure* (L x) (A x)) $\in$ *borel-measurable* M
⟨*proof*⟩

lemma *measure-measurable-subprob-algebra2*:
assumes A[*measurable*]: (*SIGMA* x:*space* M. A x) $\in$ *sets* ($M \bigotimes_M N$)
assumes L[*measurable*]: $L \in$ *measurable* M (*subprob-algebra* N)
shows (λx. *measure* (L x) (A x)) $\in$ *borel-measurable* M
⟨*proof*⟩

definition *select-sets* M = (*SOME* N. *sets* M = *sets* (*subprob-algebra* N))

lemma *select-sets1*:
sets M = *sets* (*subprob-algebra* N) $\Longrightarrow$ *sets* M = *sets* (*subprob-algebra* (*select-sets* M))
⟨*proof*⟩

lemma *sets-select-sets*[*simp*]:
assumes *sets*: *sets* M = *sets* (*subprob-algebra* N)
shows *sets* (*select-sets* M) = *sets* N
⟨*proof*⟩

lemma *space-select-sets*[*simp*]:
sets M = *sets* (*subprob-algebra* N) $\Longrightarrow$ *space* (*select-sets* M) = *space* N
⟨*proof*⟩

7.3 Join

definition *join* :: *'a measure measure ⇒ 'a measure* **where**
join M = measure-of (space (select-sets M)) (sets (select-sets M)) (λB. ∫⁺ M'. emeasure M' B ∂M)

lemma
shows *space-join*[*simp*]: *space (join M) = space (select-sets M)*
and *sets-join*[*simp*]: *sets (join M) = sets (select-sets M)*
⟨*proof*⟩

lemma *emeasure-join*:
assumes *M*[*simp*, *measurable-cong*]: *sets M = sets (subprob-algebra N)* **and** *A*: *A ∈ sets N*
shows *emeasure (join M) A = (∫⁺ M'. emeasure M' A ∂M)*
⟨*proof*⟩

lemma *measurable-join*:
join ∈ measurable (subprob-algebra (subprob-algebra N)) (subprob-algebra N)
⟨*proof*⟩

lemma *nn-integral-join*:
assumes *f*: *f ∈ borel-measurable N*
and *M*[*measurable-cong*]: *sets M = sets (subprob-algebra N)*
shows *(∫⁺x. f x ∂join M) = (∫⁺M'. ∫⁺x. f x ∂M' ∂M)*
⟨*proof*⟩

lemma *measurable-join1*:
⟦ f ∈ measurable N K; sets M = sets (subprob-algebra N) ⟧
⟹ f ∈ measurable (join M) K
⟨*proof*⟩

lemma
fixes *f* :: *- ⇒ real*
assumes *f-measurable* [*measurable*]: *f ∈ borel-measurable N*
and *f-bounded*: *⋀x. x ∈ space N ⟹ |f x| ≤ B*
and *M* [*measurable-cong*]: *sets M = sets (subprob-algebra N)*
and *fin*: *finite-measure M*
and *M-bounded*: *AE M' in M. emeasure M' (space M') ≤ ennreal B'*
shows *integrable-join*: *integrable (join M) f* (**is** *?integrable*)
and *integral-join*: *integral^L (join M) f = ∫ M'. integral^L M' f ∂M* (**is** *?integral*)
⟨*proof*⟩

lemma *join-assoc*:
assumes *M*[*measurable-cong*]: *sets M = sets (subprob-algebra (subprob-algebra N))*
shows *join (distr M (subprob-algebra N) join) = join (join M)*
⟨*proof*⟩

lemma *join-return*:

assumes *sets M* = *sets N* **and** *subprob-space M*
shows *join* (*return* (*subprob-algebra N*) *M*) = *M*
⟨*proof*⟩

lemma *join-return'*:
assumes *sets N* = *sets M*
shows *join* (*distr M* (*subprob-algebra N*) (*return N*)) = *M*
⟨*proof*⟩

lemma *join-distr-distr*:
fixes *f* :: *'a* ⇒ *'b* **and** *M* :: *'a measure measure* **and** *N* :: *'b measure*
assumes *sets M* = *sets* (*subprob-algebra R*) **and** *f* ∈ *measurable R N*
shows *join* (*distr M* (*subprob-algebra N*) (λ*M*. *distr M N f*)) = *distr* (*join M*) *N f* (**is** *?r* = *?l*)
⟨*proof*⟩

definition *bind* :: *'a measure* ⇒ (*'a* ⇒ *'b measure*) ⇒ *'b measure* **where**
bind M f = (*if space M* = {} *then count-space* {} *else*
join (*distr M* (*subprob-algebra* (*f* (*SOME x*. *x* ∈ *space M*))) *f*))

adhoc-overloading *Monad-Syntax.bind* ⇌ *bind*

lemma *bind-empty*:
space M = {} ⟹ *bind M f* = *count-space* {}
⟨*proof*⟩

lemma *bind-nonempty*:
space M ≠ {} ⟹ *bind M f* = *join* (*distr M* (*subprob-algebra* (*f* (*SOME x*. *x* ∈ *space M*))) *f*)
⟨*proof*⟩

lemma *sets-bind-empty*: *sets M* = {} ⟹ *sets* (*bind M f*) = {{}}
⟨*proof*⟩

lemma *space-bind-empty*: *space M* = {} ⟹ *space* (*bind M f*) = {}
⟨*proof*⟩

lemma *sets-bind*[*simp*, *measurable-cong*]:
assumes *f*: ⋀*x*. *x* ∈ *space M* ⟹ *sets* (*f x*) = *sets N* **and** *M*: *space M* ≠ {}
shows *sets* (*bind M f*) = *sets N*
⟨*proof*⟩

lemma *space-bind*[*simp*]:
assumes ⋀*x*. *x* ∈ *space M* ⟹ *sets* (*f x*) = *sets N* **and** *space M* ≠ {}
shows *space* (*bind M f*) = *space N*
⟨*proof*⟩

lemma *bind-cong-All*:
assumes ∀ *x* ∈ *space M*. *f x* = *g x*

shows *bind M f = bind M g*
⟨*proof*⟩

lemma *bind-cong*:
M = N ⟹ (⋀x. x ∈ space M ⟹ f x = g x) ⟹ bind M f = bind N g
⟨*proof*⟩

lemma *bind-nonempty′*:
assumes *f ∈ measurable M (subprob-algebra N) x ∈ space M*
shows *bind M f = join (distr M (subprob-algebra N) f)*
⟨*proof*⟩

lemma *bind-nonempty″*:
assumes *f ∈ measurable M (subprob-algebra N) space M ≠ {}*
shows *bind M f = join (distr M (subprob-algebra N) f)*
⟨*proof*⟩

lemma *emeasure-bind*:
⟦space M ≠ {}; f ∈ measurable M (subprob-algebra N);X ∈ sets N⟧
⟹ emeasure (M ⤜ f) X = ∫⁺x. emeasure (f x) X ∂M
⟨*proof*⟩

lemma *nn-integral-bind*:
assumes *f*: *f ∈ borel-measurable B*
assumes *N*: *N ∈ measurable M (subprob-algebra B)*
shows *(∫⁺x. f x ∂(M ⤜ N)) = (∫⁺x. ∫⁺y. f y ∂N x ∂M)*
⟨*proof*⟩

lemma *AE-bind*:
assumes *N*[*measurable*]: *N ∈ measurable M (subprob-algebra B)*
assumes *P*[*measurable*]: *Measurable.pred B P*
shows *(AE x in M ⤜ N. P x) ⟷ (AE x in M. AE y in N x. P y)*
⟨*proof*⟩

lemma *measurable-bind′*:
assumes *M1*: *f ∈ measurable M (subprob-algebra N)* **and**
M2: *case-prod g ∈ measurable (M ⨂M N) (subprob-algebra R)*
shows *(λx. bind (f x) (g x)) ∈ measurable M (subprob-algebra R)*
⟨*proof*⟩

lemma *measurable-bind*[*measurable (raw)*]:
assumes *M1*: *f ∈ measurable M (subprob-algebra N)* **and**
M2: *(λx. g (fst x) (snd x)) ∈ measurable (M ⨂M N) (subprob-algebra R)*
shows *(λx. bind (f x) (g x)) ∈ measurable M (subprob-algebra R)*
⟨*proof*⟩

lemma *measurable-bind2*:
assumes *f ∈ measurable M (subprob-algebra N)* **and** *g ∈ measurable N (subprob-algebra R)*

shows $(\lambda x.\ bind\ (f\ x)\ g) \in measurable\ M\ (subprob\text{-}algebra\ R)$
⟨*proof*⟩

lemma *subprob-space-bind*:
assumes $subprob\text{-}space\ M\ f \in measurable\ M\ (subprob\text{-}algebra\ N)$
shows $subprob\text{-}space\ (M \gg\!= f)$
⟨*proof*⟩

lemma
fixes $f :: - \Rightarrow real$
assumes *f-measurable* [*measurable*]: $f \in borel\text{-}measurable\ K$
and *f-bounded*: $\bigwedge x.\ x \in space\ K \Longrightarrow |f\ x| \leq B$
and N [*measurable*]: $N \in measurable\ M\ (subprob\text{-}algebra\ K)$
and *fin*: *finite-measure M*
and *M-bounded*: $AE\ x\ in\ M.\ emeasure\ (N\ x)\ (space\ (N\ x)) \leq ennreal\ B'$
shows *integrable-bind*: $integrable\ (bind\ M\ N)\ f$ (**is** *?integrable*)
and *integral-bind*: $integral^L\ (bind\ M\ N)\ f = \int x.\ integral^L\ (N\ x)\ f\ \partial M$ (**is** *?integral*)
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-space-bind*:
assumes *ae*: $AE\ x\ in\ M.\ prob\text{-}space\ (N\ x)$
and N[*measurable*]: $N \in measurable\ M\ (subprob\text{-}algebra\ S)$
shows $prob\text{-}space\ (M \gg\!= N)$
⟨*proof*⟩

lemma (**in** *subprob-space*) *bind-in-space*:
$A \in measurable\ M\ (subprob\text{-}algebra\ N) \Longrightarrow (M \gg\!= A) \in space\ (subprob\text{-}algebra\ N)$
⟨*proof*⟩

lemma (**in** *subprob-space*) *measure-bind*:
assumes *f*: $f \in measurable\ M\ (subprob\text{-}algebra\ N)$ **and** *X*: $X \in sets\ N$
shows $measure\ (M \gg\!= f)\ X = \int x.\ measure\ (f\ x)\ X\ \partial M$
⟨*proof*⟩

lemma *emeasure-bind-const*:
$space\ M \neq \{\} \Longrightarrow X \in sets\ N \Longrightarrow subprob\text{-}space\ N \Longrightarrow$
$emeasure\ (M \gg\!= (\lambda x.\ N))\ X = emeasure\ N\ X * emeasure\ M\ (space\ M)$
⟨*proof*⟩

lemma *emeasure-bind-const'*:
assumes *subprob-space M subprob-space N*
shows $emeasure\ (M \gg\!= (\lambda x.\ N))\ X = emeasure\ N\ X * emeasure\ M\ (space\ M)$
⟨*proof*⟩

lemma *emeasure-bind-const-prob-space*:
assumes *prob-space M subprob-space N*
shows $emeasure\ (M \gg\!= (\lambda x.\ N))\ X = emeasure\ N\ X$

⟨*proof*⟩

lemma *bind-return*:
assumes $f \in$ *measurable* M (*subprob-algebra* N) **and** $x \in$ *space* M
shows *bind* (*return* M x) $f = f\ x$
⟨*proof*⟩

lemma *bind-return'*:
shows *bind* M (*return* M) $= M$
⟨*proof*⟩

lemma *distr-bind*:
assumes N: $N \in$ *measurable* M (*subprob-algebra* K) *space* $M \neq \{\}$
assumes f: $f \in$ *measurable* K R
shows *distr* ($M \gg\!= N$) R f $= (M \gg\!= (\lambda x.\ \textit{distr}\ (N\ x)\ R\ f))$
⟨*proof*⟩

lemma *bind-distr*:
assumes f[*measurable*]: $f \in$ *measurable* M X
assumes N[*measurable*]: $N \in$ *measurable* X (*subprob-algebra* K) **and** *space* $M \neq \{\}$
shows (*distr* M X $f \gg\!= N$) $= (M \gg\!= (\lambda x.\ N\ (f\ x)))$
⟨*proof*⟩

lemma *bind-count-space-singleton*:
assumes *subprob-space* ($f\ x$)
shows *count-space* $\{x\} \gg\!= f = f\ x$
⟨*proof*⟩

lemma *restrict-space-bind*:
assumes N: $N \in$ *measurable* M (*subprob-algebra* K)
assumes *space* $M \neq \{\}$
assumes X[*simp*]: $X \in$ *sets* K $X \neq \{\}$
shows *restrict-space* (*bind* M N) X $=$ *bind* M ($\lambda x.$ *restrict-space* ($N\ x$) X)
⟨*proof*⟩

lemma *bind-restrict-space*:
assumes A: $A \cap$ *space* $M \neq \{\}$ $A \cap$ *space* $M \in$ *sets* M
and f: $f \in$ *measurable* (*restrict-space* M A) (*subprob-algebra* N)
shows *restrict-space* M $A \gg\!= f = M \gg\!= (\lambda x.$ *if* $x \in A$ *then* $f\ x$ *else null-measure* (f (*SOME* $x.\ x \in A \wedge x \in$ *space* M)))
(**is** *?lhs* $=$ *?rhs* **is** - $= M \gg\!=$ *?f*)
⟨*proof*⟩

lemma *bind-const'*: ⟦*prob-space* M; *subprob-space* N⟧ $\Longrightarrow M \gg\!= (\lambda x.\ N) = N$
⟨*proof*⟩

lemma *bind-return-distr*:
assumes *space* $M \neq \{\}$ $f \in$ *measurable* M N

shows *bind M (return N ∘ f) = distr M N f*
⟨*proof*⟩

lemma *bind-return-distr'*:
space M ≠ {} ⟹ f ∈ measurable M N ⟹ bind M (λx. return N (f x)) = distr M N f
⟨*proof*⟩

lemma *bind-assoc*:
fixes *f :: 'a ⇒ 'b measure* **and** *g :: 'b ⇒ 'c measure*
assumes *M1*: *f ∈ measurable M (subprob-algebra N)* **and** *M2*: *g ∈ measurable N (subprob-algebra R)*
shows *bind (bind M f) g = bind M (λx. bind (f x) g)*
⟨*proof*⟩

lemma *double-bind-assoc*:
assumes *Mg*: *g ∈ measurable N (subprob-algebra N')*
assumes *Mf*: *f ∈ measurable M (subprob-algebra M')*
assumes *Mh*: *case-prod h ∈ measurable (M ⨂M M') N*
shows *do {x ← M; y ← f x; g (h x y)} = do {x ← M; y ← f x; return N (h x y)} ⤜ g*
⟨*proof*⟩

lemma (**in** *prob-space*) *M-in-subprob*[*measurable* (*raw*)]: *M ∈ space (subprob-algebra M)*
⟨*proof*⟩

lemma (**in** *pair-prob-space*) *pair-measure-eq-bind*:
(M1 ⨂M M2) = (M1 ⤜ (λx. M2 ⤜ (λy. return (M1 ⨂M M2) (x, y))))
⟨*proof*⟩

lemma (**in** *pair-prob-space*) *bind-rotate*:
assumes *C*[*measurable*]: *(λ(x, y). C x y) ∈ measurable (M1 ⨂M M2) (subprob-algebra N)*
shows *(M1 ⤜ (λx. M2 ⤜ (λy. C x y))) = (M2 ⤜ (λy. M1 ⤜ (λx. C x y)))*
⟨*proof*⟩

lemma *bind-return''*: *sets M = sets N ⟹ M ⤜ return N = M*
⟨*proof*⟩

lemma (**in** *prob-space*) *distr-const*[*simp*]:
c ∈ space N ⟹ distr M N (λx. c) = return N c
⟨*proof*⟩

lemma *return-count-space-eq-density*:
return (count-space M) x = density (count-space M) (indicator {x})
⟨*proof*⟩

lemma *null-measure-in-space-subprob-algebra* [*simp*]:
null-measure M ∈ *space* (*subprob-algebra M*) ⟷ *space M* ≠ {}
⟨*proof*⟩

7.4 Giry monad on probability spaces

definition *prob-algebra* :: *'a measure* ⇒ *'a measure measure* **where**
prob-algebra K = *restrict-space* (*subprob-algebra K*) {*M*. *prob-space M*}

lemma *space-prob-algebra*: *space* (*prob-algebra M*) = {*N*. *sets N* = *sets M* ∧ *prob-space N*}
⟨*proof*⟩

lemma *measurable-measure-prob-algebra*[*measurable*]:
a ∈ *sets A* ⟹ (λ*M*. *Sigma-Algebra.measure M a*) ∈ *prob-algebra A* →$_M$ *borel*
⟨*proof*⟩

lemma *measurable-prob-algebraD*:
f ∈ *N* →$_M$ *prob-algebra M* ⟹ *f* ∈ *N* →$_M$ *subprob-algebra M*
⟨*proof*⟩

lemma *measure-measurable-prob-algebra2*:
Sigma (*space M*) *A* ∈ *sets* (*M* ⨂$_M$ *N*) ⟹ *L* ∈ *M* →$_M$ *prob-algebra N* ⟹
(λ*x*. *Sigma-Algebra.measure* (*L x*) (*A x*)) ∈ *borel-measurable M*
⟨*proof*⟩

lemma *measurable-prob-algebraI*:
(⋀*x*. *x* ∈ *space N* ⟹ *prob-space* (*f x*)) ⟹ *f* ∈ *N* →$_M$ *subprob-algebra M* ⟹
f ∈ *N* →$_M$ *prob-algebra M*
⟨*proof*⟩

lemma *measurable-distr-prob-space*:
assumes *f*: *f* ∈ *M* →$_M$ *N*
shows (λ*M'*. *distr M' N f*) ∈ *prob-algebra M* →$_M$ *prob-algebra N*
⟨*proof*⟩

lemma *measurable-return-prob-space*[*measurable*]: *return N* ∈ *N* →$_M$ *prob-algebra N*
⟨*proof*⟩

lemma *measurable-distr-prob-space2*[*measurable* (*raw*)]:
assumes *f*: *g* ∈ *L* →$_M$ *prob-algebra M* (λ(*x*, *y*). *f x y*) ∈ *L* ⨂$_M$ *M* →$_M$ *N*
shows (λ*x*. *distr* (*g x*) *N* (*f x*)) ∈ *L* →$_M$ *prob-algebra N*
⟨*proof*⟩

lemma *measurable-bind-prob-space*:
assumes *f*: *f* ∈ *M* →$_M$ *prob-algebra N* **and** *g*: *g* ∈ *N* →$_M$ *prob-algebra R*
shows (λ*x*. *bind* (*f x*) *g*) ∈ *M* →$_M$ *prob-algebra R*
⟨*proof*⟩

lemma *measurable-bind-prob-space2*[*measurable* (*raw*)]:
assumes *f*: $f \in M \to_M$ *prob-algebra N* **and** *g*: $(\lambda(x, y).\ g\ x\ y) \in (M \bigotimes_M N) \to_M$ *prob-algebra R*
shows $(\lambda x.\ bind\ (f\ x)\ (g\ x)) \in M \to_M$ *prob-algebra R*
⟨*proof*⟩

lemma *measurable-prob-algebra-generated*:
assumes *eq*: *sets N* = *sigma-sets* Ω *G* **and** *Int-stable G* $G \subseteq$ *Pow* Ω
assumes *subsp*: $\bigwedge a.\ a \in space\ M \Longrightarrow prob\text{-}space\ (K\ a)$
assumes *sets*: $\bigwedge a.\ a \in space\ M \Longrightarrow sets\ (K\ a) = sets\ N$
assumes $\bigwedge A.\ A \in G \Longrightarrow (\lambda a.\ emeasure\ (K\ a)\ A) \in borel\text{-}measurable\ M$
shows *K* ∈ *measurable M* (*prob-algebra N*)
⟨*proof*⟩

lemma *in-space-prob-algebra*:
x ∈ *space* (*prob-algebra M*) ⟹ *emeasure x* (*space M*) = *1*
⟨*proof*⟩

lemma *prob-space-pair*:
assumes *prob-space M prob-space N* **shows** *prob-space* $(M \bigotimes_M N)$
⟨*proof*⟩

lemma *measurable-pair-prob*[*measurable*]:
$f \in M \to_M$ *prob-algebra N* ⟹ $g \in M \to_M$ *prob-algebra L* ⟹ $(\lambda x.\ f\ x \bigotimes_M g\ x) \in M \to_M$ *prob-algebra* $(N \bigotimes_M L)$
⟨*proof*⟩

lemma *emeasure-bind-prob-algebra*:
assumes *A*: *A* ∈ *space* (*prob-algebra N*)
assumes *B*: *B* ∈ *N* →*M prob-algebra L*
assumes *X*: *X* ∈ *sets L*
shows *emeasure* (*bind A B*) *X* = $(\int^+ x.\ emeasure\ (B\ x)\ X\ \partial A)$
⟨*proof*⟩

lemma *prob-space-bind′*:
assumes *A*: *A* ∈ *space* (*prob-algebra M*) **and** *B*: *B* ∈ *M* →*M prob-algebra N*
shows *prob-space* (*A* ⤜ *B*)
⟨*proof*⟩

lemma *sets-bind′*:
assumes *A*: *A* ∈ *space* (*prob-algebra M*) **and** *B*: *B* ∈ *M* →*M prob-algebra N*
shows *sets* (*A* ⤜ *B*) = *sets N*
⟨*proof*⟩

lemma *bind-cong-AE′*:
assumes *M*: *M* ∈ *space* (*prob-algebra L*)
and *f*: *f* ∈ *L* →*M prob-algebra N* **and** *g*: *g* ∈ *L* →*M prob-algebra N*

and *ae*: $AE\ x\ in\ M.\ f\ x = g\ x$
shows $bind\ M\ f = bind\ M\ g$
⟨*proof*⟩

lemma *density-discrete*:
$countable\ A \Longrightarrow sets\ N = Set.Pow\ A \Longrightarrow (\bigwedge x.\ f\ x \geq 0) \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x = emeasure\ N\ \{x\}) \Longrightarrow$
$density\ (count\text{-}space\ A)\ f = N$
⟨*proof*⟩

lemma *distr-density-discrete*:
fixes f'
assumes $countable\ A$
assumes $f' \in borel\text{-}measurable\ M$
assumes $g \in measurable\ M\ (count\text{-}space\ A)$
defines $f \equiv \lambda x.\ \int^+ t.\ (if\ g\ t = x\ then\ 1\ else\ 0) * f'\ t\ \partial M$
assumes $\bigwedge x.\ \ x \in space\ M \Longrightarrow g\ x \in A$
shows $density\ (count\text{-}space\ A)\ (\lambda x.\ f\ x) = distr\ (density\ M\ f')\ (count\text{-}space\ A)\ g$
⟨*proof*⟩

lemma *bind-cong-AE*:
assumes $M = N$
assumes *f*: $f \in measurable\ N\ (subprob\text{-}algebra\ B)$
assumes *g*: $g \in measurable\ N\ (subprob\text{-}algebra\ B)$
assumes *ae*: $AE\ x\ in\ N.\ f\ x = g\ x$
shows $bind\ M\ f = bind\ N\ g$
⟨*proof*⟩

lemma *bind-cong-simp*: $M = N \Longrightarrow (\bigwedge x.\ x{\in}space\ M\ =simp{=}>\ f\ x = g\ x) \Longrightarrow bind\ M\ f = bind\ N\ g$
⟨*proof*⟩

lemma *sets-bind-measurable*:
assumes *f*: $f \in measurable\ M\ (subprob\text{-}algebra\ B)$
assumes *M*: $space\ M \neq \{\}$
shows $sets\ (M \gg\!= f) = sets\ B$
⟨*proof*⟩

lemma *space-bind-measurable*:
assumes *f*: $f \in measurable\ M\ (subprob\text{-}algebra\ B)$
assumes *M*: $space\ M \neq \{\}$
shows $space\ (M \gg\!= f) = space\ B$
⟨*proof*⟩

lemma *bind-distr-return*:
$f \in M \rightarrow_M N \Longrightarrow g \in N \rightarrow_M L \Longrightarrow space\ M \neq \{\} \Longrightarrow$
$distr\ M\ N\ f \gg\!= (\lambda x.\ return\ L\ (g\ x)) = distr\ M\ L\ (\lambda x.\ g\ (f\ x))$
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-eq-constD*:
assumes *AE x in M*. $x = y$
shows $M = return\ M\ y\ y \in space\ M$
⟨*proof*⟩

end

8 Projective Family

theory *Projective-Family*
imports *Giry-Monad*
begin

lemma *vimage-restrict-preserve-mono*:
assumes *J*: $J \subseteq I$
and *sets*: $A \subseteq (\Pi_E\ i{\in}J.\ S\ i)\ B \subseteq (\Pi_E\ i{\in}J.\ S\ i)$ **and** *ne*: $(\Pi_E\ i{\in}I.\ S\ i) \neq \{\}$
and *eq*: $(\lambda x.\ restrict\ x\ J) -` A \cap (\Pi_E\ i{\in}I.\ S\ i) \subseteq (\lambda x.\ restrict\ x\ J) -` B \cap (\Pi_E\ i{\in}I.\ S\ i)$
shows $A \subseteq B$
⟨*proof*⟩

locale *projective-family* =
fixes $I :: 'i\ set$ **and** $P :: 'i\ set \Rightarrow ('i \Rightarrow 'a)\ measure$ **and** $M :: 'i \Rightarrow 'a\ measure$
assumes *P*: $\bigwedge J\ H.\ J \subseteq H \Longrightarrow finite\ H \Longrightarrow H \subseteq I \Longrightarrow P\ J = distr\ (P\ H)\ (PiM\ J\ M)\ (\lambda f.\ restrict\ f\ J)$
assumes *prob-space-P*: $\bigwedge J.\ finite\ J \Longrightarrow J \subseteq I \Longrightarrow prob\text{-}space\ (P\ J)$
begin

lemma *sets-P*: $finite\ J \Longrightarrow J \subseteq I \Longrightarrow sets\ (P\ J) = sets\ (PiM\ J\ M)$
⟨*proof*⟩

lemma *space-P*: $finite\ J \Longrightarrow J \subseteq I \Longrightarrow space\ (P\ J) = space\ (PiM\ J\ M)$
⟨*proof*⟩

lemma *not-empty-M*: $i \in I \Longrightarrow space\ (M\ i) \neq \{\}$
⟨*proof*⟩

lemma *not-empty*: $space\ (PiM\ I\ M) \neq \{\}$
⟨*proof*⟩

abbreviation
$emb\ L\ K \equiv prod\text{-}emb\ L\ M\ K$

lemma *emb-preserve-mono*:
assumes $J \subseteq L\ L \subseteq I$ **and** *sets*: $X \in sets\ (Pi_M\ J\ M)\ Y \in sets\ (Pi_M\ J\ M)$
assumes $emb\ L\ J\ X \subseteq emb\ L\ J\ Y$
shows $X \subseteq Y$

⟨*proof*⟩

lemma *emb-injective*:
assumes *L*: $J \subseteq L$ $L \subseteq I$ **and** *X*: $X \in sets\ (Pi_M\ J\ M)$ **and** *Y*: $Y \in sets\ (Pi_M\ J\ M)$
shows $emb\ L\ J\ X = emb\ L\ J\ Y \Longrightarrow X = Y$
⟨*proof*⟩

lemma *emeasure-P*: $J \subseteq K \Longrightarrow finite\ K \Longrightarrow K \subseteq I \Longrightarrow X \in sets\ (PiM\ J\ M) \Longrightarrow P\ K\ (emb\ K\ J\ X) = P\ J\ X$
⟨*proof*⟩

inductive-set *generator* :: $('i \Rightarrow 'a)\ set\ set$ **where**
$finite\ J \Longrightarrow J \subseteq I \Longrightarrow X \in sets\ (Pi_M\ J\ M) \Longrightarrow emb\ I\ J\ X \in generator$

lemma *algebra-generator*: $algebra\ (space\ (PiM\ I\ M))\ generator$
⟨*proof*⟩

interpretation *generator*: $algebra\ space\ (PiM\ I\ M)\ generator$
⟨*proof*⟩

lemma *sets-PiM-generator*: $sets\ (PiM\ I\ M) = sigma\text{-}sets\ (space\ (PiM\ I\ M))\ generator$
⟨*proof*⟩

definition *mu-G* (‹μG›) **where**
$\mu G\ A = (THE\ x.\ \forall J{\subseteq}I.\ finite\ J \longrightarrow (\forall X{\in}sets\ (Pi_M\ J\ M).\ A = emb\ I\ J\ X \longrightarrow x = emeasure\ (P\ J)\ X))$

definition *lim* :: $('i \Rightarrow 'a)\ measure$ **where**
$lim = extend\text{-}measure\ (space\ (PiM\ I\ M))\ generator\ (\lambda x.\ x)\ \mu G$

lemma *space-lim*[*simp*]: $space\ lim = space\ (PiM\ I\ M)$
⟨*proof*⟩

lemma *sets-lim*[*simp*, *measurable*]: $sets\ lim = sets\ (PiM\ I\ M)$
⟨*proof*⟩

lemma *mu-G-spec*:
assumes *J*: $finite\ J\ J \subseteq I\ X \in sets\ (Pi_M\ J\ M)$
shows $\mu G\ (emb\ I\ J\ X) = emeasure\ (P\ J)\ X$
⟨*proof*⟩

lemma *positive-mu-G*: $positive\ generator\ \mu G$
⟨*proof*⟩

lemma *additive-mu-G*: $additive\ generator\ \mu G$
⟨*proof*⟩

lemma *emeasure-lim*:
assumes *JX*: *finite J* $J \subseteq I$ $X \in$ *sets* (*PiM J M*)
assumes *cont*: $\bigwedge J\ X.$ $(\bigwedge i.\ J\ i \subseteq I) \Longrightarrow$ *incseq J* $\Longrightarrow$ $(\bigwedge i.$ *finite* (*J i*)$) \Longrightarrow (\bigwedge i.$ *X i* $\in$ *sets* (*PiM* (*J i*) *M*)$) \Longrightarrow$
decseq $(\lambda i.$ *emb I* (*J i*) (*X i*)$) \Longrightarrow 0 <$ (*INF i. P* (*J i*) (*X i*)$) \Longrightarrow (\bigcap i.$ *emb I* (*J i*) (*X i*)$) \neq \{\}$
shows *emeasure lim* (*emb I J X*) = *P J X*
⟨*proof*⟩

end

sublocale *product-prob-space* $\subseteq$ *projective-family I* $\lambda J.$ *PiM J M M*
⟨*proof*⟩

Proof due to Ionescu Tulcea.

locale *Ionescu-Tulcea* =
fixes *P* :: *nat* ⇒ (*nat* ⇒ *'a*) ⇒ *'a measure* **and** *M* :: *nat* ⇒ *'a measure*
assumes *P*[*measurable*]: $\bigwedge i.$ *P i* $\in$ *measurable* (*PiM* {*0..<i*} *M*) (*subprob-algebra* (*M i*))
assumes *prob-space-P*: $\bigwedge i\ x.$ *x* $\in$ *space* (*PiM* {*0..<i*} *M*) $\Longrightarrow$ *prob-space* (*P i x*)
begin

lemma *non-empty*[*simp*]: *space* (*M i*) $\neq \{\}$
⟨*proof*⟩

lemma *space-PiM-not-empty*[*simp*]: *space* (*PiM UNIV M*) $\neq \{\}$
⟨*proof*⟩

lemma *space-P*: *x* $\in$ *space* (*PiM* {*0..<n*} *M*) $\Longrightarrow$ *space* (*P n x*) = *space* (*M n*)
⟨*proof*⟩

lemma *sets-P*[*measurable-cong*]: *x* $\in$ *space* (*PiM* {*0..<n*} *M*) $\Longrightarrow$ *sets* (*P n x*) = *sets* (*M n*)
⟨*proof*⟩

definition *eP* :: *nat* ⇒ (*nat* ⇒ *'a*) ⇒ (*nat* ⇒ *'a*) *measure* **where**
eP n ω = *distr* (*P n ω*) (*PiM* {*0..<Suc n*} *M*) (*fun-upd ω n*)

lemma *measurable-eP*[*measurable*]:
eP n $\in$ *measurable* (*PiM* {*0..< n*} *M*) (*subprob-algebra* (*PiM* {*0..<Suc n*} *M*))
⟨*proof*⟩

lemma *space-eP*:
x $\in$ *space* (*PiM* {*0..<n*} *M*) $\Longrightarrow$ *space* (*eP n x*) = *space* (*PiM* {*0..<Suc n*} *M*)
⟨*proof*⟩

lemma *sets-eP*[*measurable*]:
x $\in$ *space* (*PiM* {*0..<n*} *M*) $\Longrightarrow$ *sets* (*eP n x*) = *sets* (*PiM* {*0..<Suc n*} *M*)

⟨*proof*⟩

lemma *prob-space-eP*: $x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow prob\text{-}space\ (eP\ n\ x)$
⟨*proof*⟩

lemma *nn-integral-eP*:
$\omega \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow f \in borel\text{-}measurable\ (PiM\ \{0..<Suc\ n\}\ M)$
$\Longrightarrow$
$(\int^+ x.\ f\ x\ \partial eP\ n\ \omega) = (\int^+ x.\ f\ (fun\text{-}upd\ \omega\ n\ x)\ \partial P\ n\ \omega)$
⟨*proof*⟩

lemma *emeasure-eP*:
assumes ω[*simp*]: $\omega \in space\ (PiM\ \{0..<n\}\ M)$ **and** A[*measurable*]: $A \in sets\ (PiM\ \{0..<Suc\ n\}\ M)$
shows $eP\ n\ \omega\ A = P\ n\ \omega\ ((\lambda x.\ fun\text{-}upd\ \omega\ n\ x) -` A \cap space\ (M\ n))$
⟨*proof*⟩

primrec $C :: nat \Rightarrow nat \Rightarrow (nat \Rightarrow 'a) \Rightarrow (nat \Rightarrow 'a)\ measure$ **where**
$C\ n\ 0\ \omega = return\ (PiM\ \{0..<n\}\ M)\ \omega$
| $C\ n\ (Suc\ m)\ \omega = C\ n\ m\ \omega \gg= eP\ (n + m)$

lemma *measurable-C*[*measurable*]:
$C\ n\ m \in measurable\ (PiM\ \{0..<n\}\ M)\ (subprob\text{-}algebra\ (PiM\ \{0..<n + m\}\ M))$
⟨*proof*⟩

lemma *space-C*:
$x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow space\ (C\ n\ m\ x) = space\ (PiM\ \{0..<n + m\}\ M)$
⟨*proof*⟩

lemma *sets-C*[*measurable-cong*]:
$x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow sets\ (C\ n\ m\ x) = sets\ (PiM\ \{0..<n + m\}\ M)$
⟨*proof*⟩

lemma *prob-space-C*: $x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow prob\text{-}space\ (C\ n\ m\ x)$
⟨*proof*⟩

lemma *split-C*:
assumes ω: $\omega \in space\ (PiM\ \{0..<n\}\ M)$ **shows** $(C\ n\ m\ \omega \gg= C\ (n + m)\ l) = C\ n\ (m + l)\ \omega$
⟨*proof*⟩

lemma *nn-integral-C*:
assumes $m \le m'$ **and** f[*measurable*]: $f \in borel\text{-}measurable\ (PiM\ \{0..<n+m\}\ M)$
and *nonneg*: $\bigwedge x.\ x \in space\ (PiM\ \{0..<n+m\}\ M) \Longrightarrow 0 \le f\ x$
and x: $x \in space\ (PiM\ \{0..<n\}\ M)$
shows $(\int^+ x.\ f\ x\ \partial C\ n\ m\ x) = (\int^+ x.\ f\ (restrict\ x\ \{0..<n+m\})\ \partial C\ n\ m'\ x)$

⟨*proof*⟩

lemma *emeasure-C*:
assumes $m \leq m'$ **and** A[*measurable*]: $A \in sets\ (PiM\ \{0..<n+m\}\ M)$ **and** [*simp*]: $x \in space\ (PiM\ \{0..<n\}\ M)$
shows $emeasure\ (C\ n\ m'\ x)\ (prod\text{-}emb\ \{0..<n+m'\}\ M\ \{0..<n+m\}\ A) = emeasure\ (C\ n\ m\ x)\ A$
⟨*proof*⟩

lemma *distr-C*:
assumes $m \leq m'$ **and** [*simp*]: $x \in space\ (PiM\ \{0..<n\}\ M)$
shows $C\ n\ m\ x = distr\ (C\ n\ m'\ x)\ (PiM\ \{0..<n+m\}\ M)\ (\lambda x.\ restrict\ x\ \{0..<n+m\})$
⟨*proof*⟩

definition *up-to* :: *nat set* ⇒ *nat* **where**
$up\text{-}to\ J = (LEAST\ n.\ \forall i{\geq}n.\ i \notin J)$

lemma *up-to-less*: *finite* $J \Longrightarrow i \in J \Longrightarrow i < up\text{-}to\ J$
⟨*proof*⟩

lemma *up-to-iff*: *finite* $J \Longrightarrow up\text{-}to\ J \leq n \longleftrightarrow (\forall i{\in}J.\ i < n)$
⟨*proof*⟩

lemma *up-to-iff-Ico*: *finite* $J \Longrightarrow up\text{-}to\ J \leq n \longleftrightarrow J \subseteq \{0..<n\}$
⟨*proof*⟩

lemma *up-to*: *finite* $J \Longrightarrow J \subseteq \{0..<up\text{-}to\ J\}$
⟨*proof*⟩

lemma *up-to-mono*: $J \subseteq H \Longrightarrow$ *finite* $H \Longrightarrow up\text{-}to\ J \leq up\text{-}to\ H$
⟨*proof*⟩

definition *CI* :: *nat set* ⇒ (*nat* ⇒ *'a*) *measure* **where**
$CI\ J = distr\ (C\ 0\ (up\text{-}to\ J)\ (\lambda x.\ undefined))\ (PiM\ J\ M)\ (\lambda f.\ restrict\ f\ J)$

sublocale *PF*: *projective-family UNIV CI*
⟨*proof*⟩

lemma *emeasure-CI'*:
finite $J \Longrightarrow X \in sets\ (PiM\ J\ M) \Longrightarrow CI\ J\ X = C\ 0\ (up\text{-}to\ J)\ (\lambda\text{-}.\ undefined)\ (PF.emb\ \{0..<up\text{-}to\ J\}\ J\ X)$
⟨*proof*⟩

lemma *emeasure-CI*:
$J \subseteq \{0..<n\} \Longrightarrow X \in sets\ (PiM\ J\ M) \Longrightarrow CI\ J\ X = C\ 0\ n\ (\lambda\text{-}.\ undefined)\ (PF.emb\ \{0..<n\}\ J\ X)$
⟨*proof*⟩

lemma *lim*:
assumes *J*: *finite J* **and** *X*: $X \in sets\ (PiM\ J\ M)$
shows *emeasure PF.lim* $(PF.emb\ UNIV\ J\ X) = emeasure\ (CI\ J)\ X$
⟨*proof*⟩

lemma *distr-lim*: **assumes** *J*[*simp*]: *finite J* **shows** *distr PF.lim* $(PiM\ J\ M)\ (\lambda x.$ *restrict x J*$) = CI\ J$
⟨*proof*⟩

end

lemma (**in** *product-prob-space*) *emeasure-lim-emb*:
assumes ∗: *finite J* $J \subseteq I$ $X \in sets\ (PiM\ J\ M)$
shows *emeasure lim* $(emb\ I\ J\ X) = emeasure\ (Pi_M\ J\ M)\ X$
⟨*proof*⟩

end

9 Infinite Product Measure

theory *Infinite-Product-Measure*
imports *Probability-Measure Projective-Family*
begin

lemma (**in** *product-prob-space*) *distr-PiM-restrict-finite*:
assumes *finite J* $J \subseteq I$
shows *distr* $(PiM\ I\ M)\ (PiM\ J\ M)\ (\lambda x.$ *restrict x J*$) = PiM\ J\ M$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *emeasure-PiM-emb′*:
$J \subseteq I \Longrightarrow$ *finite J* $\Longrightarrow X \in sets\ (PiM\ J\ M) \Longrightarrow emeasure\ (Pi_M\ I\ M)\ (emb\ I\ J\ X) = PiM\ J\ M\ X$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *emeasure-PiM-emb*:
$J \subseteq I \Longrightarrow$ *finite J* $\Longrightarrow (\bigwedge i.\ i \in J \Longrightarrow X\ i \in sets\ (M\ i)) \Longrightarrow$
emeasure $(Pi_M\ I\ M)\ (emb\ I\ J\ (Pi_E\ J\ X)) = (\prod\ i \in J.\ emeasure\ (M\ i)\ (X\ i))$
⟨*proof*⟩

sublocale *product-prob-space* $\subseteq$ *P?*: *prob-space* $Pi_M\ I\ M$
⟨*proof*⟩

lemma *prob-space-PiM*:
assumes *M*: $\bigwedge i.\ i \in I \Longrightarrow$ *prob-space* $(M\ i)$ **shows** *prob-space* $(PiM\ I\ M)$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *emeasure-PiM-Collect*:
assumes *X*: $J \subseteq I$ *finite J* $\bigwedge i.\ i \in J \Longrightarrow X\ i \in sets\ (M\ i)$
shows *emeasure* $(Pi_M\ I\ M)\ \{x \in space\ (Pi_M\ I\ M).\ \forall\, i \in J.\ x\ i \in X\ i\} = (\prod\ i \in J.$

emeasure (*M i*) (*X i*))
⟨*proof*⟩

lemma (**in** *product-prob-space*) *emeasure-PiM-Collect-single*:
assumes *X*: *i* ∈ *I A* ∈ *sets* (*M i*)
shows *emeasure* (*Pi*$_M$ *I M*) {*x*∈*space* (*Pi*$_M$ *I M*). *x i* ∈ *A*} = *emeasure* (*M i*) *A*
⟨*proof*⟩

lemma (**in** *product-prob-space*) *measure-PiM-emb*:
assumes *J* ⊆ *I finite J* ⋀*i*. *i* ∈ *J* ⟹ *X i* ∈ *sets* (*M i*)
shows *measure* (*PiM I M*) (*emb I J* (*Pi*$_E$ *J X*)) = (∏ *i*∈*J*. *measure* (*M i*) (*X i*))
⟨*proof*⟩

lemma *sets-Collect-single'*:
i ∈ *I* ⟹ {*x*∈*space* (*M i*). *P x*} ∈ *sets* (*M i*) ⟹ {*x*∈*space* (*PiM I M*). *P* (*x i*)} ∈ *sets* (*PiM I M*)
⟨*proof*⟩

lemma (**in** *finite-product-prob-space*) *finite-measure-PiM-emb*:
(⋀*i*. *i* ∈ *I* ⟹ *A i* ∈ *sets* (*M i*)) ⟹ *measure* (*PiM I M*) (*Pi*$_E$ *I A*) = (∏ *i*∈*I*. *measure* (*M i*) (*A i*))
⟨*proof*⟩

lemma (**in** *product-prob-space*) *PiM-component*:
assumes *i* ∈ *I*
shows *distr* (*PiM I M*) (*M i*) (λω. ω *i*) = *M i*
⟨*proof*⟩

lemma (**in** *product-prob-space*) *PiM-eq*:
assumes *M'*: *sets M'* = *sets* (*PiM I M*)
assumes *eq*: ⋀*J F*. *finite J* ⟹ *J* ⊆ *I* ⟹ (⋀*j*. *j* ∈ *J* ⟹ *F j* ∈ *sets* (*M j*)) ⟹
emeasure M' (*prod-emb I M J* (Π$_E$ *j*∈*J*. *F j*)) = (∏ *j*∈*J*. *emeasure* (*M j*) (*F j*))
shows *M'* = (*PiM I M*)
⟨*proof*⟩

lemma (**in** *product-prob-space*) *AE-component*: *i* ∈ *I* ⟹ *AE x in M i*. *P x* ⟹ *AE x in PiM I M*. *P* (*x i*)
⟨*proof*⟩

lemma *emeasure-PiM-emb*:
assumes *M*: ⋀*i*. *i* ∈ *I* ⟹ *prob-space* (*M i*)
assumes *J*: *J* ⊆ *I finite J* **and** *A*: ⋀*i*. *i* ∈ *J* ⟹ *A i* ∈ *sets* (*M i*)
shows *emeasure* (*Pi*$_M$ *I M*) (*prod-emb I M J* (*Pi*$_E$ *J A*)) = (∏ *i*∈*J*. *emeasure* (*M i*) (*A i*))
⟨*proof*⟩

lemma *distr-pair-PiM-eq-PiM*:

fixes i' :: $'i$ **and** I :: $'i$ *set* **and** M :: $'i \Rightarrow 'a$ *measure*
assumes M: $\bigwedge i.\ i \in I \Longrightarrow$ *prob-space* $(M\ i)$ *prob-space* $(M\ i')$
shows *distr* $(M\ i' \bigotimes_M (\Pi_M\ i{\in}I.\ M\ i))\ (\Pi_M\ i{\in}insert\ i'\ I.\ M\ i)\ (\lambda(x,\ X).\ X(i' := x)) =$
$(\Pi_M\ i{\in}insert\ i'\ I.\ M\ i)$ (**is** *?L* = -)
⟨*proof*⟩

lemma *distr-PiM-reindex*:
assumes M: $\bigwedge i.\ i \in K \Longrightarrow$ *prob-space* $(M\ i)$
assumes f: *inj-on f I* $f \in I \to K$
shows *distr* $(Pi_M\ K\ M)\ (\Pi_M\ i{\in}I.\ M\ (f\ i))\ (\lambda\omega.\ \lambda n{\in}I.\ \omega\ (f\ n)) = (\Pi_M\ i{\in}I.\ M\ (f\ i))$
(**is** *distr ?K ?I ?t* = *?I*)
⟨*proof*⟩

lemma *distr-PiM-component*:
assumes M: $\bigwedge i.\ i \in I \Longrightarrow$ *prob-space* $(M\ i)$
assumes $i \in I$
shows *distr* $(Pi_M\ I\ M)\ (M\ i)\ (\lambda\omega.\ \omega\ i) = M\ i$
⟨*proof*⟩

lemma *AE-PiM-component*:
$(\bigwedge i.\ i \in I \Longrightarrow$ *prob-space* $(M\ i)) \Longrightarrow i \in I \Longrightarrow$ *AE x in M i. P x* $\Longrightarrow$ *AE x in PiM I M. P* $(x\ i)$
⟨*proof*⟩

lemma *decseq-emb-PiE*:
incseq J $\Longrightarrow$ *decseq* $(\lambda i.$ *prod-emb I M* $(J\ i)\ (\Pi_E\ j{\in}J\ i.\ X\ j))$
⟨*proof*⟩

9.1 Sequence space

definition *comb-seq* :: *nat* $\Rightarrow$ $(nat \Rightarrow 'a) \Rightarrow (nat \Rightarrow 'a) \Rightarrow (nat \Rightarrow 'a)$ **where**
comb-seq i $\omega\ \omega'\ j =$ (*if* $j < i$ *then* $\omega\ j$ *else* $\omega'\ (j - i)$)

lemma *split-comb-seq*: P (*comb-seq i* $\omega\ \omega'\ j) \longleftrightarrow (j < i \longrightarrow P\ (\omega\ j)) \wedge (\forall k.\ j = i + k \longrightarrow P\ (\omega'\ k))$
⟨*proof*⟩

lemma *split-comb-seq-asm*: P (*comb-seq i* $\omega\ \omega'\ j) \longleftrightarrow \neg\ ((j < i \wedge \neg\ P\ (\omega\ j)) \vee (\exists k.\ j = i + k \wedge \neg\ P\ (\omega'\ k)))$
⟨*proof*⟩

lemma *measurable-comb-seq*:
$(\lambda(\omega, \omega').$ *comb-seq i* $\omega\ \omega') \in$ *measurable* $((\Pi_M\ i{\in}UNIV.\ M) \bigotimes_M (\Pi_M\ i{\in}UNIV.\ M))\ (\Pi_M\ i{\in}UNIV.\ M)$
⟨*proof*⟩

lemma *measurable-comb-seq'*[*measurable* (*raw*)]:

assumes f: $f \in$ *measurable* N (Π_M $i \in$ *UNIV*. M) **and** g: $g \in$ *measurable* N (Π_M $i \in$ *UNIV*. M)
shows (λx. *comb-seq* i (f x) (g x)) $\in$ *measurable* N (Π_M $i \in$ *UNIV*. M)
⟨*proof*⟩

lemma *comb-seq-0*: *comb-seq* 0 ω $\omega' = \omega'$
⟨*proof*⟩

lemma *comb-seq-Suc*: *comb-seq* (*Suc* n) ω $\omega' =$ *comb-seq* n ω (*case-nat* (ω n) ω')
⟨*proof*⟩

lemma *comb-seq-Suc-0*[*simp*]: *comb-seq* (*Suc* 0) $\omega =$ *case-nat* (ω 0)
⟨*proof*⟩

lemma *comb-seq-less*: $i < n \Longrightarrow$ *comb-seq* n ω ω' $i = \omega$ i
⟨*proof*⟩

lemma *comb-seq-add*: *comb-seq* n ω ω' (i + n) = ω' i
⟨*proof*⟩

lemma *case-nat-comb-seq*: *case-nat* s' (*comb-seq* n ω ω') (i + n) = *case-nat* (*case-nat* s' ω n) ω' i
⟨*proof*⟩

lemma *case-nat-comb-seq'*:
case-nat s (*comb-seq* i ω ω') = *comb-seq* (*Suc* i) (*case-nat* s ω) ω'
⟨*proof*⟩

locale *sequence-space* = *product-prob-space* λi. M *UNIV* :: *nat set* **for** M
begin

abbreviation $S \equiv \Pi_M$ $i \in$ *UNIV*::*nat set*. M

lemma *infprod-in-sets*[*intro*]:
fixes E :: *nat* ⇒ *'a set* **assumes** E: $\bigwedge i$. E $i \in$ *sets* M
shows *Pi UNIV* $E \in$ *sets* S
⟨*proof*⟩

lemma *measure-PiM-countable*:
fixes E :: *nat* ⇒ *'a set* **assumes** E: $\bigwedge i$. E $i \in$ *sets* M
shows (λn. $\prod i \leq n$. *measure* M (E i)) ⟶ *measure* S (*Pi UNIV* E)
⟨*proof*⟩

lemma *nat-eq-diff-eq*:
fixes a b c :: *nat*
shows $c \leq b \Longrightarrow a = b - c \longleftrightarrow a + c = b$
⟨*proof*⟩

lemma *PiM-comb-seq*:

distr $(S \bigotimes_M S)$ S $(\lambda(\omega, \omega').$ *comb-seq* i ω $\omega') = S$ (**is** *?D* = -)
⟨*proof*⟩

lemma *PiM-iter*:
distr $(M \bigotimes_M S)$ S $(\lambda(s, \omega).$ *case-nat* s $\omega) = S$ (**is** *?D* = -)
⟨*proof*⟩

end

lemma *PiM-return*:
assumes *finite I*
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow \{a\ i\} \in$ *sets* $(M\ i)$
shows *PiM I* $(\lambda i.$ *return* $(M\ i)$ $(a\ i)) =$ *return* $(PiM\ I\ M)$ (*restrict a I*)
⟨*proof*⟩

lemma *distr-PiM-finite-prob-space'*:
assumes *fin*: *finite I*
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *prob-space* $(M\ i)$
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *prob-space* $(M'\ i)$
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow f \in$ *measurable* $(M\ i)$ $(M'\ i)$
shows *distr* $(PiM\ I\ M)$ $(PiM\ I\ M')$ $(compose\ I\ f) = PiM\ I$ $(\lambda i.$ *distr* $(M\ i)$ $(M'\ i)$ $f)$
⟨*proof*⟩

end

10 Independent families of events, event sets, and random variables

theory *Independent-Family*
imports *Infinite-Product-Measure*
begin

definition (**in** *prob-space*)
indep-sets $F\ I \longleftrightarrow (\forall i{\in}I.\ F\ i \subseteq$ *events*$) \wedge$
$(\forall J{\subseteq}I.\ J \neq \{\} \longrightarrow$ *finite* $J \longrightarrow (\forall A{\in}Pi\ J\ F.$ *prob* $(\bigcap j{\in}J.\ A\ j) = (\prod j{\in}J.$ *prob* $(A\ j))))$

definition (**in** *prob-space*)
indep-set $A\ B \longleftrightarrow$ *indep-sets* (*case-bool A B*) *UNIV*

definition (**in** *prob-space*)
indep-events-def-alt: *indep-events* $A\ I \longleftrightarrow$ *indep-sets* $(\lambda i.\ \{A\ i\})\ I$

lemma (**in** *prob-space*) *indep-events-def*:
indep-events $A\ I \longleftrightarrow (A`I \subseteq$ *events*$) \wedge$
$(\forall J{\subseteq}I.\ J \neq \{\} \longrightarrow$ *finite* $J \longrightarrow$ *prob* $(\bigcap j{\in}J.\ A\ j) = (\prod j{\in}J.$ *prob* $(A\ j)))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-eventsI*:
(⋀*i*. *i* ∈ *I* ⟹ *F i* ∈ *sets M*) ⟹ (⋀*J*. *J* ⊆ *I* ⟹ *finite J* ⟹ *J* ≠ {} ⟹ *prob* (⋂ *i*∈*J*. *F i*) = (∏ *i*∈*J*. *prob* (*F i*))) ⟹ *indep-events F I*
⟨*proof*⟩

definition (**in** *prob-space*)
indep-event A B ⟷ *indep-events* (*case-bool A B*) *UNIV*

lemma (**in** *prob-space*) *indep-sets-cong*:
I = *J* ⟹ (⋀*i*. *i* ∈ *I* ⟹ *F i* = *G i*) ⟹ *indep-sets F I* ⟷ *indep-sets G J*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-events-finite-index-events*:
indep-events F I ⟷ (∀ *J*⊆*I*. *J* ≠ {} ⟶ *finite J* ⟶ *indep-events F J*)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-finite-index-sets*:
indep-sets F I ⟷ (∀ *J*⊆*I*. *J* ≠ {} ⟶ *finite J* ⟶ *indep-sets F J*)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-mono-index*:
J ⊆ *I* ⟹ *indep-sets F I* ⟹ *indep-sets F J*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-mono-sets*:
assumes *indep*: *indep-sets F I*
assumes *mono*: ⋀*i*. *i*∈*I* ⟹ *G i* ⊆ *F i*
shows *indep-sets G I*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-mono*:
assumes *indep*: *indep-sets F I*
assumes *mono*: *J* ⊆ *I* ⋀*i*. *i*∈*J* ⟹ *G i* ⊆ *F i*
shows *indep-sets G J*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-setsI*:
assumes ⋀*i*. *i* ∈ *I* ⟹ *F i* ⊆ *events*
and ⋀*A J*. *J* ≠ {} ⟹ *J* ⊆ *I* ⟹ *finite J* ⟹ (∀ *j*∈*J*. *A j* ∈ *F j*) ⟹ *prob* (⋂ *j*∈*J*. *A j*) = (∏ *j*∈*J*. *prob* (*A j*))
shows *indep-sets F I*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-setsD*:
assumes *indep-sets F I* **and** *J* ⊆ *I J* ≠ {} *finite J* ∀ *j*∈*J*. *A j* ∈ *F j*
shows *prob* (⋂ *j*∈*J*. *A j*) = (∏ *j*∈*J*. *prob* (*A j*))
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-setI*:
assumes *ev*: $A \subseteq$ *events* $B \subseteq$ *events*
and *indep*: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in B \Longrightarrow$ *prob* $(a \cap b) =$ *prob* $a *$ *prob* b
shows *indep-set* $A\ B$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-setD*:
assumes *indep*: *indep-set* $A\ B$ **and** *ev*: $a \in A\ b \in B$
shows *prob* $(a \cap b) =$ *prob* $a *$ *prob* b
⟨*proof*⟩

lemma (**in** *prob-space*)
assumes *indep*: *indep-set* $A\ B$
shows *indep-setD-ev1*: $A \subseteq$ *events*
and *indep-setD-ev2*: $B \subseteq$ *events*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-Dynkin*:
assumes *indep*: *indep-sets* $F\ I$
shows *indep-sets* $(\lambda i.$ *Dynkin* $($*space* $M)$ $(F\ i))\ I$
(**is** *indep-sets* $?F\ I$)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-sigma*:
assumes *indep*: *indep-sets* $F\ I$
assumes *stable*: $\bigwedge i.\ i \in I \Longrightarrow$ *Int-stable* $(F\ i)$
shows *indep-sets* $(\lambda i.$ *sigma-sets* $($*space* $M)$ $(F\ i))\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-sigma-sets-iff*:
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *Int-stable* $(F\ i)$
shows *indep-sets* $(\lambda i.$ *sigma-sets* $($*space* $M)$ $(F\ i))\ I \longleftrightarrow$ *indep-sets* $F\ I$
⟨*proof*⟩

definition (**in** *prob-space*)
indep-vars-def2: *indep-vars* $M'\ X\ I \longleftrightarrow$
$(\forall i \in I.$ *random-variable* $(M'\ i)\ (X\ i)) \wedge$
indep-sets $(\lambda i.\ \{\ X\ i -` A \cap$ *space* $M \mid A.\ A \in$ *sets* $(M'\ i)\})\ I$

definition (**in** *prob-space*)
indep-var Ma A Mb B $\longleftrightarrow$ *indep-vars* $($*case-bool Ma Mb*$)$ $($*case-bool A B*$)$ *UNIV*

