

# Computation Laws for Cartesian Products

## Subclass Laws

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**Proposition 0.1.** Let  $A, B, C$  be classes. IF  $A \subseteq B$  then  $A \times C \subseteq B \times C$ .

*Proof.* Assume  $A \subseteq B$ . Let  $x \in A \times C$ . Take  $a \in A$  and  $c \in C$  such that  $x = (a, c)$ . Then  $a \in B$ . Hence  $(a, c) \in B \times C$ .  $\square$

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**Proposition 0.2.** Let  $A, A', B, B'$  be classes. Assume that  $A$  and  $A'$  are nonempty. Then  $(A \times A') \subseteq (B \times B')$  iff  $A \subseteq B$  and  $A' \subseteq B'$ .

*Proof.* Case  $(A \times A') \subseteq (B \times B')$ . Let us show that for all  $a \in A$  and all  $a' \in A'$  we have  $a \in B$  and  $a' \in B'$ . Let  $a \in A$  and  $a' \in A'$ . Then  $(a, a') \in A \times A'$ . Hence  $(a, a') \in B \times B'$ . Thus  $a \in B$  and  $a' \in B'$ . End.

Case  $A \subseteq B$  and  $A' \subseteq B'$ . Let  $x \in A \times A'$ . Take  $a \in A$  and  $a' \in A'$  such that  $x = (a, a')$ . Then  $a \in B$  and  $a' \in B'$ . Hence  $(a, a') \in B \times B'$ . End.  $\square$

## Distributivity of Product and Union

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**Proposition 0.3.** Let  $A, B, C$  be classes. Then  $(A \cup B) \times C = (A \times C) \cup (B \times C)$ .

*Proof.* Let us show that  $((A \cup B) \times C) \subseteq (A \times C) \cup (B \times C)$ . Let  $x \in (A \cup B) \times C$ . Take  $y \in A \cup B$  and  $c \in C$  such that  $x = (y, c)$ . Then  $y \in A$  or  $y \in B$ . If  $y \in A$  then  $x \in A \times C$  and if  $y \in B$  then  $x \in B \times C$ . Hence  $x \in A \times C$  or  $x \in B \times C$ . Thus  $x \in (A \times C) \cup (B \times C)$ . End.

Let us show that  $((A \times C) \cup (B \times C)) \subseteq (A \cup B) \times C$ . Let  $x \in (A \times C) \cup (B \times C)$ . Then  $x \in A \times C$  or  $x \in B \times C$ . Take objects  $y, c$  such that  $x = (y, c)$ . Then  $(y \in A \text{ or } y \in B)$  and  $c \in C$ . Hence  $y \in A \cup B$ . Thus  $x \in (A \cup B) \times C$ . End.  $\square$

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**Proposition 0.4.** Let  $A, B, C$  be classes. Then  $A \times (B \cup C) = (A \times B) \cup (A \times C)$ .

*Proof.* Let us show that  $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ . Let  $x \in A \times (B \cup C)$ . Take  $a \in A$  and  $y \in B \cup C$  such that  $x = (a, y)$ . Then  $y \in B$  or  $y \in C$ . Hence  $x \in A \times B$  or  $x \in A \times C$ . Indeed if  $y \in B$  then  $x \in A \times B$  and if  $y \in C$  then  $x \in A \times C$ . Thus  $x \in (A \times B) \cup (A \times C)$ . End.

Let us show that  $((A \times B) \cup (A \times C)) \subseteq A \times (B \cup C)$ . Let  $x \in (A \times B) \cup (A \times C)$ . Then  $x \in A \times B$  or  $x \in A \times C$ . Take objects  $a, y$  such that  $x = (a, y)$ . Then  $a \in A$  and  $(y \in B \text{ or } y \in C)$ . Hence  $x \in A \times (B \cup C)$ . End.  $\square$

## Distributivity of Product and Intersection

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**Proposition 0.5.** Let  $A, B, C$  be classes. Then  $(A \cap B) \times C = (A \times C) \cap (B \times C)$ .

*Proof.* Let us show that  $((A \cap B) \times C) \subseteq (A \times C) \cap (B \times C)$ . Let  $x \in (A \cap B) \times C$ . Take  $y \in A \cap B$  and  $c \in C$  such that  $x = (y, c)$ . Then  $y \in A$  and  $y \in B$ . Hence  $x \in A \times C$  and  $x \in B \times C$ . Thus  $x \in (A \times C) \cap (B \times C)$ . End.

