

## $\omega$ is a Cardinal Number

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**Proposition 0.1.**  $|\omega| = \omega$ .

*Proof.* We have  $|\omega| \leq \omega$ .

Let us show that  $|\omega|$  is not less than  $\omega$ . Assume the contrary. Then  $|\omega| \in \omega$ . Take  $n = |\omega|$  and a bijection  $f$  between  $n$  and  $\omega$ .

Define

$$g(k) = \begin{cases} \text{succ}(f(k)) & : k < n \\ 0 & : k = n \end{cases}$$

for  $k \in \text{succ}(n)$ . Then  $g$  is a map from  $\text{succ}(n)$  to  $\omega$ . Indeed we can show that  $g(k) \in \omega$  for all  $k \in \text{succ}(n)$ . Let  $k \in \text{succ}(n)$ . Case  $k < n$ . Obvious. Case  $k = n$ . Obvious. End.

$g$  is injective. Indeed we can show that for all  $k, k' \in \text{succ}(n)$  if  $k \neq k'$  then  $g(k) \neq g(k')$ .

*Proof.* Let  $k, k' \in \text{succ}(n)$ . Assume  $k \neq k'$ .

Case  $k, k' < n$ . Then  $f(k) \neq f(k')$ . Hence  $\text{succ}(f(k)) \neq \text{succ}(f(k'))$ . Thus  $g(k) \neq g(k')$ . End.

Case  $k < n$  and  $k' = n$ . We have  $\text{succ}(f(k)) \neq 0$ . Hence  $g(k) \neq g(k')$ . End.

Case  $k = n$  and  $k' < n$ . We have  $\text{succ}(f(k')) \neq 0$ . Hence  $g(k) \neq g(k')$ . End. Qed.

$g$  is surjective onto  $\omega$ . Indeed we can show that for any  $m \in \omega$  there exists a  $k \in \text{succ}(n)$  such that  $m = g(k)$ .

*Proof.* Let  $m \in \omega$ . Then  $f^{-1}(m) \in n$ .

Case  $m = 0$ . Then  $m = g(n)$ . End.

Case  $m \neq 0$ . Take  $m' \in \omega$  such that  $m = \text{succ}(m')$ . Then  $m = \text{succ}(m') = \text{succ}(f(f^{-1}(m')))) = g(f^{-1}(m'))$ . Indeed  $f(f^{-1}(m')) = m'$  and  $f^{-1}(m') < n$ . End. End.

Hence  $g$  is a bijection between  $\text{succ}(n)$  and  $\omega$ . Then we have  $n = |n| = |\text{succ}(n)| = \text{succ}(n)$ . Contradiction. End.  $\square$

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**Corollary 0.2.**  $\omega$  is a cardinal.