lemma (**in** *prob-space*) *indep-vars-def*:
indep-vars $M'\ X\ I \longleftrightarrow$
$(\forall i \in I.$ *random-variable* $(M'\ i)\ (X\ i)) \wedge$
indep-sets $(\lambda i.$ *sigma-sets* $($*space* $M)\ \{\ X\ i -` A \cap$ *space* $M \mid A.\ A \in$ *sets* $(M'$
$i)\})\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-var-eq*:
indep-var S X T Y $\longleftrightarrow$
(*random-variable S X* $\wedge$ *random-variable T Y*) $\wedge$
indep-set
(*sigma-sets* (*space M*) { *X* −‘ *A* $\cap$ *space M* | *A*. *A* $\in$ *sets S*})
(*sigma-sets* (*space M*) { *Y* −‘ *A* $\cap$ *space M* | *A*. *A* $\in$ *sets T*})
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets2-eq*:
indep-set A B $\longleftrightarrow$ *A* $\subseteq$ *events* $\wedge$ *B* $\subseteq$ *events* $\wedge$ ($\forall$ *a*$\in$*A*. $\forall$ *b*$\in$*B*. *prob* (*a* $\cap$ *b*) = *prob a* $*$ *prob b*)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-set-sigma-sets*:
assumes *indep-set A B*
assumes *A*: *Int-stable A* **and** *B*: *Int-stable B*
shows *indep-set* (*sigma-sets* (*space M*) *A*) (*sigma-sets* (*space M*) *B*)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-eventsI-indep-vars*:
assumes *indep*: *indep-vars N X I*
assumes *P*: $\bigwedge$*i*. *i* $\in$ *I* $\Longrightarrow$ {*x*$\in$*space* (*N i*). *P i x*} $\in$ *sets* (*N i*)
shows *indep-events* (λ*i*. {*x*$\in$*space M*. *P i* (*X i x*)}) *I*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-collect-sigma*:
fixes *I* :: *'j* $\Rightarrow$ *'i set* **and** *J* :: *'j set* **and** *E* :: *'i* $\Rightarrow$ *'a set set*
assumes *indep*: *indep-sets E* ($\bigcup$*j*$\in$*J*. *I j*)
assumes *Int-stable*: $\bigwedge$*i j*. *j* $\in$ *J* $\Longrightarrow$ *i* $\in$ *I j* $\Longrightarrow$ *Int-stable* (*E i*)
assumes *disjoint*: *disjoint-family-on I J*
shows *indep-sets* (λ*j*. *sigma-sets* (*space M*) ($\bigcup$ *i*$\in$*I j*. *E i*)) *J*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-restrict*:
assumes *ind*: *indep-vars M' X I* **and** *K*: $\bigwedge$*j*. *j* $\in$ *L* $\Longrightarrow$ *K j* $\subseteq$ *I* **and** *J*: *disjoint-family-on K L*
shows *indep-vars* (λ*j*. *PiM* (*K j*) *M'*) (λ*j* ω. *restrict* (λ*i*. *X i* ω) (*K j*)) *L*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-var-restrict*:
assumes *ind*: *indep-vars M' X I* **and** *AB*: *A* $\cap$ *B* = {} *A* $\subseteq$ *I B* $\subseteq$ *I*
shows *indep-var* (*PiM A M'*) ($\lambda\omega$. *restrict* (λ*i*. *X i* ω) *A*) (*PiM B M'*) ($\lambda\omega$. *restrict* (λ*i*. *X i* ω) *B*)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-subset*:
assumes *indep-vars M' X I J* $\subseteq$ *I*
shows *indep-vars M' X J*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-cong*:
$I = J \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow X\ i = Y\ i) \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow M'\ i = N'\ i) \Longrightarrow$ *indep-vars* $M'\ X\ I \longleftrightarrow$ *indep-vars* $N'\ Y\ J$
⟨*proof*⟩

definition (**in** *prob-space*) *tail-events* **where**
tail-events $A = (\bigcap n.$ *sigma-sets* (*space* M) $(\bigcup\ (A\ `\ \{n..\})))$

lemma (**in** *prob-space*) *tail-events-sets*:
assumes A: $\bigwedge i$::*nat*. $A\ i \subseteq$ *events*
shows *tail-events* $A \subseteq$ *events*
⟨*proof*⟩

lemma (**in** *prob-space*) *sigma-algebra-tail-events*:
assumes $\bigwedge i$::*nat*. *sigma-algebra* (*space* M) $(A\ i)$
shows *sigma-algebra* (*space* M) (*tail-events* A)
⟨*proof*⟩

lemma (**in** *prob-space*) *kolmogorov-0-1-law*:
fixes A :: *nat* ⇒ *'a set set*
assumes $\bigwedge i$::*nat*. *sigma-algebra* (*space* M) $(A\ i)$
assumes *indep*: *indep-sets* A *UNIV*
and X: $X \in$ *tail-events* A
shows *prob* $X = 0 \vee$ *prob* $X = 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *borel-0-1-law*:
fixes F :: *nat* ⇒ *'a set*
assumes *F2*: *indep-events* F *UNIV*
shows *prob* $(\bigcap n.\ \bigcup m{\in}\{n..\}.\ F\ m) = 0 \vee$ *prob* $(\bigcap n.\ \bigcup m{\in}\{n..\}.\ F\ m) = 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *borel-0-1-law-AE*:
fixes P :: *nat* ⇒ *'a* ⇒ *bool*
assumes *indep-events* $(\lambda m.\ \{x{\in}$*space* $M.\ P\ m\ x\})$ *UNIV* (**is** *indep-events ?P* -)
shows (*AE* x *in* M. *infinite* $\{m.\ P\ m\ x\}) \vee$ (*AE* x *in* M. *finite* $\{m.\ P\ m\ x\}$)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-finite*:
assumes I: $I \neq \{\}$ *finite* I
and F: $\bigwedge i.\ i \in I \Longrightarrow F\ i \subseteq$ *events* $\bigwedge i.\ i \in I \Longrightarrow$ *space* $M \in F\ i$
shows *indep-sets* $F\ I \longleftrightarrow (\forall A{\in}Pi\ I\ F.$ *prob* $(\bigcap j{\in}I.\ A\ j) = (\prod j{\in}I.$ *prob* $(A\ j)))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-finite*:
fixes I :: *'i set*
assumes I: $I \neq \{\}$ *finite* I

and M': $\bigwedge i.\ i \in I \Longrightarrow$ *sets* $(M'\ i) =$ *sigma-sets* (*space* $(M'\ i)$) $(E\ i)$
and *rv*: $\bigwedge i.\ i \in I \Longrightarrow$ *random-variable* $(M'\ i)\ (X\ i)$
and *Int-stable*: $\bigwedge i.\ i \in I \Longrightarrow$ *Int-stable* $(E\ i)$
and *space*: $\bigwedge i.\ i \in I \Longrightarrow$ *space* $(M'\ i) \in E\ i$ **and** *closed*: $\bigwedge i.\ i \in I \Longrightarrow E\ i \subseteq$ *Pow* (*space* $(M'\ i)$)
shows *indep-vars* $M'\ X\ I \longleftrightarrow$
$(\forall A \in (\Pi\ i \in I.\ E\ i).$ *prob* $(\bigcap j \in I.\ X\ j\ -`\ A\ j \cap$ *space* $M) = (\prod j \in I.$ *prob* $(X\ j\ -`\ A\ j \cap$ *space* $M)))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-compose*:
assumes *indep-vars* $M'\ X\ I$
assumes *rv*: $\bigwedge i.\ i \in I \Longrightarrow Y\ i \in$ *measurable* $(M'\ i)\ (N\ i)$
shows *indep-vars* $N\ (\lambda i.\ Y\ i \circ X\ i)\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-compose2*:
assumes *indep-vars* $M'\ X\ I$
assumes *rv*: $\bigwedge i.\ i \in I \Longrightarrow Y\ i \in$ *measurable* $(M'\ i)\ (N\ i)$
shows *indep-vars* $N\ (\lambda i\ x.\ Y\ i\ (X\ i\ x))\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-var-compose*:
assumes *indep-var M1 X1 M2 X2 Y1* ∈ *measurable M1 N1 Y2* ∈ *measurable M2 N2*
shows *indep-var N1* (*Y1* ∘ *X1*) *N2* (*Y2* ∘ *X2*)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-Min*:
fixes $X :: 'i \Rightarrow 'a \Rightarrow$ *real*
assumes I: *finite* $I\ i \notin I$ **and** *indep*: *indep-vars* $(\lambda\text{-}.\ borel)\ X\ (insert\ i\ I)$
shows *indep-var borel* $(X\ i)$ *borel* $(\lambda\omega.\ Min\ ((\lambda i.\ X\ i\ \omega)\ `I))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-sum*:
fixes $X :: 'i \Rightarrow 'a \Rightarrow$ *real*
assumes I: *finite* $I\ i \notin I$ **and** *indep*: *indep-vars* $(\lambda\text{-}.\ borel)\ X\ (insert\ i\ I)$
shows *indep-var borel* $(X\ i)$ *borel* $(\lambda\omega.\ \sum i \in I.\ X\ i\ \omega)$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-prod*:
fixes $X :: 'i \Rightarrow 'a \Rightarrow$ *real*
assumes I: *finite* $I\ i \notin I$ **and** *indep*: *indep-vars* $(\lambda\text{-}.\ borel)\ X\ (insert\ i\ I)$
shows *indep-var borel* $(X\ i)$ *borel* $(\lambda\omega.\ \prod i \in I.\ X\ i\ \omega)$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-varsD-finite*:
assumes X: *indep-vars* $M'\ X\ I$
assumes I: $I \neq \{\}$ *finite* $I \bigwedge i.\ i \in I \Longrightarrow A\ i \in sets\ (M'\ i)$

shows *prob* $(\bigcap i{\in}I.\ X\ i\ -`\ A\ i \cap \mathit{space}\ M) = (\prod i{\in}I.\ \mathit{prob}\ (X\ i\ -`\ A\ i \cap \mathit{space}\ M))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-varsD*:
assumes *X*: *indep-vars M' X I*
assumes *I*: $J \neq \{\}$ *finite J* $J \subseteq I$ $\bigwedge i.\ i \in J \Longrightarrow A\ i \in \mathit{sets}\ (M'\ i)$
shows *prob* $(\bigcap i{\in}J.\ X\ i\ -`\ A\ i \cap \mathit{space}\ M) = (\prod i{\in}J.\ \mathit{prob}\ (X\ i\ -`\ A\ i \cap \mathit{space}\ M))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-iff-distr-eq-PiM*:
fixes $I :: 'i\ \mathit{set}$ **and** $X :: 'i \Rightarrow 'a \Rightarrow 'b$
assumes $I \neq \{\}$
assumes *rv*: $\bigwedge i.\ \mathit{random\text{-}variable}\ (M'\ i)\ (X\ i)$
shows *indep-vars M' X I* $\longleftrightarrow$
distr M $(\Pi_M\ i{\in}I.\ M'\ i)\ (\lambda x.\ \lambda i{\in}I.\ X\ i\ x) = (\Pi_M\ i{\in}I.\ \mathit{distr}\ M\ (M'\ i)\ (X\ i))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-iff-distr-eq-PiM'*:
fixes $I :: 'i\ \mathit{set}$ **and** $X :: 'i \Rightarrow 'a \Rightarrow 'b$
assumes $I \neq \{\}$
assumes *rv*: $\bigwedge i.\ i \in I \Longrightarrow \mathit{random\text{-}variable}\ (M'\ i)\ (X\ i)$
shows *indep-vars M' X I* $\longleftrightarrow$
distr M $(\Pi_M\ i{\in}I.\ M'\ i)\ (\lambda x.\ \lambda i{\in}I.\ X\ i\ x) = (\Pi_M\ i{\in}I.\ \mathit{distr}\ M\ (M'\ i)\ (X\ i))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-varD*:
assumes *indep*: *indep-var Ma A Mb B*
assumes *sets*: *Xa* $\in$ *sets Ma Xb* $\in$ *sets Mb*
shows *prob* $((\lambda x.\ (A\ x,\ B\ x)) -` (Xa \times Xb) \cap \mathit{space}\ M) =$
prob $(A -` Xa \cap \mathit{space}\ M) * \mathit{prob}\ (B -` Xb \cap \mathit{space}\ M)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-indep-random-variable*:
assumes *ind*[*simp*]: *indep-var N X N Y*
assumes [*simp*]: *A* $\in$ *sets N B* $\in$ *sets N*
shows $\mathcal{P}(x\ \mathit{in}\ M.\ X\ x \in A \wedge Y\ x \in B) = \mathcal{P}(x\ \mathit{in}\ M.\ X\ x \in A) * \mathcal{P}(x\ \mathit{in}\ M.\ Y\ x \in B)$
⟨*proof*⟩

lemma (**in** *prob-space*)
assumes *indep-var S X T Y*
shows *indep-var-rv1*: *random-variable S X*
and *indep-var-rv2*: *random-variable T Y*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-var-distribution-eq*:

indep-var S X T Y $\longleftrightarrow$ *random-variable S X* $\wedge$ *random-variable T Y* $\wedge$
distr M S X $\bigotimes_M$ *distr M T Y* = *distr M* (*S* $\bigotimes_M$ *T*) ($\lambda x.$ (*X x*, *Y x*)) (**is** - $\longleftrightarrow$ - $\wedge$ - $\wedge$ *?S* $\bigotimes_M$ *?T* = *?J*)
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-joint-indep*:
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *X*: *distributed M S X Px* **and** *Y*: *distributed M T Y Py*
assumes *indep*: *indep-var S X T Y*
shows *distributed M* (*S* $\bigotimes_M$ *T*) ($\lambda x.$ (*X x*, *Y x*)) ($\lambda(x, y).$ *Px x* $*$ *Py y*)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-nn-integral*:
assumes *I*: *finite I indep-vars* (λ-. *borel*) *X I* $\bigwedge i\ \omega.\ i \in I \Longrightarrow 0 \leq X\ i\ \omega$
shows ($\int^+\omega.$ ($\prod i{\in}I.$ *X i* ω) ∂M) = ($\prod i{\in}I.$ $\int^+\omega.$ *X i* ω ∂M)
⟨*proof*⟩

lemma (**in** *prob-space*)
fixes *X* :: $'i \Rightarrow 'a \Rightarrow 'b$::{*real-normed-field*, *banach*, *second-countable-topology*}
assumes *I*: *finite I indep-vars* (λ-. *borel*) *X I* $\bigwedge i.\ i \in I \Longrightarrow$ *integrable M* (*X i*)
shows *indep-vars-lebesgue-integral*: ($\int \omega.$ ($\prod i{\in}I.$ *X i* ω) ∂M) = ($\prod i{\in}I.$ $\int \omega.$ *X i* ω ∂M) (**is** *?eq*)
and *indep-vars-integrable*: *integrable M* ($\lambda\omega.$ ($\prod i{\in}I.$ *X i* ω)) (**is** *?int*)
⟨*proof*⟩

lemma (**in** *prob-space*)
fixes *X1 X2* :: $'a \Rightarrow 'b$::{*real-normed-field*, *banach*, *second-countable-topology*}
assumes *indep-var borel X1 borel X2 integrable M X1 integrable M X2*
shows *indep-var-lebesgue-integral*: ($\int \omega.$ *X1* ω $*$ *X2* ω ∂M) = ($\int \omega.$ *X1* ω ∂M) $*$ ($\int \omega.$ *X2* ω ∂M) (**is** *?eq*)
and *indep-var-integrable*: *integrable M* ($\lambda\omega.$ *X1* ω $*$ *X2* ω) (**is** *?int*)
⟨*proof*⟩

end

11 Convolution Measure

theory *Convolution*
imports *Independent-Family*
begin

lemma (**in** *finite-measure*) *sigma-finite-measure*: *sigma-finite-measure M*
⟨*proof*⟩

definition *convolution* :: ($'a$ :: *ordered-euclidean-space*) *measure* $\Rightarrow$ $'a$ *measure* $\Rightarrow$ $'a$ *measure* (**infix** $\langle\star\rangle$ *50*) **where**
convolution M N = *distr* (*M* $\bigotimes_M$ *N*) *borel* ($\lambda(x, y).$ *x* + *y*)

lemma

shows *space-convolution*[*simp*]: *space* (*convolution M N*) = *space borel*
and *sets-convolution*[*simp*]: *sets* (*convolution M N*) = *sets borel*
and *measurable-convolution1*[*simp*]: *measurable A* (*convolution M N*) = *measurable A borel*
and *measurable-convolution2*[*simp*]: *measurable* (*convolution M N*) *B* = *measurable borel B*
⟨*proof*⟩

lemma *nn-integral-convolution*:
assumes *finite-measure M finite-measure N*
assumes [*measurable-cong*]: *sets N* = *sets borel sets M* = *sets borel*
assumes [*measurable*]: *f* ∈ *borel-measurable borel*
shows (∫⁺*x*. *f x* ∂*convolution M N*) = (∫⁺*x*. ∫⁺*y*. *f* (*x* + *y*) ∂*N* ∂*M*)
⟨*proof*⟩

lemma *convolution-emeasure*:
assumes *A* ∈ *sets borel finite-measure M finite-measure N*
assumes [*simp*]: *sets N* = *sets borel sets M* = *sets borel*
assumes [*simp*]: *space M* = *space N space N* = *space borel*
shows *emeasure* (*M* ⋆ *N*) *A* = ∫⁺*x*. (*emeasure N* {*a*. *a* + *x* ∈ *A*}) ∂*M*
⟨*proof*⟩

lemma *convolution-emeasure′*:
assumes [*simp*]:*A* ∈ *sets borel*
assumes [*simp*]: *finite-measure M finite-measure N*
assumes [*simp*]: *sets N* = *sets borel sets M* = *sets borel*
shows *emeasure* (*M* ⋆ *N*) *A* = ∫⁺*x*. ∫⁺*y*. (*indicator A* (*x* + *y*)) ∂*N* ∂*M*
⟨*proof*⟩

lemma *convolution-finite*:
assumes [*simp*]: *finite-measure M finite-measure N*
assumes [*measurable-cong*]: *sets N* = *sets borel sets M* = *sets borel*
shows *finite-measure* (*M* ⋆ *N*)
⟨*proof*⟩

lemma *convolution-emeasure-3*:
assumes [*simp*, *measurable*]: *A* ∈ *sets borel*
assumes [*simp*]: *finite-measure M finite-measure N finite-measure L*
assumes [*simp*]: *sets N* = *sets borel sets M* = *sets borel sets L* = *sets borel*
shows *emeasure* (*L* ⋆ (*M* ⋆ *N*)) *A* = ∫⁺*x*. ∫⁺*y*. ∫⁺*z*. *indicator A* (*x* + *y* + *z*) ∂*N* ∂*M* ∂*L*
⟨*proof*⟩

lemma *convolution-emeasure-3′*:
assumes [*simp*, *measurable*]:*A* ∈ *sets borel*
assumes [*simp*]: *finite-measure M finite-measure N finite-measure L*
assumes [*measurable-cong*, *simp*]: *sets N* = *sets borel sets M* = *sets borel sets L* = *sets borel*
shows *emeasure* ((*L* ⋆ *M*) ⋆ *N*) *A* = ∫⁺*x*. ∫⁺*y*. ∫⁺*z*. *indicator A* (*x* + *y* + *z*)

∂N ∂M ∂L
⟨*proof*⟩

lemma *convolution-commutative*:
assumes [*simp*]: *finite-measure M finite-measure N*
assumes [*measurable-cong*, *simp*]: *sets N = sets borel sets M = sets borel*
shows $(M \star N) = (N \star M)$
⟨*proof*⟩

lemma *convolution-associative*:
assumes [*simp*]: *finite-measure M finite-measure N finite-measure L*
assumes [*simp*]: *sets N = sets borel sets M = sets borel sets L = sets borel*
shows $(L \star (M \star N)) = ((L \star M) \star N)$
⟨*proof*⟩

lemma (**in** *prob-space*) *sum-indep-random-variable*:
assumes *ind*: *indep-var borel X borel Y*
assumes [*simp*, *measurable*]: *random-variable borel X*
assumes [*simp*, *measurable*]: *random-variable borel Y*
shows *distr M borel* $(\lambda x.\ X\ x + Y\ x)$ = *convolution* (*distr M borel X*) (*distr M borel Y*)
⟨*proof*⟩

lemma (**in** *prob-space*) *sum-indep-random-variable-lborel*:
assumes *ind*: *indep-var borel X borel Y*
assumes [*simp*, *measurable*]: *random-variable lborel X*
assumes [*simp*, *measurable*]:*random-variable lborel Y*
shows *distr M lborel* $(\lambda x.\ X\ x + Y\ x)$ = *convolution* (*distr M lborel X*) (*distr M lborel Y*)
⟨*proof*⟩

lemma *convolution-density*:
fixes *f g* :: *real* ⇒ *ennreal*
assumes [*measurable*]: $f \in$ *borel-measurable borel* $g \in$ *borel-measurable borel*
assumes [*simp*]:*finite-measure* (*density lborel f*) *finite-measure* (*density lborel g*)
shows *density lborel f* ⋆ *density lborel g* = *density lborel* $(\lambda x.\ \int^+ y.\ f\ (x - y) * g\ y\ \partial lborel)$
(**is** *?l* = *?r*)
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-finite-measure-density*:
distributed M N X f ⟹ *finite-measure* (*density N f*)
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-convolution*:
fixes *f* :: *real* ⇒ -
fixes *g* :: *real* ⇒ -
assumes *indep*: *indep-var borel X borel Y*

assumes *X*: *distributed M lborel X f*
assumes *Y*: *distributed M lborel Y g*
shows *distributed M lborel* (λ*x*. *X x* + *Y x*) (λ*x*. ∫⁺*y*. *f* (*x* − *y*) ∗ *g y* ∂*lborel*)
⟨*proof*⟩

lemma *prob-space-convolution-density*:
fixes *f*:: *real* ⇒ -
fixes *g*:: *real* ⇒ -
assumes [*measurable*]: *f*∈ *borel-measurable borel*
assumes [*measurable*]: *g*∈ *borel-measurable borel*
assumes *gt-0*[*simp*]: ⋀*x*. *0* ≤ *f x* ⋀*x*. *0* ≤ *g x*
assumes *prob-space* (*density lborel f*) (**is** *prob-space ?F*)
assumes *prob-space* (*density lborel g*) (**is** *prob-space ?G*)
shows *prob-space* (*density lborel* (λ*x*.∫⁺*y*. *f* (*x* − *y*) ∗ *g y* ∂*lborel*)) (**is** *prob-space ?D*)
⟨*proof*⟩

end

12 Information theory

theory *Information*
imports
Independent-Family
begin

12.1 Information theory

locale *information-space* = *prob-space* +
fixes *b* :: *real* **assumes** *b-gt-1*: *1* < *b*

Introduce some simplification rules for logarithm of base *b*.

lemmas *log-simps* = *log-mult log-inverse log-divide*

12.2 Kullback−Leibler divergence

The Kullback−Leibler divergence is also known as relative entropy or Kullback−Leibler distance.

definition
entropy-density b M N = *log b* ∘ *enn2real* ∘ *RN-deriv M N*

definition
KL-divergence b M N = *integral*^{L} *N* (*entropy-density b M N*)

lemma *measurable-entropy-density*[*measurable*]: *entropy-density b M N* ∈ *borel-measurable M*
⟨*proof*⟩

lemma (**in** *sigma-finite-measure*) *KL-density*:
fixes $f :: 'a \Rightarrow real$
assumes $1 < b$
assumes f[*measurable*]: $f \in$ *borel-measurable* M **and** *nn*: *AE* x *in* M. $0 \leq f\ x$
shows *KL-divergence* $b\ M$ (*density* $M\ f$) $= (\int x.\ f\ x * log\ b\ (f\ x)\ \partial M)$
⟨*proof*⟩

lemma (**in** *sigma-finite-measure*) *KL-density-density*:
fixes $f\ g :: 'a \Rightarrow real$
assumes $1 < b$
assumes f: $f \in$ *borel-measurable* M *AE* x *in* M. $0 \leq f\ x$
assumes g: $g \in$ *borel-measurable* M *AE* x *in* M. $0 \leq g\ x$
assumes *ac*: *AE* x *in* M. $f\ x = 0 \longrightarrow g\ x = 0$
shows *KL-divergence* b (*density* $M\ f$) (*density* $M\ g$) $= (\int x.\ g\ x * log\ b\ (g\ x\ /\ f\ x)\ \partial M)$
⟨*proof*⟩

lemma (**in** *information-space*) *KL-gt-0*:
fixes $D :: 'a \Rightarrow real$
assumes *prob-space* (*density* $M\ D$)
assumes D: $D \in$ *borel-measurable* M *AE* x *in* M. $0 \leq D\ x$
assumes *int*: *integrable* M $(\lambda x.\ D\ x * log\ b\ (D\ x))$
assumes A: *density* $M\ D \neq M$
shows $0 <$ *KL-divergence* $b\ M$ (*density* $M\ D$)
⟨*proof*⟩

lemma (**in** *sigma-finite-measure*) *KL-same-eq-0*: *KL-divergence* $b\ M\ M = 0$
⟨*proof*⟩

lemma (**in** *information-space*) *KL-eq-0-iff-eq*:
fixes $D :: 'a \Rightarrow real$
assumes *prob-space* (*density* $M\ D$)
assumes D: $D \in$ *borel-measurable* M *AE* x *in* M. $0 \leq D\ x$
assumes *int*: *integrable* M $(\lambda x.\ D\ x * log\ b\ (D\ x))$
shows *KL-divergence* $b\ M$ (*density* $M\ D$) $= 0 \longleftrightarrow$ *density* $M\ D = M$
⟨*proof*⟩

lemma (**in** *information-space*) *KL-eq-0-iff-eq-ac*:
fixes $D :: 'a \Rightarrow real$
assumes *prob-space* N
assumes *ac*: *absolutely-continuous* $M\ N$ *sets* $N =$ *sets* M
assumes *int*: *integrable* N (*entropy-density* $b\ M\ N$)
shows *KL-divergence* $b\ M\ N = 0 \longleftrightarrow N = M$
⟨*proof*⟩

lemma (**in** *information-space*) *KL-nonneg*:
assumes *prob-space* (*density* $M\ D$)
assumes D: $D \in$ *borel-measurable* M *AE* x *in* M. $0 \leq D\ x$
assumes *int*: *integrable* M $(\lambda x.\ D\ x * log\ b\ (D\ x))$

shows $0 \leq$ *KL-divergence b M* (*density M D*)
⟨*proof*⟩

lemma (**in** *sigma-finite-measure*) *KL-density-density-nonneg*:
fixes $f\ g :: 'a \Rightarrow$ *real*
assumes $1 < b$
assumes *f*: $f \in$ *borel-measurable M AE x in M*. $0 \leq f\ x$ *prob-space* (*density M f*)
assumes *g*: $g \in$ *borel-measurable M AE x in M*. $0 \leq g\ x$ *prob-space* (*density M g*)
assumes *ac*: *AE x in M*. $f\ x = 0 \longrightarrow g\ x = 0$
assumes *int*: *integrable M* (λx. $g\ x *$ *log b* ($g\ x$ / $f\ x$))
shows $0 \leq$ *KL-divergence b* (*density M f*) (*density M g*)
⟨*proof*⟩

12.3 Finite Entropy

definition (**in** *information-space*) *finite-entropy* :: *'b measure* ⇒ (*'a* ⇒ *'b*) ⇒ (*'b* ⇒ *real*) ⇒ *bool*
where
finite-entropy S X f ⟷
distributed M S X f ∧
integrable S (λx. $f\ x *$ *log b* ($f\ x$)) ∧
($\forall\, x \in$*space S*. $0 \leq f\ x$)

lemma (**in** *information-space*) *finite-entropy-simple-function*:
assumes *X*: *simple-function M X*
shows *finite-entropy* (*count-space* (*X*'*space M*)) *X* (λa. *measure M* {$x \in$ *space M*. *X x* = *a*})
⟨*proof*⟩

lemma *ac-fst*:
assumes *sigma-finite-measure T*
shows *absolutely-continuous S* (*distr* ($S \bigotimes_M T$) *S fst*)
⟨*proof*⟩

lemma *ac-snd*:
assumes *sigma-finite-measure T*
shows *absolutely-continuous T* (*distr* ($S \bigotimes_M T$) *T snd*)
⟨*proof*⟩

lemma (**in** *information-space*) *finite-entropy-integrable*:
finite-entropy S X Px ⟹ *integrable S* (λx. *Px x* $*$ *log b* (*Px x*))
⟨*proof*⟩

lemma (**in** *information-space*) *finite-entropy-distributed*:
finite-entropy S X Px ⟹ *distributed M S X Px*
⟨*proof*⟩

lemma (**in** *information-space*) *finite-entropy-nn*:
finite-entropy S X Px $\Longrightarrow$ $x \in$ *space S* $\Longrightarrow$ $0 \leq$ *Px x*
⟨*proof*⟩

lemma (**in** *information-space*) *finite-entropy-measurable*:
finite-entropy S X Px $\Longrightarrow$ *Px* $\in$ *S* $\rightarrow_M$ *borel*
⟨*proof*⟩

lemma (**in** *information-space*) *subdensity-finite-entropy*:
fixes *g* :: *'b* $\Rightarrow$ *real* **and** *f* :: *'c* $\Rightarrow$ *real*
assumes *T*: *T* $\in$ *measurable P Q*
assumes *f*: *finite-entropy P X f*
assumes *g*: *finite-entropy Q Y g*
assumes *Y*: *Y* = *T* $\circ$ *X*
shows *AE x in P. g* (*T x*) = 0 $\longrightarrow$ *f x* = 0
⟨*proof*⟩

lemma (**in** *information-space*) *finite-entropy-integrable-transform*:
finite-entropy S X Px $\Longrightarrow$ *distributed M T Y Py* $\Longrightarrow$ ($\bigwedge$*x. x* $\in$ *space T* $\Longrightarrow$ 0 $\leq$ *Py x*) $\Longrightarrow$
X = (λ*x. f* (*Y x*)) $\Longrightarrow$ *f* $\in$ *measurable T S* $\Longrightarrow$ *integrable T* (λ*x. Py x* $*$ *log b* (*Px* (*f x*)))
⟨*proof*⟩

12.4 Mutual Information

definition (**in** *prob-space*)
mutual-information b S T X Y =
KL-divergence b (*distr M S X* $\bigotimes_M$ *distr M T Y*) (*distr M* (*S* $\bigotimes_M$ *T*) (λ*x.* (*X x*, *Y x*)))

lemma (**in** *information-space*) *mutual-information-indep-vars*:
fixes *S T X Y*
defines *P* $\equiv$ *distr M S X* $\bigotimes_M$ *distr M T Y*
defines *Q* $\equiv$ *distr M* (*S* $\bigotimes_M$ *T*) (λ*x.* (*X x*, *Y x*))
shows *indep-var S X T Y* $\longleftrightarrow$
(*random-variable S X* $\wedge$ *random-variable T Y* $\wedge$
absolutely-continuous P Q $\wedge$ *integrable Q* (*entropy-density b P Q*) $\wedge$
mutual-information b S T X Y = 0)
⟨*proof*⟩

abbreviation (**in** *information-space*)
mutual-information-Pow (‹$\mathcal{I}$'(- ; -')›) **where**
$\mathcal{I}$(*X* ; *Y*) $\equiv$ *mutual-information b* (*count-space* (*X*‘*space M*)) (*count-space* (*Y*‘*space M*)) *X Y*

lemma (**in** *information-space*)
fixes *Pxy* :: *'b* $\times$ *'c* $\Rightarrow$ *real* **and** *Px* :: *'b* $\Rightarrow$ *real* **and** *Py* :: *'c* $\Rightarrow$ *real*
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*

assumes *Fx*: *finite-entropy S X Px* **and** *Fy*: *finite-entropy T Y Py*
assumes *Fxy*: *finite-entropy* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
defines $f \equiv \lambda x.\ Pxy\ x * log\ b\ (Pxy\ x\ /\ (Px\ (fst\ x) * Py\ (snd\ x)))$
shows *mutual-information-distr'*: *mutual-information b S T X Y* = $integral^L$ $(S \bigotimes_M T)\ f$ (**is** *?M* = *?R*)
and *mutual-information-nonneg'*: $0 \leq$ *mutual-information b S T X Y*
⟨*proof*⟩

lemma (**in** *information-space*)
fixes *Pxy* :: $'b \times 'c \Rightarrow$ *real* **and** *Px* :: $'b \Rightarrow$ *real* **and** *Py* :: $'c \Rightarrow$ *real*
assumes *sigma-finite-measure S sigma-finite-measure T*
assumes *Px*: *distributed M S X Px* **and** *Px-nn*: $\bigwedge x.\ x \in space\ S \Longrightarrow 0 \leq Px\ x$
and *Py*: *distributed M T Y Py* **and** *Py-nn*: $\bigwedge y.\ y \in space\ T \Longrightarrow 0 \leq Py\ y$
and *Pxy*: *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
and *Pxy-nn*: $\bigwedge x\ y.\ x \in space\ S \Longrightarrow y \in space\ T \Longrightarrow 0 \leq Pxy\ (x,\ y)$
defines $f \equiv \lambda x.\ Pxy\ x * log\ b\ (Pxy\ x\ /\ (Px\ (fst\ x) * Py\ (snd\ x)))$
shows *mutual-information-distr*: *mutual-information b S T X Y* = $integral^L$ $(S \bigotimes_M T)\ f$ (**is** *?M* = *?R*)
and *mutual-information-nonneg*: *integrable* $(S \bigotimes_M T)\ f \Longrightarrow 0 \leq$ *mutual-information b S T X Y*
⟨*proof*⟩

lemma (**in** *information-space*)
fixes *Pxy* :: $'b \times 'c \Rightarrow$ *real* **and** *Px* :: $'b \Rightarrow$ *real* **and** *Py* :: $'c \Rightarrow$ *real*
assumes *sigma-finite-measure S sigma-finite-measure T*
assumes *Px*[*measurable*]: *distributed M S X Px* **and** *Px-nn*: $\bigwedge x.\ x \in space\ S \Longrightarrow 0 \leq Px\ x$
and *Py*[*measurable*]: *distributed M T Y Py* **and** *Py-nn*: $\bigwedge x.\ x \in space\ T \Longrightarrow 0 \leq Py\ x$
and *Pxy*[*measurable*]: *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
and *Pxy-nn*: $\bigwedge x.\ x \in space\ (S \bigotimes_M T) \Longrightarrow 0 \leq Pxy\ x$
assumes *ae*: *AE x in S. AE y in T. Pxy* $(x,\ y) = Px\ x * Py\ y$
shows *mutual-information-eq-0*: *mutual-information b S T X Y* = *0*
⟨*proof*⟩

lemma (**in** *information-space*) *mutual-information-simple-distributed*:
assumes *X*: *simple-distributed M X Px* **and** *Y*: *simple-distributed M Y Py*
assumes *XY*: *simple-distributed M* $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
shows $\mathcal{I}(X\ ;\ Y) = (\sum (x,\ y) \in (\lambda x.\ (X\ x,\ Y\ x))\text{`}space\ M.\ Pxy\ (x,\ y) * log\ b\ (Pxy\ (x,\ y)\ /\ (Px\ x * Py\ y)))$
⟨*proof*⟩

lemma (**in** *information-space*)
fixes *Pxy* :: $'b \times 'c \Rightarrow$ *real* **and** *Px* :: $'b \Rightarrow$ *real* **and** *Py* :: $'c \Rightarrow$ *real*
assumes *Px*: *simple-distributed M X Px* **and** *Py*: *simple-distributed M Y Py*
assumes *Pxy*: *simple-distributed M* $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
assumes *ae*: $\forall x \in space\ M.\ Pxy\ (X\ x,\ Y\ x) = Px\ (X\ x) * Py\ (Y\ x)$
shows *mutual-information-eq-0-simple*: $\mathcal{I}(X\ ;\ Y) = 0$
⟨*proof*⟩

12.5 Entropy

definition (**in** *prob-space*) *entropy* :: *real* ⇒ *'b measure* ⇒ (*'a* ⇒ *'b*) ⇒ *real* **where**
entropy b S X = − *KL-divergence b S* (*distr M S X*)

abbreviation (**in** *information-space*)
entropy-Pow (‹ℋ'(-')›) **where**
ℋ(*X*) ≡ *entropy b* (*count-space* (*X*‘*space M*)) *X*

lemma (**in** *prob-space*) *distributed-RN-deriv*:
assumes *X*: *distributed M S X Px*
shows *AE x in S*. *RN-deriv S* (*density S Px*) *x* = *Px x*
⟨*proof*⟩

lemma (**in** *information-space*)
fixes *X* :: *'a* ⇒ *'b*
assumes *X*[*measurable*]: *distributed M MX X f* **and** *nn*: ⋀*x*. *x* ∈ *space MX* ⟹ 0 ≤ *f x*
shows *entropy-distr*: *entropy b MX X* = − (∫ *x*. *f x* ∗ *log b* (*f x*) ∂*MX*) (**is** *?eq*)
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-le*:
fixes *Px* :: *'b* ⇒ *real* **and** *MX* :: *'b measure*
assumes *X*[*measurable*]: *distributed M MX X Px* **and** *Px-nn*[*simp*]: ⋀*x*. *x* ∈ *space MX* ⟹ 0 ≤ *Px x*
and *fin*: *emeasure MX* {*x* ∈ *space MX*. *Px x* ≠ 0} ≠ *top*
and *int*: *integrable MX* (λ*x*. − *Px x* ∗ *log b* (*Px x*))
shows *entropy b MX X* ≤ *log b* (*measure MX* {*x* ∈ *space MX*. *Px x* ≠ 0})
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-le-space*:
fixes *Px* :: *'b* ⇒ *real* **and** *MX* :: *'b measure*
assumes *X*: *distributed M MX X Px* **and** *Px-nn*[*simp*]: ⋀*x*. *x* ∈ *space MX* ⟹ 0 ≤ *Px x*
and *fin*: *finite-measure MX*
and *int*: *integrable MX* (λ*x*. − *Px x* ∗ *log b* (*Px x*))
shows *entropy b MX X* ≤ *log b* (*measure MX* (*space MX*))
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-uniform*:
assumes *X*: *distributed M MX X* (λ*x*. *indicator A x* / *measure MX A*) (**is** *distributed - - - ?f*)
shows *entropy b MX X* = *log b* (*measure MX A*)
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-simple-distributed*:
simple-distributed M X f ⟹ ℋ(*X*) = − (∑ *x*∈*X*‘*space M*. *f x* ∗ *log b* (*f x*))
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-le-card-not-0*:

assumes X: *simple-distributed M X f*
shows $\mathcal{H}(X) \leq$ *log b* (*card* (X ‘ *space M* $\cap$ $\{x.\ f\ x \neq 0\}$))
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-le-card*:
assumes X: *simple-distributed M X f*
shows $\mathcal{H}(X) \leq$ *log b* (*real* (*card* (X ‘ *space M*)))
⟨*proof*⟩

12.6 Conditional Mutual Information

definition (**in** *prob-space*)
conditional-mutual-information b MX MY MZ X Y Z ≡
mutual-information b MX (*MY* $\bigotimes_M$ *MZ*) *X* (λ*x*. (*Y x*, *Z x*)) −
mutual-information b MX MZ X Z

abbreviation (**in** *information-space*)
conditional-mutual-information-Pow (‹$\mathcal{I}'$(- ; - | - ′)›) **where**
$\mathcal{I}$(*X* ; *Y* | *Z*) ≡ *conditional-mutual-information b*
(*count-space* (*X* ‘ *space M*)) (*count-space* (*Y* ‘ *space M*)) (*count-space* (*Z* ‘ *space M*)) *X Y Z*

lemma (**in** *information-space*)
assumes S: *sigma-finite-measure S* **and** T: *sigma-finite-measure T* **and** P: *sigma-finite-measure P*
assumes *Px*[*measurable*]: *distributed M S X Px*
and *Px-nn*[*simp*]: $\bigwedge x.\ x \in$ *space S* $\Longrightarrow$ *0* $\leq$ *Px x*
assumes *Pz*[*measurable*]: *distributed M P Z Pz*
and *Pz-nn*[*simp*]: $\bigwedge z.\ z \in$ *space P* $\Longrightarrow$ *0* $\leq$ *Pz z*
assumes *Pyz*[*measurable*]: *distributed M* (*T* $\bigotimes_M$ *P*) (λ*x*. (*Y x*, *Z x*)) *Pyz*
and *Pyz-nn*[*simp*]: $\bigwedge y\ z.\ y \in$ *space T* $\Longrightarrow$ $z \in$ *space P* $\Longrightarrow$ *0* $\leq$ *Pyz* (*y*, *z*)
assumes *Pxz*[*measurable*]: *distributed M* (*S* $\bigotimes_M$ *P*) (λ*x*. (*X x*, *Z x*)) *Pxz*
and *Pxz-nn*[*simp*]: $\bigwedge x\ z.\ x \in$ *space S* $\Longrightarrow$ $z \in$ *space P* $\Longrightarrow$ *0* $\leq$ *Pxz* (*x*, *z*)
assumes *Pxyz*[*measurable*]: *distributed M* (*S* $\bigotimes_M$ *T* $\bigotimes_M$ *P*) (λ*x*. (*X x*, *Y x*, *Z x*)) *Pxyz*
and *Pxyz-nn*[*simp*]: $\bigwedge x\ y\ z.\ x \in$ *space S* $\Longrightarrow$ $y \in$ *space T* $\Longrightarrow$ $z \in$ *space P* $\Longrightarrow$ *0* $\leq$ *Pxyz* (*x*, *y*, *z*)
assumes *I1*: *integrable* (*S* $\bigotimes_M$ *T* $\bigotimes_M$ *P*) (λ(*x*, *y*, *z*). *Pxyz* (*x*, *y*, *z*) ∗ *log b* (*Pxyz* (*x*, *y*, *z*) / (*Px x* ∗ *Pyz* (*y*, *z*))))
assumes *I2*: *integrable* (*S* $\bigotimes_M$ *T* $\bigotimes_M$ *P*) (λ(*x*, *y*, *z*). *Pxyz* (*x*, *y*, *z*) ∗ *log b* (*Pxz* (*x*, *z*) / (*Px x* ∗ *Pz z*)))
shows *conditional-mutual-information-generic-eq*: *conditional-mutual-information b S T P X Y Z*
= ($\int$ (*x*, *y*, *z*). *Pxyz* (*x*, *y*, *z*) ∗ *log b* (*Pxyz* (*x*, *y*, *z*) / (*Pxz* (*x*, *z*) ∗ (*Pyz* (*y*,*z*) / *Pz z*))) ∂(*S* $\bigotimes_M$ *T* $\bigotimes_M$ *P*)) (**is** *?eq*)
and *conditional-mutual-information-generic-nonneg*: *0* $\leq$ *conditional-mutual-information b S T P X Y Z* (**is** *?nonneg*)
⟨*proof*⟩

lemma (**in** *information-space*)
fixes *Px* :: - ⇒ *real*
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T* **and** *P*: *sigma-finite-measure P*
assumes *Fx*: *finite-entropy S X Px*
assumes *Fz*: *finite-entropy P Z Pz*
assumes *Fyz*: *finite-entropy* $(T \bigotimes_M P)$ $(\lambda x.\ (Y\ x,\ Z\ x))$ *Pyz*
assumes *Fxz*: *finite-entropy* $(S \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Z\ x))$ *Pxz*
assumes *Fxyz*: *finite-entropy* $(S \bigotimes_M T \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))$ *Pxyz*
shows *conditional-mutual-information-generic-eq'*: *conditional-mutual-information b S T P X Y Z*
$= (\int (x,\ y,\ z).\ Pxyz\ (x,\ y,\ z) * log\ b\ (Pxyz\ (x,\ y,\ z)\ /\ (Pxz\ (x,\ z) * (Pyz\ (y,z)\ /\ Pz\ z)))\ \partial(S \bigotimes_M T \bigotimes_M P))$ (**is** *?eq*)
and *conditional-mutual-information-generic-nonneg'*: $0 \le$ *conditional-mutual-information b S T P X Y Z* (**is** *?nonneg*)
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-mutual-information-eq*:
assumes *Pz*: *simple-distributed M Z Pz*
assumes *Pyz*: *simple-distributed M* $(\lambda x.\ (Y\ x,\ Z\ x))$ *Pyz*
assumes *Pxz*: *simple-distributed M* $(\lambda x.\ (X\ x,\ Z\ x))$ *Pxz*
assumes *Pxyz*: *simple-distributed M* $(\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))$ *Pxyz*
shows $\mathcal{I}(X\ ;\ Y\ |\ Z) =$
$(\sum (x,\ y,\ z) \in (\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))\text{`space } M.\ Pxyz\ (x,\ y,\ z) * log\ b\ (Pxyz\ (x,\ y,\ z)\ /\ (Pxz\ (x,\ z) * (Pyz\ (y,z)\ /\ Pz\ z))))$
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-mutual-information-nonneg*:
assumes *X*: *simple-function M X* **and** *Y*: *simple-function M Y* **and** *Z*: *simple-function M Z*
shows $0 \le \mathcal{I}(X\ ;\ Y\ |\ Z)$
⟨*proof*⟩

12.7 Conditional Entropy

definition (**in** *prob-space*)
conditional-entropy b S T X Y = $-$ $(\int (x,\ y).\ log\ b\ (enn2real\ (RN\text{-}deriv\ (S \bigotimes_M T)\ (distr\ M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x)))\ (x,\ y))\ /$
enn2real (*RN-deriv T* (*distr M T Y*) *y*)) *∂distr M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x)))$

abbreviation (**in** *information-space*)
conditional-entropy-Pow (‹$\mathcal{H}'$(- | -')›) **where**
$\mathcal{H}(X\ |\ Y) \equiv$ *conditional-entropy b* (*count-space* (*X*‘*space M*)) (*count-space* (*Y*‘*space M*)) *X Y*

lemma (**in** *information-space*) *conditional-entropy-generic-eq*:
fixes *Pxy* :: - ⇒ *real* **and** *Py* :: *'c* ⇒ *real*
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*

assumes *Py*[*measurable*]: *distributed M T Y Py* **and** *Py-nn*[*simp*]: $\bigwedge x.\ x \in space\ T \Longrightarrow 0 \leq Py\ x$
assumes *Pxy*[*measurable*]: *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
and *Pxy-nn*[*simp*]: $\bigwedge x\ y.\ x \in space\ S \Longrightarrow y \in space\ T \Longrightarrow 0 \leq Pxy\ (x,\ y)$
shows *conditional-entropy b S T X Y* $= -\ (\int (x,\ y).\ Pxy\ (x,\ y) * log\ b\ (Pxy\ (x,\ y)\ /\ Py\ y)\ \partial(S \bigotimes_M T))$
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-entropy-eq-entropy*:
fixes $Px :: 'b \Rightarrow real$ **and** $Py :: 'c \Rightarrow real$
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *Py*[*measurable*]: *distributed M T Y Py*
and *Py-nn*[*simp*]: $\bigwedge x.\ x \in space\ T \Longrightarrow 0 \leq Py\ x$
assumes *Pxy*[*measurable*]: *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
and *Pxy-nn*[*simp*]: $\bigwedge x\ y.\ x \in space\ S \Longrightarrow y \in space\ T \Longrightarrow 0 \leq Pxy\ (x,\ y)$
assumes *I1*: *integrable* $(S \bigotimes_M T)$ $(\lambda x.\ Pxy\ x * log\ b\ (Pxy\ x))$
assumes *I2*: *integrable* $(S \bigotimes_M T)$ $(\lambda x.\ Pxy\ x * log\ b\ (Py\ (snd\ x)))$
shows *conditional-entropy b S T X Y* $=$ *entropy b* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ $-$ *entropy b T Y*
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-entropy-eq-entropy-simple*:
assumes *X*: *simple-function M X* **and** *Y*: *simple-function M Y*
shows $\mathcal{H}(X \mid Y) =$ *entropy b* (*count-space* (*X*'*space M*) $\bigotimes_M$ *count-space* (*Y*'*space M*)) $(\lambda x.\ (X\ x,\ Y\ x))$ $-\ \mathcal{H}(Y)$
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-entropy-eq*:
assumes *Y*: *simple-distributed M Y Py*
assumes *XY*: *simple-distributed M* $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
shows $\mathcal{H}(X \mid Y) = -\ (\sum (x,\ y) \in (\lambda x.\ (X\ x,\ Y\ x))\ `\ space\ M.\ Pxy\ (x,\ y) * log\ b\ (Pxy\ (x,\ y)\ /\ Py\ y))$
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-mutual-information-eq-conditional-entropy*:
assumes *X*: *simple-function M X* **and** *Y*: *simple-function M Y*
shows $\mathcal{I}(X\ ;\ X \mid Y) = \mathcal{H}(X \mid Y)$
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-entropy-nonneg*:
assumes *X*: *simple-function M X* **and** *Y*: *simple-function M Y* **shows** $0 \leq \mathcal{H}(X \mid Y)$
⟨*proof*⟩

12.8 Equalities

lemma (**in** *information-space*) *mutual-information-eq-entropy-conditional-entropy-distr*:
fixes $Px :: 'b \Rightarrow real$ **and** $Py :: 'c \Rightarrow real$ **and** $Pxy :: ('b \times 'c) \Rightarrow real$
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*

assumes *Px*[*measurable*]: *distributed M S X Px*
and *Px-nn*[*simp*]: $\bigwedge x.\ x \in space\ S \Longrightarrow 0 \leq Px\ x$
and *Py*[*measurable*]: *distributed M T Y Py*
and *Py-nn*[*simp*]: $\bigwedge x.\ x \in space\ T \Longrightarrow 0 \leq Py\ x$
and *Pxy*[*measurable*]: *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
and *Pxy-nn*[*simp*]: $\bigwedge x\ y.\ x \in space\ S \Longrightarrow y \in space\ T \Longrightarrow 0 \leq Pxy\ (x,\ y)$
assumes *Ix*: *integrable*$(S \bigotimes_M T)$ $(\lambda x.\ Pxy\ x * log\ b\ (Px\ (fst\ x)))$
assumes *Iy*: *integrable*$(S \bigotimes_M T)$ $(\lambda x.\ Pxy\ x * log\ b\ (Py\ (snd\ x)))$
assumes *Ixy*: *integrable*$(S \bigotimes_M T)$ $(\lambda x.\ Pxy\ x * log\ b\ (Pxy\ x))$
shows *mutual-information b S T X Y* = *entropy b S X* + *entropy b T Y* − *entropy b* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$
⟨*proof*⟩

lemma (**in** *information-space*) *mutual-information-eq-entropy-conditional-entropy′*:
fixes *Px* :: $'b \Rightarrow real$ **and** *Py* :: $'c \Rightarrow real$ **and** *Pxy* :: $('b \times 'c) \Rightarrow real$
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *Px*: *distributed M S X Px* $\bigwedge x.\ x \in space\ S \Longrightarrow 0 \leq Px\ x$
and *Py*: *distributed M T Y Py* $\bigwedge x.\ x \in space\ T \Longrightarrow 0 \leq Py\ x$
assumes *Pxy*: *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
$\bigwedge x.\ x \in space\ (S \bigotimes_M T) \Longrightarrow 0 \leq Pxy\ x$
assumes *Ix*: *integrable*$(S \bigotimes_M T)$ $(\lambda x.\ Pxy\ x * log\ b\ (Px\ (fst\ x)))$
assumes *Iy*: *integrable*$(S \bigotimes_M T)$ $(\lambda x.\ Pxy\ x * log\ b\ (Py\ (snd\ x)))$
assumes *Ixy*: *integrable*$(S \bigotimes_M T)$ $(\lambda x.\ Pxy\ x * log\ b\ (Pxy\ x))$
shows *mutual-information b S T X Y* = *entropy b S X* − *conditional-entropy b S T X Y*
⟨*proof*⟩

lemma (**in** *information-space*) *mutual-information-eq-entropy-conditional-entropy*:
assumes *sf-X*: *simple-function M X* **and** *sf-Y*: *simple-function M Y*
shows $\mathcal{I}(X\ ;\ Y) = \mathcal{H}(X) - \mathcal{H}(X \mid Y)$
⟨*proof*⟩

lemma (**in** *information-space*) *mutual-information-nonneg-simple*:
assumes *sf-X*: *simple-function M X* **and** *sf-Y*: *simple-function M Y*
shows $0 \leq \mathcal{I}(X\ ;\ Y)$
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-entropy-less-eq-entropy*:
assumes *X*: *simple-function M X* **and** *Z*: *simple-function M Z*
shows $\mathcal{H}(X \mid Z) \leq \mathcal{H}(X)$
⟨*proof*⟩

lemma (**in** *information-space*)
fixes *Px* :: $'b \Rightarrow real$ **and** *Py* :: $'c \Rightarrow real$ **and** *Pxy* :: $('b \times 'c) \Rightarrow real$
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *Px*: *finite-entropy S X Px* **and** *Py*: *finite-entropy T Y Py*
assumes *Pxy*: *finite-entropy* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
shows *conditional-entropy b S T X Y* ≤ *entropy b S X*
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-chain-rule*:
assumes X: *simple-function M X* **and** Y: *simple-function M Y*
shows $\mathcal{H}(\lambda x.\ (X\ x,\ Y\ x)) = \mathcal{H}(X) + \mathcal{H}(Y|X)$
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-partition*:
assumes X: *simple-function M X*
shows $\mathcal{H}(X) = \mathcal{H}(f \circ X) + \mathcal{H}(X|f \circ X)$
⟨*proof*⟩

corollary (**in** *information-space*) *entropy-data-processing*:
assumes *simple-function M X* **shows** $\mathcal{H}(f \circ X) \leq \mathcal{H}(X)$
⟨*proof*⟩

corollary (**in** *information-space*) *entropy-of-inj*:
assumes X: *simple-function M X* **and** *inj*: *inj-on f* (*X'space M*)
shows $\mathcal{H}(f \circ X) = \mathcal{H}(X)$
⟨*proof*⟩

end

13 Properties of Various Distributions

theory *Distributions*
imports *Convolution Information*
begin

lemma (**in** *prob-space*) *distributed-affine*:
fixes f :: *real* ⇒ *ennreal*
assumes f: *distributed M lborel X f*
assumes c: $c \neq 0$
shows *distributed M lborel* $(\lambda x.\ t + c * X\ x)$ $(\lambda x.\ f\ ((x - t)\ /\ c)\ /\ |c|)$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-affineI*:
fixes f :: *real* ⇒ *ennreal* **and** c :: *real*
assumes f: *distributed M lborel* $(\lambda x.\ (X\ x - t)\ /\ c)$ $(\lambda x.\ |c| * f\ (x * c + t))$
assumes c: $c \neq 0$
shows *distributed M lborel X f*
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-AE2*:
assumes [*measurable*]: *distributed M N X f Measurable.pred N P*
shows $(AE\ x\ in\ M.\ P\ (X\ x)) \longleftrightarrow (AE\ x\ in\ N.\ 0 < f\ x \longrightarrow P\ x)$
⟨*proof*⟩

13.1 Erlang

lemma *nn-intergal-power-times-exp-Icc*:
assumes [*arith*]: $0 \leq a$
shows $(\int^+ x.\ ennreal\ (x\widehat{}k * exp\ (-x)) * indicator\ \{0\ ..\ a\}\ x\ \partial lborel) =$
$(1 - (\sum n{\leq}k.\ (a\widehat{}n * exp\ (-a))\ /\ fact\ n)) * fact\ k$ (**is** *?I* = -)
⟨*proof*⟩

lemma *nn-intergal-power-times-exp-Ici*:
shows $(\int^+ x.\ ennreal\ (x\widehat{}k * exp\ (-x)) * indicator\ \{0\ ..\}\ x\ \partial lborel) = real\text{-}of\text{-}nat\ (fact\ k)$
⟨*proof*⟩

definition *erlang-density* :: *nat* ⇒ *real* ⇒ *real* ⇒ *real* **where**
erlang-density k l x = (*if x* < *0 then 0 else* $(l\widehat{}(Suc\ k) * x\widehat{}k * exp\ (-\ l * x))\ /\ fact\ k$)

definition *erlang-CDF* :: *nat* ⇒ *real* ⇒ *real* ⇒ *real* **where**
erlang-CDF k l x = (*if x* < *0 then 0 else 1* − $(\sum n{\leq}k.\ ((l * x)\widehat{}n * exp\ (-\ l * x)\ /\ fact\ n))$)

lemma *erlang-density-nonneg*[*simp*]: $0 \leq l \Longrightarrow 0 \leq erlang\text{-}density\ k\ l\ x$
⟨*proof*⟩

lemma *borel-measurable-erlang-density*[*measurable*]: *erlang-density k l* ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *erlang-CDF-transform*: $0 < l \Longrightarrow erlang\text{-}CDF\ k\ l\ a = erlang\text{-}CDF\ k\ 1\ (l * a)$
⟨*proof*⟩

lemma *erlang-CDF-nonneg*[*simp*]: **assumes** *0* < *l* **shows** *0* ≤ *erlang-CDF k l x*
⟨*proof*⟩

lemma *nn-integral-erlang-density*:
assumes [*arith*]: *0* < *l*
shows $(\int^+ x.\ ennreal\ (erlang\text{-}density\ k\ l\ x) * indicator\ \{..\ a\}\ x\ \partial lborel) = erlang\text{-}CDF\ k\ l\ a$
⟨*proof*⟩

lemma *emeasure-erlang-density*:
0 < *l* ⟹ *emeasure* (*density lborel* (*erlang-density k l*)) {.. *a*} = *erlang-CDF k l a*
⟨*proof*⟩

lemma *nn-integral-erlang-ith-moment*:
fixes *k i* :: *nat* **and** *l* :: *real*
assumes [*arith*]: *0* < *l*
shows $(\int^+ x.\ ennreal\ (erlang\text{-}density\ k\ l\ x * x\ \widehat{}\ i)\ \partial lborel) = fact\ (k + i)\ /$

(*fact k * l ^ i*)
⟨*proof*⟩

lemma *prob-space-erlang-density*:
assumes *l*[*arith*]: *0 < l*
shows *prob-space* (*density lborel* (*erlang-density k l*)) (**is** *prob-space ?D*)
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-le*:
assumes *D*: *distributed M lborel X* (*erlang-density k l*)
assumes [*simp*, *arith*]: *0 < l 0 ≤ a*
shows $\mathcal{P}$(*x in M. X x ≤ a*) = *erlang-CDF k l a*
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-gt*:
assumes *D*[*simp*]: *distributed M lborel X* (*erlang-density k l*)
assumes [*arith*]: *0 < l 0 ≤ a*
shows $\mathcal{P}$(*x in M. a < X x*) = *1 −* (*erlang-CDF k l a*)
⟨*proof*⟩

lemma *erlang-CDF-at0*: *erlang-CDF k l 0 = 0*
⟨*proof*⟩

lemma *erlang-distributedI*:
assumes *X*[*measurable*]: *X ∈ borel-measurable M* **and** [*arith*]: *0 < l*
and *X-distr*: ⋀*a. 0 ≤ a ⟹ emeasure M* {*x*∈*space M. X x ≤ a*} = *erlang-CDF k l a*
shows *distributed M lborel X* (*erlang-density k l*)
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-iff*:
assumes [*arith*]: *0*<*l*
shows *distributed M lborel X* (*erlang-density k l*) ⟷
(*X ∈ borel-measurable M ∧ 0 < l ∧* (∀ *a*≥*0*. $\mathcal{P}$(*x in M. X x ≤ a*) = *erlang-CDF k l a*))
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-mult-const*:
assumes *erlX*: *distributed M lborel X* (*erlang-density k l*)
assumes *a-pos*[*arith*]: *0 < α 0 < l*
shows *distributed M lborel* (*λx. α * X x*) (*erlang-density k* (*l / α*))
⟨*proof*⟩

lemma (**in** *prob-space*) *has-bochner-integral-erlang-ith-moment*:
fixes *k i* :: *nat* **and** *l* :: *real*
assumes [*arith*]: *0 < l* **and** *D*: *distributed M lborel X* (*erlang-density k l*)
shows *has-bochner-integral M* (*λx. X x ^ i*) (*fact* (*k* + *i*) / (*fact k * l ^ i*))
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-ith-moment-integrable*:
0 < l ⟹ distributed M lborel X (erlang-density k l) ⟹ integrable M (λx. X x ^ i)
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-ith-moment*:
0 < l ⟹ distributed M lborel X (erlang-density k l) ⟹
*expectation (λx. X x ^ i) = fact (k + i) / (fact k * l ^ i)*
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-variance*:
assumes [*arith*]: *0 < l* **and** *distributed M lborel X (erlang-density k l)*
shows *variance X = (k + 1) / l²*
⟨*proof*⟩

