

Natural Numbers are Cardinal Numbers

SET_THEORY_07_2948332552978432

Theorem 0.1. Let $n \in \omega$. Then $|n| = n$.

Proof. Define $\Phi = \{n' \in \omega \mid |n'| = n'\}$.

(1) $0 \in \Phi$. Indeed $|0| = |\emptyset| = 0$.

(2) For all $n' \in \Phi$ we have $\text{succ}(n') \in \Phi$.

Proof. Let $n' \in \Phi$. Then $|n'| = n'$. We have $|\text{succ}(n')| \leq \text{succ}(n')$.

Let us show that $\text{succ}(n') \leq |\text{succ}(n')|$. Assume the contrary. Then $|\text{succ}(n')| < \text{succ}(n')$. Take a bijection f between $|\text{succ}(n')|$ and $\text{succ}(n')$. $|\text{succ}(n')|$ is nonzero. Hence we can take a $m \in \omega$ such that $|\text{succ}(n')| = \text{succ}(m)$. Indeed $|\text{succ}(n')| \in \text{succ}(\text{succ}(n')) \in \omega$. Then $f^{-1}(n') \leq m$.

We can show that $f^{-1}(n') < m$. Assume the contrary. Then $f^{-1}(n') = m$. $f \upharpoonright m$ is a bijection between m and $f[m]$ (by FOUNDATIONS_08_6474462 31252992). Indeed f is an injective map from $|\text{succ}(n')|$ to $\text{succ}(n')$ and $m \subseteq |\text{succ}(n')|$. We have $f[m] \subseteq n'$ and $n' \subseteq f[m]$. Hence $f[m] = n'$. Thus $f \upharpoonright m$ is a bijection between m and n' . Therefore $n' = |n'| \leq m < |\text{succ}(n')| < \text{succ}(n')$. Consequently $m = n'$. Then we have $\text{succ}(n') = |\text{succ}(n')| < \text{succ}(n')$. Contradiction. End.

Define

$$g(i) = \begin{cases} f(i) & : i \neq f^{-1}(n') \\ f(m) & : i = f^{-1}(n') \end{cases}$$

for $i \in m$.

g is a map from m to n' . Indeed we can show that $g(i) \in n'$ for each $i \in m$. Proof. Let $i \in m$. We have $g(i) \in \text{succ}(n')$. Indeed $f(i), f(m) \in \text{succ}(n')$ and $(g(i) = f(i) \text{ or } g(i) = f(m))$. Hence if $g(i) \neq n'$ then $g(i) \in n'$.

Case $i \neq f^{-1}(n')$. Then $g(i) = f(i) \in \text{succ}(n')$. If $g(i) = n'$ then $f(i) = n' = f(f^{-1}(n'))$. Hence if $g(i) = n'$ then $i = f^{-1}(n')$. Thus $g(i) \neq n'$. Therefore $g(i) \in n'$. End.

Case $i = f^{-1}(n')$. Then $g(i) = f(m) \neq f(f^{-1}(n')) = n'$. Hence $g(i) \in n'$. End. Qed.

g is surjective onto n' . Indeed we can show that for all $k \in n'$ there exists a $l \in m$ such that $k = g(l)$.

Proof. Let $k \in n'$. Then $f^{-1}(k) \neq f^{-1}(n')$.

Case $f^{-1}(k) = m$. Then $k = f(f^{-1}(k)) = f(m) = g(f^{-1}(n'))$. End.

Case $f^{-1}(k) \neq m$. Then $f^{-1}(k) \in m$. Indeed $f^{-1}(k) \in |\text{succ}(n')| = \text{succ}(m) = m \cup \{m\}$. Hence $k = f(f^{-1}(k)) = g(f^{-1}(k))$. End. Qed.

g is injective. Indeed we can show that for all $i, j \in m$ if $i \neq j$ then $g(i) \neq g(j)$.

Proof. Let $i, j \in m$. Assume $i \neq j$.

Case $i, j \neq f^{-1}(n')$. Then $g(i) = f(i) \neq f(j) = g(j)$. End.

Case $i = f^{-1}(n')$. Then $j \neq f^{-1}(n')$. Hence $g(i) = g(f^{-1}(n')) = f(m) \neq f(j) = g(j)$. Indeed $m \neq j$. End.

Case $j = f^{-1}(n')$. Then $i \neq f^{-1}(n')$. Hence $g(i) = f(i) \neq f(m) = g(f^{-1}(n')) = g(j)$. Indeed $i \neq m$. End. Qed. End. End.

Thus $\omega \subseteq \Phi$ (by SET_THEORY_03_344585425387520). Consequently $n \in \Phi$. Therefore $|n| = n$. \square

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Corollary 0.2. Every natural number is a cardinal.