

Hilbert's Paradox

Naproche formalization:

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Hilbert's Paradox, discovered around 1900 by David Hilbert, demonstrates that there cannot exist a set that is closed under powerset and under union of arbitrary subset [PeckhausKahl2002].

Theorem (Hilbert's Paradox). There exists no system of sets that is closed under powersets and closed under unions.

Proof. Assume the contrary. Consider a system of sets S that is closed under powersets and closed under unions. We have $S \subseteq S$. Hence $\bigcup S \in S$. Thus $\mathcal{P}(\bigcup S) \in S$. Contradiction. \square

Using Hilbert's Paradox it can further be shown that there exists no universal set.

Corollary. \mathbf{V} is a proper class.

Proof. Assume the contrary. Then \mathbf{V} is closed under powersets and closed under unions. Contradiction (by [Hilbert's Paradox](#)). \square