13.2 Exponential distribution

abbreviation *exponential-density* :: *real ⇒ real ⇒ real* **where**
exponential-density ≡ erlang-density 0

lemma *exponential-density-def*:
*exponential-density l x = (if x < 0 then 0 else l * exp (− x * l))*
⟨*proof*⟩

lemma *erlang-CDF-0*: *erlang-CDF 0 l a = (if 0 ≤ a then 1 − exp (− l * a) else 0)*
⟨*proof*⟩

lemma *prob-space-exponential-density*: *0 < l ⟹ prob-space (density lborel (exponential-density l))*
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributedD-le*:
assumes *D*: *distributed M lborel X (exponential-density l)* **and** *a*: *0 ≤ a* **and** *l*: *0 < l*
shows *P(x in M. X x ≤ a) = 1 − exp (− a * l)*
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributedD-gt*:
assumes *D*: *distributed M lborel X (exponential-density l)* **and** *a*: *0 ≤ a* **and** *l*: *0 < l*
shows *P(x in M. a < X x) = exp (− a * l)*
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-memoryless*:
assumes *D*: *distributed M lborel X (exponential-density l)* **and** *a*: *0 ≤ a* **and** *l*: *0 < l* **and** *t*: *0 ≤ t*
shows *P(x in M. a + t < X x | a < X x) = P(x in M. t < X x)*
⟨*proof*⟩

lemma *exponential-distributedI*:
assumes *X*[*measurable*]: $X \in$ *borel-measurable M* **and** [*arith*]: $0 < l$
and *X-distr*: $\bigwedge a.\ 0 \leq a \Longrightarrow$ *emeasure M* $\{x \in$ *space M*. $X\ x \leq a\} = 1 - exp\ (-\ a * l)$
shows *distributed M lborel X* (*exponential-density l*)
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-iff*:
assumes $0 < l$
shows *distributed M lborel X* (*exponential-density l*) $\longleftrightarrow$
$(X \in$ *borel-measurable M* $\wedge\ (\forall\, a \geq 0.\ \mathcal{P}(x$ *in M*. $X\ x \leq a) = 1 - exp\ (-\ a * l)))$
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-expectation*:
$0 < l \Longrightarrow$ *distributed M lborel X* (*exponential-density l*) $\Longrightarrow$ *expectation X* $= 1\ /\ l$
⟨*proof*⟩

lemma *exponential-density-nonneg*: $0 < l \Longrightarrow 0 \leq$ *exponential-density l x*
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-min*:
assumes $0 < l\ 0 < u$
assumes *expX*: *distributed M lborel X* (*exponential-density l*)
assumes *expY*: *distributed M lborel Y* (*exponential-density u*)
assumes *ind*: *indep-var borel X borel Y*
shows *distributed M lborel* $(\lambda x.$ *min* $(X\ x)\ (Y\ x))$ (*exponential-density* $(l + u)$)
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-Min*:
assumes *finI*: *finite I*
assumes *A*: $I \neq \{\}$
assumes *l*: $\bigwedge i.\ i \in I \Longrightarrow 0 < l\ i$
assumes *expX*: $\bigwedge i.\ i \in I \Longrightarrow$ *distributed M lborel* $(X\ i)$ (*exponential-density* $(l\ i)$)
assumes *ind*: *indep-vars* $(\lambda i.$ *borel*$)$ *X I*
shows *distributed M lborel* $(\lambda x.$ *Min* $((\lambda i.\ X\ i\ x)\ `I))$ (*exponential-density* $(\sum i \in I.\ l\ i)$)
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-variance*:
$0 < l \Longrightarrow$ *distributed M lborel X* (*exponential-density l*) $\Longrightarrow$ *variance X* $= 1\ /\ l^2$
⟨*proof*⟩

lemma *nn-integral-zero'*: *AE x in M*. $f\ x = 0 \Longrightarrow (\int^+ x.\ f\ x\ \partial M) = 0$
⟨*proof*⟩

lemma *convolution-erlang-density*:
fixes k_1 k_2 :: *nat*
assumes [*simp*, *arith*]: $0 < l$
shows $(\lambda x.\ \int^+ y.\ ennreal\ (erlang\text{-}density\ k_1\ l\ (x - y)) * ennreal\ (erlang\text{-}density\ k_2\ l\ y)\ \partial lborel) =$
$(erlang\text{-}density\ (Suc\ k_1 + Suc\ k_2 - 1)\ l)$
(**is** *?LHS* = *?RHS*)
⟨*proof*⟩

lemma (**in** *prob-space*) *sum-indep-erlang*:
assumes *indep*: *indep-var borel X borel Y*
assumes [*simp*, *arith*]: $0 < l$
assumes *erlX*: *distributed M lborel X* (*erlang-density* k_1 l)
assumes *erlY*: *distributed M lborel Y* (*erlang-density* k_2 l)
shows *distributed M lborel* $(\lambda x.\ X\ x + Y\ x)$ (*erlang-density* (*Suc* k_1 + *Suc* k_2 − 1) l)
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-sum*:
assumes *finI* : *finite I*
assumes *A*: $I \neq \{\}$
assumes [*simp*, *arith*]: $0 < l$
assumes *expX*: $\bigwedge i.\ i \in I \Longrightarrow$ *distributed M lborel* (*X i*) (*erlang-density* (*k i*) *l*)
assumes *ind*: *indep-vars* $(\lambda i.\ borel)$ *X I*
shows *distributed M lborel* $(\lambda x.\ \sum i{\in}I.\ X\ i\ x)$ (*erlang-density* $((\sum i{\in}I.\ Suc\ (k\ i)) - 1)\ l$)
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-sum*:
assumes *finI*: *finite I*
assumes *A*: $I \neq \{\}$
assumes *l*: $0 < l$
assumes *expX*: $\bigwedge i.\ i \in I \Longrightarrow$ *distributed M lborel* (*X i*) (*exponential-density l*)
assumes *ind*: *indep-vars* $(\lambda i.\ borel)$ *X I*
shows *distributed M lborel* $(\lambda x.\ \sum i{\in}I.\ X\ i\ x)$ (*erlang-density* $((card\ I) - 1)\ l$)
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-exponential*:
assumes *l*[*simp*, *arith*]: $0 < l$
assumes *D*: *distributed M lborel X* (*exponential-density l*)
shows *entropy b lborel X* = *log b* (*exp 1* / *l*)
⟨*proof*⟩

13.3 Uniform distribution

lemma *uniform-distrI*:
assumes *X*: $X \in measurable\ M\ M'$
and *A*: $A \in sets\ M'$ *emeasure* $M'\ A \neq \infty$ *emeasure* $M'\ A \neq 0$
assumes *distr*: $\bigwedge B.\ B \in sets\ M' \Longrightarrow emeasure\ M\ (X -` B \cap space\ M) =$

emeasure M' (A ∩ B) / emeasure M' A
shows *distr M M' X = uniform-measure M' A*
⟨*proof*⟩

lemma *uniform-distrI-borel*:
fixes *A* :: *real set*
assumes *X*[*measurable*]: *X* ∈ *borel-measurable M* **and** *A*: *emeasure lborel A = ennreal r 0 < r*
and [*measurable*]: *A* ∈ *sets borel*
assumes *distr*: ⋀*a*. *emeasure M* {*x*∈*space M*. *X x* ≤ *a*} = *emeasure lborel* (*A* ∩ {.. *a*}) / *r*
shows *distributed M lborel X* (λ*x*. *indicator A x / measure lborel A*)
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distrI-borel-atLeastAtMost*:
fixes *a b* :: *real*
assumes *X*: *X* ∈ *borel-measurable M* **and** *a < b*
assumes *distr*: ⋀*t*. *a* ≤ *t* ⟹ *t* ≤ *b* ⟹ *P*(*x in M*. *X x* ≤ *t*) = (*t* − *a*) / (*b* − *a*)
shows *distributed M lborel X* (λ*x*. *indicator* {*a..b*} *x / measure lborel* {*a..b*})
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-measure*:
fixes *a b* :: *real*
assumes *D*: *distributed M lborel X* (λ*x*. *indicator* {*a .. b*} *x / measure lborel* {*a .. b*})
assumes *t*: *a* ≤ *t t* ≤ *b*
shows *P*(*x in M*. *X x* ≤ *t*) = (*t* − *a*) / (*b* − *a*)
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-bounds*:
fixes *a b* :: *real*
assumes *D*: *distributed M lborel X* (λ*x*. *indicator* {*a .. b*} *x / measure lborel* {*a .. b*})
shows *a < b*
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-iff*:
fixes *a b* :: *real*
shows *distributed M lborel X* (λ*x*. *indicator* {*a..b*} *x / measure lborel* {*a..b*}) ⟷
(*X* ∈ *borel-measurable M* ∧ *a < b* ∧ (∀ *t*∈{*a .. b*}. *P*(*x in M*. *X x* ≤ *t*)= (*t* − *a*) / (*b* − *a*)))
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-expectation*:
fixes *a b* :: *real*
assumes *D*: *distributed M lborel X* (λ*x*. *indicator* {*a .. b*} *x / measure lborel* {*a .. b*})
shows *expectation X* = (*a* + *b*) / *2*

⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-variance*:
fixes *a b* :: *real*
assumes *D*: *distributed M lborel X* (λ*x*. *indicator* {*a* .. *b*} *x* / *measure lborel* {*a* .. *b*})
shows *variance X* = (*b* − *a*)² / *12*
⟨*proof*⟩

13.4 Normal distribution

definition *normal-density* :: *real* ⇒ *real* ⇒ *real* ⇒ *real* **where**
normal-density μ σ *x* = *1* / *sqrt* (*2* ∗ *pi* ∗ σ²) ∗ *exp* (−(*x* − μ)²/ (*2* ∗ σ²))

abbreviation *std-normal-density* :: *real* ⇒ *real* **where**
std-normal-density ≡ *normal-density 0 1*

lemma *std-normal-density-def*: *std-normal-density x* = (*1* / *sqrt* (*2* ∗ *pi*)) ∗ *exp* (− *x*² / *2*)
⟨*proof*⟩

lemma *normal-density-nonneg*[*simp*]: *0* ≤ *normal-density* μ σ *x*
⟨*proof*⟩

lemma *normal-density-pos*: *0* < σ ⟹ *0* < *normal-density* μ σ *x*
⟨*proof*⟩

lemma *borel-measurable-normal-density*[*measurable*]: *normal-density* μ σ ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *gaussian-moment-0*:
has-bochner-integral lborel (λ*x*. *indicator* {*0*..} *x* ∗*R* *exp* (− *x*²)) (*sqrt pi* / *2*)
⟨*proof*⟩

lemma *gaussian-moment-1*:
has-bochner-integral lborel (λ*x*::*real*. *indicator* {*0*..} *x* ∗*R* (*exp* (− *x*²) ∗ *x*)) (*1* / *2*)
⟨*proof*⟩

lemma
fixes *k* :: *nat*
shows *gaussian-moment-even-pos*:
has-bochner-integral lborel (λ*x*::*real*. *indicator* {*0*..} *x* ∗*R* (*exp* (−*x*²)∗*x*^(*2* ∗ *k*)))
((*sqrt pi* / *2*) ∗ (*fact* (*2* ∗ *k*) / (*2* ^ (*2* ∗ *k*) ∗ *fact k*)))
(**is** *?even*)
and *gaussian-moment-odd-pos*:
has-bochner-integral lborel (λ*x*::*real*. *indicator* {*0*..} *x* ∗*R* (*exp* (−*x*²)∗*x*^(*2* ∗

$k + 1$))) (*fact* k / 2)
(**is** *?odd*)
⟨*proof*⟩

context
fixes k :: *nat* **and** μ σ :: *real* **assumes** [*arith*]: $0 < \sigma$
begin

lemma *normal-moment-even*:
has-bochner-integral lborel (λx. *normal-density* μ σ x ∗ ($x - \mu$) ^ ($2 * k$)) (*fact* ($2 * k$) / ((2 / σ^2)^k ∗ *fact* k))
⟨*proof*⟩

lemma *normal-moment-abs-odd*:
has-bochner-integral lborel (λx. *normal-density* μ σ x ∗ |$x - \mu$|^($2 * k + 1$)) (2^k ∗ σ^($2 * k + 1$) ∗ *fact* k ∗ *sqrt* (2 / *pi*))
⟨*proof*⟩

lemma *normal-moment-odd*:
has-bochner-integral lborel (λx. *normal-density* μ σ x ∗ ($x - \mu$)^($2 * k + 1$)) 0
⟨*proof*⟩

lemma *integral-normal-moment-even*:
integral^L *lborel* (λx. *normal-density* μ σ x ∗ ($x - \mu$)^($2 * k$)) = *fact* ($2 * k$) / ((2 / σ^2)^k ∗ *fact* k)
⟨*proof*⟩

lemma *integral-normal-moment-abs-odd*:
integral^L *lborel* (λx. *normal-density* μ σ x ∗ |$x - \mu$|^($2 * k + 1$)) = 2 ^ k ∗ σ ^ ($2 * k + 1$) ∗ *fact* k ∗ *sqrt* (2 / *pi*)
⟨*proof*⟩

lemma *integral-normal-moment-odd*:
integral^L *lborel* (λx. *normal-density* μ σ x ∗ ($x - \mu$)^($2 * k + 1$)) = 0
⟨*proof*⟩

end

context
fixes σ :: *real*
assumes σ-*pos*[*arith*]: $0 < \sigma$
begin

lemma *normal-moment-nz-1*: *has-bochner-integral lborel* (λx. *normal-density* μ σ x ∗ x) μ
⟨*proof*⟩

lemma *integral-normal-moment-nz-1*:

integral^L *lborel* (λx. *normal-density* μ σ $x * x$) = μ
⟨*proof*⟩

lemma *integrable-normal-moment-nz-1*: *integrable lborel* (λx. *normal-density* μ σ $x * x$)
⟨*proof*⟩

lemma *integrable-normal-moment*: *integrable lborel* (λx. *normal-density* μ σ $x *$ $(x - \mu)\hat{}k$)
⟨*proof*⟩

lemma *integrable-normal-moment-abs*: *integrable lborel* (λx. *normal-density* μ σ x $* |x - \mu|\hat{}k$)
⟨*proof*⟩

lemma *integrable-normal-density*[*simp*, *intro*]: *integrable lborel* (*normal-density* μ σ)
⟨*proof*⟩

lemma *integral-normal-density*[*simp*]: ($\int$ x. *normal-density* μ σ x ∂*lborel*) = *1*
⟨*proof*⟩

lemma *prob-space-normal-density*:
prob-space (*density lborel* (*normal-density* μ σ))
⟨*proof*⟩

end

context
fixes k :: *nat*
begin

lemma *std-normal-moment-even*:
has-bochner-integral lborel (λx. *std-normal-density* $x * x\hat{}(2 * k)$) (*fact* $(2 * k)$ / $(2\hat{}k * fact\ k)$)
⟨*proof*⟩

lemma *std-normal-moment-abs-odd*:
has-bochner-integral lborel (λx. *std-normal-density* $x * |x|\hat{}(2 * k + 1)$) (*sqrt* $(2/pi) * 2\hat{}k * fact\ k$)
⟨*proof*⟩

lemma *std-normal-moment-odd*:
has-bochner-integral lborel (λx. *std-normal-density* $x * x\hat{}(2 * k + 1)$) *0*
⟨*proof*⟩

lemma *integral-std-normal-moment-even*:

integral^L lborel (*λx. std-normal-density x * x^(2*k)*) = *fact* (*2 * k*) / (*2^k * fact k*)
⟨*proof*⟩

lemma *integral-std-normal-moment-abs-odd*:
integral^L lborel (*λx. std-normal-density x * |x|^(2 * k + 1)*) = *sqrt* (*2 / pi*) * *2^k * fact k*
⟨*proof*⟩

lemma *integral-std-normal-moment-odd*:
integral^L lborel (*λx. std-normal-density x * x^(2 * k + 1)*) = *0*
⟨*proof*⟩

lemma *integrable-std-normal-moment-abs*: *integrable lborel* (*λx. std-normal-density x * |x|^k*)
⟨*proof*⟩

lemma *integrable-std-normal-moment*: *integrable lborel* (*λx. std-normal-density x * x^k*)
⟨*proof*⟩

end

lemma (**in** *prob-space*) *normal-density-affine*:
assumes *X*: *distributed M lborel X* (*normal-density μ σ*)
assumes [*simp*, *arith*]: *0 < σ α ≠ 0*
shows *distributed M lborel* (*λx. β + α * X x*) (*normal-density* (*β + α * μ*) (|*α*| * *σ*))
⟨*proof*⟩

lemma (**in** *prob-space*) *normal-standard-normal-convert*:
assumes *pos-var*[*simp*, *arith*]: *0 < σ*
shows *distributed M lborel X* (*normal-density μ σ*) = *distributed M lborel* (*λx.* (*X x − μ*) / *σ*) *std-normal-density*
⟨*proof*⟩

lemma *conv-normal-density-zero-mean*:
assumes [*simp*, *arith*]: *0 < σ 0 < τ*
shows (*λx.* ∫⁺*y. ennreal* (*normal-density 0 σ* (*x − y*) * *normal-density 0 τ y*) *∂lborel*) =
normal-density 0 (*sqrt* (*σ*² + *τ*²)) (**is** *?LHS = ?RHS*)
⟨*proof*⟩

lemma *conv-std-normal-density*:
(*λx.* ∫⁺*y. ennreal* (*std-normal-density* (*x − y*) * *std-normal-density y*) *∂lborel*) =
(*normal-density 0* (*sqrt 2*))
⟨*proof*⟩

lemma (**in** *prob-space*) *add-indep-normal*:
assumes *indep*: *indep-var borel X borel Y*
assumes *pos-var*[*arith*]: $0 < \sigma$ $0 < \tau$
assumes *normalX*[*simp*]: *distributed M lborel X* (*normal-density* μ σ)
assumes *normalY*[*simp*]: *distributed M lborel Y* (*normal-density* ν τ)
shows *distributed M lborel* ($\lambda x.$ *X x* + *Y x*) (*normal-density* (μ + ν) (*sqrt* (σ^2 + τ^2)))
⟨*proof*⟩

lemma (**in** *prob-space*) *diff-indep-normal*:
assumes *indep*[*simp*]: *indep-var borel X borel Y*
assumes [*simp*, *arith*]: $0 < \sigma$ $0 < \tau$
assumes *normalX*[*simp*]: *distributed M lborel X* (*normal-density* μ σ)
assumes *normalY*[*simp*]: *distributed M lborel Y* (*normal-density* ν τ)
shows *distributed M lborel* ($\lambda x.$ *X x* − *Y x*) (*normal-density* (μ − ν) (*sqrt* (σ^2 + τ^2)))
⟨*proof*⟩

lemma (**in** *prob-space*) *sum-indep-normal*:
assumes *finite I I* ≠ {} *indep-vars* ($\lambda i.$ *borel*) *X I*
assumes $\bigwedge i.$ *i* ∈ *I* ⟹ $0 < \sigma$ *i*
assumes *normal*: $\bigwedge i.$ *i* ∈ *I* ⟹ *distributed M lborel* (*X i*) (*normal-density* (μ *i*) (σ *i*))
shows *distributed M lborel* ($\lambda x.$ $\sum i \in I.$ *X i x*) (*normal-density* ($\sum i \in I.$ μ *i*) (*sqrt* ($\sum i \in I.$ (σ *i*)^2)))
⟨*proof*⟩

lemma (**in** *prob-space*) *standard-normal-distributed-expectation*:
assumes *D*: *distributed M lborel X std-normal-density*
shows *expectation X* = *0*
⟨*proof*⟩

lemma (**in** *prob-space*) *normal-distributed-expectation*:
assumes σ[*arith*]: $0 < \sigma$
assumes *D*: *distributed M lborel X* (*normal-density* μ σ)
shows *expectation X* = μ
⟨*proof*⟩

lemma (**in** *prob-space*) *normal-distributed-variance*:
fixes *a b* :: *real*
assumes [*simp*, *arith*]: $0 < \sigma$
assumes *D*: *distributed M lborel X* (*normal-density* μ σ)
shows *variance X* = σ^2
⟨*proof*⟩

lemma (**in** *prob-space*) *standard-normal-distributed-variance*:
distributed M lborel X std-normal-density ⟹ *variance X* = *1*
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-normal-density*:
assumes [*arith*]: $0 < \sigma$
assumes *D*: *distributed M lborel X* (*normal-density* μ σ)
shows *entropy b lborel X* = *log b* (*2* $*$ *pi* $*$ *exp 1* $*$ σ^2) / *2*
⟨*proof*⟩

end

14 Characteristic Functions

theory *Characteristic-Functions*
imports *Weak-Convergence Independent-Family Distributions*
begin

lemma *mult-min-right*: $a \geq 0 \Longrightarrow$ (*a* :: *real*) $*$ *min b c* = *min* (*a* $*$ *b*) (*a* $*$ *c*)
⟨*proof*⟩

lemma *sequentially-even-odd*:
assumes *E*: *eventually* (λn. *P* (*2* $*$ *n*)) *sequentially* **and** *O*: *eventually* (λn. *P* (*2* $*$ *n* + *1*)) *sequentially*
shows *eventually P sequentially*
⟨*proof*⟩

lemma *limseq-even-odd*:
assumes (λn. *f* (*2* $*$ *n*)) $\longrightarrow$ (*l* :: *'a* :: *topological-space*)
and (λn. *f* (*2* $*$ *n* + *1*)) $\longrightarrow$ *l*
shows *f* $\longrightarrow$ *l*
⟨*proof*⟩

14.1 Application of the FTC: integrating $e^i x$

abbreviation *iexp* :: *real* $\Rightarrow$ *complex* **where**
iexp $\equiv$ (λx. *exp* (i $*$ *complex-of-real x*))

lemma *isCont-iexp* [*simp*]: *isCont iexp x*
⟨*proof*⟩

lemma *has-vector-derivative-iexp*[*derivative-intros*]:
(*iexp has-vector-derivative* i $*$ *iexp x*) (*at x within s*)
⟨*proof*⟩

lemma *interval-integral-iexp*:
fixes *a b* :: *real*
shows (*CLBINT x*=*a*..*b*. *iexp x*) = i $*$ *iexp a* − i $*$ *iexp b*
⟨*proof*⟩

14.2 The Characteristic Function of a Real Measure.

definition

char :: *real measure* ⇒ *real* ⇒ *complex*
where *char M t* ≡ *CLINT x*|*M*. *iexp* (*t* ∗ *x*)

lemma (**in** *real-distribution*) *char-zero*: *char M 0* = *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *integrable-iexp*:
assumes *f*: *f* ∈ *borel-measurable M* ⋀*x*. *Im* (*f x*) = *0*
shows *integrable M* (λ*x*. *exp* (i ∗ (*f x*)))
⟨*proof*⟩

lemma (**in** *real-distribution*) *cmod-char-le-1*: *norm* (*char M t*) ≤ *1*
⟨*proof*⟩

lemma (**in** *real-distribution*) *isCont-char*: *isCont* (*char M*) *t*
⟨*proof*⟩

lemma (**in** *real-distribution*) *char-measurable* [*measurable*]: *char M* ∈ *borel-measurable borel*
⟨*proof*⟩

14.3 Independence

lemma (**in** *prob-space*) *char-distr-add*:
fixes *X1 X2* :: *'a* ⇒ *real* **and** *t* :: *real*
assumes *indep-var borel X1 borel X2*
shows *char* (*distr M borel* (λω. *X1* ω + *X2* ω)) *t* =
char (*distr M borel X1*) *t* ∗ *char* (*distr M borel X2*) *t*
⟨*proof*⟩

lemma (**in** *prob-space*) *char-distr-sum*:
indep-vars (λ*i*. *borel*) *X A* ⟹
char (*distr M borel* (λω. ∑ *i*∈*A*. *X i* ω)) *t* = (∏ *i*∈*A*. *char* (*distr M borel* (*X i*)) *t*)
⟨*proof*⟩

14.4 Approximations to e^{ix}

Proofs from Billingsley, page 343.

lemma *CLBINT-I0c-power-mirror-iexp*:
fixes *x* :: *real* **and** *n* :: *nat*
defines *f s m* ≡ *complex-of-real* ((*x* − *s*) ^ *m*)
shows (*CLBINT s*=*0*..*x*. *f s n* ∗ *iexp s*) =
x^*Suc n* / *Suc n* + (i / *Suc n*) ∗ (*CLBINT s*=*0*..*x*. *f s* (*Suc n*) ∗ *iexp s*)
⟨*proof*⟩

lemma *iexp-eq1*:
fixes *x* :: *real*
defines *f s m* ≡ *complex-of-real* ((*x* − *s*) ^ *m*)

shows *iexp x* =
($\sum k \leq n$. (i * *x*)^*k* / (*fact k*)) + ((i ^ (*Suc n*)) / (*fact n*)) * (*CLBINT s=0..x*. (*f s n*) * (*iexp s*)) (**is** *?P n*)
⟨*proof*⟩

lemma *iexp-eq2*:
fixes *x* :: *real*
defines *f s m* ≡ *complex-of-real* ((*x* − *s*) ^ *m*)
shows *iexp x* = ($\sum k \leq Suc\ n$. (i*x)^*k*/*fact k*) + i^*Suc n*/*fact n* * (*CLBINT s=0..x*. *f s n*(*iexp s* − *1*))
⟨*proof*⟩

lemma *abs-LBINT-I0c-abs-power-diff*:
|*LBINT s=0..x*. |(*x* − *s*)^*n*|| = |*x* ^ (*Suc n*) / (*Suc n*)|
⟨*proof*⟩

lemma *iexp-approx1*: *cmod* (*iexp x* − ($\sum k \leq n$. (i * *x*)^*k* / *fact k*)) ≤ |*x*|^(*Suc n*) / *fact* (*Suc n*)
⟨*proof*⟩

lemma *iexp-approx2*: *cmod* (*iexp x* − ($\sum k \leq n$. (i * *x*)^*k* / *fact k*)) ≤ *2* * |*x*|^*n* / *fact n*
⟨*proof*⟩

lemma (**in** *real-distribution*) *char-approx1*:
assumes *integrable-moments*: $\bigwedge k.\ k \leq n \Longrightarrow$ *integrable M* (λ*x*. *x*^*k*)
shows *cmod* (*char M t* − ($\sum k \leq n$. ((i * *t*)^*k* / *fact k*) * *expectation* (λ*x*. *x*^*k*))) ≤
(*2*|*t*|^*n* / *fact n*) * *expectation* (λ*x*. |*x*|^*n*) (**is** *cmod* (*char M t* − *?t1*) ≤ -)
⟨*proof*⟩

lemma (**in** *real-distribution*) *char-approx2*:
assumes *integrable-moments*: $\bigwedge k.\ k \leq n \Longrightarrow$ *integrable M* (λ*x*. *x* ^ *k*)
shows *cmod* (*char M t* − ($\sum k \leq n$. ((i * *t*)^*k* / *fact k*) * *expectation* (λ*x*. *x*^*k*))) ≤
(|*t*|^*n* / *fact* (*Suc n*)) * *expectation* (λ*x*. *min* (*2* * |*x*|^*n* * *Suc n*) (|*t*| * |*x*|^*Suc n*))
(**is** *cmod* (*char M t* − *?t1*) ≤ -)
⟨*proof*⟩

lemma (**in** *real-distribution*) *char-approx3*:
fixes *t*
assumes
integrable-1: *integrable M* (λ*x*. *x*) **and**
integral-1: *expectation* (λ*x*. *x*) = *0* **and**
integrable-2: *integrable M* (λ*x*. *x*^*2*) **and**
integral-2: *variance* (λ*x*. *x*) = σ*2*
shows *cmod* (*char M t* − (*1* − *t*^*2* * σ*2* / *2*)) ≤
(*t*^*2* / *6*) * *expectation* (λ*x*. *min* (*6* * *x*^*2*) (*abs t* * (*abs x*)^*3*))

⟨*proof*⟩

This is a more familiar textbook formulation in terms of random variables, but we will use the previous version for the CLT.

lemma (**in** *prob-space*) *char-approx3'*:
fixes μ :: *real measure* **and** X
assumes *rv-X* [*simp*]: *random-variable borel X*
and [*simp*]: *integrable M X integrable M* ($\lambda x.\ (X\ x)\hat{}2$) *expectation X = 0*
and *var-X*: *variance X* $= \sigma 2$
and μ*-def*: μ = *distr M borel X*
shows *cmod* (*char* μ *t* − (*1* − *t*$\hat{}$*2* ∗ $\sigma 2$ / *2*)) ≤
(*t*$\hat{}$*2* / *6*) ∗ *expectation* ($\lambda x.$ *min* (*6* ∗ (*X x*)$\hat{}$*2*) (|*t*| ∗ |*X x*|$\hat{}$*3*))
⟨*proof*⟩

this is the formulation in the book – in terms of a random variable *with* the distribution, rather the distribution itself. I don't know which is more useful, though in principal we can go back and forth between them.

lemma (**in** *prob-space*) *char-approx1'*:
fixes μ :: *real measure* **and** X
assumes *integrable-moments* : $\bigwedge k.\ k \le n \Longrightarrow$ *integrable M* ($\lambda x.\ X\ x\ \hat{}\ k$)
and *rv-X*[*measurable*]: *random-variable borel X*
and μ*-distr* : *distr M borel X* = μ
shows *cmod* (*char* μ *t* − ($\sum k \le n.$ ((i ∗ *t*)$\hat{}k$ / *fact k*) ∗ *expectation* ($\lambda x.$ (*X x*)$\hat{}k$))) ≤
(*2* ∗ |*t*|$\hat{}n$ / *fact n*) ∗ *expectation* ($\lambda x.$ |*X x*|$\hat{}n$)
⟨*proof*⟩

14.5 Calculation of the Characteristic Function of the Standard Distribution

abbreviation
std-normal-distribution ≡ *density lborel std-normal-density*

lemma *real-dist-normal-dist*: *real-distribution std-normal-distribution*
⟨*proof*⟩

lemma *std-normal-distribution-even-moments*:
fixes *k* :: *nat*
shows (*LINT x*|*std-normal-distribution*. *x*$\hat{}$(*2* ∗ *k*)) = *fact* (*2* ∗ *k*) / (*2*$\hat{}k$ ∗ *fact k*)
and *integrable std-normal-distribution* ($\lambda x.$ *x*$\hat{}$(*2* ∗ *k*))
⟨*proof*⟩

lemma *integrable-std-normal-distribution-moment*: *integrable std-normal-distribution* ($\lambda x.$ *x*$\hat{}k$)
⟨*proof*⟩

lemma *integral-std-normal-distribution-moment-odd*:

odd k ⟹ integral^L std-normal-distribution (λx. x^k) = 0
⟨*proof*⟩

lemma *std-normal-distribution-even-moments-abs*:
fixes *k* :: *nat*
shows (*LINT x|std-normal-distribution. |x|^(2 * k)) = fact (2 * k) / (2^k * fact k)*
⟨*proof*⟩

lemma *std-normal-distribution-odd-moments-abs*:
fixes *k* :: *nat*
shows (*LINT x|std-normal-distribution. |x|^(2 * k + 1)) = sqrt (2 / pi) * 2 ^ k * fact k*
⟨*proof*⟩

theorem *char-std-normal-distribution*:
char std-normal-distribution = (λt. complex-of-real (exp (− (t^2) / 2)))
⟨*proof*⟩

end

15 Helly's selection theorem

The set of bounded, monotone, right continuous functions is sequentially compact

theory *Helly-Selection*
imports *HOL−Library.Diagonal-Subsequence Weak-Convergence*
begin

lemma *minus-one-less*: *x − 1 < (x::real)*
⟨*proof*⟩

theorem *Helly-selection*:
fixes *f* :: *nat ⇒ real ⇒ real*
assumes *rcont*: *⋀n x. continuous (at-right x) (f n)*
assumes *mono*: *⋀n. mono (f n)*
assumes *bdd*: *⋀n x. |f n x| ≤ M*
shows *∃ s. strict-mono (s::nat ⇒ nat) ∧ (∃ F. (∀ x. continuous (at-right x) F) ∧ mono F ∧ (∀ x. |F x| ≤ M) ∧*
(∀ x. continuous (at x) F ⟶ (λn. f (s n) x) ⟶⟶ F x))
⟨*proof*⟩

definition
tight :: *(nat ⇒ real measure) ⇒ bool*
where
tight μ ≡ (∀ n. real-distribution (μ n)) ∧ (∀ (ε::real)>0. ∃ a b::real. a < b ∧ (∀ n.

measure $(\mu\ n)$ $\{a<..b\} > 1 - \varepsilon))$

theorem *tight-imp-convergent-subsubsequence*:
assumes μ: *tight* μ *strict-mono* s
shows $\exists r\ M.$ *strict-mono* $(r :: nat \Rightarrow nat) \wedge$ *real-distribution* $M \wedge$ *weak-conv-m* $(\mu \circ s \circ r)\ M$
⟨*proof*⟩

corollary *tight-subseq-weak-converge*:
fixes $\mu ::$ *nat* $\Rightarrow$ *real measure* **and** $M ::$ *real measure*
assumes $\bigwedge n.$ *real-distribution* $(\mu\ n)$ *real-distribution* M **and** *tight*: *tight* μ **and**
subseq: $\bigwedge s\ \nu.$ *strict-mono* $s \Longrightarrow$ *real-distribution* $\nu \Longrightarrow$ *weak-conv-m* $(\mu \circ s)\ \nu \Longrightarrow$ *weak-conv-m* $(\mu \circ s)\ M$
shows *weak-conv-m* $\mu\ M$
⟨*proof*⟩

end

16 Integral of sinc

theory *Sinc-Integral*
imports *Distributions*
begin

16.1 Various preparatory integrals

Naming convention The theorem name consists of the following parts:

- Kind of integral: *has-bochner-integral* / *integrable* / *LBINT*
- Interval: Interval (0 / infinity / open / closed) (infinity / open / closed)
- Name of the occurring constants: power, exp, m (for minus), scale, sin, ...

lemma *has-bochner-integral-I0i-power-exp-m'*:
has-bochner-integral lborel $(\lambda x.\ x\hat{}k * exp\ (-x) *$ *indicator* $\{0\ ..\}\ x$::*real*$)$ $(fact\ k)$
⟨*proof*⟩

lemma *has-bochner-integral-I0i-power-exp-m*:
has-bochner-integral lborel $(\lambda x.\ x\hat{}k * exp\ (-x) *$ *indicator* $\{0 <..\}\ x$::*real*$)$ $(fact\ k)$
⟨*proof*⟩

lemma *integrable-I0i-exp-mscale*: $0 <$ (u::*real*) $\Longrightarrow$ *set-integrable lborel* $\{0 <..\}$ $(\lambda x.\ exp\ (-(x * u)))$
⟨*proof*⟩

lemma *LBINT-I0i-exp-mscale*: $0 < (u::real) \Longrightarrow LBINT\ x{=}0..\infty.\ exp\ (-(x * u)) = 1 / u$
⟨*proof*⟩

lemma *LBINT-I0c-exp-mscale-sin*:
$LBINT\ x{=}0..t.\ exp\ (-(u * x)) * sin\ x =$
$(1 / (1 + u\hat{}2)) * (1 - exp\ (-(u * t)) * (u * sin\ t + cos\ t))$ (**is** - = *?F t*)
⟨*proof*⟩

lemma *LBINT-I0i-exp-mscale-sin*:
assumes $0 < x$
shows $LBINT\ u{=}0..\infty.\ |exp\ (-u * x) * sin\ x| = |sin\ x| / x$
⟨*proof*⟩

lemma
shows *integrable-inverse-1-plus-square*:
$set\text{-}integrable\ lborel\ (einterval\ (-\infty)\ \infty)\ (\lambda x.\ inverse\ (1 + x\hat{}2))$
and *LBINT-inverse-1-plus-square*:
$LBINT\ x{=}-\infty..\infty.\ inverse\ (1 + x\hat{}2) = pi$
⟨*proof*⟩

lemma
shows *integrable-I0i-1-div-plus-square*:
$interval\text{-}lebesgue\text{-}integrable\ lborel\ 0\ \infty\ (\lambda x.\ 1 / (1 + x\hat{}2))$
and *LBINT-I0i-1-div-plus-square*:
$LBINT\ x{=}0..\infty.\ 1 / (1 + x\hat{}2) = pi / 2$
⟨*proof*⟩

17 The sinc function, and the sine integral (Si)

abbreviation *sinc* :: *real* ⇒ *real* **where**
$sinc \equiv (\lambda x.\ if\ x = 0\ then\ 1\ else\ sin\ x / x)$

lemma *sinc-at-0*: $((\lambda x.\ sin\ x / x::real) \longrightarrow 1)\ (at\ 0)$
⟨*proof*⟩

lemma *isCont-sinc*: $isCont\ sinc\ x$
⟨*proof*⟩

lemma *continuous-on-sinc*[*continuous-intros*]:
$continuous\text{-}on\ S\ f \Longrightarrow continuous\text{-}on\ S\ (\lambda x.\ sinc\ (f\ x))$
⟨*proof*⟩

lemma *borel-measurable-sinc*[*measurable*]: $sinc \in borel\text{-}measurable\ borel$
⟨*proof*⟩

lemma *sinc-AE*: $AE\ x\ in\ lborel.\ sin\ x / x = sinc\ x$
⟨*proof*⟩

definition *Si* :: *real* ⇒ *real* **where** *Si t* ≡ *LBINT x=0..t. sin x / x*

lemma *sinc-neg* [*simp*]: *sinc* (− *x*) = *sinc x*
⟨*proof*⟩

lemma *Si-alt-def* : *Si t* = *LBINT x=0..t. sinc x*
⟨*proof*⟩

lemma *Si-neg*:
assumes *T* ≥ *0* **shows** *Si* (− *T*) = − *Si T*
⟨*proof*⟩

lemma *integrable-sinc′*:
interval-lebesgue-integrable lborel (*ereal 0*) (*ereal T*) (λ*t. sin* (*t* ∗ *ϑ*) / *t*)
⟨*proof*⟩

lemma *DERIV-Si*: (*Si has-real-derivative sinc x*) (*at x*)
⟨*proof*⟩

lemma *isCont-Si*: *isCont Si x*
⟨*proof*⟩

lemma *borel-measurable-Si*[*measurable*]: *Si* ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *Si-at-top-LBINT*:
((λ*t*. (*LBINT x=0..∞. exp* (−(*x* ∗ *t*)) ∗ (*x* ∗ *sin t* + *cos t*) / (*1* + *x*^*2*))) ⟶ *0*) *at-top*
⟨*proof*⟩

lemma *Si-at-top-integrable*:
assumes *t* ≥ *0*
shows *interval-lebesgue-integrable lborel 0* ∞ (λ*x. exp* (− (*x* ∗ *t*)) ∗ (*x* ∗ *sin t* + *cos t*) / (*1* + *x*^{2}))
⟨*proof*⟩

lemma *Si-at-top*: (*Si* ⟶ *pi* / *2*) *at-top*
⟨*proof*⟩

17.1 The final theorems: boundedness and scalability

lemma *bounded-Si*: ∃ *B*. ∀ *T*. |*Si T*| ≤ *B*
⟨*proof*⟩

lemma *LBINT-I0c-sin-scale-divide*:
assumes *T* ≥ *0*

shows *LBINT t=0..T. sin (t * ϑ) / t = sgn ϑ * Si (T * |ϑ|)*
⟨*proof*⟩

end

18 The Levy inversion theorem, and the Levy continuity theorem.

theory *Levy*
imports *Characteristic-Functions Helly-Selection Sinc-Integral*
begin

18.1 The Levy inversion theorem

lemma *Levy-Inversion-aux1*:
fixes *a b* :: *real*
assumes *a ≤ b*
shows ((λ*t*. (*iexp* (−(*t* * *a*)) − *iexp* (−(*t* * *b*))) / (i * *t*)) ⟶ *b* − *a*) (*at 0*)
(**is** (*?F* ⟶ -) (*at* -))
⟨*proof*⟩

lemma *Levy-Inversion-aux2*:
fixes *a b t* :: *real*
assumes *a ≤ b* **and** *t ≠ 0*
shows *cmod* ((*iexp* (*t* * *b*) − *iexp* (*t* * *a*)) / (i * *t*)) ≤ *b* − *a* (**is** *?F* ≤ -)
⟨*proof*⟩

theorem (**in** *real-distribution*) *Levy-Inversion*:
fixes *a b* :: *real*
assumes *a ≤ b*
defines *μ* ≡ *measure M* **and** *φ* ≡ *char M*
assumes *μ* {*a*} = *0* **and** *μ* {*b*} = *0*
shows (λ*T*. *1* / (*2* * *pi*) * (*CLBINT t*=−*T*..*T*. (*iexp* (−(*t* * *a*)) − *iexp* (−(*t* * *b*))) / (i * *t*) * *φ t*))
⟶ *μ* {*a*<..*b*}
(**is** (λ*T*. *1* / (*2* * *pi*) * (*CLBINT t*=−*T*..*T*. *?F t* * *φ t*)) ⟶ *of-real* (*μ* {*a*<..*b*}))
⟨*proof*⟩

theorem *Levy-uniqueness*:
fixes *M1 M2* :: *real measure*
assumes *real-distribution M1 real-distribution M2* **and**
char M1 = *char M2*
shows *M1* = *M2*
⟨*proof*⟩

18.2 The Levy continuity theorem

theorem *levy-continuity1*:
fixes $M :: nat \Rightarrow real\ measure$ **and** $M' :: real\ measure$
assumes $\bigwedge n.\ real\text{-}distribution\ (M\ n)\ real\text{-}distribution\ M'\ weak\text{-}conv\text{-}m\ M\ M'$
shows $(\lambda n.\ char\ (M\ n)\ t) \longrightarrow char\ M'\ t$
⟨*proof*⟩

theorem *levy-continuity*:
fixes $M :: nat \Rightarrow real\ measure$ **and** $M' :: real\ measure$
assumes *real-distr-M* : $\bigwedge n.\ real\text{-}distribution\ (M\ n)$
and *real-distr-M'*: $real\text{-}distribution\ M'$
and *char-conv*: $\bigwedge t.\ (\lambda n.\ char\ (M\ n)\ t) \longrightarrow char\ M'\ t$
shows $weak\text{-}conv\text{-}m\ M\ M'$
⟨*proof*⟩

end

19 The Central Limit Theorem

theory *Central-Limit-Theorem*
imports *Levy*
begin

theorem (**in** *prob-space*) *central-limit-theorem-zero-mean*:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$
and $\mu :: real\ measure$
and $\sigma :: real$
and $S :: nat \Rightarrow 'a \Rightarrow real$
assumes *X-indep*: $indep\text{-}vars\ (\lambda i.\ borel)\ X\ UNIV$
and *X-mean-0*: $\bigwedge n.\ expectation\ (X\ n) = 0$
and *σ-pos*: $\sigma > 0$
and *X-square-integrable*: $\bigwedge n.\ integrable\ M\ (\lambda x.\ (X\ n\ x)^2)$
and *X-variance*: $\bigwedge n.\ variance\ (X\ n) = \sigma^2$
and *X-distrib*: $\bigwedge n.\ distr\ M\ borel\ (X\ n) = \mu$
defines $S\ n \equiv \lambda x.\ \sum i{<}n.\ X\ i\ x$
shows $weak\text{-}conv\text{-}m\ (\lambda n.\ distr\ M\ borel\ (\lambda x.\ S\ n\ x\ /\ sqrt\ (n * \sigma^2)))\ std\text{-}normal\text{-}distribution$
⟨*proof*⟩

theorem (**in** *prob-space*) *central-limit-theorem*:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$
and $\mu :: real\ measure$
and $\sigma :: real$
and $S :: nat \Rightarrow 'a \Rightarrow real$
assumes *X-indep*: $indep\text{-}vars\ (\lambda i.\ borel)\ X\ UNIV$
and *X-mean*: $\bigwedge n.\ expectation\ (X\ n) = m$
and *σ-pos*: $\sigma > 0$
and *X-square-integrable*: $\bigwedge n.\ integrable\ M\ (\lambda x.\ (X\ n\ x)^2)$
and *X-variance*: $\bigwedge n.\ variance\ (X\ n) = \sigma^2$

and *X-distrib*: $\bigwedge n.$ *distr M borel* $(X\ n) = \mu$
defines $X'\ i\ x \equiv X\ i\ x - m$
shows *weak-conv-m* $(\lambda n.$ *distr M borel* $(\lambda x.\ (\sum i{<}n.\ X'\ i\ x)\ /\ sqrt\ (n{*}\sigma^2)))$
std-normal-distribution
⟨*proof*⟩

end

theory *Discrete-Topology*
imports *HOL−Analysis.Analysis*
begin

Copy of discrete types with discrete topology. This space is polish.

typedef *'a discrete* = *UNIV*::*'a set*
morphisms *of-discrete discrete*
⟨*proof*⟩

instantiation *discrete* :: (*type*) *metric-space*
begin

definition *dist-discrete* :: *'a discrete* ⇒ *'a discrete* ⇒ *real*
where *dist-discrete n m* = (*if n* = *m then 0 else 1*)

definition *uniformity-discrete* :: (*'a discrete* × *'a discrete*) *filter* **where**
(*uniformity*::(*'a discrete* × *'a discrete*) *filter*) = (*INF* $e{\in}\{0 <..\}.$ *principal* {(*x*, *y*). *dist x y* < *e*})

definition *open-discrete* :: *'a discrete set* ⇒ *bool* **where**
(*open*::*'a discrete set* ⇒ *bool*) *U* ⟷ $(\forall x{\in}U.$ *eventually* $(\lambda(x', y).\ x' = x \longrightarrow y \in U)$ *uniformity*)

instance ⟨*proof*⟩
end

lemma *open-discrete*: *open* (*S* :: *'a discrete set*)
⟨*proof*⟩

instance *discrete* :: (*type*) *complete-space*
⟨*proof*⟩

instance *discrete* :: (*countable*) *countable*
⟨*proof*⟩

instance *discrete* :: (*countable*) *second-countable-topology*
⟨*proof*⟩

instance *discrete* :: (*countable*) *polish-space* ⟨*proof*⟩

end

20 Probability mass function

theory *Probability-Mass-Function*
imports
Giry-Monad
HOL−Library.Multiset
begin

Conflicting notation from *HOL−Analysis.Infinite-Sum*

no-notation *Infinite-Sum.abs-summable-on* (**infixr** ‹*abs'-summable'-on*› *46*)

lemma *AE-emeasure-singleton*:
assumes *x*: *emeasure M* {*x*} ≠ *0* **and** *ae*: *AE x in M. P x* **shows** *P x*
⟨*proof*⟩

lemma *AE-measure-singleton*: *measure M* {*x*} ≠ *0* ⟹ *AE x in M. P x* ⟹ *P x*
⟨*proof*⟩

lemma (**in** *finite-measure*) *AE-support-countable*:
assumes [*simp*]: *sets M* = *UNIV*
shows (*AE x in M. measure M* {*x*} ≠ *0*) ⟷ (∃ *S. countable S* ∧ (*AE x in M. x* ∈ *S*))
⟨*proof*⟩

20.1 PMF as measure

typedef *'a pmf* = {*M* :: *'a measure. prob-space M* ∧ *sets M* = *UNIV* ∧ (*AE x in M. measure M* {*x*} ≠ *0*)}
morphisms *measure-pmf Abs-pmf*
⟨*proof*⟩

declare [[*coercion measure-pmf*]]

lemma *prob-space-measure-pmf*: *prob-space* (*measure-pmf p*)
⟨*proof*⟩

interpretation *measure-pmf*: *prob-space measure-pmf M* **for** *M*
⟨*proof*⟩

interpretation *measure-pmf*: *subprob-space measure-pmf M* **for** *M*
⟨*proof*⟩

lemma *subprob-space-measure-pmf*: *subprob-space* (*measure-pmf x*)
⟨*proof*⟩

locale *pmf-as-measure*
begin

setup-lifting *type-definition-pmf*

end

context
begin

interpretation *pmf-as-measure* ⟨*proof*⟩

lemma *sets-measure-pmf*[*simp*]: *sets* (*measure-pmf p*) = *UNIV*
⟨*proof*⟩

lemma *sets-measure-pmf-count-space*[*measurable-cong*]:
sets (*measure-pmf M*) = *sets* (*count-space UNIV*)
⟨*proof*⟩

lemma *space-measure-pmf*[*simp*]: *space* (*measure-pmf p*) = *UNIV*
⟨*proof*⟩

lemma *measure-pmf-UNIV* [*simp*]: *measure* (*measure-pmf p*) *UNIV* = *1*
⟨*proof*⟩

lemma *measure-pmf-in-subprob-algebra*[*measurable* (*raw*)]: *measure-pmf x* ∈ *space* (*subprob-algebra* (*count-space UNIV*))
⟨*proof*⟩

lemma *measurable-pmf-measure1*[*simp*]: *measurable* (*M* :: *'a pmf*) *N* = *UNIV* → *space N*
⟨*proof*⟩

lemma *measurable-pmf-measure2*[*simp*]: *measurable N* (*M* :: *'a pmf*) = *measurable N* (*count-space UNIV*)
⟨*proof*⟩

lemma *measurable-pair-restrict-pmf2*:
assumes *countable A*
assumes [*measurable*]: ⋀*y*. *y* ∈ *A* ⟹ (λ*x*. *f* (*x*, *y*)) ∈ *measurable M L*
shows *f* ∈ *measurable* (*M* ⨂$_M$ *restrict-space* (*measure-pmf N*) *A*) *L* (**is** *f* ∈ *measurable ?M* -)
⟨*proof*⟩

lemma *measurable-pair-restrict-pmf1*:
assumes *countable A*
assumes [*measurable*]: ⋀*x*. *x* ∈ *A* ⟹ (λ*y*. *f* (*x*, *y*)) ∈ *measurable N L*
shows *f* ∈ *measurable* (*restrict-space* (*measure-pmf M*) *A* ⨂$_M$ *N*) *L*
⟨*proof*⟩

lift-definition *pmf* :: *'a pmf* ⇒ *'a* ⇒ *real* **is** λ*M x*. *measure M* {*x*} ⟨*proof*⟩

lift-definition *set-pmf* :: *'a pmf* ⇒ *'a set* **is** λ*M*. {*x*. *measure M* {*x*} ≠ *0*} ⟨*proof*⟩
declare [[*coercion set-pmf*]]

lemma *AE-measure-pmf*: *AE x in* (*M*::*'a pmf*). *x* ∈ *M*
⟨*proof*⟩

lemma *emeasure-pmf-single-eq-zero-iff*:
fixes *M* :: *'a pmf*
shows *emeasure M* {*y*} = *0* ⟷ *y* ∉ *M*
⟨*proof*⟩

lemma *AE-measure-pmf-iff*: (*AE x in measure-pmf M*. *P x*) ⟷ (∀ *y*∈*M*. *P y*)
⟨*proof*⟩

lemma *AE-pmfI*: (⋀*y*. *y* ∈ *set-pmf M* ⟹ *P y*) ⟹ *almost-everywhere* (*measure-pmf M*) *P*
⟨*proof*⟩

lemma *countable-set-pmf* [*simp*]: *countable* (*set-pmf p*)
⟨*proof*⟩

lemma *pmf-positive*: *x* ∈ *set-pmf p* ⟹ *0* < *pmf p x*
⟨*proof*⟩

lemma *pmf-nonneg*[*simp*]: *0* ≤ *pmf p x*
⟨*proof*⟩

lemma *pmf-not-neg* [*simp*]: ¬*pmf p x* < *0*
⟨*proof*⟩

lemma *pmf-pos* [*simp*]: *pmf p x* ≠ *0* ⟹ *pmf p x* > *0*
⟨*proof*⟩

lemma *pmf-le-1*: *pmf p x* ≤ *1*
⟨*proof*⟩

lemma *set-pmf-not-empty*: *set-pmf M* ≠ {}
⟨*proof*⟩

lemma *set-pmf-iff*: *x* ∈ *set-pmf M* ⟷ *pmf M x* ≠ *0*
⟨*proof*⟩

lemma *pmf-positive-iff*: *0* < *pmf p x* ⟷ *x* ∈ *set-pmf p*
⟨*proof*⟩

lemma *set-pmf-eq*: *set-pmf M* = {*x*. *pmf M x* ≠ *0*}
⟨*proof*⟩

lemma *set-pmf-eq'*: *set-pmf p* = {*x*. *pmf p x* > *0*}
⟨*proof*⟩

lemma *emeasure-pmf-single*:
fixes *M* :: *'a pmf*
shows *emeasure M* {*x*} = *pmf M x*
⟨*proof*⟩

lemma *measure-pmf-single*: *measure* (*measure-pmf M*) {*x*} = *pmf M x*
⟨*proof*⟩

lemma *emeasure-measure-pmf-finite*: *finite S* ⟹ *emeasure* (*measure-pmf M*) *S* = (∑ *s*∈*S*. *pmf M s*)
⟨*proof*⟩

lemma *measure-measure-pmf-finite*: *finite S* ⟹ *measure* (*measure-pmf M*) *S* = *sum* (*pmf M*) *S*
⟨*proof*⟩

lemma *sum-pmf-eq-1*:
assumes *finite A set-pmf p* ⊆ *A*
shows (∑ *x*∈*A*. *pmf p x*) = *1*
⟨*proof*⟩

lemma *nn-integral-measure-pmf-support*:
fixes *f* :: *'a* ⇒ *ennreal*
assumes *f*: *finite A* **and** *nn*: ⋀*x*. *x* ∈ *A* ⟹ *0* ≤ *f x* ⋀*x*. *x* ∈ *set-pmf M* ⟹ *x* ∉ *A* ⟹ *f x* = *0*
shows (∫⁺*x*. *f x* ∂*measure-pmf M*) = (∑ *x*∈*A*. *f x* ∗ *pmf M x*)
⟨*proof*⟩

lemma *nn-integral-measure-pmf-finite*:
fixes *f* :: *'a* ⇒ *ennreal*
assumes *f*: *finite* (*set-pmf M*) **and** *nn*: ⋀*x*. *x* ∈ *set-pmf M* ⟹ *0* ≤ *f x*
shows (∫⁺*x*. *f x* ∂*measure-pmf M*) = (∑ *x*∈*set-pmf M*. *f x* ∗ *pmf M x*)
⟨*proof*⟩

lemma *integrable-measure-pmf-finite*:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}
shows *finite* (*set-pmf M*) ⟹ *integrable M f*
⟨*proof*⟩

lemma *integral-measure-pmf-real*:
assumes [*simp*]: *finite A* **and** ⋀*a*. *a* ∈ *set-pmf M* ⟹ *f a* ≠ *0* ⟹ *a* ∈ *A*
shows (∫ *x*. *f x* ∂*measure-pmf M*) = (∑ *a*∈*A*. *f a* ∗ *pmf M a*)
⟨*proof*⟩

lemma *integrable-pmf*: *integrable* (*count-space X*) (*pmf M*)
⟨*proof*⟩

lemma *integral-pmf*: $(\int x.\ pmf\ M\ x\ \partial count\text{-}space\ X) = measure\ M\ X$
⟨*proof*⟩

lemma *integral-pmf-restrict*:
$(f::'a \Rightarrow 'b::\{banach, second\text{-}countable\text{-}topology\}) \in borel\text{-}measurable\ (count\text{-}space\ UNIV) \Longrightarrow$
$(\int x.\ f\ x\ \partial measure\text{-}pmf\ M) = (\int x.\ f\ x\ \partial restrict\text{-}space\ M\ M)$
⟨*proof*⟩

lemma *emeasure-pmf*: $emeasure\ (M::'a\ pmf)\ M = 1$
⟨*proof*⟩

lemma *emeasure-pmf-UNIV* [*simp*]: $emeasure\ (measure\text{-}pmf\ M)\ UNIV = 1$
⟨*proof*⟩

lemma *in-null-sets-measure-pmfI*:
$A \cap set\text{-}pmf\ p = \{\} \Longrightarrow A \in null\text{-}sets\ (measure\text{-}pmf\ p)$
⟨*proof*⟩

lemma *measure-subprob*: $measure\text{-}pmf\ M \in space\ (subprob\text{-}algebra\ (count\text{-}space\ UNIV))$
⟨*proof*⟩