Let us show that  $((A \times C) \cap (B \times C)) \subseteq (A \cap B) \times C$ . Let  $x \in (A \times C) \cap (B \times C)$ . Then  $x \in A \times C$  and  $x \in B \times C$ . Take objects  $y, z$  such that  $x = (y, z)$ . Then  $(y \in A \text{ and } y \in B)$  and  $z \in C$ . Hence  $y \in A \cap B$ . Thus  $x \in (A \cap B) \times C$ . End.  $\square$

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**Proposition 0.6.** Let  $A, B, C$  be classes. Then  $A \times (B \cap C) = (A \times B) \cap (A \times C)$ .

*Proof.* Let us show that  $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ . Let  $x \in A \times (B \cap C)$ . Take  $a \in A$  and  $b \in B \cap C$  such that  $x = (a, b)$ . Then  $b \in B$  and  $b \in C$ . Hence  $x \in A \times B$  and  $x \in A \times C$ . Thus  $x \in (A \times B) \cap (A \times C)$ . End.

Let us show that  $((A \times B) \cap (A \times C)) \subseteq A \times (B \cap C)$ . Let  $x \in (A \times B) \cap (A \times C)$ . Then  $x \in A \times B$  and  $x \in A \times C$ . Take objects  $y, z$  such that  $x = (y, z)$ . Then  $y \in A$  and  $(z \in B \text{ and } z \in C)$ . Hence  $x \in A \times (B \cap C)$ . End.  $\square$

## Distributivity of Product and Complement

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**Proposition 0.7.** Let  $A, B, C$  be classes. Then  $(A \setminus B) \times C = (A \times C) \setminus (B \times C)$ .

*Proof.* Let us show that  $((A \setminus B) \times C) \subseteq (A \times C) \setminus (B \times C)$ . Let  $x \in (A \setminus B) \times C$ . Take  $a \in A \setminus B$  and  $c \in C$  such that  $x = (a, c)$ . Then  $a \in A$  and  $a \notin B$ . Hence  $x \in A \times C$  and  $x \notin B \times C$ . Thus  $x \in (A \times C) \setminus (B \times C)$ . End.

Let us show that  $((A \times C) \setminus (B \times C)) \subseteq (A \setminus B) \times C$ . Let  $x \in (A \times C) \setminus (B \times C)$ . Then  $x \in A \times C$  and  $x \notin B \times C$ . Take  $a \in A$  and  $c \in C$  such that  $x = (a, c)$ . Then  $a \notin B$ . Indeed if  $a \in B$  then  $x \in B \times C$ . Hence  $a \in A \setminus B$ . Thus  $x \in (A \setminus B) \times C$ . End.  $\square$

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**Proposition 0.8.** Let  $A, B, C$  be classes. Then  $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$ .

*Proof.* Let us show that  $A \times (B \setminus C) \subseteq (A \times B) \setminus (A \times C)$ . Let  $x \in A \times (B \setminus C)$ . Take  $a \in A$  and  $b \in B \setminus C$  such that  $x = (a, b)$ . Then  $b \in B$  and  $b \notin C$ . Hence  $x \in A \times B$  and  $x \notin A \times C$ . Thus  $x \in (A \times B) \setminus (A \times C)$ . End.

Let us show that  $((A \times B) \setminus (A \times C)) \subseteq A \times (B \setminus C)$ . Let  $x \in (A \times B) \setminus (A \times C)$ . Then  $x \in A \times B$  and  $x \notin A \times C$ . Take objects  $a, b$  such that  $x = (a, b)$ . Then  $a \in A$  and  $(b \in B \text{ and } b \notin C)$ . Hence  $x \in A \times (B \setminus C)$ . End.  $\square$

## Equality Law

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**Proposition 0.9.** Let  $A, A', B, B'$  be classes. Assume that  $A$  and  $A'$  are nonempty or  $B$  and  $B'$  are nonempty. Then  $(A \times A') = (B \times B')$  iff  $A = B$  and  $A' = B'$ .

*Proof.* Case  $A \times A' = B \times B'$ . Then  $A$  and  $A'$  are nonempty iff  $B$  and  $B'$  are nonempty.

Let us show that for all  $a \in A$  and all  $a' \in A'$  we have  $a \in B$  and  $a' \in B'$ . Let  $a \in A$  and  $a' \in A'$ . Then  $(a, a') \in A \times A'$ . Hence we can take  $x \in B \times B'$  such that  $x = (a, a')$ . Thus  $a \in B$  and  $a' \in B'$ . End.

Therefore  $A \subseteq B$  and  $A' \subseteq B'$ . Indeed  $A$  and  $A'$  are nonempty.

Let us show that for all  $b \in B$  and all  $b' \in B'$  we have  $b \in A$  and  $b' \in A'$ . Let  $b \in B$  and  $b' \in B'$ . Then  $(b, b') \in B \times B'$ . Hence we can take  $x \in A \times A'$  such that  $x = (b, b')$ . Thus  $(b, b') \in A \times A'$ . End.

