

Exponentiation

Definition

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Signature 0.1. Let n, m be natural numbers. n^m is a natural number.

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Axiom 0.2. Let n be a natural number. Then $n^0 = 1$.

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Axiom 0.3. Let n, m be natural numbers. Then $n^{m+1} = n^m \cdot n$.

Computation Laws

Exponentiation with 0, 1 and 2

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Proposition 0.4. Let n be a natural number. Assume $n \neq 0$. Then $0^n = 0$.

Proof. Take a natural number m such that $n = m + 1$. Then $0^n = 0^{m+1} = 0^m \cdot 0 = 0$. Indeed $0^{m+1} = 0^m \cdot 0$. \square

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Proposition 0.5. Let n be a natural number. Then $1^n = 1$.

Proof. Define $\Phi = \{n' \in \mathbb{N} \mid 1^{n'} = 1\}$.

(1) Φ contains 0.

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.

Proof. Let $n' \in \Phi$. Then $1^{n'+1} = 1^{n'} \cdot 1 = 1 \cdot 1 = 1$. Qed.

Hence every natural number is contained in Φ (by ARITHMETIC_01_4764664342773760). Thus $1^n = 1$. \square

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Proposition 0.6. Let n be a natural number. Then $n^1 = n$.

Proof. We have $n^1 = n^{0+1} = n^0 \cdot n = 1 \cdot n = n$. □

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Proposition 0.7. Let n be a natural number. Then $n^2 = n \cdot n$.

Proof. We have $n^2 = n^{1+1} = n^1 \cdot n = n \cdot n$. □

Sums as Exponents

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Proposition 0.8. Let n, m, k be natural numbers. Then $k^{n+m} = k^n \cdot k^m$.

Proof. Define $\Phi = \{m' \in \mathbb{N} \mid k^{n+m'} = k^n \cdot k^{m'}\}$.

(1) Φ contains 0.

Indeed $k^{n+0} = k^n = k^n \cdot 1 = k^n \cdot k^0$.

(2) For all $m' \in \Phi$ we have $m' + 1 \in \Phi$.

Proof. Let $m' \in \Phi$. Then

$$\begin{aligned} & k^{n+(m'+1)} \\ &= k^{(n+m')+1} \\ &= k^{n+m'} \cdot k \\ &= (k^n \cdot k^{m'}) \cdot k \\ &= k^n \cdot (k^{m'} \cdot k) \\ &= k^n \cdot k^{m'+1}. \end{aligned}$$

Qed.

Hence every natural number is contained in Φ (by ARITHMETIC_01_4764664342773760). Thus $k^{n+m} = k^n \cdot k^m$. □

Products as Exponents

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Proposition 0.9. Let n, m, k be natural numbers. Then $n^{m \cdot k} = (n^m)^k$.

Proof. Define $\Phi = \{k' \in \mathbb{N} \mid n^{m \cdot k'} = (n^m)^{k'}\}$.

(1) Φ contains 0. Indeed $(n^m)^0 = 1 = n^0 = n^{m \cdot 0}$.

(2) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.

Proof. Let $k' \in \Phi$. Then

$$\begin{aligned} & (n^m)^{k'+1} \\ &= (n^m)^{k'} \cdot n^m \\ &= n^{m \cdot k'} \cdot n^m \\ &= n^{(m \cdot k') + m} \\ &= n^{m \cdot (k'+1)}. \end{aligned}$$

Qed.

Therefore every natural number is contained in Φ (by ARITHMETIC_01_47 64664342773760). Consequently $n^{m \cdot k} = (n^m)^k$. \square

Products as Base

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Proposition 0.10. Let n, m, k be natural numbers. Then $(n \cdot m)^k = n^k \cdot m^k$.

Proof. Define $\Phi = \{k' \in \mathbb{N} \mid (n \cdot m)^{k'} = n^{k'} \cdot m^{k'}\}$.

(1) Φ contains 0. Indeed $((n \cdot m)^0) = 1 = 1 \cdot 1 = n^0 \cdot m^0$.

(2) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.

Proof. Let $k' \in \Phi$.

Let us show that $(n^{k'} \cdot m^{k'}) \cdot (n \cdot m) = (n^{k'} \cdot n) \cdot (m^{k'} \cdot m)$.

$$\begin{aligned} & (n^{k'} \cdot m^{k'}) \cdot (n \cdot m) \\ &= ((n^{k'} \cdot m^{k'}) \cdot n) \cdot m \\ &= (n^{k'} \cdot (m^{k'} \cdot n)) \cdot m \\ &= (n^{k'} \cdot (n \cdot m^{k'})) \cdot m \\ &= ((n^{k'} \cdot n) \cdot m^{k'}) \cdot m \\ &= (n^{k'} \cdot n) \cdot (m^{k'} \cdot m). \end{aligned}$$

Qed.

Hence

$$(n \cdot m)^{k'+1}$$

$$\begin{aligned}
&= (n \cdot m)^{k'} \cdot (n \cdot m) \\
&= (n^{k'} \cdot m^{k'}) \cdot (n \cdot m) \\
&= (n^{k'} \cdot n) \cdot (m^{k'} \cdot m) \\
&= n^{k'+1} \cdot m^{k'+1}.
\end{aligned}$$

Qed.

Therefore every natural number is contained in Φ (by ARITHMETIC_01_4764664342773760). Consequently $(n \cdot m)^k = n^k \cdot m^k$. \square

Zeroes of Exponentiation

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Proposition 0.11. Let n, m be natural numbers. Then $n^m = 0$ iff $n = 0$ and $m \neq 0$.

Proof. Case $n^m = 0$. Define $\Phi = \{m' \in \mathbb{N} \mid \text{if } n^{m'} = 0 \text{ then } n = 0 \text{ and } m' \neq 0\}$.

(1) Φ contains 0. Indeed if $n^0 = 0$ then we have a contradiction.

(2) For all $m' \in \Phi$ we have $m' + 1 \in \Phi$.

Proof. Let $m' \in \Phi$.

Let us show that if $n^{m'+1} = 0$ then $n = 0$ and $m' + 1 \neq 0$. Assume $n^{m'+1} = 0$. Then $0 = n^{m'+1} = n^{m'} \cdot n$. Hence $n^{m'} = 0$ or $n = 0$. We have $m' + 1 \neq 0$ and if $n^{m'} = 0$ then $n = 0$. Hence $n = 0$ and $m' + 1 \neq 0$. End. Qed.

Thus every natural number is contained in Φ (by ARITHMETIC_01_4764664342773760). Consequently $m \in \Phi$. Therefore $n = 0$ and $m \neq 0$. End.

Case $n = 0$ and $m \neq 0$. Take a natural number k such that $m = k + 1$. Then $n^m = n^{k+1} = n^k \cdot n = 0^k \cdot 0 = 0$. End. \square