20.2 Monad Interpretation

lemma *measurable-measure-pmf*[*measurable*]:
$(\lambda x.\ measure\text{-}pmf\ (M\ x)) \in measurable\ (count\text{-}space\ UNIV)\ (subprob\text{-}algebra\ (count\text{-}space\ UNIV))$
⟨*proof*⟩

lemma *bind-measure-pmf-cong*:
assumes $\bigwedge x.\ A\ x \in space\ (subprob\text{-}algebra\ N)$ $\bigwedge x.\ B\ x \in space\ (subprob\text{-}algebra\ N)$
assumes $\bigwedge i.\ i \in set\text{-}pmf\ x \Longrightarrow A\ i = B\ i$
shows $bind\ (measure\text{-}pmf\ x)\ A = bind\ (measure\text{-}pmf\ x)\ B$
⟨*proof*⟩

lift-definition *bind-pmf* :: $'a\ pmf \Rightarrow ('a \Rightarrow 'b\ pmf\ ) \Rightarrow 'b\ pmf$ **is** *bind*
⟨*proof*⟩

adhoc-overloading *Monad-Syntax.bind* ⇌ *bind-pmf*

lemma *ennreal-pmf-bind*: $pmf\ (bind\text{-}pmf\ N\ f)\ i = (\int^+ x.\ pmf\ (f\ x)\ i\ \partial measure\text{-}pmf\ N)$
⟨*proof*⟩

lemma *pmf-bind*: $pmf\ (bind\text{-}pmf\ N\ f)\ i = (\int x.\ pmf\ (f\ x)\ i\ \partial measure\text{-}pmf\ N)$
⟨*proof*⟩

lemma *bind-pmf-const*[*simp*]: *bind-pmf M* (λ*x*. *c*) = *c*
⟨*proof*⟩

lemma *set-bind-pmf*[*simp*]: *set-pmf* (*bind-pmf M N*) = (⋃ *M*∈*set-pmf M*. *set-pmf* (*N M*))
⟨*proof*⟩

lemma *bind-pmf-cong* [*fundef-cong*]:
assumes *p* = *q*
shows (⋀*x*. *x* ∈ *set-pmf q* ⟹ *f x* = *g x*) ⟹ *bind-pmf p f* = *bind-pmf q g*
⟨*proof*⟩

lemma *bind-pmf-cong-simp*:
p = *q* ⟹ (⋀*x*. *x* ∈ *set-pmf q* =*simp*=> *f x* = *g x*) ⟹ *bind-pmf p f* = *bind-pmf q g*
⟨*proof*⟩

lemma *measure-pmf-bind*: *measure-pmf* (*bind-pmf M f*) = (*measure-pmf M* ⤜ (λ*x*. *measure-pmf* (*f x*)))
⟨*proof*⟩

lemma *nn-integral-bind-pmf*[*simp*]: (∫⁺*x*. *f x* ∂*bind-pmf M N*) = (∫⁺*x*. ∫⁺*y*. *f y* ∂*N x* ∂*M*)
⟨*proof*⟩

lemma *emeasure-bind-pmf*[*simp*]: *emeasure* (*bind-pmf M N*) *X* = (∫⁺*x*. *emeasure* (*N x*) *X* ∂*M*)
⟨*proof*⟩

lift-definition *return-pmf* :: *'a* ⇒ *'a pmf* **is** *return* (*count-space UNIV*)
⟨*proof*⟩

lemma *bind-return-pmf*: *bind-pmf* (*return-pmf x*) *f* = *f x*
⟨*proof*⟩

lemma *set-return-pmf*[*simp*]: *set-pmf* (*return-pmf x*) = {*x*}
⟨*proof*⟩

lemma *bind-return-pmf'*: *bind-pmf N return-pmf* = *N*
⟨*proof*⟩

lemma *bind-assoc-pmf*: *bind-pmf* (*bind-pmf A B*) *C* = *bind-pmf A* (λ*x*. *bind-pmf* (*B x*) *C*)
⟨*proof*⟩

definition *map-pmf f M* = *bind-pmf M* (λ*x*. *return-pmf* (*f x*))

lemma *map-bind-pmf*: *map-pmf f* (*bind-pmf M g*) = *bind-pmf M* (λ*x*. *map-pmf f*

$(g\ x))$
⟨*proof*⟩

lemma *bind-map-pmf*: $bind\text{-}pmf\ (map\text{-}pmf\ f\ M)\ g = bind\text{-}pmf\ M\ (\lambda x.\ g\ (f\ x))$
⟨*proof*⟩

lemma *map-pmf-transfer*[*transfer-rule*]:
$rel\text{-}fun\ (=)\ (rel\text{-}fun\ cr\text{-}pmf\ cr\text{-}pmf)\ (\lambda f\ M.\ distr\ M\ (count\text{-}space\ UNIV)\ f)\ map\text{-}pmf\ map\text{-}pmf$
⟨*proof*⟩

lemma *map-pmf-rep-eq*:
$measure\text{-}pmf\ (map\text{-}pmf\ f\ M) = distr\ (measure\text{-}pmf\ M)\ (count\text{-}space\ UNIV)\ f$
⟨*proof*⟩

lemma *map-pmf-id*[*simp*]: $map\text{-}pmf\ id = id$
⟨*proof*⟩

lemma *map-pmf-ident*[*simp*]: $map\text{-}pmf\ (\lambda x.\ x) = (\lambda x.\ x)$
⟨*proof*⟩

lemma *map-pmf-compose*: $map\text{-}pmf\ (f \circ g) = map\text{-}pmf\ f \circ map\text{-}pmf\ g$
⟨*proof*⟩

lemma *map-pmf-comp*: $map\text{-}pmf\ f\ (map\text{-}pmf\ g\ M) = map\text{-}pmf\ (\lambda x.\ f\ (g\ x))\ M$
⟨*proof*⟩

lemma *map-pmf-cong*: $p = q \Longrightarrow (\bigwedge x.\ x \in set\text{-}pmf\ q \Longrightarrow f\ x = g\ x) \Longrightarrow map\text{-}pmf\ f\ p = map\text{-}pmf\ g\ q$
⟨*proof*⟩

lemma *pmf-set-map*: $set\text{-}pmf \circ map\text{-}pmf\ f = (`)\ f \circ set\text{-}pmf$
⟨*proof*⟩

lemma *set-map-pmf*[*simp*]: $set\text{-}pmf\ (map\text{-}pmf\ f\ M) = f`set\text{-}pmf\ M$
⟨*proof*⟩

lemma *emeasure-map-pmf*[*simp*]: $emeasure\ (map\text{-}pmf\ f\ M)\ X = emeasure\ M\ (f -` X)$
⟨*proof*⟩

lemma *measure-map-pmf*[*simp*]: $measure\ (map\text{-}pmf\ f\ M)\ X = measure\ M\ (f -` X)$
⟨*proof*⟩

lemma *nn-integral-map-pmf*[*simp*]: $(\int^+ x.\ f\ x\ \partial map\text{-}pmf\ g\ M) = (\int^+ x.\ f\ (g\ x)\ \partial M)$
⟨*proof*⟩

lemma *ennreal-pmf-map*: *pmf* (*map-pmf f p*) *x* = (∫⁺ *y*. *indicator* (*f* −‘ {*x*}) *y* ∂*measure-pmf p*)
⟨*proof*⟩

lemma *pmf-map*: *pmf* (*map-pmf f p*) *x* = *measure p* (*f* −‘ {*x*})
⟨*proof*⟩

lemma *nn-integral-pmf*: (∫⁺ *x*. *pmf p x* ∂*count-space A*) = *emeasure* (*measure-pmf p*) *A*
⟨*proof*⟩

lemma *integral-map-pmf*[*simp*]:
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}
shows *integral*^{L} (*map-pmf g p*) *f* = *integral*^{L} *p* (λ*x*. *f* (*g x*))
⟨*proof*⟩

lemma *integrable-map-pmf-eq* [*simp*]:
fixes *g* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}
shows *integrable* (*map-pmf f p*) *g* ⟷ *integrable* (*measure-pmf p*) (λ*x*. *g* (*f x*))
⟨*proof*⟩

lemma *integrable-map-pmf* [*intro*]:
fixes *g* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*}
shows *integrable* (*measure-pmf p*) (λ*x*. *g* (*f x*)) ⟹ *integrable* (*map-pmf f p*) *g*
⟨*proof*⟩

lemma *pmf-abs-summable* [*intro*]: *pmf p abs-summable-on A*
⟨*proof*⟩

lemma *measure-pmf-conv-infsetsum*: *measure* (*measure-pmf p*) *A* = *infsetsum* (*pmf p*) *A*
⟨*proof*⟩

lemma *infsetsum-pmf-eq-1*:
assumes *set-pmf p* ⊆ *A*
shows *infsetsum* (*pmf p*) *A* = *1*
⟨*proof*⟩

lemma *map-return-pmf* [*simp*]: *map-pmf f* (*return-pmf x*) = *return-pmf* (*f x*)
⟨*proof*⟩

lemma *map-pmf-const*[*simp*]: *map-pmf* (λ-. *c*) *M* = *return-pmf c*
⟨*proof*⟩

lemma *pmf-return* [*simp*]: *pmf* (*return-pmf x*) *y* = *indicator* {*y*} *x*
⟨*proof*⟩

lemma *nn-integral-return-pmf*[*simp*]: *0* ≤ *f x* ⟹ (∫⁺*x*. *f x* ∂*return-pmf x*) = *f x*
⟨*proof*⟩

lemma *emeasure-return-pmf*[*simp*]: *emeasure* (*return-pmf x*) *X* = *indicator X x*
⟨*proof*⟩

lemma *measure-return-pmf* [*simp*]: *measure-pmf.prob* (*return-pmf x*) *A* = *indicator A x*
⟨*proof*⟩

lemma *return-pmf-inj*[*simp*]: *return-pmf x* = *return-pmf y* ⟷ *x* = *y*
⟨*proof*⟩

lemma *map-pmf-eq-return-pmf-iff*:
map-pmf f p = *return-pmf x* ⟷ (∀ *y* ∈ *set-pmf p*. *f y* = *x*)
⟨*proof*⟩

definition *pair-pmf A B* = *bind-pmf A* (λ*x*. *bind-pmf B* (λ*y*. *return-pmf* (*x*, *y*)))

lemma *pmf-pair*: *pmf* (*pair-pmf M N*) (*a*, *b*) = *pmf M a* ∗ *pmf N b*
⟨*proof*⟩

lemma *set-pair-pmf*[*simp*]: *set-pmf* (*pair-pmf A B*) = *set-pmf A* × *set-pmf B*
⟨*proof*⟩

lemma *measure-pmf-in-subprob-space*[*measurable* (*raw*)]:
measure-pmf M ∈ *space* (*subprob-algebra* (*count-space UNIV*))
⟨*proof*⟩

lemma *nn-integral-pair-pmf′*: (∫⁺*x*. *f x* ∂*pair-pmf A B*) = (∫⁺*a*. ∫⁺*b*. *f* (*a*, *b*) ∂*B* ∂*A*)
⟨*proof*⟩

lemma *bind-pair-pmf*:
assumes *M*[*measurable*]: *M* ∈ *measurable* (*count-space UNIV* ⨂$_M$ *count-space UNIV*) (*subprob-algebra N*)
shows *measure-pmf* (*pair-pmf A B*) ⤜ *M* = (*measure-pmf A* ⤜ (λ*x*. *measure-pmf B* ⤜ (λ*y*. *M* (*x*, *y*))))
(**is** *?L* = *?R*)
⟨*proof*⟩

lemma *map-fst-pair-pmf*: *map-pmf fst* (*pair-pmf A B*) = *A*
⟨*proof*⟩

lemma *map-snd-pair-pmf*: *map-pmf snd* (*pair-pmf A B*) = *B*
⟨*proof*⟩

lemma *nn-integral-pmf′*:
inj-on f A ⟹ (∫⁺*x*. *pmf p* (*f x*) ∂*count-space A*) = *emeasure p* (*f* ‘ *A*)
⟨*proof*⟩

lemma *pmf-le-0-iff*[*simp*]: *pmf M p* $\leq$ *0* $\longleftrightarrow$ *pmf M p* = *0*
⟨*proof*⟩

lemma *min-pmf-0*[*simp*]: *min* (*pmf M p*) *0* = *0 min 0* (*pmf M p*) = *0*
⟨*proof*⟩

lemma *pmf-eq-0-set-pmf*: *pmf M p* = *0* $\longleftrightarrow$ *p* $\notin$ *set-pmf M*
⟨*proof*⟩

lemma *pmf-map-inj*: *inj-on f* (*set-pmf M*) $\Longrightarrow$ *x* $\in$ *set-pmf M* $\Longrightarrow$ *pmf* (*map-pmf f M*) (*f x*) = *pmf M x*
⟨*proof*⟩

lemma *pair-return-pmf* [*simp*]: *pair-pmf* (*return-pmf x*) (*return-pmf y*) = *return-pmf* (*x*, *y*)
⟨*proof*⟩

lemma *pmf-map-inj'*: *inj f* $\Longrightarrow$ *pmf* (*map-pmf f M*) (*f x*) = *pmf M x*
⟨*proof*⟩

lemma *expectation-pair-pmf-fst* [*simp*]:
fixes *f* :: *'a* $\Rightarrow$ *'b*::{*banach*, *second-countable-topology*}
shows *measure-pmf.expectation* (*pair-pmf p q*) (λ*x*. *f* (*fst x*)) = *measure-pmf.expectation p f*
⟨*proof*⟩

lemma *expectation-pair-pmf-snd* [*simp*]:
fixes *f* :: *'a* $\Rightarrow$ *'b*::{*banach*, *second-countable-topology*}
shows *measure-pmf.expectation* (*pair-pmf p q*) (λ*x*. *f* (*snd x*)) = *measure-pmf.expectation q f*
⟨*proof*⟩

lemma *pmf-map-outside*: *x* $\notin$ *f* ' *set-pmf M* $\Longrightarrow$ *pmf* (*map-pmf f M*) *x* = *0*
⟨*proof*⟩

lemma *measurable-set-pmf*[*measurable*]: *Measurable.pred* (*count-space UNIV*) (λ*x*. *x* $\in$ *set-pmf M*)
⟨*proof*⟩

20.3 PMFs as function

context
fixes *f* :: *'a* $\Rightarrow$ *real*
assumes *nonneg*: $\bigwedge$*x*. *0* $\leq$ *f x*
assumes *prob*: ($\int^+$*x*. *f x* ∂*count-space UNIV*) = *1*
begin

lift-definition *embed-pmf* :: *'a pmf* **is** *density* (*count-space UNIV*) (*ennreal* ∘ *f*)
⟨*proof*⟩

lemma *pmf-embed-pmf*: *pmf embed-pmf x = f x*
⟨*proof*⟩

lemma *set-embed-pmf*: *set-pmf embed-pmf = {x. f x ≠ 0}*
⟨*proof*⟩

end

lemma *embed-pmf-transfer*:
rel-fun (*eq-onp* (λ*f*. (∀ *x*. *0* ≤ *f x*) ∧ (∫⁺*x*. *ennreal* (*f x*) ∂*count-space UNIV*) = *1*)) *pmf-as-measure.cr-pmf* (λ*f*. *density* (*count-space UNIV*) (*ennreal* ∘ *f*)) *embed-pmf*
⟨*proof*⟩

lemma *measure-pmf-eq-density*: *measure-pmf p = density* (*count-space UNIV*) (*pmf p*)
⟨*proof*⟩

lemma *td-pmf-embed-pmf*:
type-definition pmf embed-pmf {*f*::'*a* ⇒ *real*. (∀ *x*. *0* ≤ *f x*) ∧ (∫⁺*x*. *ennreal* (*f x*) ∂*count-space UNIV*) = *1*}
⟨*proof*⟩

end

lemma *nn-integral-measure-pmf*: (∫⁺ *x*. *f x* ∂*measure-pmf p*) = ∫⁺ *x*. *ennreal* (*pmf p x*) ∗ *f x* ∂*count-space UNIV*
⟨*proof*⟩

lemma *integral-measure-pmf*:
fixes *f* :: '*a* ⇒ '*b*::{*banach*, *second-countable-topology*}
assumes *A*: *finite A*
shows (⋀*a*. *a* ∈ *set-pmf M* ⟹ *f a* ≠ *0* ⟹ *a* ∈ *A*) ⟹ (*LINT x*|*M*. *f x*) = (∑ *a*∈*A*. *pmf M a* ∗$_R$ *f a*)
⟨*proof*⟩

lemma *expectation-return-pmf* [*simp*]:
fixes *f* :: '*a* ⇒ '*b*::{*banach*, *second-countable-topology*}
shows *measure-pmf.expectation* (*return-pmf x*) *f* = *f x*
⟨*proof*⟩

lemma *pmf-expectation-bind*:
fixes *p* :: '*a pmf* **and** *f* :: '*a* ⇒ '*b pmf*
and *h* :: '*b* ⇒ '*c*::{*banach*, *second-countable-topology*}
assumes *finite A* ⋀*x*. *x* ∈ *A* ⟹ *finite* (*set-pmf* (*f x*)) *set-pmf p* ⊆ *A*
shows *measure-pmf.expectation* (*p* ⤜ *f*) *h* =
(∑ *a*∈*A*. *pmf p a* ∗$_R$ *measure-pmf.expectation* (*f a*) *h*)
⟨*proof*⟩

lemma *continuous-on-LINT-pmf*: — This is dominated convergence!?
fixes $f :: 'i \Rightarrow 'a::topological\text{-}space \Rightarrow 'b::\{banach, second\text{-}countable\text{-}topology\}$
assumes f: $\bigwedge i.\ i \in set\text{-}pmf\ M \Longrightarrow continuous\text{-}on\ A\ (f\ i)$
and *bnd*: $\bigwedge a\ i.\ a \in A \Longrightarrow i \in set\text{-}pmf\ M \Longrightarrow norm\ (f\ i\ a) \leq B$
shows *continuous-on A* $(\lambda a.\ LINT\ i|M.\ f\ i\ a)$
⟨*proof*⟩

lemma *continuous-on-LBINT*:
fixes $f :: real \Rightarrow real$
assumes f: $\bigwedge b.\ a \leq b \Longrightarrow set\text{-}integrable\ lborel\ \{a..b\}\ f$
shows *continuous-on UNIV* $(\lambda b.\ LBINT\ x{:}\{a..b\}.\ f\ x)$
⟨*proof*⟩

locale *pmf-as-function*
begin

setup-lifting *td-pmf-embed-pmf*

lemma *set-pmf-transfer*[*transfer-rule*]:
assumes *bi-total A*
shows *rel-fun* (*pcr-pmf A*) (*rel-set A*) $(\lambda f.\ \{x.\ f\ x \neq 0\})$ *set-pmf*
⟨*proof*⟩

end

context
begin

interpretation *pmf-as-function* ⟨*proof*⟩

lemma *pmf-eqI*: $(\bigwedge i.\ pmf\ M\ i = pmf\ N\ i) \Longrightarrow M = N$
⟨*proof*⟩

lemma *pmf-eq-iff*: $M = N \longleftrightarrow (\forall\, i.\ pmf\ M\ i = pmf\ N\ i)$
⟨*proof*⟩

lemma *pmf-neq-exists-less*:
assumes $M \neq N$
shows $\exists\, x.\ pmf\ M\ x < pmf\ N\ x$
⟨*proof*⟩

lemma *bind-commute-pmf*: *bind-pmf A* $(\lambda x.\ bind\text{-}pmf\ B\ (C\ x)) = bind\text{-}pmf\ B\ (\lambda y.$ *bind-pmf A* $(\lambda x.\ C\ x\ y))$
⟨*proof*⟩

lemma *pair-map-pmf1*: *pair-pmf* $(map\text{-}pmf\ f\ A)\ B = map\text{-}pmf\ (apfst\ f)\ (pair\text{-}pmf\ A\ B)$
⟨*proof*⟩

lemma *pair-map-pmf2*: *pair-pmf A* (*map-pmf f B*) = *map-pmf* (*apsnd f*) (*pair-pmf A B*)
⟨*proof*⟩

lemma *map-pair*: *map-pmf* (λ(*a*, *b*). (*f a*, *g b*)) (*pair-pmf A B*) = *pair-pmf* (*map-pmf f A*) (*map-pmf g B*)
⟨*proof*⟩

end

lemma *pair-return-pmf1*: *pair-pmf* (*return-pmf x*) *y* = *map-pmf* (*Pair x*) *y*
⟨*proof*⟩

lemma *pair-return-pmf2*: *pair-pmf x* (*return-pmf y*) = *map-pmf* (λ*x*. (*x*, *y*)) *x*
⟨*proof*⟩

lemma *pair-pair-pmf*: *pair-pmf* (*pair-pmf u v*) *w* = *map-pmf* (λ(*x*, (*y*, *z*)). ((*x*, *y*), *z*)) (*pair-pmf u* (*pair-pmf v w*))
⟨*proof*⟩

lemma *pair-commute-pmf*: *pair-pmf x y* = *map-pmf* (λ(*x*, *y*). (*y*, *x*)) (*pair-pmf y x*)
⟨*proof*⟩

lemma *set-pmf-subset-singleton*: *set-pmf p* ⊆ {*x*} ⟷ *p* = *return-pmf x*
⟨*proof*⟩

lemma *bind-eq-return-pmf*:
bind-pmf p f = *return-pmf x* ⟷ (∀ *y*∈*set-pmf p*. *f y* = *return-pmf x*)
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *pmf-False-conv-True*: *pmf p False* = *1* − *pmf p True*
⟨*proof*⟩

lemma *pmf-True-conv-False*: *pmf p True* = *1* − *pmf p False*
⟨*proof*⟩

20.4 Conditional Probabilities

lemma *measure-pmf-zero-iff*: *measure* (*measure-pmf p*) *s* = *0* ⟷ *set-pmf p* ∩ *s* = {}
⟨*proof*⟩

context
fixes *p* :: *'a pmf* **and** *s* :: *'a set*
assumes *not-empty*: *set-pmf p* ∩ *s* ≠ {}
begin

interpretation *pmf-as-measure* ⟨*proof*⟩

lemma *emeasure-measure-pmf-not-zero*: *emeasure* (*measure-pmf p*) *s* ≠ *0*
⟨*proof*⟩

lemma *measure-measure-pmf-not-zero*: *measure* (*measure-pmf p*) *s* ≠ *0*
⟨*proof*⟩

lift-definition *cond-pmf* :: *'a pmf* **is**
uniform-measure (*measure-pmf p*) *s*
⟨*proof*⟩

lemma *pmf-cond*: *pmf cond-pmf x* = (*if x* ∈ *s then pmf p x* / *measure p s else 0*)
⟨*proof*⟩

lemma *set-cond-pmf*[*simp*]: *set-pmf cond-pmf* = *set-pmf p* ∩ *s*
⟨*proof*⟩

end

lemma *measure-pmf-posI*: *x* ∈ *set-pmf p* ⟹ *x* ∈ *A* ⟹ *measure-pmf.prob p A* >
0
⟨*proof*⟩

lemma *cond-map-pmf*:
assumes *set-pmf p* ∩ *f* −‘ *s* ≠ {}
shows *cond-pmf* (*map-pmf f p*) *s* = *map-pmf f* (*cond-pmf p* (*f* −‘ *s*))
⟨*proof*⟩

lemma *bind-cond-pmf-cancel*:
assumes [*simp*]: ⋀*x*. *x* ∈ *set-pmf p* ⟹ *set-pmf q* ∩ {*y*. *R x y*} ≠ {}
assumes [*simp*]: ⋀*y*. *y* ∈ *set-pmf q* ⟹ *set-pmf p* ∩ {*x*. *R x y*} ≠ {}
assumes [*simp*]: ⋀*x y*. *x* ∈ *set-pmf p* ⟹ *y* ∈ *set-pmf q* ⟹ *R x y* ⟹ *measure*
q {*y*. *R x y*} = *measure p* {*x*. *R x y*}
shows *bind-pmf p* (λ*x*. *cond-pmf q* {*y*. *R x y*}) = *q*
⟨*proof*⟩

20.5 Relator

inductive *rel-pmf* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *'a pmf* ⇒ *'b pmf* ⇒ *bool*
for *R p q*
where
⟦ ⋀*x y*. (*x*, *y*) ∈ *set-pmf pq* ⟹ *R x y*;
map-pmf fst pq = *p*; *map-pmf snd pq* = *q* ⟧
⟹ *rel-pmf R p q*

lemma *rel-pmfI*:
assumes *R*: *rel-set R* (*set-pmf p*) (*set-pmf q*)

assumes *eq*: $\bigwedge x\ y.\ x \in$ *set-pmf* $p \Longrightarrow y \in$ *set-pmf* $q \Longrightarrow R\ x\ y \Longrightarrow$ *measure* $p\ \{x.\ R\ x\ y\} =$ *measure* $q\ \{y.\ R\ x\ y\}$
shows *rel-pmf* $R\ p\ q$
⟨*proof*⟩

lemma *rel-pmf-imp-rel-set*: *rel-pmf* $R\ p\ q \Longrightarrow$ *rel-set* R (*set-pmf* p) (*set-pmf* q)
⟨*proof*⟩

lemma *rel-pmfD-measure*:
assumes *rel-R*: *rel-pmf* $R\ p\ q$ **and** R: $\bigwedge a\ b.\ R\ a\ b \Longrightarrow R\ a\ y \longleftrightarrow R\ x\ b$
assumes $x \in$ *set-pmf* $p\ y \in$ *set-pmf* q
shows *measure* $p\ \{x.\ R\ x\ y\} =$ *measure* $q\ \{y.\ R\ x\ y\}$
⟨*proof*⟩

lemma *rel-pmf-measureD*:
assumes *rel-pmf* $R\ p\ q$
shows *measure* (*measure-pmf* p) $A \leq$ *measure* (*measure-pmf* q) $\{y.\ \exists x{\in}A.\ R\ x\ y\}$ (**is** *?lhs* $\leq$ *?rhs*)
⟨*proof*⟩

lemma *rel-pmf-iff-measure*:
assumes *symp* R *transp* R
shows *rel-pmf* $R\ p\ q \longleftrightarrow$
rel-set R (*set-pmf* p) (*set-pmf* q) $\wedge$
$(\forall x{\in}$*set-pmf* $p.\ \forall y{\in}$*set-pmf* $q.\ R\ x\ y \longrightarrow$ *measure* $p\ \{x.\ R\ x\ y\} =$ *measure* $q\ \{y.\ R\ x\ y\})$
⟨*proof*⟩

lemma *quotient-rel-set-disjoint*:
equivp $R \Longrightarrow C \in$ *UNIV* $//\ \{(x,\ y).\ R\ x\ y\} \Longrightarrow$ *rel-set* $R\ A\ B \Longrightarrow A \cap C = \{\} \longleftrightarrow B \cap C = \{\}$
⟨*proof*⟩

lemma *quotientD*: *equiv* $X\ R \Longrightarrow A \in X\ //\ R \Longrightarrow x \in A \Longrightarrow A = R\ ``\ \{x\}$
⟨*proof*⟩

lemma *rel-pmf-iff-equivp*:
assumes *equivp* R
shows *rel-pmf* $R\ p\ q \longleftrightarrow (\forall C{\in}$*UNIV* $//\ \{(x,\ y).\ R\ x\ y\}.$ *measure* $p\ C =$ *measure* $q\ C)$
(**is** *-* $\longleftrightarrow$ $(\forall C{\in}$*-*$//$*?R.* *-*))
⟨*proof*⟩

bnf *pmf*: *'a pmf map*: *map-pmf sets*: *set-pmf bd* : *card-suc natLeq rel*: *rel-pmf*
⟨*proof*⟩

lemma *map-pmf-idI*: $(\bigwedge x.\ x \in$ *set-pmf* $p \Longrightarrow f\ x = x) \Longrightarrow$ *map-pmf* $f\ p = p$
⟨*proof*⟩

lemma *rel-pmf-conj*[*simp*]:
rel-pmf (λ*x y*. *P* ∧ *Q x y*) *x y* ⟷ *P* ∧ *rel-pmf Q x y*
rel-pmf (λ*x y*. *Q x y* ∧ *P*) *x y* ⟷ *P* ∧ *rel-pmf Q x y*
⟨*proof*⟩

lemma *rel-pmf-top*[*simp*]: *rel-pmf top* = *top*
⟨*proof*⟩

lemma *rel-pmf-return-pmf1*: *rel-pmf R* (*return-pmf x*) *M* ⟷ (∀ *a*∈*M*. *R x a*)
⟨*proof*⟩

lemma *rel-pmf-return-pmf2*: *rel-pmf R M* (*return-pmf x*) ⟷ (∀ *a*∈*M*. *R a x*)
⟨*proof*⟩

lemma *rel-return-pmf*[*simp*]: *rel-pmf R* (*return-pmf x1*) (*return-pmf x2*) = *R x1 x2*
⟨*proof*⟩

lemma *rel-pmf-False*[*simp*]: *rel-pmf* (λ*x y*. *False*) *x y* = *False*
⟨*proof*⟩

lemma *rel-pmf-rel-prod*:
rel-pmf (*rel-prod R S*) (*pair-pmf A A′*) (*pair-pmf B B′*) ⟷ *rel-pmf R A B* ∧ *rel-pmf S A′ B′*
⟨*proof*⟩

lemma *rel-pmf-reflI*:
assumes ⋀*x*. *x* ∈ *set-pmf p* ⟹ *P x x*
shows *rel-pmf P p p*
⟨*proof*⟩

lemma *rel-pmf-bij-betw*:
assumes *f*: *bij-betw f* (*set-pmf p*) (*set-pmf q*)
and *eq*: ⋀*x*. *x* ∈ *set-pmf p* ⟹ *pmf p x* = *pmf q* (*f x*)
shows *rel-pmf* (λ*x y*. *f x* = *y*) *p q*
⟨*proof*⟩

context
begin

interpretation *pmf-as-measure* ⟨*proof*⟩

definition *join-pmf M* = *bind-pmf M* (λ*x*. *x*)

lemma *bind-eq-join-pmf*: *bind-pmf M f* = *join-pmf* (*map-pmf f M*)
⟨*proof*⟩

lemma *join-eq-bind-pmf*: *join-pmf M* = *bind-pmf M id*
⟨*proof*⟩

lemma *pmf-join*: *pmf* (*join-pmf N*) *i* = (∫ *M*. *pmf M i* ∂*measure-pmf N*)
⟨*proof*⟩

lemma *ennreal-pmf-join*: *ennreal* (*pmf* (*join-pmf N*) *i*) = (∫⁺ *M*. *pmf M i* ∂*measure-pmf N*)
⟨*proof*⟩

lemma *set-pmf-join-pmf*[*simp*]: *set-pmf* (*join-pmf f*) = (⋃ *p*∈*set-pmf f*. *set-pmf p*)
⟨*proof*⟩

lemma *join-return-pmf*: *join-pmf* (*return-pmf M*) = *M*
⟨*proof*⟩

lemma *map-join-pmf*: *map-pmf f* (*join-pmf AA*) = *join-pmf* (*map-pmf* (*map-pmf f*) *AA*)
⟨*proof*⟩

lemma *join-map-return-pmf*: *join-pmf* (*map-pmf return-pmf A*) = *A*
⟨*proof*⟩

end

lemma *rel-pmf-joinI*:
assumes *rel-pmf* (*rel-pmf P*) *p q*
shows *rel-pmf P* (*join-pmf p*) (*join-pmf q*)
⟨*proof*⟩

lemma *rel-pmf-bindI*:
assumes *pq*: *rel-pmf R p q*
and *fg*: ⋀*x y*. *R x y* ⟹ *rel-pmf P* (*f x*) (*g y*)
shows *rel-pmf P* (*bind-pmf p f*) (*bind-pmf q g*)
⟨*proof*⟩

Proof that *rel-pmf* preserves orders. Antisymmetry proof follows Thm. 1 in N. Saheb-Djahromi, Cpo's of measures for nondeterminism, Theoretical Computer Science 12(1):19–37, 1980, https://doi.org/10.1016/0304-3975(80)90003-1

lemma
assumes ∗: *rel-pmf R p q*
and *refl*: *reflp R* **and** *trans*: *transp R*
shows *measure-Ici*: *measure p* {*y*. *R x y*} ≤ *measure q* {*y*. *R x y*} (**is** *?thesis1*)
and *measure-Ioi*: *measure p* {*y*. *R x y* ∧ ¬ *R y x*} ≤ *measure q* {*y*. *R x y* ∧ ¬ *R y x*} (**is** *?thesis2*)
⟨*proof*⟩

lemma *rel-pmf-inf*:
fixes *p q* :: *'a pmf*

assumes *1*: *rel-pmf R p q*
assumes *2*: *rel-pmf R q p*
and *refl*: *reflp R* **and** *trans*: *transp R*
shows *rel-pmf* (*inf R* R^{-1-1}) *p q*
⟨*proof*⟩

lemma *rel-pmf-antisym*:
fixes *p q* :: *'a pmf*
assumes *1*: *rel-pmf R p q*
assumes *2*: *rel-pmf R q p*
and *refl*: *reflp R* **and** *trans*: *transp R* **and** *antisym*: *antisymp R*
shows *p* = *q*
⟨*proof*⟩

lemma *reflp-rel-pmf*: *reflp R* ⟹ *reflp* (*rel-pmf R*)
⟨*proof*⟩

lemma *antisymp-rel-pmf*:
⟦ *reflp R*; *transp R*; *antisymp R* ⟧
⟹ *antisymp* (*rel-pmf R*)
⟨*proof*⟩

lemma *transp-rel-pmf*:
assumes *transp R*
shows *transp* (*rel-pmf R*)
⟨*proof*⟩

20.6 Distributions

context
begin

interpretation *pmf-as-function* ⟨*proof*⟩

20.6.1 Bernoulli Distribution

lift-definition *bernoulli-pmf* :: *real* ⇒ *bool pmf* **is**
λ*p b*. ((λ*p*. *if b then p else 1* − *p*) ∘ *min 1* ∘ *max 0*) *p*
⟨*proof*⟩

lemma *pmf-bernoulli-True*[*simp*]: *0* ≤ *p* ⟹ *p* ≤ *1* ⟹ *pmf* (*bernoulli-pmf p*) *True* = *p*
⟨*proof*⟩

lemma *pmf-bernoulli-False*[*simp*]: *0* ≤ *p* ⟹ *p* ≤ *1* ⟹ *pmf* (*bernoulli-pmf p*) *False* = *1* − *p*
⟨*proof*⟩

lemma *set-pmf-bernoulli*[*simp*]: *0* < *p* ⟹ *p* < *1* ⟹ *set-pmf* (*bernoulli-pmf p*) = *UNIV*

⟨*proof*⟩

lemma *nn-integral-bernoulli-pmf*[*simp*]:
assumes [*simp*]: *0* ≤ *p p* ≤ *1* ⋀*x*. *0* ≤ *f x*
shows (∫⁺*x*. *f x* ∂*bernoulli-pmf p*) = *f True* ∗ *p* + *f False* ∗ (*1* − *p*)
⟨*proof*⟩

lemma *integral-bernoulli-pmf*[*simp*]:
assumes [*simp*]: *0* ≤ *p p* ≤ *1*
shows (∫ *x*. *f x* ∂*bernoulli-pmf p*) = *f True* ∗ *p* + *f False* ∗ (*1* − *p*)
⟨*proof*⟩

lemma *pmf-bernoulli-half* [*simp*]: *pmf* (*bernoulli-pmf* (*1* / *2*)) *x* = *1* / *2*
⟨*proof*⟩

lemma *measure-pmf-bernoulli-half*: *measure-pmf* (*bernoulli-pmf* (*1* / *2*)) = *uniform-count-measure UNIV*
⟨*proof*⟩

20.6.2 Geometric Distribution

context
fixes *p* :: *real* **assumes** *p*[*arith*]: *0* < *p p* ≤ *1*
begin

lift-definition *geometric-pmf* :: *nat pmf* **is** λ*n*. (*1* − *p*)^*n* ∗ *p*
⟨*proof*⟩

lemma *pmf-geometric*[*simp*]: *pmf geometric-pmf n* = (*1* − *p*)^*n* ∗ *p*
⟨*proof*⟩

end

lemma *geometric-pmf-1* [*simp*]: *geometric-pmf 1* = *return-pmf 0*
⟨*proof*⟩

lemma *set-pmf-geometric*: *0* < *p* ⟹ *p* < *1* ⟹ *set-pmf* (*geometric-pmf p*) = *UNIV*
⟨*proof*⟩

lemma *geometric-sums-times-n*:
fixes *c*::'*a*::{*banach*,*real-normed-field*}
assumes *norm c* < *1*
shows (λ*n*. *c*^*n* ∗ *of-nat n*) *sums* (*c* / (*1* − *c*)²)
⟨*proof*⟩

lemma *geometric-sums-times-norm*:
fixes *c*::'*a*::{*banach*,*real-normed-field*}
assumes *norm c* < *1*

shows $(\lambda n.\ norm\ (c \hat{}\ n) * of\text{-}nat\ n))\ sums\ (norm\ c\ /\ (1 - norm\ c)^2)$
⟨*proof*⟩

lemma *integrable-real-geometric-pmf*:
assumes $p \in \{0<..1\}$
shows *integrable* (*geometric-pmf p*) *real*
⟨*proof*⟩

lemma *expectation-geometric-pmf*:
assumes $p \in \{0<..1\}$
shows *measure-pmf.expectation* (*geometric-pmf p*) *real* $= (1 - p)\ /\ p$
⟨*proof*⟩

lemma *geometric-bind-pmf-unfold*:
assumes $p \in \{0<..1\}$
shows *geometric-pmf p* =
do {*b* ← *bernoulli-pmf p*;
if b then return-pmf 0 else map-pmf Suc (*geometric-pmf p*)}
⟨*proof*⟩

20.6.3 Uniform Multiset Distribution

context
fixes $M :: 'a\ multiset$ **assumes** *M-not-empty*: $M \neq \{\#\}$
begin

lift-definition *pmf-of-multiset* :: *'a pmf* **is** $\lambda x.\ count\ M\ x\ /\ size\ M$
⟨*proof*⟩

lemma *pmf-of-multiset*[*simp*]: *pmf pmf-of-multiset x* = *count M x / size M*
⟨*proof*⟩

lemma *set-pmf-of-multiset*[*simp*]: *set-pmf pmf-of-multiset* = *set-mset M*
⟨*proof*⟩

end

20.6.4 Uniform Distribution

context
fixes $S :: 'a\ set$ **assumes** *S-not-empty*: $S \neq \{\}$ **and** *S-finite*: *finite S*
begin

lift-definition *pmf-of-set* :: *'a pmf* **is** $\lambda x.\ indicator\ S\ x\ /\ card\ S$
⟨*proof*⟩

lemma *pmf-of-set*[*simp*]: *pmf pmf-of-set x* = *indicator S x / card S*
⟨*proof*⟩

lemma *set-pmf-of-set*[*simp*]: *set-pmf pmf-of-set* = *S*

⟨*proof*⟩

lemma *emeasure-pmf-of-set-space*[*simp*]: *emeasure pmf-of-set S = 1*
⟨*proof*⟩

lemma *nn-integral-pmf-of-set*: *nn-integral* (*measure-pmf pmf-of-set*) *f* = *sum f S* / *card S*
⟨*proof*⟩

lemma *integral-pmf-of-set*: $integral^L$ (*measure-pmf pmf-of-set*) *f* = *sum f S* / *card S*
⟨*proof*⟩

lemma *emeasure-pmf-of-set*: *emeasure* (*measure-pmf pmf-of-set*) *A* = *card* (*S* ∩ *A*) / *card S*
⟨*proof*⟩

lemma *measure-pmf-of-set*: *measure* (*measure-pmf pmf-of-set*) *A* = *card* (*S* ∩ *A*) / *card S*
⟨*proof*⟩

end

lemma *pmf-expectation-bind-pmf-of-set*:
fixes *A* :: *'a set* **and** *f* :: *'a* ⇒ *'b pmf*
and *h* :: *'b* ⇒ *'c*::{*banach*, *second-countable-topology*}
assumes *A* ≠ {} *finite A* ⋀*x*. *x* ∈ *A* ⟹ *finite* (*set-pmf* (*f x*))
shows *measure-pmf.expectation* (*pmf-of-set A* ⤜ *f*) *h* =
(∑ *a*∈*A*. *measure-pmf.expectation* (*f a*) *h* /$_R$ *real* (*card A*))
⟨*proof*⟩

lemma *map-pmf-of-set*:
assumes *finite A A* ≠ {}
shows *map-pmf f* (*pmf-of-set A*) = *pmf-of-multiset* (*image-mset f* (*mset-set A*))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *pmf-bind-pmf-of-set*:
assumes *A* ≠ {} *finite A*
shows *pmf* (*bind-pmf* (*pmf-of-set A*) *f*) *x* =
(∑ *xa*∈*A*. *pmf* (*f xa*) *x*) / *real-of-nat* (*card A*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *pmf-of-set-singleton*: *pmf-of-set* {*x*} = *return-pmf x*
⟨*proof*⟩

lemma *map-pmf-of-set-inj*:
assumes *f*: *inj-on f A*

and [*simp*]: $A \neq \{\}$ *finite A*
shows *map-pmf f* (*pmf-of-set A*) = *pmf-of-set* (*f* ' *A*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *map-pmf-of-set-bij-betw*:
assumes *bij-betw f A B A* ≠ {} *finite A*
shows *map-pmf f* (*pmf-of-set A*) = *pmf-of-set B*
⟨*proof*⟩

Choosing an element uniformly at random from the union of a disjoint family of finite non-empty sets with the same size is the same as first choosing a set from the family uniformly at random and then choosing an element from the chosen set uniformly at random.

lemma *pmf-of-set-UN*:
assumes *finite* (⋃(*f* ' *A*)) *A* ≠ {} ⋀*x*. *x* ∈ *A* ⟹ *f x* ≠ {}
⋀*x*. *x* ∈ *A* ⟹ *card* (*f x*) = *n disjoint-family-on f A*
shows *pmf-of-set* (⋃(*f* ' *A*)) = *do* {*x* ← *pmf-of-set A*; *pmf-of-set* (*f x*)}
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *bernoulli-pmf-half-conv-pmf-of-set*: *bernoulli-pmf* (*1* / *2*) = *pmf-of-set UNIV*
⟨*proof*⟩

20.6.5 Poisson Distribution

context
fixes *rate* :: *real* **assumes** *rate-pos*: *0* < *rate*
begin

lift-definition *poisson-pmf* :: *nat pmf* **is** λ*k*. *rate* ^ *k* / *fact k* ∗ *exp* (−*rate*)
⟨*proof*⟩

lemma *pmf-poisson*[*simp*]: *pmf poisson-pmf k* = *rate* ^ *k* / *fact k* ∗ *exp* (−*rate*)
⟨*proof*⟩

lemma *set-pmf-poisson*[*simp*]: *set-pmf poisson-pmf* = *UNIV*
⟨*proof*⟩

end

20.6.6 Binomial Distribution

context
fixes *n* :: *nat* **and** *p* :: *real* **assumes** *p-nonneg*: *0* ≤ *p* **and** *p-le-1*: *p* ≤ *1*
begin

lift-definition *binomial-pmf* :: *nat pmf* **is** λ*k*. (*n choose k*) ∗ *p*^*k* ∗ (*1* − *p*)^(*n* − *k*)
⟨*proof*⟩

lemma *pmf-binomial*[*simp*]: *pmf binomial-pmf k* = (*n choose k*) * *p*^*k* * (*1* − *p*)^(*n* − *k*)
⟨*proof*⟩

lemma *set-pmf-binomial-eq*: *set-pmf binomial-pmf* = (*if p* = *0 then* {*0*} *else if p* = *1 then* {*n*} *else* {.. *n*})
⟨*proof*⟩

end

end

lemma *set-pmf-binomial-0*[*simp*]: *set-pmf* (*binomial-pmf n 0*) = {*0*}
⟨*proof*⟩

lemma *set-pmf-binomial-1*[*simp*]: *set-pmf* (*binomial-pmf n 1*) = {*n*}
⟨*proof*⟩

lemma *set-pmf-binomial*[*simp*]: *0* < *p* ⟹ *p* < *1* ⟹ *set-pmf* (*binomial-pmf n p*) = {..*n*}
⟨*proof*⟩

lemma *finite-set-pmf-binomial-pmf* [*intro*]: *p* ∈ {*0..1*} ⟹ *finite* (*set-pmf* (*binomial-pmf n p*))
⟨*proof*⟩

lemma *expectation-binomial-pmf'*:
fixes *f* :: *nat* ⇒ 'a :: {*banach*, *second-countable-topology*}
assumes *p*: *p* ∈ {*0..1*}
shows *measure-pmf.expectation* (*binomial-pmf n p*) *f* =
(∑ *k*≤*n*. (*real* (*n choose k*) * *p* ^ *k* * (*1* − *p*) ^ (*n* − *k*)) *$_R$ f k*)
⟨*proof*⟩

lemma *integrable-binomial-pmf* [*simp*, *intro*]:
fixes *f* :: *nat* ⇒ 'a :: {*banach*, *second-countable-topology*}
assumes *p*: *p* ∈ {*0..1*}
shows *integrable* (*binomial-pmf n p*) *f*
⟨*proof*⟩

context includes *lifting-syntax*
begin

lemma *bind-pmf-parametric* [*transfer-rule*]:
(*rel-pmf A* ===> (*A* ===> *rel-pmf B*) ===> *rel-pmf B*) *bind-pmf bind-pmf*
⟨*proof*⟩

lemma *return-pmf-parametric* [*transfer-rule*]: (*A* ===> *rel-pmf A*) *return-pmf return-pmf*

⟨*proof*⟩

end

primrec *replicate-pmf* :: *nat* ⇒ *'a pmf* ⇒ *'a list pmf* **where**
  replicate-pmf 0 - = return-pmf []
| *replicate-pmf* (*Suc n*) *p* = *do* {*x* ← *p*; *xs* ← *replicate-pmf n p*; *return-pmf* (*x*#*xs*)}

lemma *replicate-pmf-1*: *replicate-pmf 1 p* = *map-pmf* (λ*x*. [*x*]) *p*
  ⟨*proof*⟩

lemma *set-replicate-pmf*:
  set-pmf (*replicate-pmf n p*) = {*xs*∈*lists* (*set-pmf p*). *length xs* = *n*}
  ⟨*proof*⟩

lemma *replicate-pmf-distrib*:
  replicate-pmf (*m* + *n*) *p* =
    do {*xs* ← *replicate-pmf m p*; *ys* ← *replicate-pmf n p*; *return-pmf* (*xs* @ *ys*)}
  ⟨*proof*⟩

lemma *power-diff'*:
  assumes *b* ≤ *a*
  shows *x* ^ (*a* − *b*) = (*if x* = *0* ∧ *a* = *b then 1 else x* ^ *a* / (*x*::*'a*::*field*) ^ *b*)
⟨*proof*⟩

lemma *binomial-pmf-Suc*:
  assumes *p* ∈ {*0..1*}
  shows *binomial-pmf* (*Suc n*) *p* =
          do {*b* ← *bernoulli-pmf p*;
              k ← *binomial-pmf n p*;
              return-pmf ((*if b then 1 else 0*) + *k*)} (**is** - = *?rhs*)
⟨*proof*⟩

lemma *binomial-pmf-0*: *p* ∈ {*0..1*} ⟹ *binomial-pmf 0 p* = *return-pmf 0*
  ⟨*proof*⟩

lemma *binomial-pmf-altdef*:
  assumes *p* ∈ {*0..1*}
  shows *binomial-pmf n p* = *map-pmf* (*length* ∘ *filter id*) (*replicate-pmf n* (*bernoulli-pmf p*))
  ⟨*proof*⟩

20.7 Negative Binomial distribution

The negative binomial distribution counts the number of times a weighted coin comes up tails before having come up heads n times. In other words: how many failures do we see before seeing the n-th success?

An alternative view is that the negative binomial distribution is the sum of n i.i.d. geometric variables (this is the definition that we use).

Note that there are sometimes different conventions for this distributions in the literature; for instance, sometimes the number of *attempts* is counted instead of the number of failures. This only shifts the entire distribution by a constant number and is thus not a big difference. I think that the convention we use is the most natural one since the support of the distribution starts at 0, whereas for the other convention it starts at n.

primrec *neg-binomial-pmf* :: *nat* ⇒ *real* ⇒ *nat pmf* **where**
  neg-binomial-pmf 0 p = return-pmf 0
| *neg-binomial-pmf (Suc n) p =*
    map-pmf (λ(x,y). (x + y)) (pair-pmf (geometric-pmf p) (neg-binomial-pmf n p))

lemma *neg-binomial-pmf-Suc-0* [*simp*]: *neg-binomial-pmf (Suc 0) p = geometric-pmf p*
  ⟨*proof*⟩

lemmas *neg-binomial-pmf-Suc* [*simp del*] = *neg-binomial-pmf.simps*(*2*)

lemma *neg-binomial-prob-1* [*simp*]: *neg-binomial-pmf n 1 = return-pmf 0*
  ⟨*proof*⟩

We can now show the aforementioned intuition about counting the failures before the n-th success with the following recurrence:

lemma *neg-binomial-pmf-unfold*:
  assumes *p*: *p* ∈ {*0<..1*}
  shows *neg-binomial-pmf (Suc n) p =*
        do {b ← bernoulli-pmf p;
            if b then neg-binomial-pmf n p else map-pmf Suc (neg-binomial-pmf (Suc n) p)}
  (**is** *- = ?rhs*)
  ⟨*proof*⟩

Next, we show an explicit formula for the probability mass function of the negative binomial distribution:

lemma *pmf-neg-binomial*:
  assumes *p*: *p* ∈ {*0<..1*}
  shows *pmf (neg-binomial-pmf n p) k = real ((k + n − 1) choose k) ∗ p ^ n ∗ (1 − p) ^ k*
⟨*proof*⟩

lemma *gbinomial-0-left*: *0 gchoose k = (if k = 0 then 1 else 0)*
  ⟨*proof*⟩

The following alternative formula highlights why it is called 'negative binomial distribution':

lemma *pmf-neg-binomial'*:
assumes *p*: $p \in \{0<..1\}$
shows *pmf* (*neg-binomial-pmf n p*) *k* = (−1) ^ *k* ∗ ((−*real n*) *gchoose k*) ∗ *p* ^ *n* ∗ (1 − *p*) ^ *k*
⟨*proof*⟩

The cumulative distribution function of the negative binomial distribution can be expressed in terms of that of the 'normal' binomial distribution.

lemma *prob-neg-binomial-pmf-atMost*:
assumes *p*: $p \in \{0<..1\}$
shows *measure-pmf.prob* (*neg-binomial-pmf n p*) {..*k*} =
measure-pmf.prob (*binomial-pmf* (*n* + *k*) (1 − *p*)) {..*k*}
⟨*proof*⟩

lemma *prob-neg-binomial-pmf-lessThan*:
assumes *p*: $p \in \{0<..1\}$
shows *measure-pmf.prob* (*neg-binomial-pmf n p*) {..<*k*} =
measure-pmf.prob (*binomial-pmf* (*n* + *k* − 1) (1 − *p*)) {..<*k*}
⟨*proof*⟩

The expected value of the negative binomial distribution is $n(1-p)/p$:

lemma *nn-integral-neg-binomial-pmf-real*:
assumes *p*: $p \in \{0<..1\}$
shows *nn-integral* (*measure-pmf* (*neg-binomial-pmf n p*)) *of-nat* = *ennreal* (*n* ∗ (1 − *p*) / *p*)
⟨*proof*⟩

lemma *integrable-neg-binomial-pmf-real*:
assumes *p*: $p \in \{0<..1\}$
shows *integrable* (*measure-pmf* (*neg-binomial-pmf n p*)) *real*
⟨*proof*⟩

lemma *expectation-neg-binomial-pmf*:
assumes *p*: $p \in \{0<..1\}$
shows *measure-pmf.expectation* (*neg-binomial-pmf n p*) *real* = *n* ∗ (1 − *p*) / *p*
⟨*proof*⟩

20.8 PMFs from association lists

definition *pmf-of-list* :: ('*a* × *real*) *list* ⇒ '*a pmf* **where**
pmf-of-list xs = *embed-pmf* (λ*x*. *sum-list* (*map snd* (*filter* (λ*z*. *fst z* = *x*) *xs*)))

definition *pmf-of-list-wf* **where**
pmf-of-list-wf xs ⟷ (∀ *x*∈*set* (*map snd xs*) . *x* ≥ 0) ∧ *sum-list* (*map snd xs*) = 1

lemma *pmf-of-list-wfI*:
(⋀*x*. *x* ∈ *set* (*map snd xs*) ⟹ *x* ≥ 0) ⟹ *sum-list* (*map snd xs*) = 1 ⟹ *pmf-of-list-wf xs*

⟨*proof*⟩

context
begin

private lemma *pmf-of-list-aux*:
assumes $\bigwedge x.\ x \in set\ (map\ snd\ xs) \Longrightarrow x \geq 0$
assumes $sum\text{-}list\ (map\ snd\ xs) = 1$
shows $(\int^+ x.\ ennreal\ (sum\text{-}list\ (map\ snd\ [z \leftarrow xs\ .\ fst\ z = x]))\ \partial count\text{-}space\ UNIV) = 1$
⟨*proof*⟩

lemma *pmf-pmf-of-list*:
assumes *pmf-of-list-wf xs*
shows $pmf\ (pmf\text{-}of\text{-}list\ xs)\ x = sum\text{-}list\ (map\ snd\ (filter\ (\lambda z.\ fst\ z = x)\ xs))$
⟨*proof*⟩

end

lemma *set-pmf-of-list*:
assumes *pmf-of-list-wf xs*
shows $set\text{-}pmf\ (pmf\text{-}of\text{-}list\ xs) \subseteq set\ (map\ fst\ xs)$
⟨*proof*⟩

lemma *finite-set-pmf-of-list*:
assumes *pmf-of-list-wf xs*
shows $finite\ (set\text{-}pmf\ (pmf\text{-}of\text{-}list\ xs))$
⟨*proof*⟩

lemma *emeasure-Int-set-pmf*:
$emeasure\ (measure\text{-}pmf\ p)\ (A \cap set\text{-}pmf\ p) = emeasure\ (measure\text{-}pmf\ p)\ A$
⟨*proof*⟩

lemma *measure-Int-set-pmf*:
$measure\ (measure\text{-}pmf\ p)\ (A \cap set\text{-}pmf\ p) = measure\ (measure\text{-}pmf\ p)\ A$
⟨*proof*⟩

lemma *measure-prob-cong-0*:
assumes $\bigwedge x.\ x \in A - B \Longrightarrow pmf\ p\ x = 0$
assumes $\bigwedge x.\ x \in B - A \Longrightarrow pmf\ p\ x = 0$
shows $measure\ (measure\text{-}pmf\ p)\ A = measure\ (measure\text{-}pmf\ p)\ B$
⟨*proof*⟩

lemma *emeasure-pmf-of-list*:
assumes *pmf-of-list-wf xs*
shows $emeasure\ (pmf\text{-}of\text{-}list\ xs)\ A = ennreal\ (sum\text{-}list\ (map\ snd\ (filter\ (\lambda x.\ fst\ x \in A)\ xs)))$
⟨*proof*⟩

lemma *measure-pmf-of-list*:
assumes *pmf-of-list-wf xs*
shows *measure* (*pmf-of-list xs*) *A* = *sum-list* (*map snd* (*filter* (λ*x*. *fst x* ∈ *A*) *xs*))
⟨*proof*⟩

lemma *sum-list-nonneg-eq-zero-iff*:
fixes *xs* :: 'a :: *linordered-ab-group-add list*
shows (⋀*x*. *x* ∈ *set xs* ⟹ *x* ≥ *0*) ⟹ *sum-list xs* = *0* ⟷ *set xs* ⊆ {*0*}
⟨*proof*⟩

lemma *sum-list-filter-nonzero*:
sum-list (*filter* (λ*x*. *x* ≠ *0*) *xs*) = *sum-list xs*
⟨*proof*⟩

lemma *set-pmf-of-list-eq*:
assumes *pmf-of-list-wf xs* ⋀*x*. *x* ∈ *snd* ' *set xs* ⟹ *x* > *0*
shows *set-pmf* (*pmf-of-list xs*) = *fst* ' *set xs*
⟨*proof*⟩

lemma *pmf-of-list-remove-zeros*:
assumes *pmf-of-list-wf xs*
defines *xs'* ≡ *filter* (λ*z*. *snd z* ≠ *0*) *xs*
shows *pmf-of-list-wf xs' pmf-of-list xs'* = *pmf-of-list xs*
⟨*proof*⟩

end

21 Code generation for PMFs

theory *PMF-Impl*
imports *Probability-Mass-Function HOL−Library.AList-Mapping*
begin