Therefore  $B \subseteq A$  and  $B' \subseteq A'$ . Indeed  $B$  and  $B'$  are nonempty. End.

Case  $A = B$  and  $A' = B'$ . Trivial.  $\square$

## Intersections of Products

**Proposition 0.10.** Let  $A, A', B, B'$  be classes. Then  $(A \times B) \cap (A' \times B') = (A \cap A') \times (B \cap B')$ .

*Proof.* Let us show that  $((A \times B) \cap (A' \times B')) \subseteq (A \cap A') \times (B \cap B')$ . Let  $x \in (A \times B) \cap (A' \times B')$ . Then  $x \in A \times B$  and  $x \in A' \times B'$ . Take objects  $a, b$  such that  $x = (a, b)$ . Then  $a \in A, A'$  and  $b \in B, B'$ . Hence  $a \in A \cap A'$  and  $b \in B \cap B'$ . Thus  $x \in (A \cap A') \times (B \cap B')$ . End.

Let us show that  $(A \cap A') \times (B \cap B') \subseteq (A \times B) \cap (A' \times B')$ . Let  $x \in (A \cap A') \times (B \cap B')$ . Take elements  $a, b$  such that  $x = (a, b)$ . Then  $a \in A \cap A'$  and  $b \in B \cap B'$ . Hence  $a \in A, A'$  and  $b \in B, B'$ . Thus  $x \in A \times B$  and  $x \in A' \times B'$ . Therefore  $x \in (A \times B) \cap (A' \times B')$ . End.  $\square$

## Unions of Products

**Proposition 0.11.** Let  $A, A', B, B'$  be classes. Then  $(A \times B) \cup (A' \times B') \subseteq (A \cup A') \times (B \cup B')$ .

*Proof.* Let  $x \in (A \times B) \cup (A' \times B')$ . Then  $x \in A \times B$  or  $x \in A' \times B'$ . Take objects  $a, b$  such that  $x = (a, b)$ . Then  $(a \in A \text{ or } a \in A')$  and  $(b \in B \text{ or } b \in B')$ . Hence  $a \in A \cup A'$  and  $b \in B \cup B'$ . Thus  $x \in (A \cup A') \times (B \cup B')$ .  $\square$

## Complements of Products

**Proposition 0.12.** Let  $A, A', B, B'$  be classes. Then  $(A \times B) \setminus (A' \times B') = (A \times (B \setminus B')) \cup ((A \setminus A') \times B)$ .

*Proof.* Let us show that  $((A \times B) \setminus (A' \times B')) \subseteq (A \times (B \setminus B')) \cup ((A \setminus A') \times B)$ . Let  $x \in (A \times B) \setminus (A' \times B')$ . Then  $x \in A \times B$  and  $x \notin A' \times B'$ . Take  $a \in A$  and  $b \in B$  such that  $x = (a, b)$ . Then it is wrong that  $a \in A'$  and  $b \in B'$ . Hence  $a \notin A'$  or  $b \notin B'$ . Thus  $a \in A \setminus A'$  or  $b \in B \setminus B'$ . Therefore  $x \in A \times (B \setminus B')$  or  $x \in (A \setminus A') \times B$ . Hence we have  $x \in (A \times (B \setminus B')) \cup ((A \setminus A') \times B)$ . End.

Let us show that  $(A \times (B \setminus B')) \cup ((A \setminus A') \times B) \subseteq (A \times B) \setminus (A' \times B')$ . Let  $x \in (A \times (B \setminus B')) \cup ((A \setminus A') \times B)$ . Then  $x \in (A \times (B \setminus B'))$  or  $x \in ((A \setminus A') \times B)$ . Take elements  $a, b$  such that  $x = (a, b)$  (by FOUNDATIONS\_04\_2877806274936832). Then  $(a \in A \text{ and } b \in B \setminus B')$  or  $(a \in A \setminus A' \text{ and } b \in B)$ .

Case  $a \in A$  and  $b \in B \setminus B'$ . Then  $a \in A$  and  $b \in B$ . Hence  $x \in A \times B$ . We have  $b \notin B'$ . Thus  $x \notin A' \times B'$ . Therefore  $x \in (A \times B) \setminus (A' \times B')$ . End.

Case  $a \in A \setminus A'$  and  $b \in B$ . Then  $a \in A$  and  $b \in B$ . Hence  $x \in A \times B$ . We have  $a \notin A'$ . Thus  $x \notin A' \times B'$ . Therefore  $x \in (A \times B) \setminus (A' \times B')$ . End.  $\square$