21.1 General code generation setup

definition *pmf-of-mapping* :: ('a, *real*) *mapping* ⇒ 'a *pmf* **where**
pmf-of-mapping m = *embed-pmf* (*Mapping.lookup-default 0 m*)

lemma *nn-integral-lookup-default*:
fixes *m* :: ('a, *real*) *mapping*
assumes *finite* (*Mapping.keys m*) *All-mapping m* (λ- *x*. *x* ≥ *0*)
shows *nn-integral* (*count-space UNIV*) (λ*k*. *ennreal* (*Mapping.lookup-default 0 m k*)) =
ennreal (∑ *k*∈*Mapping.keys m*. *Mapping.lookup-default 0 m k*)
⟨*proof*⟩

lemma *pmf-of-mapping*:
assumes *finite* (*Mapping.keys m*) *All-mapping m* (λ- *p*. *p* ≥ *0*)
assumes (∑ *x*∈*Mapping.keys m*. *Mapping.lookup-default 0 m x*) = *1*
shows *pmf* (*pmf-of-mapping m*) *x* = *Mapping.lookup-default 0 m x*
⟨*proof*⟩

lemma *pmf-of-set-pmf-of-mapping*:
assumes *A* ≠ {} *set xs* = *A distinct xs*
shows *pmf-of-set A* = *pmf-of-mapping* (*Mapping.tabulate xs* (λ-. *1* / *real* (*length xs*)))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lift-definition *mapping-of-pmf* :: *'a pmf* ⇒ (*'a*, *real*) *mapping* **is**
λ*p x*. *if pmf p x* = *0 then None else Some* (*pmf p x*) ⟨*proof*⟩

lemma *lookup-default-mapping-of-pmf*:
Mapping.lookup-default 0 (*mapping-of-pmf p*) *x* = *pmf p x*
⟨*proof*⟩

context
begin

interpretation *pmf-as-function* ⟨*proof*⟩

lemma *nn-integral-pmf-eq-1*: (∫⁺ *x*. *ennreal* (*pmf p x*) ∂*count-space UNIV*) = *1*
⟨*proof*⟩
end

lemma *pmf-of-mapping-mapping-of-pmf* [*code abstype*]:
pmf-of-mapping (*mapping-of-pmf p*) = *p*
⟨*proof*⟩

lemma *mapping-of-pmfI*:
assumes ⋀*x*. *x* ∈ *Mapping.keys m* ⟹ *Mapping.lookup m x* = *Some* (*pmf p x*)
assumes *Mapping.keys m* = *set-pmf p*
shows *mapping-of-pmf p* = *m*
⟨*proof*⟩

lemma *mapping-of-pmfI'*:
assumes ⋀*x*. *x* ∈ *Mapping.keys m* ⟹ *Mapping.lookup-default 0 m x* = *pmf p x*

assumes *Mapping.keys m* = *set-pmf p*
shows *mapping-of-pmf p* = *m*
⟨*proof*⟩

lemma *return-pmf-code* [*code abstract*]:
mapping-of-pmf (*return-pmf x*) = *Mapping.update x 1 Mapping.empty*
⟨*proof*⟩

lemma *pmf-of-set-code-aux*:
assumes $A \neq \{\}$ *set xs* = *A distinct xs*
shows *mapping-of-pmf* (*pmf-of-set A*) = *Mapping.tabulate xs* (λ-. *1 / real* (*length xs*))
⟨*proof*⟩

definition *pmf-of-set-impl* **where**
pmf-of-set-impl A = *mapping-of-pmf* (*pmf-of-set A*)

lemma *pmf-of-set-impl-code-alt*:
assumes $A \neq \{\}$ *finite A*
shows *pmf-of-set-impl A* =
(*let p* = *1 / real* (*card A*)
in Finite-Set.fold (λ*x. Mapping.update x p*) *Mapping.empty A*)
⟨*proof*⟩

lemma *pmf-of-set-impl-code* [*code*]:
pmf-of-set-impl (*set xs*) =
(*if xs* = [] *then*
Code.abort (*STR ''pmf-of-set of empty set''*) (λ-. *mapping-of-pmf* (*pmf-of-set* (*set xs*)))
else let xs′ = *remdups xs*; *p* = *1 / real* (*length xs′*) *in*
Mapping.tabulate xs′ (λ-. *p*))
⟨*proof*⟩

lemma *pmf-of-set-code* [*code abstract*]:
mapping-of-pmf (*pmf-of-set A*) = *pmf-of-set-impl A*
⟨*proof*⟩

lemma *pmf-of-multiset-pmf-of-mapping*:
assumes $A \neq \{\#\}$ *set xs* = *set-mset A distinct xs*
shows *mapping-of-pmf* (*pmf-of-multiset A*) = *Mapping.tabulate xs* (λ*x. count A x / real* (*size A*))
⟨*proof*⟩

definition *pmf-of-multiset-impl* **where**
pmf-of-multiset-impl A = *mapping-of-pmf* (*pmf-of-multiset A*)

lemma *pmf-of-multiset-impl-code-alt*:
assumes $A \neq \{\#\}$
shows *pmf-of-multiset-impl A* =
(*let p* = *1 / real* (*size A*)
in fold-mset (λ*x. Mapping.map-default x 0* ((+) *p*)) *Mapping.empty A*)
⟨*proof*⟩

lemma *pmf-of-multiset-impl-code* [*code*]:

pmf-of-multiset-impl (*mset xs*) =
(*if xs* = [] *then*
Code.abort (*STR ''pmf-of-multiset of empty multiset''*)
(λ-. *mapping-of-pmf* (*pmf-of-multiset* (*mset xs*)))
else let xs' = *remdups xs*; *p* = *1* / *real* (*length xs*) *in*
Mapping.tabulate xs' (λ*x*. *real* (*count* (*mset xs*) *x*) ∗ *p*))
⟨*proof*⟩

lemma *pmf-of-multiset-code* [*code abstract*]:
mapping-of-pmf (*pmf-of-multiset A*) = *pmf-of-multiset-impl A*
⟨*proof*⟩

lemma *bernoulli-pmf-code* [*code abstract*]:
mapping-of-pmf (*bernoulli-pmf p*) =
(*if p* ≤ *0 then Mapping.update False 1 Mapping.empty*
else if p ≥ *1 then Mapping.update True 1 Mapping.empty*
else Mapping.update False (*1* − *p*) (*Mapping.update True p Mapping.empty*))
⟨*proof*⟩

lemma *pmf-code* [*code*]: *pmf p x* = *Mapping.lookup-default 0* (*mapping-of-pmf p*) *x*
⟨*proof*⟩

lemma *set-pmf-code* [*code*]: *set-pmf p* = *Mapping.keys* (*mapping-of-pmf p*)
⟨*proof*⟩

lemma *keys-mapping-of-pmf* [*simp*]: *Mapping.keys* (*mapping-of-pmf p*) = *set-pmf p*
⟨*proof*⟩

definition *fold-combine-plus* **where**
fold-combine-plus = *comm-monoid-set.F* (*Mapping.combine* ((+) :: *real* ⇒ -)) *Mapping.empty*

context
begin

interpretation *fold-combine-plus*: *combine-mapping-abel-semigroup* (+) :: *real* ⇒ -
⟨*proof*⟩ **lemma** *lookup-default-fold-combine-plus*:
fixes *A* :: *'b set* **and** *f* :: *'b* ⇒ (*'a*, *real*) *mapping*
assumes *finite A*
shows *Mapping.lookup-default 0* (*fold-combine-plus f A*) *x* =
(∑ *y*∈*A*. *Mapping.lookup-default 0* (*f y*) *x*)
⟨*proof*⟩ **lemma** *keys-fold-combine-plus*:

finite A ⟹ Mapping.keys (fold-combine-plus f A) = (⋃x∈A. Mapping.keys (f x))
⟨*proof*⟩ **lemma** *fold-combine-plus-code* [*code*]:
fold-combine-plus g (set xs) = foldr (λx. Mapping.combine (+) (g x)) (remdups xs) Mapping.empty
⟨*proof*⟩ **lemma** *lookup-default-0-map-values*:
assumes *f x 0 = 0*
shows *Mapping.lookup-default 0 (Mapping.map-values f m) x = f x (Mapping.lookup-default 0 m x)*
⟨*proof*⟩ **lemma** *mapping-of-bind-pmf*:
assumes *finite (set-pmf p)*
shows *mapping-of-pmf (bind-pmf p f) =*
fold-combine-plus (λx. Mapping.map-values (λ-. (∗) (pmf p x))
(mapping-of-pmf (f x))) (set-pmf p)
⟨*proof*⟩

lift-definition *bind-pmf-aux :: 'a pmf ⇒ ('a ⇒ 'b pmf) ⇒ 'a set ⇒ ('b, real) mapping* **is**
λ(p :: 'a pmf) (f :: 'a ⇒ 'b pmf) (A::'a set) (x::'b).
if x ∈ (⋃y∈A. set-pmf (f y)) then
Some (measure-pmf.expectation p (λy. indicator A y ∗ pmf (f y) x))
else None ⟨*proof*⟩

lemma *keys-bind-pmf-aux* [*simp*]:
Mapping.keys (bind-pmf-aux p f A) = (⋃x∈A. set-pmf (f x))
⟨*proof*⟩

lemma *lookup-default-bind-pmf-aux*:
Mapping.lookup-default 0 (bind-pmf-aux p f A) x =
(if x ∈ (⋃y∈A. set-pmf (f y)) then
measure-pmf.expectation p (λy. indicator A y ∗ pmf (f y) x) else 0)
⟨*proof*⟩

lemma *lookup-default-bind-pmf-aux'* [*simp*]:
Mapping.lookup-default 0 (bind-pmf-aux p f (set-pmf p)) x = pmf (bind-pmf p f) x
⟨*proof*⟩

lemma *bind-pmf-aux-correct*:
mapping-of-pmf (bind-pmf p f) = bind-pmf-aux p f (set-pmf p)
⟨*proof*⟩

lemma *bind-pmf-aux-code-aux*:
assumes *finite A*
shows *bind-pmf-aux p f A =*
fold-combine-plus (λx. Mapping.map-values (λ-. (∗) (pmf p x))
(mapping-of-pmf (f x))) A (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *bind-pmf-aux-code* [*code*]:
bind-pmf-aux p f (set xs) =
fold-combine-plus (λx. Mapping.map-values (λ-. (∗) (pmf p x))
(mapping-of-pmf (f x))) (set xs)
⟨*proof*⟩

lemmas *bind-pmf-code* [*code abstract*] = *bind-pmf-aux-correct*

end

hide-const (**open**) *fold-combine-plus*

lift-definition *cond-pmf-impl* :: *'a pmf ⇒ 'a set ⇒ ('a, real) mapping option* **is**
λp A. if A ∩ set-pmf p = {} then None else
Some (λx. if x ∈ A ∩ set-pmf p then Some (pmf p x / measure-pmf.prob p A)
else None) ⟨*proof*⟩

lemma *cond-pmf-impl-code-alt*:
assumes *finite A*
shows *cond-pmf-impl p A = (*
let C = A ∩ set-pmf p;
prob = (∑ x∈C. pmf p x)
in if prob = 0 then
None
else
Some (Mapping.map-values (λ- y. y / prob)
(Mapping.filter (λk -. k ∈ C) (mapping-of-pmf p))))
⟨*proof*⟩

lemma *cond-pmf-impl-code* [*code*]:
cond-pmf-impl p (set xs) = (
let C = set xs ∩ set-pmf p;
prob = (∑ x∈C. pmf p x)
in if prob = 0 then
None
else
Some (Mapping.map-values (λ- y. y / prob)
(Mapping.filter (λk -. k ∈ C) (mapping-of-pmf p))))
⟨*proof*⟩

lemma *cond-pmf-code* [*code abstract*]:
mapping-of-pmf (cond-pmf p A) =
(case cond-pmf-impl p A of
None ⇒ Code.abort (STR ''cond-pmf with set of probability 0'')
(λ-. mapping-of-pmf (cond-pmf p A))
| Some m ⇒ m)
⟨*proof*⟩

lemma *binomial-pmf-code* [*code abstract*]:
mapping-of-pmf (*binomial-pmf n p*) = (
if p < *0* ∨ *p* > *1 then*
Code.abort (*STR ''binomial-pmf with invalid probability''*)
(λ-. *mapping-of-pmf* (*binomial-pmf n p*))
else if p = *0 then Mapping.update 0 1 Mapping.empty*
else if p = *1 then Mapping.update n 1 Mapping.empty*
else Mapping.tabulate [*0..*<*Suc n*] (λ*k*. *real* (*n choose k*) ∗ *p* ^ *k* ∗ (*1* − *p*) ^
(*n* − *k*)))
⟨*proof*⟩

lemma *pred-pmf-code* [*code*]:
pred-pmf P p = (∀ *x*∈*set-pmf p*. *P x*)
⟨*proof*⟩

lemma *mapping-of-pmf-pmf-of-list*:
assumes ⋀*x*. *x* ∈ *snd* ' *set xs* ⟹ *x* > *0 sum-list* (*map snd xs*) = *1*
shows *mapping-of-pmf* (*pmf-of-list xs*) =
Mapping.tabulate (*remdups* (*map fst xs*))
(λ*x*. *sum-list* (*map snd* (*filter* (λ*z*. *fst z* = *x*) *xs*)))
⟨*proof*⟩

lemma *mapping-of-pmf-pmf-of-list'*:
assumes *pmf-of-list-wf xs*
defines *xs'* ≡ *filter* (λ*z*. *snd z* ≠ *0*) *xs*
shows *mapping-of-pmf* (*pmf-of-list xs*) =
Mapping.tabulate (*remdups* (*map fst xs'*))
(λ*x*. *sum-list* (*map snd* (*filter* (λ*z*. *fst z* = *x*) *xs'*))) (**is** - = *?rhs*)
⟨*proof*⟩

lemma *pmf-of-list-wf-code* [*code*]:
pmf-of-list-wf xs ⟷ *list-all* (λ*z*. *snd z* ≥ *0*) *xs* ∧ *sum-list* (*map snd xs*) = *1*
⟨*proof*⟩

lemma *pmf-of-list-code* [*code abstract*]:
mapping-of-pmf (*pmf-of-list xs*) = (
if pmf-of-list-wf xs then
let xs' = *filter* (λ*z*. *snd z* ≠ *0*) *xs*
in Mapping.tabulate (*remdups* (*map fst xs'*))
(λ*x*. *sum-list* (*map snd* (*filter* (λ*z*. *fst z* = *x*) *xs'*)))
else
Code.abort (*STR ''Invalid list for pmf-of-list''*) (λ-. *mapping-of-pmf* (*pmf-of-list*
xs)))
⟨*proof*⟩

lemma *mapping-of-pmf-eq-iff* [*simp*]:

mapping-of-pmf p = mapping-of-pmf q ⟷ p = (q :: 'a pmf)
⟨*proof*⟩

21.2 Code abbreviations for integrals and probabilities

Integrals and probabilities are defined for general measures, so we cannot give any code equations directly. We can, however, specialise these constants them to PMFs, give code equations for these specialised constants, and tell the code generator to unfold the original constants to the specialised ones whenever possible.

definition *pmf-integral* **where**
pmf-integral p f = lebesgue-integral (measure-pmf p) (f :: - ⇒ real)

definition *pmf-set-integral* **where**
pmf-set-integral p f A = lebesgue-integral (measure-pmf p) (λx. indicator A x ∗ f x :: real)

definition *pmf-prob* **where**
pmf-prob p A = measure-pmf.prob p A

lemma *pmf-prob-compl*: *pmf-prob p (−A) = 1 − pmf-prob p A*
⟨*proof*⟩

lemma *pmf-integral-pmf-set-integral* [*code*]:
pmf-integral p f = pmf-set-integral p f (set-pmf p)
⟨*proof*⟩

lemma *pmf-prob-pmf-set-integral*:
pmf-prob p A = pmf-set-integral p (λ-. 1) A
⟨*proof*⟩

lemma *pmf-set-integral-code-alt-finite*:
finite A ⟹ pmf-set-integral p f A = (∑ x∈A. pmf p x ∗ f x)
⟨*proof*⟩

lemma *pmf-set-integral-code* [*code*]:
pmf-set-integral p f (set xs) = (∑ x∈set xs. pmf p x ∗ f x)
⟨*proof*⟩

lemma *pmf-prob-code-alt-finite*:
finite A ⟹ pmf-prob p A = (∑ x∈A. pmf p x)
⟨*proof*⟩

lemma *pmf-prob-code* [*code*]:
pmf-prob p (set xs) = (∑ x∈set xs. pmf p x)
pmf-prob p (List.coset xs) = 1 − (∑ x∈set xs. pmf p x)
⟨*proof*⟩

lemma *pmf-prob-code-unfold* [*code-abbrev*]: *pmf-prob p = measure-pmf.prob p*
⟨*proof*⟩

lemma *pmf-integral-code-unfold* [*code-abbrev*]: *pmf-integral p = measure-pmf.expectation p*
⟨*proof*⟩

definition *pmf-of-alist xs = embed-pmf* (λ*x. case map-of xs x of Some p ⇒ p | None ⇒ 0*)

lemma *pmf-of-mapping-Mapping* [*code-post*]:
pmf-of-mapping (*Mapping xs*) = *pmf-of-alist xs*
⟨*proof*⟩

instantiation *pmf* :: (*equal*) *equal*
begin

definition *equal-pmf p q* = (*mapping-of-pmf p = mapping-of-pmf* (*q* :: *'a pmf*))

instance ⟨*proof*⟩
end

definition *single* :: *'a ⇒ 'a multiset* **where**
single s = {#*s*#}

instantiation *pmf* :: (*random*) *random*
begin

context
includes *state-combinator-syntax* **and** *term-syntax*
begin

definition
pmfify :: (*'b*::*typerep multiset* × (*unit ⇒ Code-Evaluation.term*)) ⇒
'b × (*unit ⇒ Code-Evaluation.term*) ⇒
'b pmf × (*unit ⇒ Code-Evaluation.term*) **where**
[*code-unfold*]: *pmfify A x* =
Code-Evaluation.valtermify pmf-of-multiset {·}
(*Code-Evaluation.valtermify* (+) {·} *A* {·}
(*Code-Evaluation.valtermify single* {·} *x*))

definition
Quickcheck-Random.random i =

Quickcheck-Random.random i $\circ\!\!\rightarrow$ (λ*A*.
Quickcheck-Random.random i $\circ\!\!\rightarrow$ (λ*x*. *Pair* (*pmfify A x*)))

instance ⟨*proof*⟩

end

end

instantiation *pmf* :: (*full-exhaustive*) *full-exhaustive*
begin

definition *full-exhaustive-pmf* :: ('*a pmf* × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒ *natural* ⇒ (*bool* × *term list*) *option*
where
full-exhaustive-pmf f i =
Quickcheck-Exhaustive.full-exhaustive (λ*A*.
Quickcheck-Exhaustive.full-exhaustive (λ*x*. *f* (*pmfify A x*)) *i*) *i*

instance ⟨*proof*⟩

end

end

22 Finite Maps

theory *Fin-Map*
imports *HOL−Analysis.Finite-Product-Measure HOL−Library.Finite-Map*
begin

The *fmap* type can be instantiated to *polish-space*, needed for the proof of projective limit. *extensional* functions are used for the representation in order to stay close to the developments of (finite) products Pi_E and their sigma-algebra Pi_M.

type-notation *fmap* (‹(‹*notation*=‹*infix fmap*››- ⇒$_F$ /-)› [*22*, *21*] *21*)

unbundle *fmap.lifting*

22.1 Domain and Application

lift-definition *domain*::('*i* ⇒$_F$ '*a*) ⇒ '*i set* **is** *dom* ⟨*proof*⟩

lemma *finite-domain*[*simp*, *intro*]: *finite* (*domain P*)
⟨*proof*⟩

lift-definition *proj* :: ('*i* ⇒$_F$ '*a*) ⇒ '*i* ⇒ '*a*
(‹(‹*indent=1 notation*=‹*mixfix proj*››'(-')$_F$)› [*0*] *1000*) **is**

$\lambda f\ x.$ *if* $x \in$ *dom* f *then the* $(f\ x)$ *else undefined* ⟨*proof*⟩

declare [[*coercion proj*]]

lemma *extensional-proj*[*simp*, *intro*]: $(P)_F \in$ *extensional* (*domain* P)
⟨*proof*⟩

lemma *proj-undefined*[*simp*, *intro*]: $i \notin$ *domain* $P \Longrightarrow P\ i =$ *undefined*
⟨*proof*⟩

lemma *finmap-eq-iff*: $P = Q \longleftrightarrow$ (*domain* $P =$ *domain* $Q \wedge (\forall\, i{\in}$*domain* $P.\ P\ i = Q\ i)$)
⟨*proof*⟩

22.2 Constructor of Finite Maps

lift-definition *finmap-of*::$'i$ *set* $\Rightarrow$ $('i \Rightarrow 'a) \Rightarrow ('i \Rightarrow_F 'a)$ **is**
$\lambda I\ f\ x.$ *if* $x \in I \wedge$ *finite* I *then Some* $(f\ x)$ *else None*
⟨*proof*⟩

lemma *proj-finmap-of*[*simp*]:
assumes *finite inds*
shows $(\mathit{finmap\text{-}of}\ \mathit{inds}\ f)_F = \mathit{restrict}\ f\ \mathit{inds}$
⟨*proof*⟩

lemma *domain-finmap-of*[*simp*]:
assumes *finite inds*
shows *domain* (*finmap-of inds* f) = *inds*
⟨*proof*⟩

lemma *finmap-of-eq-iff*[*simp*]:
assumes *finite* i *finite* j
shows *finmap-of* $i\ m =$ *finmap-of* $j\ n \longleftrightarrow i = j \wedge (\forall\, k{\in}i.\ m\ k{=}\ n\ k)$
⟨*proof*⟩

lemma *finmap-of-inj-on-extensional-finite*:
assumes *finite* K
assumes $S \subseteq$ *extensional* K
shows *inj-on* (*finmap-of* K) S
⟨*proof*⟩

22.3 Product set of Finite Maps

This is *Pi* for Finite Maps, most of this is copied

definition $Pi' :: 'i\ set \Rightarrow ('i \Rightarrow 'a\ set) \Rightarrow ('i \Rightarrow_F 'a)\ set$ **where**
$Pi'\ I\ A = \{\ P.\ \mathit{domain}\ P = I \wedge (\forall\, i.\ i \in I \longrightarrow (P)_F\ i \in A\ i)\ \}$

syntax
-Pi' :: [*pttrn*, $'a$ *set*, $'b$ *set*] => $('a$ => $'b)$ *set*

(‹(‹*indent=3 notation=‹binder* Π′››Π′′ -∈-./ -)› *10*)
syntax-consts
-Pi′ == *Pi′*
translations
Π′ *x*∈*A*. *B* == *CONST Pi′ A* (λ*x*. *B*)

22.3.1 Basic Properties of *Pi′*

lemma *Pi′-I*[*intro*!]: *domain f* = *A* ⟹ (⋀*x*. *x* ∈ *A* ⟹ *f x* ∈ *B x*) ⟹ *f* ∈ *Pi′ A B*
⟨*proof*⟩

lemma *Pi′-I′*[*simp*]: *domain f* = *A* ⟹ (⋀*x*. *x* ∈ *A* ⟶ *f x* ∈ *B x*) ⟹ *f* ∈ *Pi′ A B*
⟨*proof*⟩

lemma *Pi′-mem*: *f*∈ *Pi′ A B* ⟹ *x* ∈ *A* ⟹ *f x* ∈ *B x*
⟨*proof*⟩

lemma *Pi′-iff*: *f* ∈ *Pi′ I X* ⟷ *domain f* = *I* ∧ (∀ *i*∈*I*. *f i* ∈ *X i*)
⟨*proof*⟩

lemma *Pi′E* [*elim*]:
f ∈ *Pi′ A B* ⟹ (*f x* ∈ *B x* ⟹ *domain f* = *A* ⟹ *Q*) ⟹ (*x* ∉ *A* ⟹ *Q*) ⟹ *Q*
⟨*proof*⟩

lemma *in-Pi′-cong*:
domain f = *domain g* ⟹ (⋀ *w*. *w* ∈ *A* ⟹ *f w* = *g w*) ⟹ *f* ∈ *Pi′ A B* ⟷ *g* ∈ *Pi′ A B*
⟨*proof*⟩

lemma *Pi′-eq-empty*[*simp*]:
assumes *finite A* **shows** (*Pi′ A B*) = {} ⟷ (∃ *x*∈*A*. *B x* = {})
⟨*proof*⟩

lemma *Pi′-mono*: (⋀*x*. *x* ∈ *A* ⟹ *B x* ⊆ *C x*) ⟹ *Pi′ A B* ⊆ *Pi′ A C*
⟨*proof*⟩

lemma *Pi-Pi′*: *finite A* ⟹ (*Pi*$_E$ *A B*) = *proj* ‘ *Pi′ A B*
⟨*proof*⟩

22.4 Topological Space of Finite Maps

instantiation *fmap* :: (*type*, *topological-space*) *topological-space*
begin

definition *open-fmap* :: ('*a* ⇒$_F$ '*b*) *set* ⇒ *bool* **where**
[*code drop*]: *open-fmap* = *generate-topology* {*Pi′ a b*|*a b*. ∀ *i*∈*a*. *open* (*b i*)}

lemma *open-Pi'I*: $(\bigwedge i.\ i \in I \Longrightarrow open\ (A\ i)) \Longrightarrow open\ (Pi'\ I\ A)$
⟨*proof*⟩

instance ⟨*proof*⟩

end

lemma *open-restricted-space*:
shows *open* $\{m.\ P\ (domain\ m)\}$
⟨*proof*⟩

lemma *closed-restricted-space*:
shows *closed* $\{m.\ P\ (domain\ m)\}$
⟨*proof*⟩

lemma *tendsto-proj*: $((\lambda x.\ x) \longrightarrow a)\ F \Longrightarrow ((\lambda x.\ (x)_F\ i) \longrightarrow (a)_F\ i)\ F$
⟨*proof*⟩

lemma *continuous-proj*:
shows *continuous-on s* $(\lambda x.\ (x)_F\ i)$
⟨*proof*⟩

instance *fmap* :: (*type*, *first-countable-topology*) *first-countable-topology*
⟨*proof*⟩

22.5 Metric Space of Finite Maps

instantiation *fmap* :: (*type*, *metric-space*) *dist*
begin

definition *dist-fmap* **where**
dist P Q = *Max* (*range* (λ*i*. *dist* ((*P*)$_F$ *i*) ((*Q*)$_F$ *i*))) + (*if domain P* = *domain Q then 0 else 1*)

instance ⟨*proof*⟩
end

instantiation *fmap* :: (*type*, *metric-space*) *uniformity-dist*
begin

definition
(*uniformity* :: (('*a*, '*b*) *fmap* × ('*a* ⇒$_F$ '*b*)) *filter*) =
(*INF e*∈{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

instance
⟨*proof*⟩
end

instantiation *fmap* :: (*type*, *metric-space*) *metric-space*

begin

lemma *finite-proj-image'*: $x \notin$ *domain* $P \Longrightarrow$ *finite* $((P)_F$ ' $S)$
⟨*proof*⟩

lemma *finite-proj-image*: *finite* $((P)_F$ ' $S)$
⟨*proof*⟩

lemma *finite-proj-diag*: *finite* $((\lambda i.\ d\ ((P)_F\ i)\ ((Q)_F\ i))$ ' $S)$
⟨*proof*⟩

lemma *dist-le-1-imp-domain-eq*:
shows *dist* $P\ Q < 1 \Longrightarrow$ *domain* $P =$ *domain* Q
⟨*proof*⟩

lemma *dist-proj*:
shows *dist* $((x)_F\ i)\ ((y)_F\ i) \leq$ *dist* $x\ y$
⟨*proof*⟩

lemma *dist-finmap-lessI*:
assumes *domain* $P =$ *domain* Q
assumes $0 < e$
assumes $\bigwedge i.\ i \in$ *domain* $P \Longrightarrow$ *dist* $(P\ i)\ (Q\ i) < e$
shows *dist* $P\ Q < e$
⟨*proof*⟩

instance
⟨*proof*⟩

end

22.6 Complete Space of Finite Maps

lemma *tendsto-finmap*:
fixes f::*nat* $\Rightarrow$ ('*i* $\Rightarrow_F$ ('*a*::*metric-space*))
assumes *ind-f*: $\bigwedge n.$ *domain* $(f\ n) =$ *domain* g
assumes *proj-g*: $\bigwedge i.\ i \in$ *domain* $g \Longrightarrow (\lambda n.\ (f\ n)\ i) \longrightarrow g\ i$
shows $f \longrightarrow g$
⟨*proof*⟩

instance *fmap* :: (*type*, *complete-space*) *complete-space*
⟨*proof*⟩

22.7 Second Countable Space of Finite Maps

instantiation *fmap* :: (*countable*, *second-countable-topology*) *second-countable-topology*
begin

definition *basis-proj*::'*b set set*
where *basis-proj* = (*SOME B*. *countable B* $\wedge$ *topological-basis B*)

lemma *countable-basis-proj*: *countable basis-proj* **and** *basis-proj*: *topological-basis basis-proj*
⟨*proof*⟩

definition *basis-finmap*::('*a* ⇒$_F$ '*b*) *set set*
where *basis-finmap* = {*Pi'* *I S*|*I S*. *finite I* ∧ (∀ *i* ∈ *I*. *S i* ∈ *basis-proj*)}

lemma *in-basis-finmapI*:
assumes *finite I* **assumes** ⋀*i*. *i* ∈ *I* ⟹ *S i* ∈ *basis-proj*
shows *Pi'* *I S* ∈ *basis-finmap*
⟨*proof*⟩

lemma *basis-finmap-eq*:
assumes *basis-proj* ≠ {}
shows *basis-finmap* = (λ*f*. *Pi'* (*domain f*) (λ*i*. *from-nat-into basis-proj* ((*f*)$_F$ *i*))) ‘
(*UNIV*::('*a* ⇒$_F$ *nat*) *set*) (**is** - = *?f* ‘ -)
⟨*proof*⟩

lemma *basis-finmap-eq-empty*: *basis-proj* = {} ⟹ *basis-finmap* = {*Pi'* {} *undefined*}
⟨*proof*⟩

lemma *countable-basis-finmap*: *countable basis-finmap*
⟨*proof*⟩

lemma *finmap-topological-basis*:
topological-basis basis-finmap
⟨*proof*⟩

lemma *range-enum-basis-finmap-imp-open*:
assumes *x* ∈ *basis-finmap*
shows *open x*
⟨*proof*⟩

instance ⟨*proof*⟩

end

22.8 Polish Space of Finite Maps

instance *fmap* :: (*countable*, *polish-space*) *polish-space* ⟨*proof*⟩

22.9 Product Measurable Space of Finite Maps

definition *PiF I M* ≡
sigma (⋃ *J* ∈ *I*. (Π' *j*∈*J*. *space* (*M j*))) {(Π' *j*∈*J*. *X j*) |*X J*. *J* ∈ *I* ∧ *X* ∈ (Π *j*∈*J*. *sets* (*M j*))}

abbreviation
$Pi_F\ I\ M \equiv PiF\ I\ M$

syntax
$\text{-}PiF :: pttrn \Rightarrow 'i\ set \Rightarrow 'a\ measure \Rightarrow ('i => 'a)\ measure$
$(\langle(\langle indent=3\ notation=\langle binder\ \Pi_F\rangle\rangle\Pi_F\ \text{-}\in\text{-}./\ \text{-})\rangle\ \ 10)$
syntax-consts
$\text{-}PiF == PiF$
translations
$\Pi_F\ x{\in}I.\ M == CONST\ PiF\ I\ (\%x.\ M)$

lemma *PiF-gen-subset*: $\{(\Pi'\ j{\in}J.\ X\ j)\ |X\ J.\ J \in I \wedge X \in (\Pi\ j{\in}J.\ sets\ (M\ j))\}$
$\subseteq$
$Pow\ (\bigcup J \in I.\ (\Pi'\ j{\in}J.\ space\ (M\ j)))$
⟨*proof*⟩

lemma *space-PiF*: $space\ (PiF\ I\ M) = (\bigcup J \in I.\ (\Pi'\ j{\in}J.\ space\ (M\ j)))$
⟨*proof*⟩

lemma *sets-PiF*:
$sets\ (PiF\ I\ M) = sigma\text{-}sets\ (\bigcup J \in I.\ (\Pi'\ j{\in}J.\ space\ (M\ j)))$
$\{(\Pi'\ j{\in}J.\ X\ j)\ |X\ J.\ J \in I \wedge X \in (\Pi\ j{\in}J.\ sets\ (M\ j))\}$
⟨*proof*⟩

lemma *sets-PiF-singleton*:
$sets\ (PiF\ \{I\}\ M) = sigma\text{-}sets\ (\Pi'\ j{\in}I.\ space\ (M\ j))$
$\{(\Pi'\ j{\in}I.\ X\ j)\ |X.\ X \in (\Pi\ j{\in}I.\ sets\ (M\ j))\}$
⟨*proof*⟩

lemma *in-sets-PiFI*:
assumes $X = (Pi'\ J\ S)\ J \in I\ \bigwedge i.\ i{\in}J \Longrightarrow S\ i \in sets\ (M\ i)$
shows $X \in sets\ (PiF\ I\ M)$
⟨*proof*⟩

lemma *product-in-sets-PiFI*:
assumes $J \in I\ \bigwedge i.\ i{\in}J \Longrightarrow S\ i \in sets\ (M\ i)$
shows $(Pi'\ J\ S) \in sets\ (PiF\ I\ M)$
⟨*proof*⟩

lemma *singleton-space-subset-in-sets*:
fixes J
assumes $J \in I$
assumes *finite* J
shows $space\ (PiF\ \{J\}\ M) \in sets\ (PiF\ I\ M)$
⟨*proof*⟩

lemma *singleton-subspace-set-in-sets*:
assumes A: $A \in sets\ (PiF\ \{J\}\ M)$
assumes *finite* J

assumes $J \in I$
shows $A \in sets\ (PiF\ I\ M)$
⟨*proof*⟩

lemma *finite-measurable-singletonI*:
assumes *finite I*
assumes $\bigwedge J.\ J \in I \Longrightarrow finite\ J$
assumes *MN*: $\bigwedge J.\ J \in I \Longrightarrow A \in measurable\ (PiF\ \{J\}\ M)\ N$
shows $A \in measurable\ (PiF\ I\ M)\ N$
⟨*proof*⟩

lemma *countable-finite-comprehension*:
fixes $f :: 'a::countable\ set \Rightarrow$ -
assumes $\bigwedge s.\ P\ s \Longrightarrow finite\ s$
assumes $\bigwedge s.\ P\ s \Longrightarrow f\ s \in sets\ M$
shows $\bigcup \{f\ s | s.\ P\ s\} \in sets\ M$
⟨*proof*⟩

lemma *space-subset-in-sets*:
fixes $J::'a::countable\ set\ set$
assumes $J \subseteq I$
assumes $\bigwedge j.\ j \in J \Longrightarrow finite\ j$
shows $space\ (PiF\ J\ M) \in sets\ (PiF\ I\ M)$
⟨*proof*⟩

lemma *subspace-set-in-sets*:
fixes $J::'a::countable\ set\ set$
assumes *A*: $A \in sets\ (PiF\ J\ M)$
assumes $J \subseteq I$
assumes $\bigwedge j.\ j \in J \Longrightarrow finite\ j$
shows $A \in sets\ (PiF\ I\ M)$
⟨*proof*⟩

lemma *countable-measurable-PiFI*:
fixes $I::'a::countable\ set\ set$
assumes *MN*: $\bigwedge J.\ J \in I \Longrightarrow finite\ J \Longrightarrow A \in measurable\ (PiF\ \{J\}\ M)\ N$
shows $A \in measurable\ (PiF\ I\ M)\ N$
⟨*proof*⟩

lemma *measurable-PiF*:
assumes *f*: $\bigwedge x.\ x \in space\ N \Longrightarrow domain\ (f\ x) \in I \wedge (\forall\, i \in domain\ (f\ x).\ (f\ x)\ i \in space\ (M\ i))$
assumes *S*: $\bigwedge J\ S.\ J \in I \Longrightarrow (\bigwedge i.\ i \in J \Longrightarrow S\ i \in sets\ (M\ i)) \Longrightarrow$
$f\ -`\ (Pi'\ J\ S) \cap space\ N \in sets\ N$
shows $f \in measurable\ N\ (PiF\ I\ M)$
⟨*proof*⟩

lemma *restrict-sets-measurable*:
assumes *A*: $A \in sets\ (PiF\ I\ M)$ **and** $J \subseteq I$

shows $A \cap \{m.\ domain\ m \in J\} \in sets\ (PiF\ J\ M)$
⟨*proof*⟩

lemma *measurable-finmap-of*:
assumes f: $\bigwedge i.\ (\exists\, x \in space\ N.\ i \in J\ x) \Longrightarrow (\lambda x.\ f\ x\ i) \in measurable\ N\ (M\ i)$
assumes J: $\bigwedge x.\ x \in space\ N \Longrightarrow J\ x \in I\ \bigwedge x.\ x \in space\ N \Longrightarrow finite\ (J\ x)$
assumes JN: $\bigwedge S.\ \{x.\ J\ x = S\} \cap space\ N \in sets\ N$
shows $(\lambda x.\ finmap\text{-}of\ (J\ x)\ (f\ x)) \in measurable\ N\ (PiF\ I\ M)$
⟨*proof*⟩

lemma *measurable-PiM-finmap-of*:
assumes *finite J*
shows $finmap\text{-}of\ J \in measurable\ (Pi_M\ J\ M)\ (PiF\ \{J\}\ M)$
⟨*proof*⟩

lemma *proj-measurable-singleton*:
assumes $A \in sets\ (M\ i)$
shows $(\lambda x.\ (x)_F\ i) -` A \cap space\ (PiF\ \{I\}\ M) \in sets\ (PiF\ \{I\}\ M)$
⟨*proof*⟩

lemma *measurable-proj-singleton*:
assumes $i \in I$
shows $(\lambda x.\ (x)_F\ i) \in measurable\ (PiF\ \{I\}\ M)\ (M\ i)$
⟨*proof*⟩

lemma *measurable-proj-countable*:
fixes I::'a::*countable set set*
assumes $y \in space\ (M\ i)$
shows $(\lambda x.\ if\ i \in domain\ x\ then\ (x)_F\ i\ else\ y) \in measurable\ (PiF\ I\ M)\ (M\ i)$
⟨*proof*⟩

lemma *measurable-restrict-proj*:
assumes $J \in II\ finite\ J$
shows $finmap\text{-}of\ J \in measurable\ (PiM\ J\ M)\ (PiF\ II\ M)$
⟨*proof*⟩

lemma *measurable-proj-PiM*:
fixes $J\ K$ ::'a::*countable set* **and** I::'a *set set*
assumes *finite J* $J \in I$
assumes $x \in space\ (PiM\ J\ M)$
shows $proj \in measurable\ (PiF\ \{J\}\ M)\ (PiM\ J\ M)$
⟨*proof*⟩

lemma *space-PiF-singleton-eq-product*:
assumes *finite I*
shows $space\ (PiF\ \{I\}\ M) = (\Pi'\ i{\in}I.\ space\ (M\ i))$
⟨*proof*⟩

adapted from $sets\ (Pi_M\ ?I\ ?M) = sigma\text{-}sets\ (\Pi_E\ i{\in}?I.\ space\ (?M\ i))\ \{\{f$

$\in \Pi_E\ i{\in}?I.\ space\ (?M\ i).\ f\ i \in A\}\ |i\ A.\ i \in ?I \wedge A \in sets\ (?M\ i)\}$

lemma *sets-PiF-single*:
assumes *finite I* $I \neq \{\}$
shows *sets* (*PiF* $\{I\}$ *M*) =
sigma-sets ($\Pi'\ i{\in}I.\ space\ (M\ i)$)
$\{\{f{\in}\Pi'\ i{\in}I.\ space\ (M\ i).\ f\ i \in A\} \mid i\ A.\ i \in I \wedge A \in sets\ (M\ i)\}$
(**is** - = *sigma-sets* ?Ω ?*R*)
⟨*proof*⟩

adapted from $(\bigwedge i.\ i \in ?I \Longrightarrow ?A\ i = ?B\ i) \Longrightarrow Pi_E\ ?I\ ?A = Pi_E\ ?I\ ?B$

lemma *Pi'-cong*:
assumes *finite I*
assumes $\bigwedge i.\ i \in I \Longrightarrow f\ i = g\ i$
shows *Pi'* *I f* = *Pi'* *I g*
⟨*proof*⟩

adapted from $[\![finite\ ?I;\ \bigwedge i\ n\ m.\ [\![i \in ?I;\ n \leq m]\!] \Longrightarrow ?A\ n\ i \subseteq ?A\ m\ i]\!] \Longrightarrow (\bigcup_n Pi\ ?I\ (?A\ n)) = (\Pi\ i{\in}?I.\ \bigcup_n ?A\ n\ i)$

lemma *Pi'-UN*:
fixes *A* :: *nat* ⇒ *'i* ⇒ *'a set*
assumes *finite I*
assumes *mono*: $\bigwedge i\ n\ m.\ i \in I \Longrightarrow n \leq m \Longrightarrow A\ n\ i \subseteq A\ m\ i$
shows $(\bigcup n.\ Pi'\ I\ (A\ n)) = Pi'\ I\ (\lambda i.\ \bigcup n.\ A\ n\ i)$
⟨*proof*⟩

adapted from $[\![\bigwedge i.\ i \in ?I \Longrightarrow \exists S{\subseteq}?E\ i.\ countable\ S \wedge ?\Omega\ i = \bigcup S;\ \bigwedge i.\ i \in ?I \Longrightarrow ?E\ i \subseteq Pow\ (?\Omega\ i);\ \bigwedge j.\ j \in ?J \Longrightarrow finite\ j;\ \bigcup ?J = ?I]\!] \Longrightarrow sets\ (Pi_M\ ?I\ (\lambda i.\ sigma\ (?\Omega\ i)\ (?E\ i))) = sets\ (sigma\ (Pi_E\ ?I\ ?\Omega)\ \{\{f \in Pi_E\ ?I\ ?\Omega.\ \forall i{\in}j.\ f\ i \in A\ i\} |A\ j.\ j \in ?J \wedge A \in Pi\ j\ ?E\})$

lemma *sigma-fprod-algebra-sigma-eq*:
fixes *E* :: *'i* ⇒ *'a set set* **and** *S* :: *'i* ⇒ *nat* ⇒ *'a set*
assumes [*simp*]: *finite I* $I \neq \{\}$
and *S-union*: $\bigwedge i.\ i \in I \Longrightarrow (\bigcup j.\ S\ i\ j) = space\ (M\ i)$
and *S-in-E*: $\bigwedge i.\ i \in I \Longrightarrow range\ (S\ i) \subseteq E\ i$
assumes *E-closed*: $\bigwedge i.\ i \in I \Longrightarrow E\ i \subseteq Pow\ (space\ (M\ i))$
and *E-generates*: $\bigwedge i.\ i \in I \Longrightarrow sets\ (M\ i) = sigma\text{-}sets\ (space\ (M\ i))\ (E\ i)$
defines *P* == { *Pi'* *I F* | *F*. ∀ *i*∈*I*. *F i* ∈ *E i* }
shows *sets* (*PiF* $\{I\}$ *M*) = *sigma-sets* (*space* (*PiF* $\{I\}$ *M*)) *P*
⟨*proof*⟩

lemma *product-open-generates-sets-PiF-single*:
assumes $I \neq \{\}$
assumes [*simp*]: *finite I*
shows *sets* (*PiF* $\{I\}$ (λ-. *borel*::*'b*::*second-countable-topology measure*)) =
sigma-sets (*space* (*PiF* $\{I\}$ (λ-. *borel*))) {*Pi'* *I F* |*F*. (∀ *i*∈*I*. *F i* ∈ *Collect open*)}
⟨*proof*⟩

lemma *finmap-UNIV*[*simp*]: $(\bigcup J{\in}Collect\ finite.\ \Pi'\ j{\in}J.\ UNIV) = UNIV$ ⟨*proof*⟩

lemma *borel-eq-PiF-borel*:
shows $(borel :: ('i{::}countable \Rightarrow_F\ 'a{::}polish\text{-}space)\ measure) =$
$PiF\ (Collect\ finite)\ (\lambda\text{-}.\ borel :: 'a\ measure)$
⟨*proof*⟩

22.10 Isomorphism between Functions and Finite Maps

lemma *measurable-finmap-compose*:
shows $(\lambda m.\ compose\ J\ m\ f) \in measurable\ (PiM\ (f\ `\ J)\ (\lambda\text{-}.\ M))\ (PiM\ J\ (\lambda\text{-}.\ M))$
⟨*proof*⟩

lemma *measurable-compose-inv*:
assumes *inj*: $\bigwedge j.\ j \in J \Longrightarrow f'\ (f\ j) = j$
shows $(\lambda m.\ compose\ (f\ `\ J)\ m\ f') \in measurable\ (PiM\ J\ (\lambda\text{-}.\ M))\ (PiM\ (f\ `\ J)\ (\lambda\text{-}.\ M))$
⟨*proof*⟩

locale *function-to-finmap* =
fixes $J{::}'a\ set$ **and** $f :: 'a \Rightarrow 'b{::}countable$ **and** f'
assumes [*simp*]: *finite J*
assumes *inv*: $i \in J \Longrightarrow f'\ (f\ i) = i$
begin

to measure finmaps

definition $fm = (finmap\text{-}of\ (f\ `\ J))\ o\ (\lambda g.\ compose\ (f\ `\ J)\ g\ f')$

lemma *domain-fm*[*simp*]: $domain\ (fm\ x) = f\ `\ J$
⟨*proof*⟩

lemma *fm-restrict*[*simp*]: $fm\ (restrict\ y\ J) = fm\ y$
⟨*proof*⟩

lemma *fm-product*:
assumes $\bigwedge i.\ space\ (M\ i) = UNIV$
shows $fm\ -`\ Pi'\ (f\ `\ J)\ S \cap space\ (Pi_M\ J\ M) = (\Pi_E\ j \in J.\ S\ (f\ j))$
⟨*proof*⟩

lemma *fm-measurable*:
assumes $f\ `\ J \in N$
shows $fm \in measurable\ (Pi_M\ J\ (\lambda\text{-}.\ M))\ (Pi_F\ N\ (\lambda\text{-}.\ M))$
⟨*proof*⟩

lemma *proj-fm*:
assumes $x \in J$
shows $fm\ m\ (f\ x) = m\ x$
⟨*proof*⟩

lemma *inj-on-compose-f'*: *inj-on* (λ*g*. *compose* (*f* ‘ *J*) *g f'*) (*extensional J*)
⟨*proof*⟩

lemma *inj-on-fm*:
assumes ⋀*i*. *space* (*M i*) = *UNIV*
shows *inj-on fm* (*space* (*Pi*$_M$ *J M*))
⟨*proof*⟩

to measure functions

definition *mf* = (λ*g*. *compose J g f*) *o proj*

lemma *mf-fm*:
assumes *x* ∈ *space* (*Pi*$_M$ *J* (λ-. *M*))
shows *mf* (*fm x*) = *x*
⟨*proof*⟩

lemma *mf-measurable*:
assumes *space M* = *UNIV*
shows *mf* ∈ *measurable* (*PiF* {*f* ‘ *J*} (λ-. *M*)) (*PiM J* (λ-. *M*))
⟨*proof*⟩

lemma *fm-image-measurable*:
assumes *space M* = *UNIV*
assumes *X* ∈ *sets* (*Pi*$_M$ *J* (λ-. *M*))
shows *fm* ‘ *X* ∈ *sets* (*PiF* {*f* ‘ *J*} (λ-. *M*))
⟨*proof*⟩

lemma *fm-image-measurable-finite*:
assumes *space M* = *UNIV*
assumes *X* ∈ *sets* (*Pi*$_M$ *J* (λ-. *M*::'*c measure*))
shows *fm* ‘ *X* ∈ *sets* (*PiF* (*Collect finite*) (λ-. *M*::'*c measure*))
⟨*proof*⟩

measure on finmaps

definition *mapmeasure M N* = *distr M* (*PiF* (*Collect finite*) *N*) (*fm*)

lemma *sets-mapmeasure*[*simp*]: *sets* (*mapmeasure M N*) = *sets* (*PiF* (*Collect finite*) *N*)
⟨*proof*⟩

lemma *space-mapmeasure*[*simp*]: *space* (*mapmeasure M N*) = *space* (*PiF* (*Collect finite*) *N*)
⟨*proof*⟩

lemma *mapmeasure-PiF*:
assumes *s1*: *space M* = *space* (*Pi*$_M$ *J* (λ-. *N*))
assumes *s2*: *sets M* = *sets* (*Pi*$_M$ *J* (λ-. *N*))
assumes *space N* = *UNIV*

assumes $X \in sets\ (PiF\ (Collect\ finite)\ (\lambda\text{-}.\ N))$
shows $emeasure\ (mapmeasure\ M\ (\lambda\text{-}.\ N))\ X = emeasure\ M\ ((fm - `\ X \cap extensional\ J))$
⟨*proof*⟩

lemma *mapmeasure-PiM*:
fixes $N::'c\ measure$
assumes *s1*: $space\ M = space\ (Pi_M\ J\ (\lambda\text{-}.\ N))$
assumes *s2*: $sets\ M = (Pi_M\ J\ (\lambda\text{-}.\ N))$
assumes *N*: $space\ N = UNIV$
assumes *X*: $X \in sets\ M$
shows $emeasure\ M\ X = emeasure\ (mapmeasure\ M\ (\lambda\text{-}.\ N))\ (fm\ `\ X)$
⟨*proof*⟩

end

end

23 Projective Limit

theory *Projective-Limit*
imports
Fin-Map
Infinite-Product-Measure
HOL−Library.Diagonal-Subsequence
begin

23.1 Sequences of Finite Maps in Compact Sets

locale *finmap-seqs-into-compact* =
fixes $K::nat \Rightarrow (nat \Rightarrow_F\ 'a::metric\text{-}space)\ set$ **and** $f::nat \Rightarrow (nat \Rightarrow_F\ 'a)$ **and** *M*
assumes *compact*: $\bigwedge n.\ compact\ (K\ n)$
assumes *f-in-K*: $\bigwedge n.\ K\ n \neq \{\}$
assumes *domain-K*: $\bigwedge n.\ k \in K\ n \Longrightarrow domain\ k = domain\ (f\ n)$
assumes *proj-in-K*:
$\bigwedge t\ n\ m.\ m \geq n \Longrightarrow t \in domain\ (f\ n) \Longrightarrow (f\ m)_F\ t \in (\lambda k.\ (k)_F\ t)\ `\ K\ n$
begin

lemma *proj-in-K'*: $(\exists n.\ \forall m \geq n.\ (f\ m)_F\ t \in (\lambda k.\ (k)_F\ t)\ `\ K\ n)$
⟨*proof*⟩

lemma *proj-in-KE*:
obtains n **where** $\bigwedge m.\ m \geq n \Longrightarrow (f\ m)_F\ t \in (\lambda k.\ (k)_F\ t)\ `\ K\ n$
⟨*proof*⟩

lemma *compact-projset*:
shows $compact\ ((\lambda k.\ (k)_F\ i)\ `\ K\ n)$
⟨*proof*⟩

end

lemma *compactE'*:
fixes $S :: 'a :: metric\text{-}space\ set$
assumes *compact* S $\forall n{\geq}m.\ f\ n \in S$
obtains $l\ r$ **where** $l \in S$ *strict-mono* $(r{::}nat{\Rightarrow}nat)$ $((f \circ r) \longrightarrow l)$ *sequentially*
⟨*proof*⟩

sublocale *finmap-seqs-into-compact* $\subseteq$ *subseqs* $\lambda n\ s.\ (\exists l.\ (\lambda i.\ ((f\ o\ s)\ i)_F\ n) \longrightarrow l)$
⟨*proof*⟩

lemma (**in** *finmap-seqs-into-compact*) *diagonal-tendsto*: $\exists l.\ (\lambda i.\ (f\ (diagseq\ i))_F\ n) \longrightarrow l$
⟨*proof*⟩

23.2 Daniell-Kolmogorov Theorem

Existence of Projective Limit

locale *polish-projective* = *projective-family I P* λ-. *borel*::$'a$::*polish-space measure*
for I::$'i$ *set* **and** P
begin

lemma *emeasure-lim-emb*:
assumes X: $J \subseteq I$ *finite* J $X \in sets\ (\Pi_M\ i{\in}J.\ borel)$
shows *lim* $(emb\ I\ J\ X) = P\ J\ X$
⟨*proof*⟩

lemma *measure-lim-emb*:
$J \subseteq I \Longrightarrow finite\ J \Longrightarrow X \in sets\ (\Pi_M\ i{\in}J.\ borel) \Longrightarrow measure\ lim\ (emb\ I\ J\ X) = measure\ (P\ J)\ X$
⟨*proof*⟩

end

hide-const (**open**) *PiF*
hide-const (**open**) Pi_F
hide-const (**open**) Pi'
hide-const (**open**) *finmap-of*
hide-const (**open**) *proj*
hide-const (**open**) *domain*
hide-const (**open**) *basis-finmap*

sublocale *polish-projective* $\subseteq$ *P*: *prob-space lim*
⟨*proof*⟩

locale *polish-product-prob-space* =
product-prob-space λ-. *borel*::($'a$::*polish-space*) *measure I* **for** I::$'i$ *set*

sublocale *polish-product-prob-space* ⊆ *P*: *polish-projective I* λ*J*. *PiM J* (λ-. *borel*::('*a*) *measure*)
⟨*proof*⟩

lemma (**in** *polish-product-prob-space*) *limP-eq-PiM*: *lim* = *PiM I* (λ-. *borel*)
⟨*proof*⟩

end

24 Random Permutations

theory *Random-Permutations*
imports
HOL−Combinatorics.Multiset-Permutations
Probability-Mass-Function
begin

Choosing a set permutation (i.e. a distinct list with the same elements as the set) uniformly at random is the same as first choosing the first element of the list and then choosing the rest of the list as a permutation of the remaining set.

lemma *random-permutation-of-set*:
assumes *finite A A* ≠ {}
shows *pmf-of-set* (*permutations-of-set A*) =
do {
x ← *pmf-of-set A*;
xs ← *pmf-of-set* (*permutations-of-set* (*A* − {*x*}));
return-pmf (*x*#*xs*)
} (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

A generic fold function that takes a function, an initial state, and a set and chooses a random order in which it then traverses the set in the same fashion as a left fold over a list. We first give a recursive definition.

function *fold-random-permutation* :: ('*a* ⇒ '*b* ⇒ '*b*) ⇒ '*b* ⇒ '*a set* ⇒ '*b pmf*
where
fold-random-permutation f x {} = *return-pmf x*
| ¬*finite A* ⟹ *fold-random-permutation f x A* = *return-pmf x*
| *finite A* ⟹ *A* ≠ {} ⟹
fold-random-permutation f x A =
pmf-of-set A ⤜ (λ*a*. *fold-random-permutation f* (*f a x*) (*A* − {*a*}))
⟨*proof*⟩
termination ⟨*proof*⟩

We can now show that the above recursive definition is equivalent to choosing a random set permutation and folding over it (in any direction).

lemma *fold-random-permutation-foldl*:

assumes *finite A*
shows *fold-random-permutation f x A =*
map-pmf (foldl (λx y. f y x) x) (pmf-of-set (permutations-of-set A))
⟨*proof*⟩

lemma *fold-random-permutation-foldr*:
assumes *finite A*
shows *fold-random-permutation f x A =*
map-pmf (λxs. foldr f xs x) (pmf-of-set (permutations-of-set A))
⟨*proof*⟩

lemma *fold-random-permutation-fold*:
assumes *finite A*
shows *fold-random-permutation f x A =*
map-pmf (λxs. fold f xs x) (pmf-of-set (permutations-of-set A))
⟨*proof*⟩

lemma *fold-random-permutation-code* [*code*]:
fold-random-permutation f x (set xs) =
map-pmf (foldl (λx y. f y x) x) (pmf-of-set (permutations-of-set (set xs)))
⟨*proof*⟩

We now introduce a slightly generalised version of the above fold operation that does not simply return the result in the end, but applies a monadic bind to it. This may seem somewhat arbitrary, but it is a common use case, e.g. in the Social Decision Scheme of Random Serial Dictatorship, where voters narrow down a set of possible winners in a random order and the winner is chosen from the remaining set uniformly at random.

function *fold-bind-random-permutation*
:: *('a ⇒ 'b ⇒ 'b) ⇒ ('b ⇒ 'c pmf) ⇒ 'b ⇒ 'a set ⇒ 'c pmf* **where**
fold-bind-random-permutation f g x {} = g x
| *¬finite A ⟹ fold-bind-random-permutation f g x A = g x*
| *finite A ⟹ A ≠ {} ⟹*
fold-bind-random-permutation f g x A =
pmf-of-set A ⤜ (λa. fold-bind-random-permutation f g (f a x) (A − {a}))
⟨*proof*⟩
termination ⟨*proof*⟩

We now show that the recursive definition is equivalent to a random fold followed by a monadic bind.

lemma *fold-bind-random-permutation-altdef* [*code*]:
fold-bind-random-permutation f g x A = fold-random-permutation f x A ⤜ g
⟨*proof*⟩

We can now derive the following nice monadic representations of the combined fold-and-bind:

lemma *fold-bind-random-permutation-foldl*:
assumes *finite A*

shows *fold-bind-random-permutation f g x A* =
do {*xs* ← *pmf-of-set* (*permutations-of-set A*); *g* (*foldl* (λ*x y*. *f y x*) *x xs*)}
⟨*proof*⟩

lemma *fold-bind-random-permutation-foldr*:
assumes *finite A*
shows *fold-bind-random-permutation f g x A* =
do {*xs* ← *pmf-of-set* (*permutations-of-set A*); *g* (*foldr f xs x*)}
⟨*proof*⟩

lemma *fold-bind-random-permutation-fold*:
assumes *finite A*
shows *fold-bind-random-permutation f g x A* =
do {*xs* ← *pmf-of-set* (*permutations-of-set A*); *g* (*fold f xs x*)}
⟨*proof*⟩

The following useful lemma allows us to swap partitioning a set w. r. t. a predicate and drawing a random permutation of that set.

lemma *partition-random-permutations*:
assumes *finite A*
shows *map-pmf* (*partition P*) (*pmf-of-set* (*permutations-of-set A*)) =
pair-pmf (*pmf-of-set* (*permutations-of-set* {*x*∈*A*. *P x*}))
(*pmf-of-set* (*permutations-of-set* {*x*∈*A*. ¬*P x*})) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

end

25 Discrete subprobability distribution

theory *SPMF* **imports**
Probability-Mass-Function
HOL−Library.Complete-Partial-Order2
HOL−Library.Rewrite
begin

25.1 Auxiliary material

lemma *cSUP-singleton* [*simp*]: (*SUP x*∈{*x*}. *f x* :: - :: *conditionally-complete-lattice*) = *f x*
⟨*proof*⟩

25.1.1 More about extended reals

lemma [*simp*]:
shows *ennreal-max-0*: *ennreal* (*max 0 x*) = *ennreal x*
and *ennreal-max-0'*: *ennreal* (*max x 0*) = *ennreal x*
⟨*proof*⟩

lemma *e2ennreal-0* [*simp*]: *e2ennreal 0* = *0*

⟨*proof*⟩

lemma *enn2real-bot* [*simp*]: *enn2real* ⊥ = *0*
⟨*proof*⟩

lemma *continuous-at-ennreal*[*continuous-intros*]: *continuous F f* ⟹ *continuous F* (λ*x*. *ennreal* (*f x*))
⟨*proof*⟩

lemma *ennreal-Sup*:
assumes ∗: (*SUP a*∈*A*. *ennreal a*) ≠ ⊤
and *A* ≠ {}
shows *ennreal* (*Sup A*) = (*SUP a*∈*A*. *ennreal a*)
⟨*proof*⟩

lemma *ennreal-SUP*:
⟦ (*SUP a*∈*A*. *ennreal* (*f a*)) ≠ ⊤; *A* ≠ {} ⟧ ⟹ *ennreal* (*SUP a*∈*A*. *f a*) = (*SUP a*∈*A*. *ennreal* (*f a*))
⟨*proof*⟩

lemma *ennreal-lt-0*: *x* < *0* ⟹ *ennreal x* = *0*
⟨*proof*⟩

25.1.2 More about *'a option*

lemma *None-in-map-option-image* [*simp*]: *None* ∈ *map-option f* ‘ *A* ⟷ *None* ∈ *A*
⟨*proof*⟩

lemma *Some-in-map-option-image* [*simp*]: *Some x* ∈ *map-option f* ‘ *A* ⟷ (∃ *y*. *x* = *f y* ∧ *Some y* ∈ *A*)
⟨*proof*⟩

lemma *case-option-collapse*: *case-option x* (λ-. *x*) = (λ-. *x*)
⟨*proof*⟩

lemma *case-option-id*: *case-option None Some* = *id*
⟨*proof*⟩

inductive *ord-option* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *'a option* ⇒ *'b option* ⇒ *bool*
for *ord* :: *'a* ⇒ *'b* ⇒ *bool*
where
None: *ord-option ord None x*
| *Some*: *ord x y* ⟹ *ord-option ord* (*Some x*) (*Some y*)

inductive-simps *ord-option-simps* [*simp*]:
ord-option ord None x
ord-option ord x None
ord-option ord (*Some x*) (*Some y*)

ord-option ord (*Some x*) *None*

inductive-simps *ord-option-eq-simps* [*simp*]:
ord-option (=) *None y*
ord-option (=) (*Some x*) *y*

lemma *ord-option-reflI*: (⋀*y*. *y* ∈ *set-option x* ⟹ *ord y y*) ⟹ *ord-option ord x x*
⟨*proof*⟩

lemma *reflp-ord-option*: *reflp ord* ⟹ *reflp* (*ord-option ord*)
⟨*proof*⟩

lemma *ord-option-trans*:
⟦ *ord-option ord x y*; *ord-option ord y z*;
⋀*a b c*. ⟦ *a* ∈ *set-option x*; *b* ∈ *set-option y*; *c* ∈ *set-option z*; *ord a b*; *ord b c* ⟧ ⟹ *ord a c* ⟧
⟹ *ord-option ord x z*
⟨*proof*⟩

lemma *transp-ord-option*: *transp ord* ⟹ *transp* (*ord-option ord*)
⟨*proof*⟩

lemma *antisymp-ord-option*: *antisymp ord* ⟹ *antisymp* (*ord-option ord*)
⟨*proof*⟩

lemma *ord-option-chainD*:
Complete-Partial-Order.chain (*ord-option ord*) *Y*
⟹ *Complete-Partial-Order.chain ord* {*x*. *Some x* ∈ *Y*}
⟨*proof*⟩

definition *lub-option* :: ('*a set* ⇒ '*b*) ⇒ '*a option set* ⇒ '*b option*
where *lub-option lub Y* = (*if Y* ⊆ {*None*} *then None else Some* (*lub* {*x*. *Some x* ∈ *Y*}))

lemma *map-lub-option*: *map-option f* (*lub-option lub Y*) = *lub-option* (*f* ∘ *lub*) *Y*
⟨*proof*⟩

lemma *lub-option-upper*:
assumes *Complete-Partial-Order.chain* (*ord-option ord*) *Y x* ∈ *Y*
and *lub-upper*: ⋀*Y x*. ⟦ *Complete-Partial-Order.chain ord Y*; *x* ∈ *Y* ⟧ ⟹ *ord x* (*lub Y*)
shows *ord-option ord x* (*lub-option lub Y*)
⟨*proof*⟩

lemma *lub-option-least*:
assumes *Y*: *Complete-Partial-Order.chain* (*ord-option ord*) *Y*
and *upper*: ⋀*x*. *x* ∈ *Y* ⟹ *ord-option ord x y*
assumes *lub-least*: ⋀*Y y*. ⟦ *Complete-Partial-Order.chain ord Y*; ⋀*x*. *x* ∈ *Y* ⟹

ord x y ⟧ ⟹ *ord* (*lub Y*) *y*
shows *ord-option ord* (*lub-option lub Y*) *y*
⟨*proof*⟩

lemma *lub-map-option*: *lub-option lub* (*map-option f ' Y*) = *lub-option* (*lub* ∘ (') *f*) *Y*
⟨*proof*⟩

lemma *ord-option-mono*: ⟦ *ord-option A x y*; ⋀*x y*. *A x y* ⟹ *B x y* ⟧ ⟹ *ord-option B x y*
⟨*proof*⟩

lemma *ord-option-mono'* [*mono*]:
(⋀*x y*. *A x y* ⟶ *B x y*) ⟹ *ord-option A x y* ⟶ *ord-option B x y*
⟨*proof*⟩

lemma *ord-option-compp*: *ord-option* (*A OO B*) = *ord-option A OO ord-option B*
⟨*proof*⟩

lemma *ord-option-inf*: *inf* (*ord-option A*) (*ord-option B*) = *ord-option* (*inf A B*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *ord-option-map2*: *ord-option ord x* (*map-option f y*) = *ord-option* (λ*x y*. *ord x* (*f y*)) *x y*
⟨*proof*⟩

lemma *ord-option-map1*: *ord-option ord* (*map-option f x*) *y* = *ord-option* (λ*x y*. *ord* (*f x*) *y*) *x y*
⟨*proof*⟩

lemma *option-ord-Some1-iff*: *option-ord* (*Some x*) *y* ⟷ *y* = *Some x*
⟨*proof*⟩

25.1.3 A relator for sets that treats sets like predicates

context includes *lifting-syntax*
begin

definition *rel-pred* :: ('*a* ⇒ '*b* ⇒ *bool*) ⇒ '*a set* ⇒ '*b set* ⇒ *bool*
where *rel-pred R A B* = (*R* ===> (=)) (λ*x*. *x* ∈ *A*) (λ*y*. *y* ∈ *B*)

lemma *rel-predI*: (*R* ===> (=)) (λ*x*. *x* ∈ *A*) (λ*y*. *y* ∈ *B*) ⟹ *rel-pred R A B*
⟨*proof*⟩

lemma *rel-predD*: ⟦ *rel-pred R A B*; *R x y* ⟧ ⟹ *x* ∈ *A* ⟷ *y* ∈ *B*
⟨*proof*⟩

lemma *Collect-parametric*: ((*A* ===> (=)) ===> *rel-pred A*) *Collect Collect*

— Declare this rule as *transfer-rule* only locally because it blows up the search space for *transfer* (in combination with *Collect-transfer*)
⟨*proof*⟩

end

25.1.4 Monotonicity rules

lemma *monotone-gfp-eadd1*: *monotone* (≥) (≥) (λ*x*. *x* + *y* :: *enat*)
⟨*proof*⟩

lemma *monotone-gfp-eadd2*: *monotone* (≥) (≥) (λ*y*. *x* + *y* :: *enat*)
⟨*proof*⟩

lemma *mono2mono-gfp-eadd*[*THEN gfp.mono2mono2*, *cont-intro*, *simp*]:
shows *monotone-eadd*: *monotone* (*rel-prod* (≥) (≥)) (≥) (λ(*x*, *y*). *x* + *y* :: *enat*)
⟨*proof*⟩

lemma *eadd-gfp-partial-function-mono* [*partial-function-mono*]:
⟦ *monotone* (*fun-ord* (≥)) (≥) *f*; *monotone* (*fun-ord* (≥)) (≥) *g* ⟧
⟹ *monotone* (*fun-ord* (≥)) (≥) (λ*x*. *f x* + *g x* :: *enat*)
⟨*proof*⟩

lemma *mono2mono-ereal*[*THEN lfp.mono2mono*]:
shows *monotone-ereal*: *monotone* (≤) (≤) *ereal*
⟨*proof*⟩

lemma *mono2mono-ennreal*[*THEN lfp.mono2mono*]:
shows *monotone-ennreal*: *monotone* (≤) (≤) *ennreal*
⟨*proof*⟩

25.1.5 Bijections

lemma *bi-unique-rel-set-bij-betw*:
assumes *unique*: *bi-unique R*
and *rel*: *rel-set R A B*
shows ∃*f*. *bij-betw f A B* ∧ (∀ *x*∈*A*. *R x* (*f x*))
⟨*proof*⟩

lemma *bij-betw-rel-setD*: *bij-betw f A B* ⟹ *rel-set* (λ*x y*. *y* = *f x*) *A B*
⟨*proof*⟩

25.2 Subprobability mass function

type-synonym *'a spmf* = *'a option pmf*
translations (*type*) *'a spmf* ↽ (*type*) *'a option pmf*

definition *measure-spmf* :: *'a spmf* ⇒ *'a measure*
where *measure-spmf p* = *distr* (*restrict-space* (*measure-pmf p*) (*range Some*)) (*count-space UNIV*) *the*

abbreviation *spmf* :: *'a spmf* ⇒ *'a* ⇒ *real*
where *spmf p x* ≡ *pmf p* (*Some x*)

lemma *space-measure-spmf*: *space* (*measure-spmf p*) = *UNIV*
⟨*proof*⟩

lemma *sets-measure-spmf* [*simp*, *measurable-cong*]: *sets* (*measure-spmf p*) = *sets* (*count-space UNIV*)
⟨*proof*⟩

lemma *measure-spmf-not-bot* [*simp*]: *measure-spmf p* ≠ ⊥
⟨*proof*⟩

lemma *measurable-the-measure-pmf-Some* [*measurable*, *simp*]:
the ∈ *measurable* (*restrict-space* (*measure-pmf p*) (*range Some*)) (*count-space UNIV*)
⟨*proof*⟩

lemma *measurable-spmf-measure1*[*simp*]: *measurable* (*measure-spmf M*) *N* = *UNIV* → *space N*
⟨*proof*⟩

lemma *measurable-spmf-measure2*[*simp*]: *measurable N* (*measure-spmf M*) = *measurable N* (*count-space UNIV*)
⟨*proof*⟩

lemma *subprob-space-measure-spmf* [*simp*, *intro*!]: *subprob-space* (*measure-spmf p*)
⟨*proof*⟩

interpretation *measure-spmf*: *subprob-space measure-spmf p* **for** *p*
⟨*proof*⟩

lemma *finite-measure-spmf* [*simp*]: *finite-measure* (*measure-spmf p*)
⟨*proof*⟩

lemma *spmf-conv-measure-spmf*: *spmf p x* = *measure* (*measure-spmf p*) {*x*}
⟨*proof*⟩

lemma *emeasure-measure-spmf-conv-measure-pmf*:
emeasure (*measure-spmf p*) *A* = *emeasure* (*measure-pmf p*) (*Some* ‘ *A*)
⟨*proof*⟩

lemma *measure-measure-spmf-conv-measure-pmf*:
measure (*measure-spmf p*) *A* = *measure* (*measure-pmf p*) (*Some* ‘ *A*)
⟨*proof*⟩

lemma *emeasure-spmf-map-pmf-Some* [*simp*]:
emeasure (*measure-spmf* (*map-pmf Some p*)) *A* = *emeasure* (*measure-pmf p*) *A*

⟨*proof*⟩

lemma *measure-spmf-map-pmf-Some* [*simp*]:
measure (*measure-spmf* (*map-pmf Some p*)) *A* = *measure* (*measure-pmf p*) *A*
⟨*proof*⟩

lemma *nn-integral-measure-spmf*: (∫⁺ *x*. *f x* ∂*measure-spmf p*) = ∫⁺ *x*. *ennreal* (*spmf p x*) ∗ *f x* ∂*count-space UNIV*
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *integral-measure-spmf*:
assumes *integrable* (*measure-spmf p*) *f*
shows (∫ *x*. *f x* ∂*measure-spmf p*) = ∫ *x*. *spmf p x* ∗ *f x* ∂*count-space UNIV*
⟨*proof*⟩

lemma *emeasure-spmf-single*: *emeasure* (*measure-spmf p*) {*x*} = *spmf p x*
⟨*proof*⟩

lemma *measurable-measure-spmf*[*measurable*]:
(λ*x*. *measure-spmf* (*M x*)) ∈ *measurable* (*count-space UNIV*) (*subprob-algebra* (*count-space UNIV*))
⟨*proof*⟩

lemma *nn-integral-measure-spmf-conv-measure-pmf*:
assumes [*measurable*]: *f* ∈ *borel-measurable* (*count-space UNIV*)
shows *nn-integral* (*measure-spmf p*) *f* = *nn-integral* (*restrict-space* (*measure-pmf p*) (*range Some*)) (*f* ∘ *the*)
⟨*proof*⟩

lemma *measure-spmf-in-space-subprob-algebra* [*simp*]:
measure-spmf p ∈ *space* (*subprob-algebra* (*count-space UNIV*))
⟨*proof*⟩

lemma *nn-integral-spmf-neq-top*: (∫⁺ *x*. *spmf p x* ∂*count-space UNIV*) ≠ ⊤
⟨*proof*⟩

lemma *SUP-spmf-neq-top'*: (*SUP p*∈*Y*. *ennreal* (*spmf p x*)) ≠ ⊤
⟨*proof*⟩

lemma *SUP-spmf-neq-top*: (*SUP i*. *ennreal* (*spmf* (*Y i*) *x*)) ≠ ⊤
⟨*proof*⟩

lemma *SUP-emeasure-spmf-neq-top*: (*SUP p*∈*Y*. *emeasure* (*measure-spmf p*) *A*) ≠ ⊤
⟨*proof*⟩

25.3 Support

definition *set-spmf* :: *'a spmf* ⇒ *'a set*
where *set-spmf p* = *set-pmf p* ⤜ *set-option*

lemma *set-spmf-rep-eq*: *set-spmf p* = {*x*. *measure* (*measure-spmf p*) {*x*} ≠ *0*}
⟨*proof*⟩

lemma *in-set-spmf*: *x* ∈ *set-spmf p* ⟷ *Some x* ∈ *set-pmf p*
⟨*proof*⟩

lemma *AE-measure-spmf-iff* [*simp*]: (*AE x in measure-spmf p*. *P x*) ⟷ (∀ *x*∈*set-spmf p*. *P x*)
⟨*proof*⟩

lemma *spmf-eq-0-set-spmf*: *spmf p x* = *0* ⟷ *x* ∉ *set-spmf p*
⟨*proof*⟩

lemma *in-set-spmf-iff-spmf*: *x* ∈ *set-spmf p* ⟷ *spmf p x* ≠ *0*
⟨*proof*⟩

lemma *set-spmf-return-pmf-None* [*simp*]: *set-spmf* (*return-pmf None*) = {}
⟨*proof*⟩

lemma *countable-set-spmf* [*simp*]: *countable* (*set-spmf p*)
⟨*proof*⟩

lemma *spmf-eqI*:
assumes ⋀*i*. *spmf p i* = *spmf q i*
shows *p* = *q*
⟨*proof*⟩

lemma *integral-measure-spmf-restrict*:
fixes *f* :: *'a* ⇒ *'b* :: {*banach*, *second-countable-topology*}
shows (∫ *x*. *f x* ∂*measure-spmf M*) = (∫ *x*. *f x* ∂*restrict-space* (*measure-spmf M*) (*set-spmf M*))
⟨*proof*⟩

lemma *nn-integral-measure-spmf'*:
(∫⁺ *x*. *f x* ∂*measure-spmf p*) = ∫⁺ *x*. *ennreal* (*spmf p x*) ∗ *f x* ∂*count-space* (*set-spmf p*)
⟨*proof*⟩

25.4 Functorial structure

abbreviation *map-spmf* :: (*'a* ⇒ *'b*) ⇒ *'a spmf* ⇒ *'b spmf*
where *map-spmf f* ≡ *map-pmf* (*map-option f*)

context begin
⟨*ML*⟩

lemma *map-comp*: *map-spmf f* (*map-spmf g p*) = *map-spmf* (*f* ∘ *g*) *p*
⟨*proof*⟩

lemma *map-id0*: *map-spmf id* = *id*
⟨*proof*⟩

lemma *map-id* [*simp*]: *map-spmf id p* = *p*
⟨*proof*⟩

lemma *map-ident* [*simp*]: *map-spmf* (λ*x*. *x*) *p* = *p*
⟨*proof*⟩

end

lemma *set-map-spmf* [*simp*]: *set-spmf* (*map-spmf f p*) = *f* ‘ *set-spmf p*
⟨*proof*⟩

lemma *map-spmf-cong*:
⟦ *p* = *q*; ⋀*x*. *x* ∈ *set-spmf q* ⟹ *f x* = *g x* ⟧ ⟹ *map-spmf f p* = *map-spmf g q*
⟨*proof*⟩

lemma *map-spmf-cong-simp*:
⟦ *p* = *q*; ⋀*x*. *x* ∈ *set-spmf q* =*simp*=> *f x* = *g x* ⟧
⟹ *map-spmf f p* = *map-spmf g q*
⟨*proof*⟩

lemma *map-spmf-idI*: (⋀*x*. *x* ∈ *set-spmf p* ⟹ *f x* = *x*) ⟹ *map-spmf f p* = *p*
⟨*proof*⟩

lemma *emeasure-map-spmf*:
emeasure (*measure-spmf* (*map-spmf f p*)) *A* = *emeasure* (*measure-spmf p*) (*f* −‘ *A*)
⟨*proof*⟩

lemma *measure-map-spmf*: *measure* (*measure-spmf* (*map-spmf f p*)) *A* = *measure* (*measure-spmf p*) (*f* −‘ *A*)
⟨*proof*⟩

lemma *measure-map-spmf-conv-distr*:
measure-spmf (*map-spmf f p*) = *distr* (*measure-spmf p*) (*count-space UNIV*) *f*
⟨*proof*⟩

lemma *spmf-map-pmf-Some* [*simp*]: *spmf* (*map-pmf Some p*) *i* = *pmf p i*
⟨*proof*⟩

lemma *spmf-map-inj*: ⟦ *inj-on f* (*set-spmf M*); *x* ∈ *set-spmf M* ⟧ ⟹ *spmf* (*map-spmf f M*) (*f x*) = *spmf M x*
⟨*proof*⟩

lemma *spmf-map-inj'*: *inj f* $\Longrightarrow$ *spmf* (*map-spmf f M*) (*f x*) = *spmf M x*
⟨*proof*⟩

lemma *spmf-map-outside*: *x* $\notin$ *f* ‘ *set-spmf M* $\Longrightarrow$ *spmf* (*map-spmf f M*) *x* = *0*
⟨*proof*⟩

lemma *ennreal-spmf-map*: *ennreal* (*spmf* (*map-spmf f p*) *x*) = *emeasure* (*measure-spmf p*) (*f* −‘ {*x*})
⟨*proof*⟩

lemma *spmf-map*: *spmf* (*map-spmf f p*) *x* = *measure* (*measure-spmf p*) (*f* −‘ {*x*})
⟨*proof*⟩

lemma *ennreal-spmf-map-conv-nn-integral*:
ennreal (*spmf* (*map-spmf f p*) *x*) = $integral^N$ (*measure-spmf p*) (*indicator* (*f* −‘ {*x*}))
⟨*proof*⟩

25.5 Monad operations

25.5.1 Return

abbreviation *return-spmf* :: *'a* ⇒ *'a spmf*
where *return-spmf x* ≡ *return-pmf* (*Some x*)

lemma *pmf-return-spmf*: *pmf* (*return-spmf x*) *y* = *indicator* {*y*} (*Some x*)
⟨*proof*⟩

lemma *measure-spmf-return-spmf*: *measure-spmf* (*return-spmf x*) = *Giry-Monad.return* (*count-space UNIV*) *x*
⟨*proof*⟩

lemma *measure-spmf-return-pmf-None* [*simp*]: *measure-spmf* (*return-pmf None*) = *null-measure* (*count-space UNIV*)
⟨*proof*⟩

lemma *set-return-spmf* [*simp*]: *set-spmf* (*return-spmf x*) = {*x*}
⟨*proof*⟩

25.5.2 Bind

definition *bind-spmf* :: *'a spmf* ⇒ (*'a* ⇒ *'b spmf*) ⇒ *'b spmf*
where *bind-spmf x f* = *bind-pmf x* (λ*a*. *case a of None* ⇒ *return-pmf None* | *Some a'* ⇒ *f a'*)

adhoc-overloading *Monad-Syntax.bind* ⇌ *bind-spmf*

lemma *return-None-bind-spmf* [*simp*]: *return-pmf None* ⤜ (*f* :: *'a* ⇒ -) = *return-pmf None*

⟨*proof*⟩

lemma *return-bind-spmf* [*simp*]: *return-spmf x* ⤜ *f* = *f x*
⟨*proof*⟩

lemma *bind-return-spmf* [*simp*]: *x* ⤜ *return-spmf* = *x*
⟨*proof*⟩

lemma *bind-spmf-assoc* [*simp*]:
fixes *x* :: *'a spmf* **and** *f* :: *'a* ⇒ *'b spmf* **and** *g* :: *'b* ⇒ *'c spmf*
shows (*x* ⤜ *f*) ⤜ *g* = *x* ⤜ (λ*y*. *f y* ⤜ *g*)
⟨*proof*⟩

lemma *pmf-bind-spmf-None*: *pmf* (*p* ⤜ *f*) *None* = *pmf p None* + ∫ *x*. *pmf* (*f x*) *None* ∂*measure-spmf p*
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *spmf-bind*: *spmf* (*p* ⤜ *f*) *y* = ∫ *x*. *spmf* (*f x*) *y* ∂*measure-spmf p*
⟨*proof*⟩

lemma *ennreal-spmf-bind*: *ennreal* (*spmf* (*p* ⤜ *f*) *x*) = ∫⁺ *y*. *spmf* (*f y*) *x* ∂*measure-spmf p*
⟨*proof*⟩

lemma *measure-spmf-bind-pmf*: *measure-spmf* (*p* ⤜ *f*) = *measure-pmf p* ⤜ *measure-spmf* ∘ *f*
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *measure-spmf-bind*: *measure-spmf* (*p* ⤜ *f*) = *measure-spmf p* ⤜ *measure-spmf* ∘ *f*
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *map-spmf-bind-spmf*: *map-spmf f* (*bind-spmf p g*) = *bind-spmf p* (*map-spmf f* ∘ *g*)
⟨*proof*⟩

lemma *bind-map-spmf*: *map-spmf f p* ⤜ *g* = *p* ⤜ *g* ∘ *f*
⟨*proof*⟩

lemma *spmf-bind-leI*:
assumes ⋀*y*. *y* ∈ *set-spmf p* ⟹ *spmf* (*f y*) *x* ≤ *r*
and *0* ≤ *r*
shows *spmf* (*bind-spmf p f*) *x* ≤ *r*
⟨*proof*⟩

lemma *map-spmf-conv-bind-spmf*: *map-spmf f p* = (*p* ⤜ (λ*x*. *return-spmf* (*f x*)))

⟨*proof*⟩

lemma *bind-spmf-cong*:
⟦ $p = q$; $\bigwedge x.\ x \in$ *set-spmf* $q \Longrightarrow f\ x = g\ x$ ⟧ $\Longrightarrow$ *bind-spmf* $p\ f =$ *bind-spmf* $q\ g$
⟨*proof*⟩

lemma *bind-spmf-cong-simp*:
⟦ $p = q$; $\bigwedge x.\ x \in$ *set-spmf* q =*simp*=> $f\ x = g\ x$ ⟧
$\Longrightarrow$ *bind-spmf* $p\ f =$ *bind-spmf* $q\ g$
⟨*proof*⟩

lemma *set-bind-spmf*: *set-spmf* $(M \gg\!= f) =$ *set-spmf* $M \gg\!=$ $(\textit{set-spmf} \circ f)$
⟨*proof*⟩

lemma *bind-spmf-const-return-None* [*simp*]: *bind-spmf* p $(\lambda\text{-}.$ *return-pmf None*$) =$ *return-pmf None*
⟨*proof*⟩

lemma *bind-commute-spmf*:
bind-spmf p $(\lambda x.$ *bind-spmf* q $(f\ x)) =$ *bind-spmf* q $(\lambda y.$ *bind-spmf* p $(\lambda x.\ f\ x\ y))$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

25.6 Relator

abbreviation *rel-spmf* :: $('a \Rightarrow 'b \Rightarrow$ *bool*$) \Rightarrow$ *'a spmf* $\Rightarrow$ *'b spmf* $\Rightarrow$ *bool*
where *rel-spmf* $R \equiv$ *rel-pmf* $(\textit{rel-option}\ R)$

lemma *rel-spmf-mono*:
⟦*rel-spmf* $A\ f\ g$; $\bigwedge x\ y.\ A\ x\ y \Longrightarrow B\ x\ y$ ⟧ $\Longrightarrow$ *rel-spmf* $B\ f\ g$
⟨*proof*⟩

lemma *rel-spmf-mono-strong*:
⟦ *rel-spmf* $A\ f\ g$; $\bigwedge x\ y.$ ⟦ $A\ x\ y$; $x \in$ *set-spmf* f; $y \in$ *set-spmf* g ⟧ $\Longrightarrow B\ x\ y$ ⟧ $\Longrightarrow$ *rel-spmf* $B\ f\ g$
⟨*proof*⟩

lemma *rel-spmf-reflI*: $(\bigwedge x.\ x \in$ *set-spmf* $p \Longrightarrow P\ x\ x) \Longrightarrow$ *rel-spmf* $P\ p\ p$
⟨*proof*⟩

lemma *rel-spmfI* [*intro?*]:
⟦ $\bigwedge x\ y.\ (x, y) \in$ *set-spmf* $pq \Longrightarrow P\ x\ y$; *map-spmf fst* $pq = p$; *map-spmf snd* $pq = q$ ⟧
$\Longrightarrow$ *rel-spmf* $P\ p\ q$
⟨*proof*⟩

lemma *rel-spmfE* [*elim?*, *consumes 1*, *case-names rel-spmf*]:
assumes *rel-spmf* $P\ p\ q$
obtains *pq* **where**

$\bigwedge x\ y.\ (x,\ y) \in \mathit{set\text{-}spmf}\ pq \Longrightarrow P\ x\ y$
$p = \mathit{map\text{-}spmf}\ \mathit{fst}\ pq$
$q = \mathit{map\text{-}spmf}\ \mathit{snd}\ pq$
⟨*proof*⟩

lemma *rel-spmf-simps*:
$\mathit{rel\text{-}spmf}\ R\ p\ q \longleftrightarrow (\exists\, pq.\ (\forall\, (x,\ y) \in \mathit{set\text{-}spmf}\ pq.\ R\ x\ y) \wedge \mathit{map\text{-}spmf}\ \mathit{fst}\ pq = p \wedge \mathit{map\text{-}spmf}\ \mathit{snd}\ pq = q)$
⟨*proof*⟩

lemma *spmf-rel-map*:
shows *spmf-rel-map1*: $\bigwedge R\ f\ x.\ \mathit{rel\text{-}spmf}\ R\ (\mathit{map\text{-}spmf}\ f\ x) = \mathit{rel\text{-}spmf}\ (\lambda x.\ R\ (f\ x))\ x$
and *spmf-rel-map2*: $\bigwedge R\ x\ g\ y.\ \mathit{rel\text{-}spmf}\ R\ x\ (\mathit{map\text{-}spmf}\ g\ y) = \mathit{rel\text{-}spmf}\ (\lambda x\ y.\ R\ x\ (g\ y))\ x\ y$
⟨*proof*⟩

lemma *spmf-rel-conversep*: $\mathit{rel\text{-}spmf}\ R^{-1-1} = (\mathit{rel\text{-}spmf}\ R)^{-1-1}$
⟨*proof*⟩

lemma *spmf-rel-eq*: $\mathit{rel\text{-}spmf}\ (=) = (=)$
⟨*proof*⟩

context includes *lifting-syntax*
begin

lemma *bind-spmf-parametric* [*transfer-rule*]:
$(\mathit{rel\text{-}spmf}\ A ===> (A ===> \mathit{rel\text{-}spmf}\ B) ===> \mathit{rel\text{-}spmf}\ B)\ \mathit{bind\text{-}spmf}\ \mathit{bind\text{-}spmf}$
⟨*proof*⟩

lemma *return-spmf-parametric*: $(A ===> \mathit{rel\text{-}spmf}\ A)\ \mathit{return\text{-}spmf}\ \mathit{return\text{-}spmf}$
⟨*proof*⟩

lemma *map-spmf-parametric*: $((A ===> B) ===> \mathit{rel\text{-}spmf}\ A ===> \mathit{rel\text{-}spmf}\ B)\ \mathit{map\text{-}spmf}\ \mathit{map\text{-}spmf}$
⟨*proof*⟩

lemma *rel-spmf-parametric*:
$((A ===> B ===> (=)) ===> \mathit{rel\text{-}spmf}\ A ===> \mathit{rel\text{-}spmf}\ B ===> (=))\ \mathit{rel\text{-}spmf}\ \mathit{rel\text{-}spmf}$
⟨*proof*⟩

lemma *set-spmf-parametric* [*transfer-rule*]:
$(\mathit{rel\text{-}spmf}\ A ===> \mathit{rel\text{-}set}\ A)\ \mathit{set\text{-}spmf}\ \mathit{set\text{-}spmf}$
⟨*proof*⟩

lemma *return-spmf-None-parametric*:
$(\mathit{rel\text{-}spmf}\ A)\ (\mathit{return\text{-}pmf}\ \mathit{None})\ (\mathit{return\text{-}pmf}\ \mathit{None})$
⟨*proof*⟩

end

lemma *rel-spmf-bindI*:
⟦ *rel-spmf R p q*; ⋀*x y*. *R x y* ⟹ *rel-spmf P* (*f x*) (*g y*) ⟧
⟹ *rel-spmf P* (*p* ⤜ *f*) (*q* ⤜ *g*)
⟨*proof*⟩

lemma *rel-spmf-bind-reflI*:
(⋀*x*. *x* ∈ *set-spmf p* ⟹ *rel-spmf P* (*f x*) (*g x*)) ⟹ *rel-spmf P* (*p* ⤜ *f*) (*p* ⤜ *g*)
⟨*proof*⟩

lemma *rel-pmf-return-pmfI*: *P x y* ⟹ *rel-pmf P* (*return-pmf x*) (*return-pmf y*)
⟨*proof*⟩

context includes *lifting-syntax*
begin

We do not yet have a relator for *'a measure*, so we combine *Sigma-Algebra.measure* and *measure-pmf*

lemma *measure-pmf-parametric*:
(*rel-pmf A* ===> *rel-pred A* ===> (=)) (λ*p*. *measure* (*measure-pmf p*)) (λ*q*. *measure* (*measure-pmf q*))
⟨*proof*⟩

lemma *measure-spmf-parametric*:
(*rel-spmf A* ===> *rel-pred A* ===> (=)) (λ*p*. *measure* (*measure-spmf p*)) (λ*q*. *measure* (*measure-spmf q*))
⟨*proof*⟩

end

25.7 From *'a pmf* to *'a spmf*

definition *spmf-of-pmf* :: *'a pmf* ⇒ *'a spmf*
where *spmf-of-pmf* = *map-pmf Some*

lemma *set-spmf-spmf-of-pmf* [*simp*]: *set-spmf* (*spmf-of-pmf p*) = *set-pmf p*
⟨*proof*⟩

lemma *spmf-spmf-of-pmf* [*simp*]: *spmf* (*spmf-of-pmf p*) *x* = *pmf p x*
⟨*proof*⟩

lemma *pmf-spmf-of-pmf-None* [*simp*]: *pmf* (*spmf-of-pmf p*) *None* = *0*
⟨*proof*⟩

lemma *emeasure-spmf-of-pmf* [*simp*]: *emeasure* (*measure-spmf* (*spmf-of-pmf p*)) *A* = *emeasure* (*measure-pmf p*) *A*

⟨*proof*⟩

lemma *measure-spmf-spmf-of-pmf* [*simp*]: *measure-spmf* (*spmf-of-pmf p*) = *measure-pmf p*
⟨*proof*⟩

lemma *map-spmf-of-pmf* [*simp*]: *map-spmf f* (*spmf-of-pmf p*) = *spmf-of-pmf* (*map-pmf f p*)
⟨*proof*⟩

lemma *rel-spmf-spmf-of-pmf* [*simp*]: *rel-spmf R* (*spmf-of-pmf p*) (*spmf-of-pmf q*) = *rel-pmf R p q*
⟨*proof*⟩

lemma *spmf-of-pmf-return-pmf* [*simp*]: *spmf-of-pmf* (*return-pmf x*) = *return-spmf x*
⟨*proof*⟩

lemma *bind-spmf-of-pmf* [*simp*]: *bind-spmf* (*spmf-of-pmf p*) *f* = *bind-pmf p f*
⟨*proof*⟩

lemma *set-spmf-bind-pmf*: *set-spmf* (*bind-pmf p f*) = *Set.bind* (*set-pmf p*) (*set-spmf* ∘ *f*)
⟨*proof*⟩

lemma *spmf-of-pmf-bind*: *spmf-of-pmf* (*bind-pmf p f*) = *bind-pmf p* (λ*x*. *spmf-of-pmf* (*f x*))
⟨*proof*⟩

lemma *bind-pmf-return-spmf*: *p* ⤜ (λ*x*. *return-spmf* (*f x*)) = *spmf-of-pmf* (*map-pmf f p*)
⟨*proof*⟩

25.8 Weight of a subprobability

abbreviation *weight-spmf* :: *'a spmf* ⇒ *real*
where *weight-spmf p* ≡ *measure* (*measure-spmf p*) (*space* (*measure-spmf p*))

lemma *weight-spmf-def*: *weight-spmf p* = *measure* (*measure-spmf p*) *UNIV*
⟨*proof*⟩

lemma *weight-spmf-le-1*: *weight-spmf p* ≤ *1*
⟨*proof*⟩

lemma *weight-return-spmf* [*simp*]: *weight-spmf* (*return-spmf x*) = *1*
⟨*proof*⟩

lemma *weight-return-pmf-None* [*simp*]: *weight-spmf* (*return-pmf None*) = *0*
⟨*proof*⟩

lemma *weight-map-spmf* [*simp*]: *weight-spmf* (*map-spmf f p*) = *weight-spmf p*
⟨*proof*⟩

lemma *weight-spmf-of-pmf* [*simp*]: *weight-spmf* (*spmf-of-pmf p*) = *1*
⟨*proof*⟩

lemma *weight-spmf-nonneg*: *weight-spmf p* ≥ *0*
⟨*proof*⟩

lemma (**in** *finite-measure*) *integrable-weight-spmf* [*simp*]:
(λ*x*. *weight-spmf* (*f x*)) ∈ *borel-measurable M* ⟹ *integrable M* (λ*x*. *weight-spmf* (*f x*))
⟨*proof*⟩

lemma *weight-spmf-eq-nn-integral-spmf*: *weight-spmf p* = ∫⁺ *x*. *spmf p x* ∂*count-space UNIV*
⟨*proof*⟩

lemma *weight-spmf-eq-nn-integral-support*:
weight-spmf p = ∫⁺ *x*. *spmf p x* ∂*count-space* (*set-spmf p*)
⟨*proof*⟩

lemma *pmf-None-eq-weight-spmf*: *pmf p None* = *1* − *weight-spmf p*
⟨*proof*⟩

lemma *weight-spmf-conv-pmf-None*: *weight-spmf p* = *1* − *pmf p None*
⟨*proof*⟩

lemma *weight-spmf-lt-0*: ¬ *weight-spmf p* < *0*
⟨*proof*⟩

lemma *spmf-le-weight*: *spmf p x* ≤ *weight-spmf p*
⟨*proof*⟩

lemma *weight-spmf-eq-0*: *weight-spmf p* = *0* ⟷ *p* = *return-pmf None*
⟨*proof*⟩

lemma *weight-bind-spmf*: *weight-spmf* (*x* ⤜ *f*) = *lebesgue-integral* (*measure-spmf x*) (*weight-spmf* ∘ *f*)
⟨*proof*⟩

lemma *rel-spmf-weightD*: *rel-spmf A p q* ⟹ *weight-spmf p* = *weight-spmf q*
⟨*proof*⟩

lemma *rel-spmf-bij-betw*:
assumes *f*: *bij-betw f* (*set-spmf p*) (*set-spmf q*)
and *eq*: ⋀*x*. *x* ∈ *set-spmf p* ⟹ *spmf p x* = *spmf q* (*f x*)
shows *rel-spmf* (λ*x y*. *f x* = *y*) *p q*

⟨*proof*⟩

25.9 From density to spmfs

context fixes $f :: 'a \Rightarrow real$ **begin**

definition *embed-spmf* :: *'a spmf*
  where *embed-spmf* = *embed-pmf* (λx. *case x of None* ⇒ *1* − *enn2real* ($\int^+ x$. *ennreal* (*f x*) ∂*count-space UNIV*) | *Some x'* ⇒ *max 0* (*f x'*))

context
  assumes *prob*: ($\int^+ x$. *ennreal* (*f x*) ∂*count-space UNIV*) ≤ *1*
begin

lemma *nn-integral-embed-spmf-eq-1*:
  ($\int^+ x$. *ennreal* (*case x of None* ⇒ *1* − *enn2real* ($\int^+ x$. *ennreal* (*f x*) ∂*count-space UNIV*) | *Some x'* ⇒ *max 0* (*f x'*)) ∂*count-space UNIV*) = *1*
  (**is** *?lhs* = - **is** ($\int^+ x$. *?f x* ∂*?M*) = -)
⟨*proof*⟩

lemma *pmf-embed-spmf-None*: *pmf embed-spmf None* = *1* − *enn2real* ($\int^+ x$. *ennreal* (*f x*) ∂*count-space UNIV*)
⟨*proof*⟩

lemma *spmf-embed-spmf* [*simp*]: *spmf embed-spmf x* = *max 0* (*f x*)
  ⟨*proof*⟩

end

end

lemma *embed-spmf-K-0*[*simp*]: *embed-spmf* (λ-. *0*) = *return-pmf None*
  ⟨*proof*⟩

25.10 Ordering on spmfs

rel-pmf does not preserve a ccpo structure. Counterexample by Saheb-Djahromi: Take prefix order over *bool llist* and the set *range* (λn :: *nat*. *uniform* (*llist-n n*)) where *llist-n* is the set of all *llist*s of length *n* and *uniform* returns a uniform distribution over the given set. The set forms a chain in *ord-pmf lprefix*, but it has not an upper bound. Any upper bound may contain only infinite lists in its support because otherwise it is not greater than the *n+1*-st element in the chain where *n* is the length of the finite list. Moreover its support must contain all infinite lists, because otherwise there is a finite list all of whose finite extensions are not in the support - a contradiction to the upper bound property. Hence, the support is uncountable, but pmf's only have countable support.

However, if all chains in the ccpo are finite, then it should preserve the ccpo

structure.

abbreviation *ord-spmf* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a\ spmf \Rightarrow 'a\ spmf \Rightarrow bool$
  where *ord-spmf ord* $\equiv$ *rel-pmf* (*ord-option ord*)

locale *ord-spmf-syntax* **begin**
notation *ord-spmf* (**infix** $\langle\sqsubseteq_1\rangle$ *60*)
end

lemma *ord-spmf-map-spmf1*: *ord-spmf R* (*map-spmf f p*) = *ord-spmf* ($\lambda x.$ *R* (*f x*)) *p*
  ⟨*proof*⟩

lemma *ord-spmf-map-spmf2*: *ord-spmf R p* (*map-spmf f q*) = *ord-spmf* ($\lambda x\ y.$ *R x* (*f y*)) *p q*
  ⟨*proof*⟩

lemma *ord-spmf-map-spmf12*: *ord-spmf R* (*map-spmf f p*) (*map-spmf f q*) = *ord-spmf* ($\lambda x\ y.$ *R* (*f x*) (*f y*)) *p q*
  ⟨*proof*⟩

lemmas *ord-spmf-map-spmf* = *ord-spmf-map-spmf1 ord-spmf-map-spmf2 ord-spmf-map-spmf12*

context fixes *ord* :: $'a \Rightarrow 'a \Rightarrow bool$ (**structure**) **begin**
interpretation *ord-spmf-syntax* ⟨*proof*⟩

lemma *ord-spmfI*:
  ⟦ $\bigwedge x\ y.$ (*x*, *y*) ∈ *set-spmf pq* $\Longrightarrow$ *ord x y*; *map-spmf fst pq* = *p*; *map-spmf snd pq* = *q* ⟧
  $\Longrightarrow$ *p* $\sqsubseteq$ *q*
  ⟨*proof*⟩

lemma *ord-spmf-None* [*simp*]: *return-pmf None* $\sqsubseteq$ *x*
  ⟨*proof*⟩

lemma *ord-spmf-reflI*: ($\bigwedge x.$ *x* ∈ *set-spmf p* $\Longrightarrow$ *ord x x*) $\Longrightarrow$ *p* $\sqsubseteq$ *p*
  ⟨*proof*⟩

lemma *rel-spmf-inf*:
  assumes *p* $\sqsubseteq$ *q*
  and *q* $\sqsubseteq$ *p*
  and *refl*: *reflp ord*
  and *trans*: *transp ord*
  shows *rel-spmf* (*inf ord ord*$^{-1-1}$) *p q*
⟨*proof*⟩

end

lemma *ord-spmf-return-spmf2*: *ord-spmf R p* (*return-spmf y*) $\longleftrightarrow$ ($\forall x{\in}$*set-spmf p*. *R x y*)

⟨*proof*⟩

lemma *ord-spmf-mono*: ⟦ *ord-spmf A p q*; ⋀*x y*. *A x y* ⟹ *B x y* ⟧ ⟹ *ord-spmf B p q*
⟨*proof*⟩

lemma *ord-spmf-compp*: *ord-spmf* (*A OO B*) = *ord-spmf A OO ord-spmf B*
⟨*proof*⟩

lemma *ord-spmf-bindI*:
assumes *pq*: *ord-spmf R p q*
and *fg*: ⋀*x y*. *R x y* ⟹ *ord-spmf P* (*f x*) (*g y*)
shows *ord-spmf P* (*p* ⤜ *f*) (*q* ⤜ *g*)
⟨*proof*⟩

lemma *ord-spmf-bind-reflI*:
(⋀*x*. *x* ∈ *set-spmf p* ⟹ *ord-spmf R* (*f x*) (*g x*)) ⟹ *ord-spmf R* (*p* ⤜ *f*) (*p* ⤜ *g*)
⟨*proof*⟩

lemma *ord-pmf-increaseI*:
assumes *le*: ⋀*x*. *spmf p x* ≤ *spmf q x*
and *refl*: ⋀*x*. *x* ∈ *set-spmf p* ⟹ *R x x*
shows *ord-spmf R p q*
⟨*proof*⟩

lemma *ord-spmf-eq-leD*:
assumes *ord-spmf* (=) *p q*
shows *spmf p x* ≤ *spmf q x*
⟨*proof*⟩

lemma *ord-spmf-eqD-set-spmf*: *ord-spmf* (=) *p q* ⟹ *set-spmf p* ⊆ *set-spmf q*
⟨*proof*⟩

lemma *ord-spmf-eqD-emeasure*:
ord-spmf (=) *p q* ⟹ *emeasure* (*measure-spmf p*) *A* ≤ *emeasure* (*measure-spmf q*) *A*
⟨*proof*⟩

lemma *ord-spmf-eqD-measure-spmf*: *ord-spmf* (=) *p q* ⟹ *measure-spmf p* ≤ *measure-spmf q*
⟨*proof*⟩

25.11 CCPO structure for the flat ccpo *ord-option* (=)

context fixes *Y* :: *'a spmf set* **begin**

definition *lub-spmf* :: *'a spmf*
where *lub-spmf* = *embed-spmf* (λ*x*. *enn2real* (*SUP p* ∈ *Y*. *ennreal* (*spmf p x*)))

— We go through *ennreal* to have a sensible definition even if Y is empty.

lemma *lub-spmf-empty* [*simp*]: *SPMF.lub-spmf* {} = *return-pmf None*
⟨*proof*⟩

context assumes *chain*: *Complete-Partial-Order.chain* (*ord-spmf* (=)) *Y*
begin

lemma *chain-ord-spmf-eqD*: *Complete-Partial-Order.chain* (≤) ((λ*p x*. *ennreal* (*spmf p x*)) ‘ *Y*)
(**is** *Complete-Partial-Order.chain* - (*?f* ‘ -))
⟨*proof*⟩

lemma *ord-spmf-eq-pmf-None-eq*:
assumes *le*: *ord-spmf* (=) *p q*
and *None*: *pmf p None* = *pmf q None*
shows *p* = *q*
⟨*proof*⟩

lemma *ord-spmf-eqD-pmf-None*:
assumes *ord-spmf* (=) *x y*
shows *pmf x None* ≥ *pmf y None*
⟨*proof*⟩

Chains on *'a spmf* maintain countable support. Thanks to Johannes Hölzl for the proof idea.

lemma *spmf-chain-countable*: *countable* (⋃*p*∈*Y*. *set-spmf p*)
⟨*proof*⟩

lemma *lub-spmf-subprob*: (∫$^+$ *x*. (*SUP p* ∈ *Y*. *ennreal* (*spmf p x*)) ∂*count-space UNIV*) ≤ *1*
⟨*proof*⟩

lemma *spmf-lub-spmf*:
assumes *Y* ≠ {}
shows *spmf lub-spmf x* = (*SUP p* ∈ *Y*. *spmf p x*)
⟨*proof*⟩

lemma *ennreal-spmf-lub-spmf*: *Y* ≠ {} ⟹ *ennreal* (*spmf lub-spmf x*) = (*SUP p*∈*Y*. *ennreal* (*spmf p x*))
⟨*proof*⟩

lemma *lub-spmf-upper*:
assumes *p*: *p* ∈ *Y*
shows *ord-spmf* (=) *p lub-spmf*
⟨*proof*⟩

lemma *lub-spmf-least*:
assumes *z*: ⋀*x*. *x* ∈ *Y* ⟹ *ord-spmf* (=) *x z*

shows *ord-spmf* (=) *lub-spmf z*
⟨*proof*⟩

lemma *set-lub-spmf*: *set-spmf lub-spmf* = (⋃ *p*∈*Y*. *set-spmf p*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *emeasure-lub-spmf*:
assumes *Y*: *Y* ≠ {}
shows *emeasure* (*measure-spmf lub-spmf*) *A* = (*SUP y*∈*Y*. *emeasure* (*measure-spmf y*) *A*)
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *measure-lub-spmf*:
assumes *Y*: *Y* ≠ {}
shows *measure* (*measure-spmf lub-spmf*) *A* = (*SUP y*∈*Y*. *measure* (*measure-spmf y*) *A*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *weight-lub-spmf*:
assumes *Y*: *Y* ≠ {}
shows *weight-spmf lub-spmf* = (*SUP y*∈*Y*. *weight-spmf y*)
⟨*proof*⟩

lemma *measure-spmf-lub-spmf*:
assumes *Y*: *Y* ≠ {}
shows *measure-spmf lub-spmf* = (*SUP p*∈*Y*. *measure-spmf p*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

end

end

lemma *partial-function-definitions-spmf*: *partial-function-definitions* (*ord-spmf* (=)) *lub-spmf*
(**is** *partial-function-definitions ?R* -)
⟨*proof*⟩

lemma *ccpo-spmf*: *class.ccpo lub-spmf* (*ord-spmf* (=)) (*mk-less* (*ord-spmf* (=)))
⟨*proof*⟩

interpretation *spmf*: *partial-function-definitions ord-spmf* (=) *lub-spmf*
rewrites *lub-spmf* {} ≡ *return-pmf None*
⟨*proof*⟩

⟨*ML*⟩

declare *spmf.leq-refl*[*simp*]
declare *admissible-leI*[*OF ccpo-spmf*, *cont-intro*]

abbreviation *mono-spmf* ≡ *monotone* (*fun-ord* (*ord-spmf* (=))) (*ord-spmf* (=))

lemma *lub-spmf-const* [*simp*]: *lub-spmf* {*p*} = *p*
⟨*proof*⟩

lemma *bind-spmf-mono′*:
assumes *fg*: *ord-spmf* (=) *f g*
and *hk*: ⋀*x* :: *′a*. *ord-spmf* (=) (*h x*) (*k x*)
shows *ord-spmf* (=) (*f* ⤜ *h*) (*g* ⤜ *k*)
⟨*proof*⟩

lemma *bind-spmf-mono* [*partial-function-mono*]:
assumes *mf*: *mono-spmf B* **and** *mg*: ⋀*y*. *mono-spmf* (λ*f*. *C y f*)
shows *mono-spmf* (λ*f*. *bind-spmf* (*B f*) (λ*y*. *C y f*))
⟨*proof*⟩

lemma *monotone-bind-spmf1*: *monotone* (*ord-spmf* (=)) (*ord-spmf* (=)) (λ*y*. *bind-spmf y g*)
⟨*proof*⟩

lemma *monotone-bind-spmf2*:
assumes *g*: ⋀*x*. *monotone ord* (*ord-spmf* (=)) (λ*y*. *g y x*)
shows *monotone ord* (*ord-spmf* (=)) (λ*y*. *bind-spmf p* (*g y*))
⟨*proof*⟩

lemma *bind-lub-spmf*:
assumes *chain*: *Complete-Partial-Order.chain* (*ord-spmf* (=)) *Y*
shows *bind-spmf* (*lub-spmf Y*) *f* = *lub-spmf* ((λ*p*. *bind-spmf p f*) ‘ *Y*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *map-lub-spmf*:
Complete-Partial-Order.chain (*ord-spmf* (=)) *Y*
⟹ *map-spmf f* (*lub-spmf Y*) = *lub-spmf* (*map-spmf f* ‘ *Y*)
⟨*proof*⟩

lemma *mcont-bind-spmf1*: *mcont lub-spmf* (*ord-spmf* (=)) *lub-spmf* (*ord-spmf* (=)) (λ*y*. *bind-spmf y f*)
⟨*proof*⟩

lemma *bind-lub-spmf2*:
assumes *chain*: *Complete-Partial-Order.chain ord Y*
and *g*: ⋀*y*. *monotone ord* (*ord-spmf* (=)) (*g y*)
shows *bind-spmf x* (λ*y*. *lub-spmf* (*g y* ‘ *Y*)) = *lub-spmf* ((λ*p*. *bind-spmf x* (λ*y*. *g y p*)) ‘ *Y*)
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *mcont-bind-spmf* [*cont-intro*]:
assumes *f*: *mcont luba orda lub-spmf* (*ord-spmf* (=)) *f*
and *g*: ⋀*y*. *mcont luba orda lub-spmf* (*ord-spmf* (=)) (*g y*)
shows *mcont luba orda lub-spmf* (*ord-spmf* (=)) (λ*x*. *bind-spmf* (*f x*) (λ*y*. *g y x*))
⟨*proof*⟩

lemma *bind-pmf-mono* [*partial-function-mono*]:
(⋀*y*. *mono-spmf* (λ*f*. *C y f*)) ⟹ *mono-spmf* (λ*f*. *bind-pmf p* (λ*x*. *C x f*))
⟨*proof*⟩

lemma *map-spmf-mono* [*partial-function-mono*]: *mono-spmf B* ⟹ *mono-spmf* (λ*g*. *map-spmf f* (*B g*))
⟨*proof*⟩

lemma *mcont-map-spmf* [*cont-intro*]:
mcont luba orda lub-spmf (*ord-spmf* (=)) *g*
⟹ *mcont luba orda lub-spmf* (*ord-spmf* (=)) (λ*x*. *map-spmf f* (*g x*))
⟨*proof*⟩

lemma *monotone-set-spmf*: *monotone* (*ord-spmf* (=)) (⊆) *set-spmf*
⟨*proof*⟩

lemma *cont-set-spmf*: *cont lub-spmf* (*ord-spmf* (=)) *Union* (⊆) *set-spmf*
⟨*proof*⟩

lemma *mcont2mcont-set-spmf*[*THEN mcont2mcont*, *cont-intro*]:
shows *mcont-set-spmf*: *mcont lub-spmf* (*ord-spmf* (=)) *Union* (⊆) *set-spmf*
⟨*proof*⟩

lemma *monotone-spmf*: *monotone* (*ord-spmf* (=)) (≤) (λ*p*. *spmf p x*)
⟨*proof*⟩

lemma *cont-spmf*: *cont lub-spmf* (*ord-spmf* (=)) *Sup* (≤) (λ*p*. *spmf p x*)
⟨*proof*⟩

lemma *mcont-spmf*: *mcont lub-spmf* (*ord-spmf* (=)) *Sup* (≤) (λ*p*. *spmf p x*)
⟨*proof*⟩

lemma *cont-ennreal-spmf*: *cont lub-spmf* (*ord-spmf* (=)) *Sup* (≤) (λ*p*. *ennreal* (*spmf p x*))
⟨*proof*⟩

lemma *mcont2mcont-ennreal-spmf* [*THEN mcont2mcont*, *cont-intro*]:
shows *mcont-ennreal-spmf*: *mcont lub-spmf* (*ord-spmf* (=)) *Sup* (≤) (λ*p*. *ennreal* (*spmf p x*))
⟨*proof*⟩

lemma *nn-integral-map-spmf* [*simp*]: *nn-integral* (*measure-spmf* (*map-spmf f p*))

g = *nn-integral* (*measure-spmf p*) ($g \circ f$)
⟨*proof*⟩

25.11.1 Admissibility of *rel-spmf*

lemma *rel-spmf-measureD*:
assumes *rel-spmf R p q*
shows *measure* (*measure-spmf p*) $A \leq$ *measure* (*measure-spmf q*) {y. $\exists x{\in}A$. R x y} (**is** *?lhs* $\leq$ *?rhs*)
⟨*proof*⟩

locale *rel-spmf-characterisation* =
assumes *rel-pmf-measureI*:
$\bigwedge(R :: 'a$ *option* $\Rightarrow$ $'b$ *option* $\Rightarrow$ *bool*) *p q*.
($\bigwedge A$. *measure* (*measure-pmf p*) $A \leq$ *measure* (*measure-pmf q*) {y. $\exists x{\in}A$. R x y})
$\Longrightarrow$ *rel-pmf R p q*
— This assumption is shown to hold in general in the AFP entry *MFMC-Countable*.
begin

context fixes $R :: 'a \Rightarrow 'b \Rightarrow$ *bool* **begin**

lemma *rel-spmf-measureI*:
assumes *eq1*: $\bigwedge A$. *measure* (*measure-spmf p*) $A \leq$ *measure* (*measure-spmf q*) {y. $\exists x{\in}A$. R x y}
assumes *eq2*: *weight-spmf q* $\leq$ *weight-spmf p*
shows *rel-spmf R p q*
⟨*proof*⟩

lemma *admissible-rel-spmf*:
ccpo.admissible (*prod-lub lub-spmf lub-spmf*) (*rel-prod* (*ord-spmf* (=)) (*ord-spmf* (=))) (*case-prod* (*rel-spmf R*))
(**is** *ccpo.admissible ?lub ?ord ?P*)
⟨*proof*⟩

lemma *admissible-rel-spmf-mcont* [*cont-intro*]:
⟦ *mcont lub ord lub-spmf* (*ord-spmf* (=)) f; *mcont lub ord lub-spmf* (*ord-spmf* (=)) g ⟧
$\Longrightarrow$ *ccpo.admissible lub ord* (λx. *rel-spmf R* (f x) (g x))
⟨*proof*⟩

context includes *lifting-syntax*
begin

lemma *fixp-spmf-parametric'*:
assumes f: $\bigwedge x$. *monotone* (*ord-spmf* (=)) (*ord-spmf* (=)) F
and g: $\bigwedge x$. *monotone* (*ord-spmf* (=)) (*ord-spmf* (=)) G
and *param*: (*rel-spmf R* ===> *rel-spmf R*) F G
shows (*rel-spmf R*) (*ccpo.fixp lub-spmf* (*ord-spmf* (=)) F) (*ccpo.fixp lub-spmf*

(*ord-spmf* (=)) *G*)
⟨*proof*⟩

lemma *fixp-spmf-parametric*:
assumes *f*: ⋀*x*. *mono-spmf* (λ*f*. *F f x*)
and *g*: ⋀*x*. *mono-spmf* (λ*f*. *G f x*)
and *param*: ((*A* ===> *rel-spmf R*) ===> *A* ===> *rel-spmf R*) *F G*
shows (*A* ===> *rel-spmf R*) (*spmf.fixp-fun F*) (*spmf.fixp-fun G*)
⟨*proof*⟩

end

end

end

25.12 Restrictions on spmfs

definition *restrict-spmf* :: *'a spmf* ⇒ *'a set* ⇒ *'a spmf* (**infixl** ‹⟃› *110*)
where *p* ⟃ *A* = *map-pmf* (λ*x*. *x* ⤜ (λ*y*. *if y* ∈ *A then Some y else None*)) *p*

lemma *set-restrict-spmf* [*simp*]: *set-spmf* (*p* ⟃ *A*) = *set-spmf p* ∩ *A*
⟨*proof*⟩

lemma *restrict-map-spmf*: *map-spmf f p* ⟃ *A* = *map-spmf f* (*p* ⟃ (*f* −‘ *A*))
⟨*proof*⟩

lemma *restrict-restrict-spmf* [*simp*]: *p* ⟃ *A* ⟃ *B* = *p* ⟃ (*A* ∩ *B*)
⟨*proof*⟩

lemma *restrict-spmf-empty* [*simp*]: *p* ⟃ {} = *return-pmf None*
⟨*proof*⟩

lemma *restrict-spmf-UNIV* [*simp*]: *p* ⟃ *UNIV* = *p*
⟨*proof*⟩

lemma *spmf-restrict-spmf-outside* [*simp*]: *x* ∉ *A* ⟹ *spmf* (*p* ⟃ *A*) *x* = *0*
⟨*proof*⟩

lemma *emeasure-restrict-spmf* [*simp*]: *emeasure* (*measure-spmf* (*p* ⟃ *A*)) *X* = *emeasure* (*measure-spmf p*) (*X* ∩ *A*)
⟨*proof*⟩

lemma *measure-restrict-spmf* [*simp*]:
measure (*measure-spmf* (*p* ⟃ *A*)) *X* = *measure* (*measure-spmf p*) (*X* ∩ *A*)
⟨*proof*⟩

lemma *spmf-restrict-spmf*: *spmf* (*p* ⟃ *A*) *x* = (*if x* ∈ *A then spmf p x else 0*)
⟨*proof*⟩

lemma *spmf-restrict-spmf-inside* [*simp*]: $x \in A \Longrightarrow spmf\ (p \restriction A)\ x = spmf\ p\ x$
⟨*proof*⟩

lemma *pmf-restrict-spmf-None*: $pmf\ (p \restriction A)\ None = pmf\ p\ None + measure\ (measure\text{-}spmf\ p)\ (-A)$
⟨*proof*⟩

lemma *restrict-spmf-trivial*: $(\bigwedge x.\ x \in set\text{-}spmf\ p \Longrightarrow x \in A) \Longrightarrow p \restriction A = p$
⟨*proof*⟩

lemma *restrict-spmf-trivial'*: $set\text{-}spmf\ p \subseteq A \Longrightarrow p \restriction A = p$
⟨*proof*⟩

lemma *restrict-return-spmf*: $return\text{-}spmf\ x \restriction A = (if\ x \in A\ then\ return\text{-}spmf\ x\ else\ return\text{-}pmf\ None)$
⟨*proof*⟩

lemma *restrict-return-spmf-inside* [*simp*]: $x \in A \Longrightarrow return\text{-}spmf\ x \restriction A = return\text{-}spmf\ x$
⟨*proof*⟩

lemma *restrict-return-spmf-outside* [*simp*]: $x \notin A \Longrightarrow return\text{-}spmf\ x \restriction A = return\text{-}pmf\ None$
⟨*proof*⟩

lemma *restrict-spmf-return-pmf-None* [*simp*]: $return\text{-}pmf\ None \restriction A = return\text{-}pmf\ None$
⟨*proof*⟩

lemma *restrict-bind-pmf*: $bind\text{-}pmf\ p\ g \restriction A = p \gg\!= (\lambda x.\ g\ x \restriction A)$
⟨*proof*⟩

lemma *restrict-bind-spmf*: $bind\text{-}spmf\ p\ g \restriction A = p \gg\!= (\lambda x.\ g\ x \restriction A)$
⟨*proof*⟩

lemma *bind-restrict-pmf*: $bind\text{-}pmf\ (p \restriction A)\ g = p \gg\!= (\lambda x.\ if\ x \in Some\ `\ A\ then\ g\ x\ else\ g\ None)$
⟨*proof*⟩

lemma *bind-restrict-spmf*: $bind\text{-}spmf\ (p \restriction A)\ g = p \gg\!= (\lambda x.\ if\ x \in A\ then\ g\ x\ else\ return\text{-}pmf\ None)$
⟨*proof*⟩

lemma *spmf-map-restrict*: $spmf\ (map\text{-}spmf\ fst\ (p \restriction (snd -` \{y\})))\ x = spmf\ p\ (x, y)$
⟨*proof*⟩

lemma *measure-eqI-restrict-spmf*:

assumes *rel-spmf R* (*restrict-spmf p A*) (*restrict-spmf q B*)
shows *measure* (*measure-spmf p*) *A* = *measure* (*measure-spmf q*) *B*
⟨*proof*⟩

25.13 Subprobability distributions of sets

definition *spmf-of-set* :: *'a set* ⇒ *'a spmf*
where
spmf-of-set A = (*if finite A* ∧ *A* ≠ {} *then spmf-of-pmf* (*pmf-of-set A*) *else return-pmf None*)

lemma *spmf-of-set*: *spmf* (*spmf-of-set A*) *x* = *indicator A x* / *card A*
⟨*proof*⟩

lemma *pmf-spmf-of-set-None* [*simp*]: *pmf* (*spmf-of-set A*) *None* = *indicator* {*A*. *infinite A* ∨ *A* = {}} *A*
⟨*proof*⟩

lemma *set-spmf-of-set*: *set-spmf* (*spmf-of-set A*) = (*if finite A then A else* {})
⟨*proof*⟩

lemma *set-spmf-of-set-finite* [*simp*]: *finite A* ⟹ *set-spmf* (*spmf-of-set A*) = *A*
⟨*proof*⟩

lemma *spmf-of-set-singleton*: *spmf-of-set* {*x*} = *return-spmf x*
⟨*proof*⟩

lemma *map-spmf-of-set-inj-on* [*simp*]:
inj-on f A ⟹ *map-spmf f* (*spmf-of-set A*) = *spmf-of-set* (*f* ‘ *A*)
⟨*proof*⟩

lemma *spmf-of-pmf-pmf-of-set* [*simp*]:
⟦ *finite A*; *A* ≠ {} ⟧ ⟹ *spmf-of-pmf* (*pmf-of-set A*) = *spmf-of-set A*
⟨*proof*⟩

lemma *weight-spmf-of-set*:
weight-spmf (*spmf-of-set A*) = (*if finite A* ∧ *A* ≠ {} *then 1 else 0*)
⟨*proof*⟩

lemma *weight-spmf-of-set-finite* [*simp*]: ⟦ *finite A*; *A* ≠ {} ⟧ ⟹ *weight-spmf* (*spmf-of-set A*) = *1*
⟨*proof*⟩

lemma *weight-spmf-of-set-infinite* [*simp*]: *infinite A* ⟹ *weight-spmf* (*spmf-of-set A*) = *0*
⟨*proof*⟩

lemma *measure-spmf-spmf-of-set*:
measure-spmf (*spmf-of-set A*) = (*if finite A* ∧ *A* ≠ {} *then measure-pmf* (*pmf-of-set*

A) *else null-measure* (*count-space UNIV*))
⟨*proof*⟩

lemma *emeasure-spmf-of-set*:
emeasure (*measure-spmf* (*spmf-of-set S*)) *A* = *card* (*S* ∩ *A*) / *card S*
⟨*proof*⟩

lemma *measure-spmf-of-set*:
measure (*measure-spmf* (*spmf-of-set S*)) *A* = *card* (*S* ∩ *A*) / *card S*
⟨*proof*⟩

lemma *nn-integral-spmf-of-set*: *nn-integral* (*measure-spmf* (*spmf-of-set A*)) *f* = *sum f A* / *card A*
⟨*proof*⟩

lemma *integral-spmf-of-set*: $integral^L$ (*measure-spmf* (*spmf-of-set A*)) *f* = *sum f A* / *card A*
⟨*proof*⟩

notepad begin — *pmf-of-set* is not fully parametric.
⟨*proof*⟩
end

lemma *rel-pmf-of-set-bij*:
assumes *f*: *bij-betw f A B*
and *A*: *A* ≠ {} *finite A*
and *B*: *B* ≠ {} *finite B*
and *R*: ⋀*x*. *x* ∈ *A* ⟹ *R x* (*f x*)
shows *rel-pmf R* (*pmf-of-set A*) (*pmf-of-set B*)
⟨*proof*⟩

lemma *rel-spmf-of-set-bij*:
assumes *f*: *bij-betw f A B*
and *R*: ⋀*x*. *x* ∈ *A* ⟹ *R x* (*f x*)
shows *rel-spmf R* (*spmf-of-set A*) (*spmf-of-set B*)
⟨*proof*⟩

context includes *lifting-syntax*
begin

lemma *rel-spmf-of-set*:
assumes *bi-unique R*
shows (*rel-set R* ===> *rel-spmf R*) *spmf-of-set spmf-of-set*
⟨*proof*⟩

end

lemma *map-mem-spmf-of-set*:
assumes *finite B B* ≠ {}

shows *map-spmf* (λ*x*. *x* ∈ *A*) (*spmf-of-set B*) = *spmf-of-pmf* (*bernoulli-pmf* (*card* (*A* ∩ *B*) / *card B*))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

abbreviation *coin-spmf* :: *bool spmf*
where *coin-spmf* ≡ *spmf-of-set UNIV*

lemma *map-eq-const-coin-spmf*: *map-spmf* ((=) *c*) *coin-spmf* = *coin-spmf*
⟨*proof*⟩

lemma *bind-coin-spmf-eq-const*: *coin-spmf* ⤜ (λ*x* :: *bool*. *return-spmf* (*b* = *x*)) = *coin-spmf*
⟨*proof*⟩

lemma *bind-coin-spmf-eq-const'*: *coin-spmf* ⤜ (λ*x* :: *bool*. *return-spmf* (*x* = *b*)) = *coin-spmf*
⟨*proof*⟩

25.14 Losslessness

definition *lossless-spmf* :: *'a spmf* ⇒ *bool*
where *lossless-spmf p* ⟷ *weight-spmf p* = *1*

lemma *lossless-iff-pmf-None*: *lossless-spmf p* ⟷ *pmf p None* = *0*
⟨*proof*⟩

lemma *lossless-return-spmf* [*iff*]: *lossless-spmf* (*return-spmf x*)
⟨*proof*⟩

lemma *lossless-return-pmf-None* [*iff*]: ¬ *lossless-spmf* (*return-pmf None*)
⟨*proof*⟩

lemma *lossless-map-spmf* [*simp*]: *lossless-spmf* (*map-spmf f p*) ⟷ *lossless-spmf p*
⟨*proof*⟩

lemma *lossless-bind-spmf* [*simp*]:
lossless-spmf (*p* ⤜ *f*) ⟷ *lossless-spmf p* ∧ (∀ *x*∈*set-spmf p*. *lossless-spmf* (*f x*))
⟨*proof*⟩

lemma *lossless-weight-spmfD*: *lossless-spmf p* ⟹ *weight-spmf p* = *1*
⟨*proof*⟩

lemma *lossless-iff-set-pmf-None*:
lossless-spmf p ⟷ *None* ∉ *set-pmf p*
⟨*proof*⟩

lemma *lossless-spmf-of-set* [*simp*]: *lossless-spmf* (*spmf-of-set A*) ⟷ *finite A* ∧ *A* ≠ {}
⟨*proof*⟩

lemma *lossless-spmf-spmf-of-spmf* [*simp*]: *lossless-spmf* (*spmf-of-pmf p*)
⟨*proof*⟩

lemma *lossless-spmf-bind-pmf* [*simp*]:
lossless-spmf (*bind-pmf p f*) ⟷ (∀ *x*∈*set-pmf p*. *lossless-spmf* (*f x*))
⟨*proof*⟩

lemma *lossless-spmf-conv-spmf-of-pmf*: *lossless-spmf p* ⟷ (∃ *p'*. *p* = *spmf-of-pmf p'*)
⟨*proof*⟩

lemma *spmf-False-conv-True*: *lossless-spmf p* ⟹ *spmf p False* = *1* − *spmf p True*
⟨*proof*⟩

lemma *spmf-True-conv-False*: *lossless-spmf p* ⟹ *spmf p True* = *1* − *spmf p False*
⟨*proof*⟩

lemma *bind-eq-return-spmf*:
bind-spmf p f = *return-spmf x* ⟷ (∀ *y*∈*set-spmf p*. *f y* = *return-spmf x*) ∧ *lossless-spmf p*
⟨*proof*⟩

lemma *rel-spmf-return-spmf2*:
rel-spmf R p (*return-spmf x*) ⟷ *lossless-spmf p* ∧ (∀ *a*∈*set-spmf p*. *R a x*)
⟨*proof*⟩

lemma *rel-spmf-return-spmf1*:
rel-spmf R (*return-spmf x*) *p* ⟷ *lossless-spmf p* ∧ (∀ *a*∈*set-spmf p*. *R x a*)
⟨*proof*⟩

lemma *rel-spmf-bindI1*:
assumes *f*: ⋀*x*. *x* ∈ *set-spmf p* ⟹ *rel-spmf R* (*f x*) *q*
and *p*: *lossless-spmf p*
shows *rel-spmf R* (*bind-spmf p f*) *q*
⟨*proof*⟩

lemma *rel-spmf-bindI2*:
⟦ ⋀*x*. *x* ∈ *set-spmf q* ⟹ *rel-spmf R p* (*f x*); *lossless-spmf q* ⟧
⟹ *rel-spmf R p* (*bind-spmf q f*)
⟨*proof*⟩

25.15 Scaling

definition *scale-spmf* :: *real* ⇒ *'a spmf* ⇒ *'a spmf*
where

*scale-spmf r p = embed-spmf (λx. min (inverse (weight-spmf p)) (max 0 r) * spmf p x)*

lemma *scale-spmf-le-1*:
*(∫⁺ x. min (inverse (weight-spmf p)) (max 0 r) * spmf p x ∂count-space UNIV) ≤ 1* (**is** *?lhs ≤ -*)
⟨*proof*⟩

lemma *spmf-scale-spmf*: *spmf (scale-spmf r p) x = max 0 (min (inverse (weight-spmf p)) r) * spmf p x* (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *real-inverse-le-1-iff*: **fixes** *x :: real*
shows *⟦ 0 ≤ x; x ≤ 1 ⟧ ⟹ 1 / x ≤ 1 ⟷ x = 1 ∨ x = 0*
⟨*proof*⟩

lemma *spmf-scale-spmf'*: *r ≤ 1 ⟹ spmf (scale-spmf r p) x = max 0 r * spmf p x*
⟨*proof*⟩

lemma *scale-spmf-neg*: *r ≤ 0 ⟹ scale-spmf r p = return-pmf None*
⟨*proof*⟩

lemma *scale-spmf-return-None* [*simp*]: *scale-spmf r (return-pmf None) = return-pmf None*
⟨*proof*⟩

lemma *scale-spmf-conv-bind-bernoulli*:
assumes *r ≤ 1*
shows *scale-spmf r p = bind-pmf (bernoulli-pmf r) (λb. if b then p else return-pmf None)* (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *nn-integral-spmf*: *(∫⁺ x. spmf p x ∂count-space A) = emeasure (measure-spmf p) A*
⟨*proof*⟩

lemma *measure-spmf-scale-spmf*: *measure-spmf (scale-spmf r p) = scale-measure (min (inverse (weight-spmf p)) r) (measure-spmf p)*
⟨*proof*⟩

lemma *measure-spmf-scale-spmf'*:
assumes *r ≤ 1*
shows *measure-spmf (scale-spmf r p) = scale-measure r (measure-spmf p)*
⟨*proof*⟩

lemma *scale-spmf-1* [*simp*]: *scale-spmf 1 p = p*
⟨*proof*⟩

lemma *scale-spmf-0* [*simp*]: *scale-spmf 0 p = return-pmf None*
⟨*proof*⟩

lemma *bind-scale-spmf*:
assumes *r*: *r ≤ 1*
shows *bind-spmf* (*scale-spmf r p*) *f* = *bind-spmf p* (λ*x*. *scale-spmf r* (*f x*))
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *scale-bind-spmf*:
assumes *r ≤ 1*
shows *scale-spmf r* (*bind-spmf p f*) = *bind-spmf p* (λ*x*. *scale-spmf r* (*f x*))
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *bind-spmf-const*: *bind-spmf p* (λ*x*. *q*) = *scale-spmf* (*weight-spmf p*) *q* (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *map-scale-spmf*: *map-spmf f* (*scale-spmf r p*) = *scale-spmf r* (*map-spmf f p*) (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *set-scale-spmf*: *set-spmf* (*scale-spmf r p*) = (*if r > 0 then set-spmf p else* {})
⟨*proof*⟩

lemma *set-scale-spmf′* [*simp*]: *0 < r ⟹ set-spmf* (*scale-spmf r p*) = *set-spmf p*
⟨*proof*⟩

lemma *rel-spmf-scaleI*:
assumes *r > 0 ⟹ rel-spmf A p q*
shows *rel-spmf A* (*scale-spmf r p*) (*scale-spmf r q*)
⟨*proof*⟩

lemma *weight-scale-spmf*: *weight-spmf* (*scale-spmf r p*) = *min 1* (*max 0 r ∗ weight-spmf p*)
⟨*proof*⟩

lemma *weight-scale-spmf′* [*simp*]:
⟦ *0 ≤ r*; *r ≤ 1* ⟧ ⟹ *weight-spmf* (*scale-spmf r p*) = *r ∗ weight-spmf p*
⟨*proof*⟩

lemma *pmf-scale-spmf-None*:
pmf (*scale-spmf k p*) *None* = *1 − min 1* (*max 0 k ∗* (*1 − pmf p None*))
⟨*proof*⟩

lemma *scale-scale-spmf*:
scale-spmf r (*scale-spmf r′ p*) = *scale-spmf* (*r ∗ max 0* (*min* (*inverse* (*weight-spmf*

$p))\ r'))\ p$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *scale-scale-spmf'* [*simp*]:
assumes $0 \leq r\ r \leq 1\ 0 \leq r'\ r' \leq 1$
shows *scale-spmf r* (*scale-spmf r' p*) = *scale-spmf* (*r* ∗ *r'*) *p*
⟨*proof*⟩

lemma *scale-spmf-eq-same*: *scale-spmf r p* = *p* ⟷ *weight-spmf p* = *0* ∨ *r* = *1* ∨ *r* ≥ *1* ∧ *weight-spmf p* = *1*
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *map-const-spmf-of-set*:
⟦ *finite A*; *A* ≠ {} ⟧ ⟹ *map-spmf* (λ-. *c*) (*spmf-of-set A*) = *return-spmf c*
⟨*proof*⟩

25.16 Conditional spmfs

lemma *set-pmf-Int-Some*: *set-pmf p* ∩ *Some* ‘ *A* = {} ⟷ *set-spmf p* ∩ *A* = {}
⟨*proof*⟩

lemma *measure-spmf-zero-iff*: *measure* (*measure-spmf p*) *A* = *0* ⟷ *set-spmf p* ∩ *A* = {}
⟨*proof*⟩

definition *cond-spmf* :: *'a spmf* ⇒ *'a set* ⇒ *'a spmf*
where *cond-spmf p A* = (*if set-spmf p* ∩ *A* = {} *then return-pmf None else cond-pmf p* (*Some* ‘ *A*))

lemma *set-cond-spmf* [*simp*]: *set-spmf* (*cond-spmf p A*) = *set-spmf p* ∩ *A*
⟨*proof*⟩

lemma *cond-map-spmf* [*simp*]: *cond-spmf* (*map-spmf f p*) *A* = *map-spmf f* (*cond-spmf p* (*f* −‘ *A*))
⟨*proof*⟩

lemma *spmf-cond-spmf* [*simp*]:
spmf (*cond-spmf p A*) *x* = (*if x* ∈ *A then spmf p x* / *measure* (*measure-spmf p*) *A else 0*)
⟨*proof*⟩

lemma *bind-eq-return-pmf-None*:
bind-spmf p f = *return-pmf None* ⟷ (∀ *x*∈*set-spmf p*. *f x* = *return-pmf None*)
⟨*proof*⟩

lemma *return-pmf-None-eq-bind*:
return-pmf None = *bind-spmf p f* ⟷ (∀ *x*∈*set-spmf p*. *f x* = *return-pmf None*)

⟨*proof*⟩

25.17 Product spmf

definition *pair-spmf* :: *'a spmf ⇒ 'b spmf ⇒ ('a × 'b) spmf*
where *pair-spmf p q = bind-pmf (pair-pmf p q) (λxy. case xy of (Some x, Some y) ⇒ return-spmf (x, y) | - ⇒ return-pmf None)*

lemma *map-fst-pair-spmf* [*simp*]: *map-spmf fst (pair-spmf p q) = scale-spmf (weight-spmf q) p*
⟨*proof*⟩

lemma *map-snd-pair-spmf* [*simp*]: *map-spmf snd (pair-spmf p q) = scale-spmf (weight-spmf p) q*
⟨*proof*⟩

lemma *set-pair-spmf* [*simp*]: *set-spmf (pair-spmf p q) = set-spmf p × set-spmf q*
⟨*proof*⟩

lemma *spmf-pair* [*simp*]: *spmf (pair-spmf p q) (x, y) = spmf p x * spmf q y* (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *pair-map-spmf2*: *pair-spmf p (map-spmf f q) = map-spmf (apsnd f) (pair-spmf p q)*
⟨*proof*⟩

lemma *pair-map-spmf1*: *pair-spmf (map-spmf f p) q = map-spmf (apfst f) (pair-spmf p q)*
⟨*proof*⟩

lemma *pair-map-spmf*: *pair-spmf (map-spmf f p) (map-spmf g q) = map-spmf (map-prod f g) (pair-spmf p q)*
⟨*proof*⟩

lemma *pair-spmf-alt-def*: *pair-spmf p q = bind-spmf p (λx. bind-spmf q (λy. return-spmf (x, y)))*
⟨*proof*⟩

lemma *weight-pair-spmf* [*simp*]: *weight-spmf (pair-spmf p q) = weight-spmf p * weight-spmf q*
⟨*proof*⟩

lemma *pair-scale-spmf1*:
r ≤ 1 ⟹ pair-spmf (scale-spmf r p) q = scale-spmf r (pair-spmf p q)
⟨*proof*⟩

lemma *pair-scale-spmf2*:
r ≤ 1 ⟹ pair-spmf p (scale-spmf r q) = scale-spmf r (pair-spmf p q)

⟨*proof*⟩

lemma *pair-spmf-return-None1* [*simp*]: *pair-spmf* (*return-pmf None*) *p* = *return-pmf None*
⟨*proof*⟩

lemma *pair-spmf-return-None2* [*simp*]: *pair-spmf p* (*return-pmf None*) = *return-pmf None*
⟨*proof*⟩

lemma *pair-spmf-return-spmf1*: *pair-spmf* (*return-spmf x*) *q* = *map-spmf* (*Pair x*) *q*
⟨*proof*⟩

lemma *pair-spmf-return-spmf2*: *pair-spmf p* (*return-spmf y*) = *map-spmf* (λ*x*. (*x*, *y*)) *p*
⟨*proof*⟩

lemma *pair-spmf-return-spmf* [*simp*]: *pair-spmf* (*return-spmf x*) (*return-spmf y*) = *return-spmf* (*x*, *y*)
⟨*proof*⟩

lemma *rel-pair-spmf-prod*:
rel-spmf (*rel-prod A B*) (*pair-spmf p q*) (*pair-spmf p′ q′*) ⟷
rel-spmf A (*scale-spmf* (*weight-spmf q*) *p*) (*scale-spmf* (*weight-spmf q′*) *p′*) ∧
rel-spmf B (*scale-spmf* (*weight-spmf p*) *q*) (*scale-spmf* (*weight-spmf p′*) *q′*)
(**is** *?lhs* ⟷ *?rhs* **is** - ⟷ *?A* ∧ *?B* **is** - ⟷ *rel-spmf* - *?p ?p′* ∧ *rel-spmf* - *?q ?q′*)
⟨*proof*⟩

lemma *pair-pair-spmf*:
pair-spmf (*pair-spmf p q*) *r* = *map-spmf* (λ(*x*, (*y*, *z*)). ((*x*, *y*), *z*)) (*pair-spmf p* (*pair-spmf q r*))
⟨*proof*⟩

lemma *pair-commute-spmf*:
pair-spmf p q = *map-spmf* (λ(*y*, *x*). (*x*, *y*)) (*pair-spmf q p*)
⟨*proof*⟩

25.18 Assertions

definition *assert-spmf* :: *bool* ⇒ *unit spmf*
where *assert-spmf b* = (*if b then return-spmf* () *else return-pmf None*)

lemma *assert-spmf-simps* [*simp*]:
assert-spmf True = *return-spmf* ()
assert-spmf False = *return-pmf None*
⟨*proof*⟩

lemma *in-set-assert-spmf* [*simp*]: $x \in$ *set-spmf* (*assert-spmf p*) $\longleftrightarrow$ *p*
⟨*proof*⟩

lemma *set-spmf-assert-spmf-eq-empty* [*simp*]: *set-spmf* (*assert-spmf b*) = {} $\longleftrightarrow$ ¬ *b*
⟨*proof*⟩

lemma *lossless-assert-spmf* [*iff*]: *lossless-spmf* (*assert-spmf b*) $\longleftrightarrow$ *b*
⟨*proof*⟩

25.19 Try

definition *try-spmf* :: *'a spmf* ⇒ *'a spmf* ⇒ *'a spmf*
(‹(‹*open-block notation=‹mixfix try-spmf›› TRY - ELSE -*)› [*0,60*] *59*)
where *TRY p ELSE q* = *bind-pmf p* (λ*x*. *case x of None* ⇒ *q* | *Some y* ⇒ *return-spmf y*)

lemma *try-spmf-lossless* [*simp*]:
assumes *lossless-spmf p*
shows *TRY p ELSE q* = *p*
⟨*proof*⟩

lemma *try-spmf-return-spmf1*: *TRY return-spmf x ELSE q* = *return-spmf x*
⟨*proof*⟩

lemma *try-spmf-return-None* [*simp*]: *TRY return-pmf None ELSE q* = *q*
⟨*proof*⟩

lemma *try-spmf-return-pmf-None2* [*simp*]: *TRY p ELSE return-pmf None* = *p*
⟨*proof*⟩

lemma *map-try-spmf*: *map-spmf f* (*try-spmf p q*) = *try-spmf* (*map-spmf f p*) (*map-spmf f q*)
⟨*proof*⟩

lemma *try-spmf-bind-pmf*: *TRY* (*bind-pmf p f*) *ELSE q* = *bind-pmf p* (λ*x*. *TRY* (*f x*) *ELSE q*)
⟨*proof*⟩

lemma *try-spmf-bind-spmf-lossless*:
lossless-spmf p ⟹ *TRY* (*bind-spmf p f*) *ELSE q* = *bind-spmf p* (λ*x*. *TRY* (*f x*) *ELSE q*)
⟨*proof*⟩

lemma *try-spmf-bind-out*:
lossless-spmf p ⟹ *bind-spmf p* (λ*x*. *TRY* (*f x*) *ELSE q*) = *TRY* (*bind-spmf p f*) *ELSE q*
⟨*proof*⟩

lemma *lossless-try-spmf* [*simp*]:
lossless-spmf (*TRY p ELSE q*) ⟷ *lossless-spmf p* ∨ *lossless-spmf q*
⟨*proof*⟩

context includes *lifting-syntax*
begin

lemma *try-spmf-parametric* [*transfer-rule*]:
(*rel-spmf A* ===> *rel-spmf A* ===> *rel-spmf A*) *try-spmf try-spmf*
⟨*proof*⟩

end

lemma *try-spmf-cong*:
⟦ *p* = *p'*; ¬ *lossless-spmf p'* ⟹ *q* = *q'* ⟧ ⟹ *TRY p ELSE q* = *TRY p' ELSE q'*
⟨*proof*⟩

lemma *rel-spmf-try-spmf*:
⟦ *rel-spmf R p p'*; ¬ *lossless-spmf p'* ⟹ *rel-spmf R q q'* ⟧
⟹ *rel-spmf R* (*TRY p ELSE q*) (*TRY p' ELSE q'*)
⟨*proof*⟩

lemma *spmf-try-spmf*:
spmf (*TRY p ELSE q*) *x* = *spmf p x* + *pmf p None* ∗ *spmf q x*
⟨*proof*⟩

lemma *try-scale-spmf-same* [*simp*]: *lossless-spmf p* ⟹ *TRY scale-spmf k p ELSE p* = *p*
⟨*proof*⟩

lemma *pmf-try-spmf-None* [*simp*]: *pmf* (*TRY p ELSE q*) *None* = *pmf p None* ∗ *pmf q None* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *try-bind-spmf-lossless2*:
lossless-spmf q ⟹ *TRY* (*bind-spmf p f*) *ELSE q* = *TRY* (*p* ⤜ (λ*x*. *TRY* (*f x*) *ELSE q*)) *ELSE q*
⟨*proof*⟩

lemma *try-bind-spmf-lossless2'*:
fixes *f* :: *'a* ⇒ *'b spmf* **shows**
⟦ *NO-MATCH* (λ*x* :: *'a*. *try-spmf* (*g x* :: *'b spmf*) (*h x*)) *f*; *lossless-spmf q* ⟧
⟹ *TRY* (*bind-spmf p f*) *ELSE q* = *TRY* (*p* ⤜ (λ*x* :: *'a*. *TRY* (*f x*) *ELSE q*)) *ELSE q*
⟨*proof*⟩

lemma *try-bind-assert-spmf*:
TRY (*assert-spmf b* ⤜ *f*) *ELSE q* = (*if b then TRY* (*f* ()) *ELSE q else q*)
⟨*proof*⟩

25.20 Miscellaneous

lemma assumes *rel-spmf* ($\lambda x\ y.\ bad1\ x = bad2\ y \wedge (\neg\ bad2\ y \longrightarrow A\ x \longleftrightarrow B\ y)$) *p q* (**is** *rel-spmf ?A - -*)
shows *fundamental-lemma-bad*: *measure* (*measure-spmf p*) {*x*. *bad1 x*} = *measure* (*measure-spmf q*) {*y*. *bad2 y*} (**is** *?bad*)
and *fundamental-lemma*: |*measure* (*measure-spmf p*) {*x*. *A x*} − *measure* (*measure-spmf q*) {*y*. *B y*}| ≤
measure (*measure-spmf p*) {*x*. *bad1 x*} (**is** *?fundamental*)
⟨*proof*⟩

end

26 Indexed products of PMFs

theory *Product-PMF*
imports *Probability-Mass-Function Independent-Family*
begin

Conflicting notation from *HOL−Analysis.Infinite-Sum*

no-notation *Infinite-Sum.abs-summable-on* (**infixr** ‹*abs′-summable′-on*› *46*)

26.1 Preliminaries

lemma *pmf-expectation-eq-infsetsum*: *measure-pmf.expectation p f* = *infsetsum* (λ*x*. *pmf p x* ∗ *f x*) *UNIV*
⟨*proof*⟩

lemma *measure-pmf-prob-product*:
assumes *countable A countable B*
shows *measure-pmf.prob* (*pair-pmf M N*) (*A* × *B*) = *measure-pmf.prob M A* ∗ *measure-pmf.prob N B*
⟨*proof*⟩

26.2 Definition

In analogy to Pi_M, we define an indexed product of PMFs. In the literature, this is typically called taking a vector of independent random variables. Note that the components do not have to be identically distributed.

The operation takes an explicit index set A and a function f that maps each element from A to a PMF and defines the product measure $\bigotimes_{i\in A} f(i)$, which is represented as a (′*a* ⇒ ′*b*) *pmf*.

Note that unlike Pi_M, this only works for *finite* index sets. It could be extended to countable sets and beyond, but the construction becomes somewhat more involved.

definition *Pi-pmf* :: ′*a set* ⇒ ′*b* ⇒ (′*a* ⇒ ′*b pmf*) ⇒ (′*a* ⇒ ′*b*) *pmf* **where**
Pi-pmf A dflt p =

embed-pmf ($\lambda f.$ *if* ($\forall x.\ x \notin A \longrightarrow f\ x = dflt$) *then* $\prod x{\in}A.$ *pmf* ($p\ x$) ($f\ x$) *else 0*)

A technical subtlety that needs to be addressed is this: Intuitively, the functions in the support of a product distribution have domain A. However, since HOL is a total logic, these functions must still return *some* value for inputs outside A. The product measure Pi_M simply lets these functions return *undefined* in these cases. We chose a different solution here, which is to supply a default value *dflt* that is returned in these cases.

As one possible application, one could model the result of n different independent coin tosses as *Pi-pmf* $\{0..<n\}$ *False* (λ-. *bernoulli-pmf* (*1 / 2*)). This returns a function of type $nat \Rightarrow bool$ that maps every natural number below n to the result of the corresponding coin toss, and every other natural number to *False*.

lemma *pmf-Pi*:
assumes A: *finite A*
shows *pmf* (*Pi-pmf A dflt p*) f =
(*if* ($\forall x.\ x \notin A \longrightarrow f\ x = dflt$) *then* $\prod x{\in}A.$ *pmf* ($p\ x$) ($f\ x$) *else 0*)
⟨*proof*⟩

lemma *Pi-pmf-cong*:
assumes $A = A'$ $dflt = dflt'$ $\bigwedge x.\ x \in A \Longrightarrow f\ x = f'\ x$
shows *Pi-pmf A dflt f* = *Pi-pmf A' dflt' f'*
⟨*proof*⟩

lemma *pmf-Pi'*:
assumes *finite A* $\bigwedge x.\ x \notin A \Longrightarrow f\ x = dflt$
shows *pmf* (*Pi-pmf A dflt p*) f = ($\prod x{\in}A.$ *pmf* ($p\ x$) ($f\ x$))
⟨*proof*⟩

lemma *pmf-Pi-outside*:
assumes *finite A* $\exists x.\ x \notin A \wedge f\ x \neq dflt$
shows *pmf* (*Pi-pmf A dflt p*) $f = 0$
⟨*proof*⟩

lemma *pmf-Pi-empty* [*simp*]: *Pi-pmf* {} *dflt p* = *return-pmf* (λ-. *dflt*)
⟨*proof*⟩

lemma *set-Pi-pmf-subset*: *finite A* $\Longrightarrow$ *set-pmf* (*Pi-pmf A dflt p*) $\subseteq$ {$f.\ \forall x.\ x \notin A \longrightarrow f\ x = dflt$}
⟨*proof*⟩

26.3 Dependent product sets with a default

The following describes a dependent product of sets where the functions are required to return the default value *dflt* outside their domain, in analogy to Pi_E, which uses *undefined*.

definition *PiE-dflt*
where *PiE-dflt A dflt B* = $\{f.\ \forall x.\ (x \in A \longrightarrow f\ x \in B\ x) \wedge (x \notin A \longrightarrow f\ x = dflt)\}$

lemma *restrict-PiE-dflt*: $(\lambda h.\ restrict\ h\ A)$ ' *PiE-dflt A dflt B* = *PiE A B*
⟨*proof*⟩

lemma *dflt-image-PiE*: $(\lambda h\ x.\ if\ x \in A\ then\ h\ x\ else\ dflt)$ ' *PiE A B* = *PiE-dflt A dflt B*
(**is** *?f* ' *?X* = *?Y*)
⟨*proof*⟩

lemma *finite-PiE-dflt* [*intro*]:
assumes *finite A* $\bigwedge x.\ x \in A \Longrightarrow$ *finite* (*B x*)
shows *finite* (*PiE-dflt A d B*)
⟨*proof*⟩

lemma *card-PiE-dflt*:
assumes *finite A* $\bigwedge x.\ x \in A \Longrightarrow$ *finite* (*B x*)
shows *card* (*PiE-dflt A d B*) = $(\prod x \in A.\ card\ (B\ x))$
⟨*proof*⟩

lemma *PiE-dflt-empty-iff* [*simp*]: *PiE-dflt A dflt B* = {} $\longleftrightarrow$ $(\exists x \in A.\ B\ x = \{\})$
⟨*proof*⟩

lemma *set-Pi-pmf-subset'*:
assumes *finite A*
shows *set-pmf* (*Pi-pmf A dflt p*) $\subseteq$ *PiE-dflt A dflt* (*set-pmf* $\circ$ *p*)
⟨*proof*⟩

lemma *set-Pi-pmf*:
assumes *finite A*
shows *set-pmf* (*Pi-pmf A dflt p*) = *PiE-dflt A dflt* (*set-pmf* $\circ$ *p*)
⟨*proof*⟩

The probability of an independent combination of events is precisely the product of the probabilities of each individual event.

lemma *measure-Pi-pmf-PiE-dflt*:
assumes [*simp*]: *finite A*
shows *measure-pmf.prob* (*Pi-pmf A dflt p*) (*PiE-dflt A dflt B*) =
$(\prod x \in A.\ measure\text{-}pmf.prob\ (p\ x)\ (B\ x))$
⟨*proof*⟩

lemma *measure-Pi-pmf-Pi*:
fixes *t*::*nat*
assumes [*simp*]: *finite A*
shows *measure-pmf.prob* (*Pi-pmf A dflt p*) (*Pi A B*) =
$(\prod x \in A.\ measure\text{-}pmf.prob\ (p\ x)\ (B\ x))$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

26.4 Common PMF operations on products

Pi-pmf distributes over the 'bind' operation in the Giry monad:

lemma *Pi-pmf-bind*:
assumes *finite A*
shows *Pi-pmf A d* (λx. *bind-pmf* (*p x*) (*q x*)) =
do {*f* ← *Pi-pmf A d' p*; *Pi-pmf A d* (λx. *q x* (*f x*))} (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *Pi-pmf-return-pmf* [*simp*]:
assumes *finite A*
shows *Pi-pmf A dflt* (λx. *return-pmf* (*f x*)) = *return-pmf* (λx. *if x* ∈ *A then f x else dflt*)
⟨*proof*⟩

Analogously any componentwise mapping can be pulled outside the product:

lemma *Pi-pmf-map*:
assumes [*simp*]: *finite A* **and** *f dflt* = *dflt'*
shows *Pi-pmf A dflt'* (λx. *map-pmf f* (*g x*)) = *map-pmf* (λh. *f* ∘ *h*) (*Pi-pmf A dflt g*)
⟨*proof*⟩

We can exchange the default value in a product of PMFs like this:

lemma *Pi-pmf-default-swap*:
assumes *finite A*
shows *map-pmf* (λf x. *if x* ∈ *A then f x else dflt'*) (*Pi-pmf A dflt p*) =
Pi-pmf A dflt' p (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

The following rule allows reindexing the product:

lemma *Pi-pmf-bij-betw*:
assumes *finite A bij-betw h A B* ⋀x. *x* ∉ *A* ⟹ *h x* ∉ *B*
shows *Pi-pmf A dflt* (λ-. *f*) = *map-pmf* (λg. *g* ∘ *h*) (*Pi-pmf B dflt* (λ-. *f*))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

A product of uniform random choices is again a uniform distribution.

lemma *Pi-pmf-of-set*:
assumes *finite A* ⋀x. *x* ∈ *A* ⟹ *finite* (*B x*) ⋀x. *x* ∈ *A* ⟹ *B x* ≠ {}
shows *Pi-pmf A d* (λx. *pmf-of-set* (*B x*)) = *pmf-of-set* (*PiE-dflt A d B*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

26.5 Merging and splitting PMF products

The following lemma shows that we can add a single PMF to a product:

lemma *Pi-pmf-insert*:

assumes *finite A* $x \notin A$
shows *Pi-pmf (insert x A) dflt p = map-pmf (λ(y,f). f(x:=y)) (pair-pmf (p x) (Pi-pmf A dflt p))*
⟨*proof*⟩

lemma *Pi-pmf-insert'*:
assumes *finite A* $x \notin A$
shows *Pi-pmf (insert x A) dflt p =*
do {y ← p x; f ← Pi-pmf A dflt p; return-pmf (f(x := y))}
⟨*proof*⟩

lemma *Pi-pmf-singleton*:
Pi-pmf {x} dflt p = map-pmf (λa b. if b = x then a else dflt) (p x)
⟨*proof*⟩

Projecting a product of PMFs onto a component yields the expected result:

lemma *Pi-pmf-component*:
assumes *finite A*
shows *map-pmf (λf. f x) (Pi-pmf A dflt p) = (if x ∈ A then p x else return-pmf dflt)*
⟨*proof*⟩

We can take merge two PMF products on disjoint sets like this:

lemma *Pi-pmf-union*:
assumes *finite A finite B* $A \cap B = \{\}$
shows *Pi-pmf (A ∪ B) dflt p =*
map-pmf (λ(f,g) x. if x ∈ A then f x else g x)
(pair-pmf (Pi-pmf A dflt p) (Pi-pmf B dflt p)) (**is** *- = map-pmf (?h A) (?q A)*)
⟨*proof*⟩

We can also project a product to a subset of the indices by mapping all the other indices to the default value:

lemma *Pi-pmf-subset*:
assumes *finite A* $A' \subseteq A$
shows *Pi-pmf A' dflt p = map-pmf (λf x. if x ∈ A' then f x else dflt) (Pi-pmf A dflt p)*
⟨*proof*⟩

lemma *Pi-pmf-subset'*:
fixes $f :: 'a \Rightarrow 'b$ *pmf*
assumes *finite A* $B \subseteq A$ $\bigwedge x.\ x \in A - B \Longrightarrow f\ x = $ *return-pmf dflt*
shows *Pi-pmf A dflt f = Pi-pmf B dflt f*
⟨*proof*⟩

lemma *Pi-pmf-if-set*:
assumes *finite A*
shows *Pi-pmf A dflt (λx. if b x then f x else return-pmf dflt) =*
Pi-pmf {x∈A. b x} dflt f

⟨*proof*⟩

lemma *Pi-pmf-if-set'*:
assumes *finite A*
shows *Pi-pmf A dflt* (λ*x*. *if b x then return-pmf dflt else f x*) =
Pi-pmf {*x*∈*A*. ¬*b x*} *dflt f*
⟨*proof*⟩

Lastly, we can delete a single component from a product:

lemma *Pi-pmf-remove*:
assumes *finite A*
shows *Pi-pmf* (*A* − {*x*}) *dflt p* = *map-pmf* (λ*f*. *f*(*x* := *dflt*)) (*Pi-pmf A dflt p*)
⟨*proof*⟩

26.6 Additional properties

lemma *nn-integral-prod-Pi-pmf*:
assumes *finite A*
shows *nn-integral* (*Pi-pmf A dflt p*) (λ*y*. ∏ *x*∈*A*. *f x* (*y x*)) = (∏ *x*∈*A*. *nn-integral* (*p x*) (*f x*))
⟨*proof*⟩

lemma *integrable-prod-Pi-pmf*:
fixes *f* :: *'a* ⇒ *'b* ⇒ *'c* :: {*real-normed-field*, *second-countable-topology*, *banach*}
assumes *finite A* **and** ⋀*x*. *x* ∈ *A* ⟹ *integrable* (*measure-pmf* (*p x*)) (*f x*)
shows *integrable* (*measure-pmf* (*Pi-pmf A dflt p*)) (λ*h*. ∏ *x*∈*A*. *f x* (*h x*))
⟨*proof*⟩

lemma *expectation-prod-Pi-pmf*:
fixes *f* :: - ⇒ - ⇒ *real*
assumes *finite A*
assumes ⋀*x*. *x* ∈ *A* ⟹ *integrable* (*measure-pmf* (*p x*)) (*f x*)
assumes ⋀*x y*. *x* ∈ *A* ⟹ *y* ∈ *set-pmf* (*p x*) ⟹ *f x y* ≥ *0*
shows *measure-pmf.expectation* (*Pi-pmf A dflt p*) (λ*y*. ∏ *x*∈*A*. *f x* (*y x*)) =
(∏ *x*∈*A*. *measure-pmf.expectation* (*p x*) (λ*v*. *f x v*))
⟨*proof*⟩

lemma *indep-vars-Pi-pmf*:
assumes *fin*: *finite I*
shows *prob-space.indep-vars* (*measure-pmf* (*Pi-pmf I dflt p*))
(λ-. *count-space UNIV*) (λ*x f*. *f x*) *I*
⟨*proof*⟩

lemma
fixes *h* :: *'a* :: *comm-monoid-add* ⇒ *'b*::{*banach*, *second-countable-topology*}
assumes *fin*: *finite I*
assumes *integrable*: ⋀*i*. *i* ∈ *I* ⟹ *integrable* (*measure-pmf* (*D i*)) *h*
shows *integrable-sum-Pi-pmf*: *integrable* (*Pi-pmf I dflt D*) (λ*g*. ∑ *i*∈*I*. *h* (*g i*))

and *expectation-sum-Pi-pmf*:
measure-pmf.expectation (*Pi-pmf I dflt D*) (λ*g*. ∑ *i*∈*I*. *h* (*g i*)) =
(∑ *i*∈*I*. *measure-pmf.expectation* (*D i*) *h*)
⟨*proof*⟩

26.7 Applications

Choosing a subset of a set uniformly at random is equivalent to tossing a fair coin independently for each element and collecting all the elements that came up heads.

lemma *pmf-of-set-Pow-conv-bernoulli*:
assumes *finite* (*A* :: *'a set*)
shows *map-pmf* (λ*b*. {*x*∈*A*. *b x*}) (*Pi-pmf A P* (λ-. *bernoulli-pmf* (*1*/*2*))) = *pmf-of-set* (*Pow A*)
⟨*proof*⟩

A binomial distribution can be seen as the number of successes in n independent coin tosses.

lemma *binomial-pmf-altdef'*:
fixes *A* :: *'a set*
assumes *finite A* **and** *card A* = *n* **and** *p*: *p* ∈ {*0..1*}
shows *binomial-pmf n p* =
map-pmf (λ*f*. *card* {*x*∈*A*. *f x*}) (*Pi-pmf A dflt* (λ-. *bernoulli-pmf p*)) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

end

27 Hoeffding's Lemma and Hoeffding's Inequality

theory *Hoeffding*
imports *Product-PMF Independent-Family*
begin

Hoeffding's inequality shows that a sum of bounded independent random variables is concentrated around its mean, with an exponential decay of the tail probabilities.

27.1 Hoeffding's Lemma

lemma *convex-on-exp*:
fixes *l* :: *real*
assumes *l* ≥ *0*
shows *convex-on UNIV* (λ*x*. *exp*(*l*∗*x*))
⟨*proof*⟩

lemma *mult-const-minus-self-real-le*:

fixes $x :: real$
shows $x * (c - x) \leq c^2 \ / \ 4$
⟨*proof*⟩

lemma *Hoeffdings-lemma-aux*:
fixes $h\ p :: real$
assumes $h \geq 0$ **and** $p \geq 0$
defines $L \equiv (\lambda h.\ -h * p + ln\ (1 + p * (exp\ h - 1)))$
shows $L\ h \leq h^2 \ / \ 8$
⟨*proof*⟩

locale *interval-bounded-random-variable* = *prob-space* +
fixes $f :: 'a \Rightarrow real$ **and** $a\ b :: real$
assumes *random-variable* [*measurable*]: *random-variable borel f*
assumes *AE-in-interval*: *AE x in M*. $f\ x \in \{a..b\}$
begin

lemma *integrable* [*intro*]: *integrable M f*
⟨*proof*⟩

We first show Hoeffding's lemma for distributions whose expectation is 0. The general case will easily follow from this later.

lemma *Hoeffdings-lemma-nn-integral-0*:
assumes $l > 0$ **and** *E0*: *expectation f* $= 0$
shows *nn-integral M* $(\lambda x.\ exp\ (l * f\ x)) \leq ennreal\ (exp\ (l^2 * (b - a)^2 \ / \ 8))$
⟨*proof*⟩

context
begin

interpretation *shift*: *interval-bounded-random-variable M* $\lambda x.\ f\ x - \mu\ a - \mu\ b - \mu$
rewrites $b - \mu - (a - \mu) \equiv b - a$
⟨*proof*⟩

lemma *expectation-shift*: *expectation* $(\lambda x.\ f\ x -$ *expectation f*$) = 0$
⟨*proof*⟩

lemmas *Hoeffdings-lemma-nn-integral* = *shift.Hoeffdings-lemma-nn-integral-0*[*OF - expectation-shift*]

end

end

27.2 Hoeffding's Inequality

Consider n independent real random variables $X_1, \ldots, X_n$ that each almost surely lie in a compact interval $[a_i, b_i]$. Hoeffding's inequality states that the distribution of the sum of the X_i is tightly concentrated around the sum of the expected values: the probability of it being above or below the sum of the expected values by more than some ε decreases exponentially with ε.

locale *indep-interval-bounded-random-variables = prob-space +*
  fixes *I :: 'b set* **and** *X :: 'b ⇒ 'a ⇒ real*
  fixes *a b :: 'b ⇒ real*
  assumes *fin*: *finite I*
  assumes *indep*: *indep-vars* (*λ-. borel*) *X I*
  assumes *AE-in-interval*: *⋀i. i ∈ I ⟹ AE x in M. X i x ∈ {a i..b i}*
begin

lemma *random-variable* [*measurable*]:
  assumes *i*: *i ∈ I*
  shows *random-variable borel* (*X i*)
  ⟨*proof*⟩

lemma *bounded-random-variable* [*intro*]:
  assumes *i*: *i ∈ I*
  shows *interval-bounded-random-variable M* (*X i*) (*a i*) (*b i*)
  ⟨*proof*⟩

end

locale *Hoeffding-ineq = indep-interval-bounded-random-variables +*
  fixes *μ :: real*
  defines *μ ≡* (*∑ i∈I. expectation* (*X i*))
begin

theorem *Hoeffding-ineq-ge*:
  assumes *ε ≥ 0*
  assumes (*∑ i∈I.* (*b i − a i*)²) *> 0*
  shows *prob* {*x∈space M.* (*∑ i∈I. X i x*) *≥ μ + ε*} *≤ exp* (*−2 ∗ ε²* / (*∑ i∈I.* (*b i − a i*)²))
proof (*cases ε = 0*)
  case [*simp*]: *True*
  have *prob* {*x∈space M.* (*∑ i∈I. X i x*) *≥ μ + ε*} *≤ 1*
    by *simp*
  thus *?thesis* **by** *simp*
next
  case *False*
  with ‹*ε ≥ 0*› **have** *ε*: *ε > 0*
    by *auto*

define d **where** $d = (\sum i{\in}I.\ (b\ i - a\ i)^2)$
define $l :: real$ **where** $l = 4 * \varepsilon\ /\ d$
have d: $d > 0$
using *assms* **by** (*simp add*: *d-def*)
have l: $l > 0$
using ε d **by** (*simp add*: *l-def*)
define μ' **where** $\mu' = (\lambda i.\ expectation\ (X\ i))$

have $\{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \geq \mu + \varepsilon\} = \{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) - \mu \geq \varepsilon\}$
by (*simp add*: *algebra-simps*)
hence $ennreal\ (prob\ \{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \geq \mu + \varepsilon\}) = emeasure\ M\ \ldots$
by (*simp add*: *emeasure-eq-measure*)
also have $\ldots \leq ennreal\ (exp\ (-l{*}\varepsilon)) * (\int^+ x{\in}space\ M.\ exp\ (l * ((\sum i{\in}I.\ X\ i\ x) - \mu))\ \partial M)$
by (*intro Chernoff-ineq-nn-integral-ge l*) *auto*
also have $(\lambda x.\ (\sum i{\in}I.\ X\ i\ x) - \mu) = (\lambda x.\ (\sum i{\in}I.\ X\ i\ x - \mu'\ i))$
by (*simp add*: *μ-def sum-subtractf μ'-def*)
also have $(\int^+ x{\in}space\ M.\ exp\ (l * ((\sum i{\in}I.\ X\ i\ x - \mu'\ i)))\ \partial M) =$
$(\int^+ x.\ (\prod i{\in}I.\ ennreal\ (exp\ (l * (X\ i\ x - \mu'\ i))))\ \partial M)$
by (*intro nn-integral-cong*)
(*simp-all add*: *sum-distrib-left ring-distribs exp-diff exp-sum fin prod-ennreal*)
also have $\ldots = (\prod i{\in}I.\ \int^+ x.\ ennreal\ (exp\ (l * (X\ i\ x - \mu'\ i)))\ \partial M)$
by (*intro indep-vars-nn-integral fin indep-vars-compose2*[*OF indep*]) *auto*
also have $ennreal\ (exp\ (-l * \varepsilon)) * \ldots \leq$
$ennreal\ (exp\ (-l * \varepsilon)) * (\prod i{\in}I.\ ennreal\ (exp\ (l^2 * (b\ i - a\ i)^2\ /\ 8)))$
proof (*intro mult-left-mono prod-mono-ennreal*)
fix i **assume** i: $i \in I$
from i **interpret** *interval-bounded-random-variable M X i a i b i* **..**
show $(\int^+ x.\ ennreal\ (exp\ (l * (X\ i\ x - \mu'\ i)))\ \partial M) \leq ennreal\ (exp\ (l^2 * (b\ i - a\ i)^2\ /\ 8))$
unfolding *μ'-def* **by** (*rule Hoeffdings-lemma-nn-integral*) *fact+*
qed *auto*
also have $\ldots = ennreal\ (exp\ (-l{*}\varepsilon) * (\prod i{\in}I.\ exp\ (l^2 * (b\ i - a\ i)^2\ /\ 8)))$
by (*simp add*: *prod-ennreal prod-nonneg flip*: *ennreal-mult*)
also have $exp\ (-l{*}\varepsilon) * (\prod i{\in}I.\ exp\ (l^2 * (b\ i - a\ i)^2\ /\ 8)) = exp\ (d * l^2\ /\ 8 - l * \varepsilon)$
by (*simp add*: *exp-diff exp-minus sum-divide-distrib sum-distrib-left*
sum-distrib-right exp-sum fin divide-simps mult-ac d-def)
also have $d * l^2\ /\ 8 - l * \varepsilon = -2 * \varepsilon^2\ /\ d$
using d **by** (*simp add*: *l-def field-simps power2-eq-square*)
finally show *?thesis*
by (*subst* (*asm*) *ennreal-le-iff*) (*simp-all add*: *d-def*)
qed

corollary *Hoeffding-ineq-le*:
assumes ε: $\varepsilon \geq 0$
assumes $(\sum i{\in}I.\ (b\ i - a\ i)^2) > 0$
shows $prob\ \{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \leq \mu - \varepsilon\} \leq exp\ (-2 * \varepsilon^2\ /\ (\sum i{\in}I.$

$(b\ i - a\ i)^2))$
⟨*proof*⟩

corollary *Hoeffding-ineq-abs-ge*:
assumes ε: $\varepsilon \geq 0$
assumes $(\sum i{\in}I.\ (b\ i - a\ i)^2) > 0$
shows $prob\ \{x{\in}space\ M.\ |(\sum i{\in}I.\ X\ i\ x) - \mu| \geq \varepsilon\} \leq 2 * exp\ (-2 * \varepsilon^2\ /\ (\sum i{\in}I.\ (b\ i - a\ i)^2))$
⟨*proof*⟩

end

27.3 Hoeffding's inequality for i.i.d. bounded random variables

If we have n even identically-distributed random variables, the statement of Hoeffding's lemma simplifies a bit more: it shows that the probability that the average of the X_i is more than ε above the expected value is no greater than $e^{\frac{-2ny^2}{(b-a)^2}}$.

This essentially gives us a more concrete version of the weak law of large numbers: the law states that the probability vanishes for $n \to \infty$ for any $\varepsilon > 0$. Unlike Hoeffding's inequality, it does not assume the variables to have bounded support, but it does not provide concrete bounds.

locale *iid-interval-bounded-random-variables* = *prob-space* +
fixes I :: $'b$ *set* **and** X :: $'b \Rightarrow 'a \Rightarrow$ *real* **and** Y :: $'a \Rightarrow$ *real*
fixes $a\ b$ :: *real*
assumes *fin*: *finite I*
assumes *indep*: *indep-vars* (λ-. *borel*) *X I*
assumes *distr-X*: $i \in I \Longrightarrow$ *distr M borel* (*X i*) = *distr M borel Y*
assumes *rv-Y* [*measurable*]: *random-variable borel Y*
assumes *AE-in-interval*: *AE x in M. Y x* ∈ {*a*..*b*}
begin

lemma *random-variable* [*measurable*]:
assumes *i*: $i \in I$
shows *random-variable borel* (*X i*)
⟨*proof*⟩

sublocale *X*: *indep-interval-bounded-random-variables M I X* λ-. *a* λ-. *b*
⟨*proof*⟩

lemma *expectation-X* [*simp*]:
assumes *i*: $i \in I$
shows *expectation* (*X i*) = *expectation Y*
⟨*proof*⟩

end

locale *Hoeffding-ineq-iid* = *iid-interval-bounded-random-variables* +
  fixes μ :: *real*
  defines $\mu \equiv$ *expectation Y*
begin

sublocale *X*: *Hoeffding-ineq M I X* λ-. *a* λ-. *b real* (*card I*) $*$ μ
  ⟨*proof*⟩

corollary
  assumes ε: $\varepsilon \geq 0$
  assumes $a < b$ $I \neq \{\}$
  defines $n \equiv$ *card I*
  shows *Hoeffding-ineq-ge*:
        prob $\{x \in$ *space M*. $(\sum i \in I.\ X\ i\ x) \geq n * \mu + \varepsilon\} \leq$
          exp $(-2 * \varepsilon^2 / (n * (b - a)^2))$ (**is** *?le*)
    and *Hoeffding-ineq-le*:
        prob $\{x \in$ *space M*. $(\sum i \in I.\ X\ i\ x) \leq n * \mu - \varepsilon\} \leq$
          exp $(-2 * \varepsilon^2 / (n * (b - a)^2))$ (**is** *?ge*)
    and *Hoeffding-ineq-abs-ge*:
        prob $\{x \in$ *space M*. $|(\sum i \in I.\ X\ i\ x) - n * \mu| \geq \varepsilon\} \leq$
          2 $*$ *exp* $(-2 * \varepsilon^2 / (n * (b - a)^2))$ (**is** *?abs-ge*)
⟨*proof*⟩

lemma
  assumes ε: $\varepsilon \geq 0$
  assumes $a < b$ $I \neq \{\}$
  defines $n \equiv$ *card I*
  shows *Hoeffding-ineq-ge′*:
        prob $\{x \in$ *space M*. $(\sum i \in I.\ X\ i\ x) / n \geq \mu + \varepsilon\} \leq$
          exp $(-2 * n * \varepsilon^2 / (b - a)^2)$ (**is** *?ge*)
    and *Hoeffding-ineq-le′*:
        prob $\{x \in$ *space M*. $(\sum i \in I.\ X\ i\ x) / n \leq \mu - \varepsilon\} \leq$
          exp $(-2 * n * \varepsilon^2 / (b - a)^2)$ (**is** *?le*)
    and *Hoeffding-ineq-abs-ge′*:
        prob $\{x \in$ *space M*. $|(\sum i \in I.\ X\ i\ x) / n - \mu| \geq \varepsilon\} \leq$
          2 $*$ *exp* $(-2 * n * \varepsilon^2 / (b - a)^2)$ (**is** *?abs-ge*)
⟨*proof*⟩

end

27.4 Hoeffding's Inequality for the Binomial distribution

We can now apply Hoeffding's inequality to the Binomial distribution, which can be seen as the sum of n i.i.d. coin flips (the support of each of which is contained in $[0, 1]$).

locale *binomial-distribution* =
  fixes *n* :: *nat* **and** *p* :: *real*

assumes p: $p \in \{0..1\}$
begin

context
fixes *coins* :: $(nat \Rightarrow bool)$ *pmf* **and** μ
assumes n: $n > 0$
defines *coins* $\equiv$ *Pi-pmf* $\{..<n\}$ *False* $(\lambda$-. *bernoulli-pmf p*$)$
begin

lemma *coins-component*:
assumes i: $i < n$
shows *distr* (*measure-pmf coins*) *borel* $(\lambda f.$ *if f i then 1 else 0*$)$ =
distr (*measure-pmf* (*bernoulli-pmf p*)) *borel* $(\lambda b.$ *if b then 1 else 0*$)$
⟨*proof*⟩

lemma *prob-binomial-pmf-conv-coins*:
measure-pmf.prob (*binomial-pmf n p*) $\{x.$ *P* (*real x*)$\}$ =
measure-pmf.prob coins $\{x.$ *P* $(\sum i<n.$ *if x i then 1 else 0*$)\}$
⟨*proof*⟩

interpretation *Hoeffding-ineq-iid*
coins $\{..<n\}$ $\lambda i\ f.$ *if f i then 1 else 0* $\lambda f.$ *if f 0 then 1 else 0 0 1 p*
⟨*proof*⟩

corollary
fixes ε :: *real*
assumes ε: $\varepsilon \geq 0$
shows *prob-ge*: *measure-pmf.prob* (*binomial-pmf n p*) $\{x.\ x \geq n * p + \varepsilon\} \leq exp$ $(-2 * \varepsilon^2 / n)$
and *prob-le*: *measure-pmf.prob* (*binomial-pmf n p*) $\{x.\ x \leq n * p - \varepsilon\} \leq exp$ $(-2 * \varepsilon^2 / n)$
and *prob-abs-ge*:
measure-pmf.prob (*binomial-pmf n p*) $\{x.\ |x - n * p| \geq \varepsilon\} \leq 2 * exp\ (-2$ $* \varepsilon^2 / n)$
⟨*proof*⟩

corollary
fixes ε :: *real*
assumes ε: $\varepsilon \geq 0$
shows *prob-ge′*: *measure-pmf.prob* (*binomial-pmf n p*) $\{x.\ x / n \geq p + \varepsilon\} \leq exp$ $(-2 * n * \varepsilon^2)$
and *prob-le′*: *measure-pmf.prob* (*binomial-pmf n p*) $\{x.\ x / n \leq p - \varepsilon\} \leq exp$ $(-2 * n * \varepsilon^2)$
and *prob-abs-ge′*:
measure-pmf.prob (*binomial-pmf n p*) $\{x.\ |x / n - p| \geq \varepsilon\} \leq 2 * exp\ (-2$ $* n * \varepsilon^2)$
⟨*proof*⟩

end

end

27.5 Tail bounds for the negative binomial distribution

Since the tail probabilities of a negative Binomial distribution are equal to the tail probabilities of some Binomial distribution, we can obtain tail bounds for the negative Binomial distribution through the Hoeffding tail bounds for the Binomial distribution.

context
fixes p q :: *real*
assumes *p*: $p \in \{0<..<1\}$
defines $q \equiv 1 - p$
begin

lemma *prob-neg-binomial-pmf-ge-bound*:
fixes n :: *nat* **and** k :: *real*
defines $\mu \equiv real\ n * q\ /\ p$
assumes *k*: $k \geq 0$
shows *measure-pmf.prob* (*neg-binomial-pmf n p*) $\{x.\ real\ x \geq \mu + k\}$
$\leq exp\ (-\ 2 * p\ \hat{}\ 3 * k^2\ /\ (n + p * k))$
⟨*proof*⟩

lemma *prob-neg-binomial-pmf-le-bound*:
fixes n :: *nat* **and** k :: *real*
defines $\mu \equiv real\ n * q\ /\ p$
assumes *k*: $k \geq 0$
shows *measure-pmf.prob* (*neg-binomial-pmf n p*) $\{x.\ real\ x \leq \mu - k\}$
$\leq exp\ (-2 * p\ \hat{}\ 3 * k^2\ /\ (n - p * k))$
⟨*proof*⟩

Due to the function $exp(-l/x)$ being concave for $x \geq \frac{l}{2}$, the above two bounds can be combined into the following one for moderate values of k. (cf. https://math.stackexchange.com/questions/1565559)

lemma *prob-neg-binomial-pmf-abs-ge-bound*:
fixes n :: *nat* **and** k :: *real*
defines $\mu \equiv real\ n * q\ /\ p$
assumes $k \geq 0$ **and** *n-ge*: $n \geq p * k * (p^2 * k + 1)$
shows *measure-pmf.prob* (*neg-binomial-pmf n p*) $\{x.\ |real\ x - \mu| \geq k\} \leq$
$2 * exp\ (-2 * p\ \hat{}\ 3 * k\ \hat{}\ 2\ /\ n)$
⟨*proof*⟩

end

end

theory *Stream-Space*

imports
Infinite-Product-Measure
HOL−Library.Stream
HOL−Library.Linear-Temporal-Logic-on-Streams
begin

lemma *stream-eq-Stream-iff*: $s = x \mathbin{\#\#} t \longleftrightarrow (shd\ s = x \wedge stl\ s = t)$
⟨*proof*⟩

lemma *Stream-snth*: $(x \mathbin{\#\#} s) !! n = (case\ n\ of\ 0 \Rightarrow x \mid Suc\ n \Rightarrow s !! n)$
⟨*proof*⟩

definition *to-stream* :: $(nat \Rightarrow 'a) \Rightarrow 'a\ stream$ **where**
$to\text{-}stream\ X = smap\ X\ nats$

lemma *to-stream-nat-case*: $to\text{-}stream\ (case\text{-}nat\ x\ X) = x \mathbin{\#\#} to\text{-}stream\ X$
⟨*proof*⟩

lemma *to-stream-in-streams*: $to\text{-}stream\ X \in streams\ S \longleftrightarrow (\forall n.\ X\ n \in S)$
⟨*proof*⟩

definition *stream-space* :: $'a\ measure \Rightarrow 'a\ stream\ measure$ **where**
$stream\text{-}space\ M =$
$distr\ (\Pi_M\ i{\in}UNIV.\ M)\ (vimage\text{-}algebra\ (streams\ (space\ M))\ snth\ (\Pi_M\ i{\in}UNIV.\ M))\ to\text{-}stream$

lemma *space-stream-space*: $space\ (stream\text{-}space\ M) = streams\ (space\ M)$
⟨*proof*⟩

lemma *streams-stream-space*[*intro*]: $streams\ (space\ M) \in sets\ (stream\text{-}space\ M)$
⟨*proof*⟩

lemma *stream-space-Stream*:
$x \mathbin{\#\#} \omega \in space\ (stream\text{-}space\ M) \longleftrightarrow x \in space\ M \wedge \omega \in space\ (stream\text{-}space\ M)$
⟨*proof*⟩

lemma *stream-space-eq-distr*: $stream\text{-}space\ M = distr\ (\Pi_M\ i{\in}UNIV.\ M)\ (stream\text{-}space\ M)\ to\text{-}stream$
⟨*proof*⟩

lemma *sets-stream-space-cong*[*measurable-cong*]:
$sets\ M = sets\ N \Longrightarrow sets\ (stream\text{-}space\ M) = sets\ (stream\text{-}space\ N)$
⟨*proof*⟩

lemma *measurable-snth-PiM*: $(\lambda\omega\ n.\ \omega !! n) \in measurable\ (stream\text{-}space\ M)\ (\Pi_M\ i{\in}UNIV.\ M)$
⟨*proof*⟩

lemma *measurable-snth*[*measurable*]: $(\lambda\omega.\ \omega\ !!\ n) \in$ *measurable* (*stream-space M*) *M*
⟨*proof*⟩

lemma *measurable-shd*[*measurable*]: *shd* ∈ *measurable* (*stream-space M*) *M*
⟨*proof*⟩

lemma *measurable-stream-space2*:
assumes *f-snth*: $\bigwedge n.\ (\lambda x.\ f\ x\ !!\ n) \in$ *measurable N M*
shows *f* ∈ *measurable N* (*stream-space M*)
⟨*proof*⟩

lemma *measurable-stream-coinduct*[*consumes 1*, *case-names shd stl*, *coinduct set*: *measurable*]:
assumes *F f*
assumes *h*: $\bigwedge f.\ F\ f \Longrightarrow (\lambda x.\ shd\ (f\ x)) \in$ *measurable N M*
assumes *t*: $\bigwedge f.\ F\ f \Longrightarrow F\ (\lambda x.\ stl\ (f\ x))$
shows *f* ∈ *measurable N* (*stream-space M*)
⟨*proof*⟩

lemma *measurable-sdrop*[*measurable*]: *sdrop n* ∈ *measurable* (*stream-space M*) (*stream-space M*)
⟨*proof*⟩

lemma *measurable-stl*[*measurable*]: $(\lambda\omega.\ stl\ \omega) \in$ *measurable* (*stream-space M*) (*stream-space M*)
⟨*proof*⟩

lemma *measurable-to-stream*[*measurable*]: *to-stream* ∈ *measurable* ($\Pi_M\ i \in UNIV.\ M$) (*stream-space M*)
⟨*proof*⟩

lemma *measurable-Stream*[*measurable* (*raw*)]:
assumes *f*[*measurable*]: *f* ∈ *measurable N M*
assumes *g*[*measurable*]: *g* ∈ *measurable N* (*stream-space M*)
shows $(\lambda x.\ f\ x\ \#\#\ g\ x) \in$ *measurable N* (*stream-space M*)
⟨*proof*⟩

lemma *measurable-smap*[*measurable*]:
assumes *X*[*measurable*]: *X* ∈ *measurable N M*
shows *smap X* ∈ *measurable* (*stream-space N*) (*stream-space M*)
⟨*proof*⟩

lemma *measurable-stake*[*measurable*]:
stake i ∈ *measurable* (*stream-space* (*count-space UNIV*)) (*count-space* (*UNIV* :: *'a*::*countable list set*))
⟨*proof*⟩

lemma *measurable-shift*[*measurable*]:

assumes *f*: $f \in$ *measurable N* (*stream-space M*)
assumes [*measurable*]: $g \in$ *measurable N* (*stream-space M*)
shows ($\lambda x.$ *stake n* (*f x*) @− *g x*) ∈ *measurable N* (*stream-space M*)
⟨*proof*⟩

lemma *measurable-case-stream-replace*[*measurable* (*raw*)]:
($\lambda x.$ *f x* (*shd* (*g x*)) (*stl* (*g x*))) ∈ *measurable M N* ⟹ ($\lambda x.$ *case-stream* (*f x*) (*g x*)) ∈ *measurable M N*
⟨*proof*⟩

lemma *measurable-ev-at*[*measurable*]:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) *P*
shows *Measurable.pred* (*stream-space M*) (*ev-at P n*)
⟨*proof*⟩

lemma *measurable-alw*[*measurable*]:
Measurable.pred (*stream-space M*) *P* ⟹ *Measurable.pred* (*stream-space M*) (*alw P*)
⟨*proof*⟩

lemma *measurable-ev*[*measurable*]:
Measurable.pred (*stream-space M*) *P* ⟹ *Measurable.pred* (*stream-space M*) (*ev P*)
⟨*proof*⟩

lemma *measurable-until*:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) φ *Measurable.pred* (*stream-space M*) ψ
shows *Measurable.pred* (*stream-space M*) (φ *until* ψ)
⟨*proof*⟩

lemma *measurable-holds* [*measurable*]: *Measurable.pred M P* ⟹ *Measurable.pred* (*stream-space M*) (*holds P*)
⟨*proof*⟩

lemma *measurable-hld*[*measurable*]: **assumes** [*measurable*]: *t* ∈ *sets M* **shows** *Measurable.pred* (*stream-space M*) (*HLD t*)
⟨*proof*⟩

lemma *measurable-nxt*[*measurable* (*raw*)]:
Measurable.pred (*stream-space M*) *P* ⟹ *Measurable.pred* (*stream-space M*) (*nxt P*)
⟨*proof*⟩

lemma *measurable-suntil*[*measurable*]:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) *Q Measurable.pred* (*stream-space M*) *P*
shows *Measurable.pred* (*stream-space M*) (*Q suntil P*)
⟨*proof*⟩

lemma *measurable-szip*:
(λ(*ω1*, *ω2*). *szip ω1 ω2*) ∈ *measurable* (*stream-space M* ⨂$_M$ *stream-space N*) (*stream-space* (*M* ⨂$_M$ *N*))
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-space-stream-space*: *prob-space* (*stream-space M*)
⟨*proof*⟩

lemma (**in** *prob-space*) *nn-integral-stream-space*:
assumes [*measurable*]: *f* ∈ *borel-measurable* (*stream-space M*)
shows (∫$^+$*X*. *f X* ∂*stream-space M*) = (∫$^+$*x*. (∫$^+$*X*. *f* (*x* ## *X*) ∂*stream-space M*) ∂*M*)
⟨*proof*⟩

lemma (**in** *prob-space*) *emeasure-stream-space*:
assumes *X*[*measurable*]: *X* ∈ *sets* (*stream-space M*)
shows *emeasure* (*stream-space M*) *X* = (∫$^+$*t*. *emeasure* (*stream-space M*) {*x*∈*space* (*stream-space M*). *t* ## *x* ∈ *X* } ∂*M*)
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-stream-space*:
assumes *P*[*measurable*]: {*x*∈*space* (*stream-space M*). *P x*} ∈ *sets* (*stream-space M*)
shows *P*(*x in stream-space M*. *P x*) = (∫$^+$*t*. *P*(*x in stream-space M*. *P* (*t* ## *x*)) ∂*M*)
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-stream-space*:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) *P*
shows (*AE X in stream-space M*. *P X*) = (*AE x in M*. *AE X in stream-space M*. *P* (*x* ## *X*))
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-stream-all*:
assumes [*measurable*]: *Measurable.pred M P* **and** *P*: *AE x in M*. *P x*
shows *AE x in stream-space M*. *stream-all P x*
⟨*proof*⟩

lemma *streams-sets*:
assumes *X*[*measurable*]: *X* ∈ *sets M* **shows** *streams X* ∈ *sets* (*stream-space M*)
⟨*proof*⟩

lemma *sets-stream-space-in-sets*:
assumes *space*: *space N* = *streams* (*space M*)
assumes *sets*: ⋀*i*. (λ*x*. *x* !! *i*) ∈ *measurable N M*
shows *sets* (*stream-space M*) ⊆ *sets N*
⟨*proof*⟩

lemma *sets-stream-space-eq*: *sets* (*stream-space M*) =
sets (*SUP i∈UNIV*. *vimage-algebra* (*streams* (*space M*)) (λ*s*. *s* !! *i*) *M*)
⟨*proof*⟩

lemma *sets-restrict-stream-space*:
assumes *S*[*measurable*]: *S* ∈ *sets M*
shows *sets* (*restrict-space* (*stream-space M*) (*streams S*)) = *sets* (*stream-space* (*restrict-space M S*))
⟨*proof*⟩

primrec *sstart* :: *'a set* ⇒ *'a list* ⇒ *'a stream set* **where**
sstart S [] = *streams S*
| [*simp del*]: *sstart S* (*x* # *xs*) = (##) *x* ' *sstart S xs*

lemma *in-sstart*[*simp*]: *s* ∈ *sstart S* (*x* # *xs*) ⟷ *shd s* = *x* ∧ *stl s* ∈ *sstart S xs*
⟨*proof*⟩

lemma *sstart-in-streams*: *xs* ∈ *lists S* ⟹ *sstart S xs* ⊆ *streams S*
⟨*proof*⟩

lemma *sstart-eq*: *x* ∈ *streams S* ⟹ *x* ∈ *sstart S xs* = (∀ *i*<*length xs*. *x* !! *i* = *xs* ! *i*)
⟨*proof*⟩

lemma *sstart-sets*: *sstart S xs* ∈ *sets* (*stream-space* (*count-space UNIV*))
⟨*proof*⟩

lemma *sigma-sets-singletons*:
assumes *countable S*
shows *sigma-sets S* ((λ*s*. {*s*})'*S*) = *Pow S*
⟨*proof*⟩

lemma *sets-count-space-eq-sigma*:
countable S ⟹ *sets* (*count-space S*) = *sets* (*sigma S* ((λ*s*. {*s*})'*S*))
⟨*proof*⟩

lemma *sets-stream-space-sstart*:
assumes *S*[*simp*]: *countable S*
shows *sets* (*stream-space* (*count-space S*)) = *sets* (*sigma* (*streams S*) (*sstart S*'*lists S* ∪ {{}}))
⟨*proof*⟩

lemma *Int-stable-sstart*: *Int-stable* (*sstart S*'*lists S* ∪ {{}})
⟨*proof*⟩

lemma *stream-space-eq-sstart*:
assumes *S*[*simp*]: *countable S*
assumes *P*: *prob-space M prob-space N*
assumes *ae*: *AE x in M*. *x* ∈ *streams S AE x in N*. *x* ∈ *streams S*

assumes *sets-M*: *sets M = sets (stream-space (count-space UNIV))*
assumes *sets-N*: *sets N = sets (stream-space (count-space UNIV))*
assumes ∗: ⋀*xs*. *xs* ≠ [] ⟹ *xs* ∈ *lists S* ⟹ *emeasure M* (*sstart S xs*) = *emeasure N* (*sstart S xs*)
shows *M = N*
⟨*proof*⟩

lemma *sets-sstart*[*measurable*]: *sstart* Ω *xs* ∈ *sets* (*stream-space* (*count-space UNIV*))
⟨*proof*⟩

primrec *scylinder* :: *'a set* ⇒ *'a set list* ⇒ *'a stream set*
where
scylinder S [] = *streams S*
| *scylinder S* (*A* # *As*) = {ω∈*streams S*. *shd* ω ∈ *A* ∧ *stl* ω ∈ *scylinder S As*}

lemma *scylinder-streams*: *scylinder S xs* ⊆ *streams S*
⟨*proof*⟩

lemma *sets-scylinder*: (∀ *x*∈*set xs*. *x* ∈ *sets S*) ⟹ *scylinder* (*space S*) *xs* ∈ *sets* (*stream-space S*)
⟨*proof*⟩

lemma *stream-space-eq-scylinder*:
assumes *P*: *prob-space M prob-space N*
assumes *Int-stable G* **and** *S*: *sets S = sets* (*sigma* (*space S*) *G*)
and *C*: *countable C C* ⊆ *G* ⋃ *C = space S* **and** *G*: *G* ⊆ *Pow* (*space S*)
assumes *sets-M*: *sets M = sets* (*stream-space S*)
assumes *sets-N*: *sets N = sets* (*stream-space S*)
assumes ∗: ⋀*xs*. *xs* ≠ [] ⟹ *xs* ∈ *lists G* ⟹ *emeasure M* (*scylinder* (*space S*) *xs*) = *emeasure N* (*scylinder* (*space S*) *xs*)
shows *M = N*
⟨*proof*⟩

lemma *stream-space-coinduct*:
fixes *R* :: *'a stream measure* ⇒ *'a stream measure* ⇒ *bool*
assumes *R A B*
assumes *R*: ⋀*A B*. *R A B* ⟹ ∃ *K*∈*space* (*prob-algebra M*).
∃ *A'*∈*M* →*M prob-algebra* (*stream-space M*). ∃ *B'*∈*M* →*M prob-algebra* (*stream-space M*).
(*AE y in K*. *R* (*A' y*) (*B' y*) ∨ *A' y* = *B' y*) ∧
A = *do* { *y* ← *K*; ω ← *A' y*; *return* (*stream-space M*) (*y* ## ω) } ∧
B = *do* { *y* ← *K*; ω ← *B' y*; *return* (*stream-space M*) (*y* ## ω) }
shows *A = B*
⟨*proof*⟩

end

theory *Tree-Space*

imports *HOL−Analysis.Analysis HOL−Library.Tree*
begin

lemma *countable-lfp*:
assumes *step*: $\bigwedge Y.$ *countable* $Y \Longrightarrow$ *countable* $(F\ Y)$
and *cont*: *Order-Continuity.sup-continuous* F
shows *countable* $(lfp\ F)$
⟨*proof*⟩

lemma *countable-lfp-apply*:
assumes *step*: $\bigwedge Y\ x.\ (\bigwedge x.$ *countable* $(Y\ x)) \Longrightarrow$ *countable* $(F\ Y\ x)$
and *cont*: *Order-Continuity.sup-continuous* F
shows *countable* $(lfp\ F\ x)$
⟨*proof*⟩

inductive-set *trees* :: *'a set* ⇒ *'a tree set* **for** S :: *'a set* **where**
[*intro*!]: *Leaf* ∈ *trees S*
| $l \in$ *trees* $S \Longrightarrow r \in$ *trees* $S \Longrightarrow v \in S \Longrightarrow$ *Node* $l\ v\ r \in$ *trees* S

lemma *Node-in-trees-iff*[*simp*]: *Node* $l\ v\ r \in$ *trees* $S \longleftrightarrow (l \in$ *trees* $S \wedge v \in S \wedge$ $r \in$ *trees* $S)$
⟨*proof*⟩

lemma *trees-sub-lfp*: *trees* $S \subseteq$ *lfp* $(\lambda T.\ T \cup \{Leaf\} \cup (\bigcup l{\in}T.\ (\bigcup v{\in}S.\ (\bigcup r{\in}T.$ $\{Node\ l\ v\ r\}))))$
⟨*proof*⟩

lemma *countable-trees*: *countable* $A \Longrightarrow$ *countable* $(trees\ A)$
⟨*proof*⟩

lemma *trees-UNIV*[*simp*]: *trees UNIV* = *UNIV*
⟨*proof*⟩

instance *tree* :: (*countable*) *countable*
⟨*proof*⟩

lemma *map-in-trees*[*intro*]: $(\bigwedge x.\ x \in$ *set-tree* $t \Longrightarrow f\ x \in S) \Longrightarrow$ *map-tree* $f\ t \in$ *trees* S
⟨*proof*⟩

primrec *trees-cyl* :: *'a set tree* ⇒ *'a tree set* **where**
trees-cyl Leaf = {*Leaf*}
| *trees-cyl* (*Node* $l\ v\ r$) = $(\bigcup l'{\in}$*trees-cyl* $l.\ (\bigcup v'{\in}v.\ (\bigcup r'{\in}$*trees-cyl* $r.\ \{$*Node* $l'\ v'$ $r'\}))))$

definition *tree-sigma* :: *'a measure* ⇒ *'a tree measure*
where
tree-sigma M = *sigma* (*trees* (*space* M)) (*trees-cyl* ‘ *trees* (*sets* M))

lemma *Node-in-trees-cyl*: *Node* l' v' $r' \in$ *trees-cyl* $t \longleftrightarrow$
$(\exists\, l\ v\ r.\ t =$ *Node* $l\ v\ r \wedge l' \in$ *trees-cyl* $l \wedge r' \in$ *trees-cyl* $r \wedge v' \in v)$
⟨*proof*⟩

lemma *trees-cyl-sub-trees*:
assumes $t \in$ *trees* A $A \subseteq$ *Pow* B **shows** *trees-cyl* $t \subseteq$ *trees* B
⟨*proof*⟩

lemma *trees-cyl-sets-in-space*: *trees-cyl* ‘ *trees* (*sets* M) $\subseteq$ *Pow* (*trees* (*space* M))
⟨*proof*⟩

lemma *space-tree-sigma*: *space* (*tree-sigma* M) = *trees* (*space* M)
⟨*proof*⟩

lemma *sets-tree-sigma-eq*: *sets* (*tree-sigma* M) = *sigma-sets* (*trees* (*space* M)) (*trees-cyl* ‘ *trees* (*sets* M))
⟨*proof*⟩

lemma *Leaf-in-space-tree-sigma* [*measurable*, *simp*, *intro*]: *Leaf* $\in$ *space* (*tree-sigma* M)
⟨*proof*⟩

lemma *Leaf-in-tree-sigma* [*measurable*, *simp*, *intro*]: {*Leaf*} $\in$ *sets* (*tree-sigma* M)
⟨*proof*⟩

lemma *trees-cyl-map-treeI*: $t \in$ *trees-cyl* (*map-tree* ($\lambda x.\ A$) t) **if** $*$: $t \in$ *trees* A
⟨*proof*⟩

lemma *trees-cyl-map-in-sets*:
$(\bigwedge x.\ x \in$ *set-tree* $t \Longrightarrow f\, x \in$ *sets* $M) \Longrightarrow$ *trees-cyl* (*map-tree* $f\, t$) $\in$ *sets* (*tree-sigma* M)
⟨*proof*⟩

lemma *Node-in-tree-sigma*:
assumes L: $X \in$ *sets* ($M \bigotimes_M$ (*tree-sigma* $M \bigotimes_M$ *tree-sigma* M))
shows {*Node* $l\ v\ r \mid l\ v\ r.\ (v, l, r) \in X$} $\in$ *sets* (*tree-sigma* M)
⟨*proof*⟩

lemma *measurable-left*[*measurable*]: *left* $\in$ *tree-sigma* $M \to_M$ *tree-sigma* M
⟨*proof*⟩

lemma *measurable-right*[*measurable*]: *right* $\in$ *tree-sigma* $M \to_M$ *tree-sigma* M
⟨*proof*⟩

lemma *measurable-value'*: *value* $\in$ *restrict-space* (*tree-sigma* M) ($-${*Leaf*}) $\to_M$ M
⟨*proof*⟩

lemma *measurable-value*[*measurable* (*raw*)]:

assumes $f \in X \to_M$ *tree-sigma* M
and $\bigwedge x.\ x \in$ *space* $X \Longrightarrow f\ x \neq$ *Leaf*
shows $(\lambda\omega.$ *value* $(f\ \omega)) \in X \to_M M$
⟨*proof*⟩

lemma *measurable-Node* [*measurable*]:
$(\lambda(l,x,r).$ *Node* $l\ x\ r) \in$ *tree-sigma* $M \bigotimes_M M \bigotimes_M$ *tree-sigma* $M \to_M$ *tree-sigma* M
⟨*proof*⟩

lemma *measurable-Node'* [*measurable* (*raw*)]:
assumes [*measurable*]: $l \in B \to_M$ *tree-sigma* A
assumes [*measurable*]: $x \in B \to_M A$
assumes [*measurable*]: $r \in B \to_M$ *tree-sigma* A
shows $(\lambda y.$ *Node* $(l\ y)\ (x\ y)\ (r\ y)) \in B \to_M$ *tree-sigma* A
⟨*proof*⟩

lemma *measurable-rec-tree*[*measurable* (*raw*)]:
assumes *t*: $t \in B \to_M$ *tree-sigma* M
assumes *l*: $l \in B \to_M A$
assumes *n*: $(\lambda(x,\ l,\ v,\ r,\ al,\ ar).\ n\ x\ l\ v\ r\ al\ ar) \in$
$(B \bigotimes_M$ *tree-sigma* $M \bigotimes_M M \bigotimes_M$ *tree-sigma* $M \bigotimes_M A \bigotimes_M A) \to_M A$ (**is** *?N* $\in$ *?M* $\to_M A$)
shows $(\lambda x.$ *rec-tree* $(l\ x)\ (n\ x)\ (t\ x)) \in B \to_M A$
⟨*proof*⟩

lemma *measurable-case-tree* [*measurable* (*raw*)]:
assumes $t \in B \to_M$ *tree-sigma* M
assumes $l \in B \to_M A$
assumes $(\lambda(x,\ l,\ v,\ r).\ n\ x\ l\ v\ r)$
$\in B \bigotimes_M$ *tree-sigma* $M \bigotimes_M M \bigotimes_M$ *tree-sigma* $M \to_M A$
shows $(\lambda x.$ *case-tree* $(l\ x)\ (n\ x)\ (t\ x)) \in B \to_M (A :: 'a$ *measure*$)$
⟨*proof*⟩

hide-const (**open**) *left*
hide-const (**open**) *right*

end

28 Conditional Expectation

theory *Conditional-Expectation*
imports *Probability-Measure*
begin

28.1 Restricting a measure to a sub-sigma-algebra

definition *subalgebra*::$'a$ *measure* $\Rightarrow$ $'a$ *measure* $\Rightarrow$ *bool* **where**

subalgebra M F = ((*space F* = *space M*) ∧ (*sets F* ⊆ *sets M*))

lemma *sub-measure-space*:
assumes *i*: *subalgebra M F*
shows *measure-space* (*space M*) (*sets F*) (*emeasure M*)
⟨*proof*⟩

definition *restr-to-subalg*::'*a measure* ⇒ '*a measure* ⇒ '*a measure* **where**
restr-to-subalg M F = *measure-of* (*space M*) (*sets F*) (*emeasure M*)

lemma *space-restr-to-subalg*:
space (*restr-to-subalg M F*) = *space M*
⟨*proof*⟩

lemma *sets-restr-to-subalg* [*measurable-cong*]:
assumes *subalgebra M F*
shows *sets* (*restr-to-subalg M F*) = *sets F*
⟨*proof*⟩

lemma *emeasure-restr-to-subalg*:
assumes *subalgebra M F*
A ∈ *sets F*
shows *emeasure* (*restr-to-subalg M F*) *A* = *emeasure M A*
⟨*proof*⟩

lemma *null-sets-restr-to-subalg*:
assumes *subalgebra M F*
shows *null-sets* (*restr-to-subalg M F*) = *null-sets M* ∩ *sets F*
⟨*proof*⟩

lemma *AE-restr-to-subalg*:
assumes *subalgebra M F*
AE x in (*restr-to-subalg M F*). *P x*
shows *AE x in M*. *P x*
⟨*proof*⟩

lemma *AE-restr-to-subalg2*:
assumes *subalgebra M F*
AE x in M. *P x* **and** [*measurable*]: *P* ∈ *measurable F* (*count-space UNIV*)
shows *AE x in* (*restr-to-subalg M F*). *P x*
⟨*proof*⟩

lemma *prob-space-restr-to-subalg*:
assumes *subalgebra M F*
prob-space M
shows *prob-space* (*restr-to-subalg M F*)
⟨*proof*⟩

lemma *finite-measure-restr-to-subalg*:

assumes *subalgebra M F*
finite-measure M
shows *finite-measure* (*restr-to-subalg M F*) ⟨*proof*⟩

lemma *measurable-in-subalg*:
assumes *subalgebra M F*
$f \in$ *measurable F N*
shows $f \in$ *measurable* (*restr-to-subalg M F*) *N* ⟨*proof*⟩

lemma *measurable-in-subalg′*:
assumes *subalgebra M F*
$f \in$ *measurable* (*restr-to-subalg M F*) *N*
shows $f \in$ *measurable F N* ⟨*proof*⟩

lemma *measurable-from-subalg*:
assumes *subalgebra M F*
$f \in$ *measurable F N*
shows $f \in$ *measurable M N* ⟨*proof*⟩

The following is the direct transposition of `nn_integral_subalgebra` (from `Nonnegative_Lebesgue_Integration`) in the current notations, with the removal of the useless assumption $f \geq 0$.

lemma *nn-integral-subalgebra2*:
assumes *subalgebra M F* **and** [*measurable*]: $f \in$ *borel-measurable F*
shows $(\int^+ x.\ f\ x\ \partial(\textit{restr-to-subalg}\ M\ F)) = (\int^+ x.\ f\ x\ \partial M)$ ⟨*proof*⟩

The following is the direct transposition of `integral_subalgebra` (from `Bochner_Integration`) in the current notations.

lemma *integral-subalgebra2*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes *subalgebra M F* **and**
[*measurable*]: $f \in$ *borel-measurable F*
shows $(\int x.\ f\ x\ \partial(\textit{restr-to-subalg}\ M\ F)) = (\int x.\ f\ x\ \partial M)$ ⟨*proof*⟩

lemma *integrable-from-subalg*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes *subalgebra M F*
integrable (*restr-to-subalg M F*) *f*
shows *integrable M f* ⟨*proof*⟩

lemma *integrable-in-subalg*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}

assumes [*measurable*]: *subalgebra M F*
f ∈ *borel-measurable F*
integrable M f
shows *integrable* (*restr-to-subalg M F*) *f*
⟨*proof*⟩

28.2 Nonnegative conditional expectation

The conditional expectation of a function f, on a measure space M, with respect to a sub sigma algebra F, should be a function g which is F-measurable whose integral on any F-set coincides with the integral of f. Such a function is uniquely defined almost everywhere. The most direct construction is to use the measure fdM, restrict it to the sigma-algebra F, and apply the Radon-Nikodym theorem to write it as $gdM_{|F}$ for some F-measurable function g. Another classical construction for L^2 functions is done by orthogonal projection on F-measurable functions, and then extending by density to L^1. The Radon-Nikodym point of view avoids the L^2 machinery, and works for all positive functions.

In this paragraph, we develop the definition and basic properties for nonnegative functions, as the basics of the general case. As in the definition of integrals, the nonnegative case is done with ennreal-valued functions, without any integrability assumption.

definition *nn-cond-exp* :: *'a measure* ⇒ *'a measure* ⇒ (*'a* ⇒ *ennreal*) ⇒ (*'a* ⇒ *ennreal*)
where
nn-cond-exp M F f =
(*if f* ∈ *borel-measurable M* ∧ *subalgebra M F*
then RN-deriv (*restr-to-subalg M F*) (*restr-to-subalg* (*density M f*) *F*)
else (λ-. *0*))

lemma
shows *borel-measurable-nn-cond-exp* [*measurable*]: *nn-cond-exp M F f* ∈ *borel-measurable F*
and *borel-measurable-nn-cond-exp2* [*measurable*]: *nn-cond-exp M F f* ∈ *borel-measurable M*
⟨*proof*⟩

The good setting for conditional expectations is the situation where the subalgebra F gives rise to a sigma-finite measure space. To see what goes wrong if it is not sigma-finite, think of $\mathbb{R}$ with the trivial sigma-algebra $\{\emptyset, \mathbb{R}\}$. In this case, conditional expectations have to be constant functions, so they have integral 0 or ∞. This means that a positive integrable function can have no meaningful conditional expectation.

locale *sigma-finite-subalgebra* =
fixes *M F*::*'a measure*
assumes *subalg*: *subalgebra M F*

and *sigma-fin-subalg*: *sigma-finite-measure* (*restr-to-subalg M F*)

lemma *sigma-finite-subalgebra-is-sigma-finite*:
assumes *sigma-finite-subalgebra M F*
shows *sigma-finite-measure M*
⟨*proof*⟩

sublocale *sigma-finite-subalgebra* ⊆ *sigma-finite-measure*
⟨*proof*⟩

Conditional expectations are very often used in probability spaces. This is a special case of the previous one, as we prove now.

locale *finite-measure-subalgebra* = *finite-measure* +
fixes *F*::'*a measure*
assumes *subalg*: *subalgebra M F*

lemma *finite-measure-subalgebra-is-sigma-finite*:
assumes *finite-measure-subalgebra M F*
shows *sigma-finite-subalgebra M F*
⟨*proof*⟩

sublocale *finite-measure-subalgebra* ⊆ *sigma-finite-subalgebra*
⟨*proof*⟩

context *sigma-finite-subalgebra*
begin

The next lemma is arguably the most fundamental property of conditional expectation: when computing an expectation against an F-measurable function, it is equivalent to work with a function or with its F-conditional expectation.

This property (even for bounded test functions) characterizes conditional expectations, as the second lemma below shows. From this point on, we will only work with it, and forget completely about the definition using Radon-Nikodym derivatives.

lemma *nn-cond-exp-intg*:
assumes [*measurable*]: *f* ∈ *borel-measurable F g* ∈ *borel-measurable M*
shows (∫⁺ *x*. *f x* ∗ *nn-cond-exp M F g x* ∂*M*) = (∫⁺ *x*. *f x* ∗ *g x* ∂*M*)
⟨*proof*⟩

lemma *nn-cond-exp-charact*:
assumes ⋀*A*. *A* ∈ *sets F* ⟹ (∫⁺ *x* ∈ *A*. *f x* ∂*M*) = (∫⁺ *x* ∈ *A*. *g x* ∂*M*) **and**
[*measurable*]: *f* ∈ *borel-measurable M g* ∈ *borel-measurable F*
shows *AE x in M*. *g x* = *nn-cond-exp M F f x*
⟨*proof*⟩

lemma *nn-cond-exp-F-meas*:
assumes *f* ∈ *borel-measurable F*

shows *AE x in M. f x = nn-cond-exp M F f x*
⟨*proof*⟩

lemma *nn-cond-exp-prod*:
assumes [*measurable*]: *f ∈ borel-measurable F g ∈ borel-measurable M*
shows *AE x in M. f x * nn-cond-exp M F g x = nn-cond-exp M F (λx. f x * g x) x*
⟨*proof*⟩

lemma *nn-cond-exp-sum*:
assumes [*measurable*]: *f ∈ borel-measurable M g ∈ borel-measurable M*
shows *AE x in M. nn-cond-exp M F f x + nn-cond-exp M F g x = nn-cond-exp M F (λx. f x + g x) x*
⟨*proof*⟩

lemma *nn-cond-exp-cong*:
assumes *AE x in M. f x = g x*
and [*measurable*]: *f ∈ borel-measurable M g ∈ borel-measurable M*
shows *AE x in M. nn-cond-exp M F f x = nn-cond-exp M F g x*
⟨*proof*⟩

lemma *nn-cond-exp-mono*:
assumes *AE x in M. f x ≤ g x*
and [*measurable*]: *f ∈ borel-measurable M g ∈ borel-measurable M*
shows *AE x in M. nn-cond-exp M F f x ≤ nn-cond-exp M F g x*
⟨*proof*⟩

lemma *nested-subalg-is-sigma-finite*:
assumes *subalgebra M G subalgebra G F*
shows *sigma-finite-subalgebra M G*
⟨*proof*⟩

lemma *nn-cond-exp-nested-subalg*:
assumes *subalgebra M G subalgebra G F*
and [*measurable*]: *f ∈ borel-measurable M*
shows *AE x in M. nn-cond-exp M F f x = nn-cond-exp M F (nn-cond-exp M G f) x*
⟨*proof*⟩

end

28.3 Real conditional expectation

Once conditional expectations of positive functions are defined, the definition for real-valued functions follows readily, by taking the difference of positive and negative parts. One could also define a conditional expectation of vector-space valued functions, as in `Bochner_Integral`, but since the real-valued case is the most important, and quicker to formalize, I con-

centrate on it. (It is also essential for the case of the most general Pettis integral.)

definition *real-cond-exp* :: *'a measure* ⇒ *'a measure* ⇒ (*'a* ⇒ *real*) ⇒ (*'a* ⇒ *real*)
where
real-cond-exp M F f =
(λ*x*. *enn2real*(*nn-cond-exp M F* (λ*x*. *ennreal* (*f x*)) *x*) − *enn2real*(*nn-cond-exp M F* (λ*x*. *ennreal* (−*f x*)) *x*))

lemma
shows *borel-measurable-cond-exp* [*measurable*]: *real-cond-exp M F f* ∈ *borel-measurable F*
and *borel-measurable-cond-exp2* [*measurable*]: *real-cond-exp M F f* ∈ *borel-measurable M*
⟨*proof*⟩

context *sigma-finite-subalgebra*
begin

lemma *real-cond-exp-abs*:
assumes [*measurable*]: *f* ∈ *borel-measurable M*
shows *AE x in M*. *abs*(*real-cond-exp M F f x*) ≤ *nn-cond-exp M F* (λ*x*. *ennreal* (*abs*(*f x*))) *x*
⟨*proof*⟩

The next lemma shows that the conditional expectation is an F-measurable function whose average against an F-measurable function f coincides with the average of the original function against f. It is obtained as a consequence of the same property for the positive conditional expectation, taking the difference of the positive and the negative part. The proof is given first assuming $f \geq 0$ for simplicity, and then extended to the general case in the subsequent lemma. The idea of the proof is essentially trivial, but the implementation is slightly tedious as one should check all the integrability properties of the different parts, and go back and forth between positive integral and signed integrals, and between real-valued functions and ennreal-valued functions.

Once this lemma is available, we will use it to characterize the conditional expectation, and never come back to the original technical definition, as we did in the case of the nonnegative conditional expectation.

lemma *real-cond-exp-intg-fpos*:
assumes *integrable M* (λ*x*. *f x* ∗ *g x*) **and** *f-pos*[*simp*]: ⋀*x*. *f x* ≥ *0* **and**
[*measurable*]: *f* ∈ *borel-measurable F g* ∈ *borel-measurable M*
shows *integrable M* (λ*x*. *f x* ∗ *real-cond-exp M F g x*)
(∫ *x*. *f x* ∗ *real-cond-exp M F g x* ∂*M*) = (∫ *x*. *f x* ∗ *g x* ∂*M*)
⟨*proof*⟩

lemma *real-cond-exp-intg*:
assumes *integrable M* (λ*x*. *f x* ∗ *g x*) **and**

[*measurable*]: $f \in$ *borel-measurable* F $g \in$ *borel-measurable* M
shows *integrable* M $(\lambda x.\ f\ x * $ *real-cond-exp* $M\ F\ g\ x)$
$(\int\ x.\ f\ x * $ *real-cond-exp* $M\ F\ g\ x\ \partial M) = (\int\ x.\ f\ x * g\ x\ \partial M)$
⟨*proof*⟩

lemma *real-cond-exp-intA*:
assumes [*measurable*]: *integrable* $M\ f\ A \in$ *sets* F
shows $(\int\ x \in A.\ f\ x\ \partial M) = (\int x \in A.$ *real-cond-exp* $M\ F\ f\ x\ \partial M)$
⟨*proof*⟩

lemma *real-cond-exp-int* [*intro*]:
assumes *integrable* $M\ f$
shows *integrable* M (*real-cond-exp* $M\ F\ f$) $(\int x.$ *real-cond-exp* $M\ F\ f\ x\ \partial M) = (\int x.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma *real-cond-exp-charact*:
assumes $\bigwedge A.\ A \in$ *sets* $F \Longrightarrow (\int\ x \in A.\ f\ x\ \partial M) = (\int\ x \in A.\ g\ x\ \partial M)$
and [*measurable*]: *integrable* $M\ f$ *integrable* $M\ g$
$g \in$ *borel-measurable* F
shows *AE* x *in* M. *real-cond-exp* $M\ F\ f\ x = g\ x$
⟨*proof*⟩

lemma *real-cond-exp-F-meas* [*intro*, *simp*]:
assumes *integrable* $M\ f$
$f \in$ *borel-measurable* F
shows *AE* x *in* M. *real-cond-exp* $M\ F\ f\ x = f\ x$
⟨*proof*⟩

lemma *real-cond-exp-mult*:
assumes [*measurable*]:$f \in$ *borel-measurable* F $g \in$ *borel-measurable* M *integrable* M $(\lambda x.\ f\ x * g\ x)$
shows *AE* x *in* M. *real-cond-exp* $M\ F$ $(\lambda x.\ f\ x * g\ x)\ x = f\ x *$ *real-cond-exp* $M\ F\ g\ x$
⟨*proof*⟩

lemma *real-cond-exp-add* [*intro*]:
assumes [*measurable*]: *integrable* $M\ f$ *integrable* $M\ g$
shows *AE* x *in* M. *real-cond-exp* $M\ F$ $(\lambda x.\ f\ x + g\ x)\ x =$ *real-cond-exp* $M\ F\ f\ x$ $+$ *real-cond-exp* $M\ F\ g\ x$
⟨*proof*⟩

lemma *real-cond-exp-cong*:
assumes *ae*: *AE* x *in* M. $f\ x = g\ x$ **and** [*measurable*]: $f \in$ *borel-measurable* M $g \in$ *borel-measurable* M
shows *AE* x *in* M. *real-cond-exp* $M\ F\ f\ x =$ *real-cond-exp* $M\ F\ g\ x$
⟨*proof*⟩

lemma *real-cond-exp-cmult* [*intro*, *simp*]:

fixes *c*::*real*
assumes *integrable M f*
shows *AE x in M. real-cond-exp M F* (λ*x. c* ∗ *f x*) *x* = *c* ∗ *real-cond-exp M F f x*
⟨*proof*⟩

lemma *real-cond-exp-cdiv* [*intro*, *simp*]:
fixes *c*::*real*
assumes *integrable M f*
shows *AE x in M. real-cond-exp M F* (λ*x. f x* / *c*) *x* = *real-cond-exp M F f x* / *c*
⟨*proof*⟩

lemma *real-cond-exp-diff* [*intro*, *simp*]:
assumes [*measurable*]: *integrable M f integrable M g*
shows *AE x in M. real-cond-exp M F* (λ*x. f x* − *g x*) *x* = *real-cond-exp M F f x* − *real-cond-exp M F g x*
⟨*proof*⟩

lemma *real-cond-exp-pos* [*intro*]:
assumes *AE x in M. f x* ≥ *0* **and** [*measurable*]: *f* ∈ *borel-measurable M*
shows *AE x in M. real-cond-exp M F f x* ≥ *0*
⟨*proof*⟩

lemma *real-cond-exp-mono*:
assumes *AE x in M. f x* ≤ *g x* **and** [*measurable*]: *integrable M f integrable M g*
shows *AE x in M. real-cond-exp M F f x* ≤ *real-cond-exp M F g x*
⟨*proof*⟩

lemma (**in** −) *measurable-P-restriction* [*measurable* (*raw*)]:
assumes [*measurable*]: *Measurable.pred M P A* ∈ *sets M*
shows {*x* ∈ *A. P x*} ∈ *sets M*
⟨*proof*⟩

lemma *real-cond-exp-gr-c*:
assumes [*measurable*]: *integrable M f*
and *AE*: *AE x in M. f x* > *c*
shows *AE x in M. real-cond-exp M F f x* > *c*
⟨*proof*⟩

lemma *real-cond-exp-less-c*:
assumes [*measurable*]: *integrable M f*
and *AE x in M. f x* < *c*
shows *AE x in M. real-cond-exp M F f x* < *c*
⟨*proof*⟩

lemma *real-cond-exp-ge-c*:
assumes [*measurable*]: *integrable M f*
and *AE x in M. f x* ≥ *c*

shows *AE x in M. real-cond-exp M F f x ≥ c*
⟨*proof*⟩

lemma *real-cond-exp-le-c*:
assumes [*measurable*]: *integrable M f*
and *AE x in M. f x ≤ c*
shows *AE x in M. real-cond-exp M F f x ≤ c*
⟨*proof*⟩

lemma *real-cond-exp-mono-strict*:
assumes *AE x in M. f x < g x* **and** [*measurable*]: *integrable M f integrable M g*
shows *AE x in M. real-cond-exp M F f x < real-cond-exp M F g x*
⟨*proof*⟩

lemma *real-cond-exp-nested-subalg* [*intro*, *simp*]:
assumes *subalgebra M G subalgebra G F*
and [*measurable*]: *integrable M f*
shows *AE x in M. real-cond-exp M F (real-cond-exp M G f) x = real-cond-exp M F f x*
⟨*proof*⟩

lemma *real-cond-exp-sum* [*intro*, *simp*]:
fixes *f*::*'b ⇒ 'a ⇒ real*
assumes [*measurable*]: *⋀i. integrable M (f i)*
shows *AE x in M. real-cond-exp M F (λx. ∑ i∈I. f i x) x = (∑ i∈I. real-cond-exp M F (f i) x)*
⟨*proof*⟩

Jensen's inequality, describing the behavior of the integral under a convex function, admits a version for the conditional expectation, as follows.

theorem *real-cond-exp-jensens-inequality*:
fixes *q :: real ⇒ real*
assumes *X*: *integrable M X AE x in M. X x ∈ I*
assumes *I*: *I = {a <..< b} ∨ I = {a <..} ∨ I = {..< b} ∨ I = UNIV*
assumes *q*: *integrable M (λx. q (X x)) convex-on I q q ∈ borel-measurable borel*
shows *AE x in M. real-cond-exp M F X x ∈ I*
AE x in M. q (real-cond-exp M F X x) ≤ real-cond-exp M F (λx. q (X x)) x
⟨*proof*⟩

Jensen's inequality does not imply that $q(E(X|F))$ is integrable, as it only proves an upper bound for it. Indeed, this is not true in general, as the following counterexample shows:

on $[1, \infty)$ with Lebesgue measure, let F be the sigma-algebra generated by the intervals $[n, n+1)$ for integer n. Let $q(x) = -\sqrt{x}$ for $x \geq 0$. Define X which is equal to $1/n$ over $[n, n+1/n)$ and 2^{-n} on $[n+1/n, n+1)$. Then X is integrable as $\sum 1/n^2 < \infty$, and $q(X)$ is integrable as $\sum 1/n^{3/2} < \infty$. On the other hand, $E(X|F)$ is essentially equal to $1/n^2$ on $[n, n+1)$ (we neglect the term 2^{-n}, we only put it there because X should take its values

in $I = (0, \infty)$). Hence, $q(E(X|F))$ is equal to $-1/n$ on $[n, n+1)$, hence it is not integrable.

However, this counterexample is essentially the only situation where this function is not integrable, as shown by the next lemma.

lemma *integrable-convex-cond-exp*:
fixes q :: *real* ⇒ *real*
assumes X: *integrable M X AE x in M. X x ∈ I*
assumes I: $I = \{a <..< b\} \vee I = \{a <..\} \vee I = \{..< b\} \vee I = UNIV$
assumes q: *integrable M* (λ*x. q* (*X x*)) *convex-on I q q ∈ borel-measurable borel*
assumes H: *emeasure M* (*space M*) = ∞ ⟹ *0 ∈ I*
shows *integrable M* (λ*x. q* (*real-cond-exp M F X x*))
⟨*proof*⟩

end

end

theory *Essential-Supremum*
imports *HOL−Analysis.Analysis*
begin

lemma *ae-filter-eq-bot-iff*: *ae-filter M = bot* ⟷ *emeasure M* (*space M*) = *0*
⟨*proof*⟩

29 The essential supremum

In this paragraph, we define the essential supremum and give its basic properties. The essential supremum of a function is its maximum value if one is allowed to throw away a set of measure 0. It is convenient to define it to be infinity for non-measurable functions, as it allows for neater statements in general. This is a prerequisiste to define the space L^∞.

definition *esssup*::*'a measure* ⇒ (*'a* ⇒ *'b*::{*second-countable-topology*, *dense-linorder*, *linorder-topology*, *complete-linorder*}) ⇒ *'b*
where *esssup M f* = (*if f ∈ borel-measurable M then Limsup* (*ae-filter M*) *f else top*)

lemma *esssup-non-measurable*: *f ∉ M →M borel* ⟹ *esssup M f = top*
⟨*proof*⟩

lemma *esssup-eq-AE*:
assumes *f*: *f ∈ M →M borel* **shows** *esssup M f = Inf* {*z. AE x in M. f x ≤ z*}
⟨*proof*⟩

lemma *esssup-eq*: *f ∈ M →M borel* ⟹ *esssup M f = Inf* {*z. emeasure M* {*x ∈ space M. f x > z*} = *0*}
⟨*proof*⟩

lemma *esssup-zero-measure*:
emeasure M {*x* ∈ *space M*. *f x* > *esssup M f*} = *0*
⟨*proof*⟩

lemma *esssup-AE*: *AE x in M*. *f x* ≤ *esssup M f*
⟨*proof*⟩

lemma *esssup-pos-measure*:
f ∈ *borel-measurable M* ⟹ *z* < *esssup M f* ⟹ *emeasure M* {*x* ∈ *space M*. *f x* > *z*} > *0*
⟨*proof*⟩

lemma *esssup-I* [*intro*]: *f* ∈ *borel-measurable M* ⟹ *AE x in M*. *f x* ≤ *c* ⟹ *esssup M f* ≤ *c*
⟨*proof*⟩

lemma *esssup-AE-mono*: *f* ∈ *borel-measurable M* ⟹ *AE x in M*. *f x* ≤ *g x* ⟹ *esssup M f* ≤ *esssup M g*
⟨*proof*⟩

lemma *esssup-mono*: *f* ∈ *borel-measurable M* ⟹ (⋀*x*. *f x* ≤ *g x*) ⟹ *esssup M f* ≤ *esssup M g*
⟨*proof*⟩

lemma *esssup-AE-cong*:
f ∈ *borel-measurable M* ⟹ *g* ∈ *borel-measurable M* ⟹ *AE x in M*. *f x* = *g x* ⟹ *esssup M f* = *esssup M g*
⟨*proof*⟩

lemma *esssup-const*: *emeasure M* (*space M*) ≠ *0* ⟹ *esssup M* (λ*x*. *c*) = *c*
⟨*proof*⟩

lemma *esssup-cmult*: **assumes** *c* > (*0*::*real*) **shows** *esssup M* (λ*x*. *c* ∗ *f x*::*ereal*) = *c* ∗ *esssup M f*
⟨*proof*⟩

lemma *esssup-add*:
esssup M (λ*x*. *f x* + *g x*::*ereal*) ≤ *esssup M f* + *esssup M g*
⟨*proof*⟩

lemma *esssup-zero-space*:
emeasure M (*space M*) = *0* ⟹ *f* ∈ *borel-measurable M* ⟹ *esssup M f* = (−∞::*ereal*)
⟨*proof*⟩

end

30 Stopping times

theory *Stopping-Time*
imports *HOL−Analysis.Analysis*
begin

30.1 Stopping Time

This is also called strong stopping time. Then stopping time is T with alternative is $T\ x < t$ measurable.

definition *stopping-time* :: ('*t*::*linorder* ⇒ '*a measure*) ⇒ ('*a* ⇒ '*t*) ⇒ *bool*
where
stopping-time F T = (∀ *t*. *Measurable.pred* (*F t*) (λ*x*. *T x* ≤ *t*))

lemma *stopping-time-cong*: (⋀*t x*. *x* ∈ *space* (*F t*) ⟹ *T x* = *S x*) ⟹ *stopping-time F T* = *stopping-time F S*
⟨*proof*⟩

lemma *stopping-timeD*: *stopping-time F T* ⟹ *Measurable.pred* (*F t*) (λ*x*. *T x* ≤ *t*)
⟨*proof*⟩

lemma *stopping-timeD2*: *stopping-time F T* ⟹ *Measurable.pred* (*F t*) (λ*x*. *t* < *T x*)
⟨*proof*⟩

lemma *stopping-timeI*[*intro?*]: (⋀*t*. *Measurable.pred* (*F t*) (λ*x*. *T x* ≤ *t*)) ⟹ *stopping-time F T*
⟨*proof*⟩

lemma *measurable-stopping-time*:
fixes *T* :: '*a* ⇒ '*t*::{*linorder-topology*, *second-countable-topology*}
assumes *T*: *stopping-time F T*
and *M*: ⋀*t*. *sets* (*F t*) ⊆ *sets M* ⋀*t*. *space* (*F t*) = *space M*
shows *T* ∈ *M* →$_M$ *borel*
⟨*proof*⟩

lemma *stopping-time-const*: *stopping-time F* (λ*x*. *c*)
⟨*proof*⟩

lemma *stopping-time-min*:
stopping-time F T ⟹ *stopping-time F S* ⟹ *stopping-time F* (λ*x*. *min* (*T x*) (*S x*))
⟨*proof*⟩

lemma *stopping-time-max*:
stopping-time F T ⟹ *stopping-time F S* ⟹ *stopping-time F* (λ*x*. *max* (*T x*) (*S x*))
⟨*proof*⟩

31 Filtration

locale *filtration* =
  fixes Ω :: *'a set* **and** F :: *'t*::{*linorder-topology*, *second-countable-topology*} $\Rightarrow$ *'a measure*
  assumes *space-F*: $\bigwedge i.$ *space* $(F\ i) = \Omega$
  assumes *sets-F-mono*: $\bigwedge i\ j.\ i \leq j \Longrightarrow$ *sets* $(F\ i) \leq$ *sets* $(F\ j)$
begin

31.1 σ-algebra of a Stopping Time

definition *pre-sigma* :: $('a \Rightarrow 't) \Rightarrow$ *'a measure*
where
  pre-sigma $T =$ *sigma* Ω $\{A.\ \forall t.\ \{\omega{\in}A.\ T\ \omega \leq t\} \in$ *sets* $(F\ t)\}$

lemma *space-pre-sigma*: *space* (*pre-sigma* T) $= \Omega$
  ⟨*proof*⟩

lemma *measure-pre-sigma*[*simp*]: *emeasure* (*pre-sigma* T) $= (\lambda\text{-}.\ 0)$
  ⟨*proof*⟩

lemma *sigma-algebra-pre-sigma*:
  assumes T: *stopping-time* $F\ T$
  shows *sigma-algebra* Ω $\{A.\ \forall t.\ \{\omega{\in}A.\ T\ \omega \leq t\} \in$ *sets* $(F\ t)\}$
  ⟨*proof*⟩

lemma *sets-pre-sigma*: *stopping-time* $F\ T \Longrightarrow$ *sets* (*pre-sigma* T) $= \{A.\ \forall t.\ \{\omega{\in}A.\ T\ \omega \leq t\} \in$ *sets* $(F\ t)\}$
  ⟨*proof*⟩

lemma *sets-pre-sigmaI*: *stopping-time* $F\ T \Longrightarrow (\bigwedge t.\ \{\omega{\in}A.\ T\ \omega \leq t\} \in$ *sets* $(F\ t)) \Longrightarrow A \in$ *sets* (*pre-sigma* T)
  ⟨*proof*⟩

lemma *pred-pre-sigmaI*:
  assumes T: *stopping-time* $F\ T$
  shows $(\bigwedge t.$ *Measurable.pred* $(F\ t)\ (\lambda\omega.\ P\ \omega \wedge T\ \omega \leq t)) \Longrightarrow$ *Measurable.pred* (*pre-sigma* T) P
  ⟨*proof*⟩

lemma *sets-pre-sigmaD*: *stopping-time* $F\ T \Longrightarrow A \in$ *sets* (*pre-sigma* T) $\Longrightarrow \{\omega{\in}A.\ T\ \omega \leq t\} \in$ *sets* $(F\ t)$
  ⟨*proof*⟩

lemma *stopping-time-le-const*: *stopping-time* $F\ T \Longrightarrow s \leq t \Longrightarrow$ *Measurable.pred* $(F\ t)\ (\lambda\omega.\ T\ \omega \leq s)$
  ⟨*proof*⟩

lemma *measurable-stopping-time-pre-sigma*:
  assumes T: *stopping-time* $F\ T$ **shows** $T \in$ *pre-sigma* $T \rightarrow_M$ *borel*

⟨*proof*⟩

lemma *mono-pre-sigma*:
assumes T: *stopping-time F T* **and** S: *stopping-time F S*
and *le*: $\bigwedge\omega.\ \omega \in \Omega \Longrightarrow T\ \omega \leq S\ \omega$
shows *sets* (*pre-sigma T*) ⊆ *sets* (*pre-sigma S*)
⟨*proof*⟩

lemma *stopping-time-less-const*:
assumes T: *stopping-time F T* **shows** *Measurable.pred* (*F t*) (λω. *T ω* < *t*)
⟨*proof*⟩

lemma *stopping-time-eq-const*: *stopping-time F T* ⟹ *Measurable.pred* (*F t*) (λω. *T ω* = *t*)
⟨*proof*⟩

lemma *stopping-time-less*:
assumes T: *stopping-time F T* **and** S: *stopping-time F S*
shows *Measurable.pred* (*pre-sigma T*) (λω. *T ω* < *S ω*)
⟨*proof*⟩

end

lemma *stopping-time-SUP-enat*:
fixes *T* :: *nat* ⇒ (′*a* ⇒ *enat*)
shows (⋀*i*. *stopping-time F* (*T i*)) ⟹ *stopping-time F* (*SUP i*. *T i*)
⟨*proof*⟩

lemma *less-eSuc-iff*: *a* < *eSuc b* ⟷ (*a* ≤ *b* ∧ *a* ≠ ∞)
⟨*proof*⟩

lemma *stopping-time-Inf-enat*:
fixes *F* :: *enat* ⇒ ′*a measure*
assumes *F*: *filtration* Ω *F*
assumes *P*: ⋀*i*. *Measurable.pred* (*F i*) (*P i*)
shows *stopping-time F* (λω. *Inf* {*i*. *P i ω*})
⟨*proof*⟩

lemma *stopping-time-Inf-nat*:
fixes *F* :: *nat* ⇒ ′*a measure*
assumes *F*: *filtration* Ω *F*
assumes *P*: ⋀*i*. *Measurable.pred* (*F i*) (*P i*) **and** *wf*: ⋀*i ω*. *ω* ∈ Ω ⟹ ∃ *n*. *P n ω*
shows *stopping-time F* (λω. *Inf* {*i*. *P i ω*})
⟨*proof*⟩

end

theory *Probability*
imports
Central-Limit-Theorem
Discrete-Topology
PMF-Impl
Projective-Limit
Random-Permutations
SPMF
Product-PMF
Hoeffding
Stream-Space
Tree-Space
Conditional-Expectation
Essential-Supremum
Stopping-Time
begin

end