The Cantor-Schröder-Bernstein Theoremin ℕaproche

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2025

This is a formalization of the *Cantor-Schröder-Bernstein Theorem*, i.e. of the fact that two sets are equinumerous iff they can be embedded into each other [1]. Its proof of is based on the Knaster-Tarski Fixed Point Theorem.

Theorem 1 (Cantor-Schröder-Bernstein). Let x, y be sets. x and y are equinumerous iff there exists a injective map from x to y and there exists an injective map from y to x.

Proof.

Case x and y are equinumerous. Take a bijection f between x and y. Then f^{-1} is a bijection between y and x. Hence f is an injective map from x to y and f^{-1} is an injective map from y to x. □

Case there exists an injective map from x to y and there exists an injective map from y to x. Take an injective map f from x to y. Take an injective map g from y to x. We have $y \setminus f[a] \subset y$ for any $a \in \mathcal{P}(x)$.

(1) Define $h(a) := x \setminus g[y \setminus f[a]]$ for $a \in \mathcal{P}(x)$.

h is a map from $\mathcal{P}(x)$ to $\mathcal{P}(x)$. Indeed $h(a)$ is a subset of x for each subset a of x.

Let us show that h preserves subsets. Let u, v be subsets of x. Assume $u \subset v$. Then $f[u] \subset f[v]$. Hence $y \setminus f[v] \subset y \setminus f[u]$. Thus $g[y \setminus f[v]] \subset g[y \setminus f[u]]$. Indeed $y \setminus f[v]$ and $y \setminus f[u]$ are subsets of y. Therefore $x \setminus g[y \setminus f[u]] \subset x \setminus g[y \setminus f[v]]$. Consequently $h[u] \subset h[v]$. End.

Hence we can take a fixed point c of h (by Knaster-Tarski).

(2) Define $F(u) := f(u)$ for $u \in c$.

We have $c = h(c)$ iff $x \setminus c = g[y \setminus f[c]]$. g^{-1} is a bijection between range(g) and y. Thus $x \setminus c = g[y \setminus f[c]] \subset$ range(g). Therefore $x \setminus c$ is a subset of dom(g^{-1}).

(3) Define $G(u) := g^{-1}(u)$ for $u \in x \setminus c$.

F is a bijection between c and range(F). G is a bijection between $x \setminus c$ and range(G).

Define

$$H(u) := \begin{cases} F(u) & : u \in c \\ G(u) & : u \notin c \end{cases}$$

for $u \in x$.

Let us show that H is a map to y. $\mathrm{dom}(H)$ is a set and every value of H is an object. Hence H is a map.

Let us show that every value of H lies in y. Let v be a value of H. Take $u \in x$ such that $H(u) = v$. If $u \in c$ then $v = H(u) = F(u) = f(u) \in y$. If $u \notin c$ then $v = H(u) = G(u) = g^{-1}(u) \in y$. End. End.

(4) H is surjective onto y. Indeed we can show that every element of y is a value of H. Let $v \in y$.

Case $v \in f[c]$. Take $u \in c$ such that $f(u) = v$. Then $F(u) = v$. □

Case $v \notin f[c]$. Then $v \in y \setminus f[c]$. Hence $g(v) \in g[y \setminus f[c]]$. Thus $g(v) \in x \setminus h(c)$. We have $g(v) \in x \setminus c$. Therefore we can take $u \in x \setminus c$ such that $G(u) = v$. Then $v = H(u)$. □

End.

(5) H is injective. Indeed we can show that for all $u, v \in x$ if $u \neq v$ then $H(u) \neq H(v)$. Let $u, v \in x$. Assume $u \neq v$.

Case $u, v \in c$. Then $H(u) = F(u)$ and $H(v) = F(v)$. We have $F(u) \neq F(v)$. Hence $H(u) \neq H(v)$. □

Case $u, v \notin c$. Then $H(u) = G(u)$ and $H(v) = G(v)$. We have $G(u) \neq G(v)$. Hence $H(u) \neq H(v)$. □

Case $u \in c$ *and* $v \notin c$. Then $H(u) = F(u)$ and $H(v) = G(v)$. Hence $v \in g[y \setminus F[c]]$. We have $G(v) = g^{-1}(v) \in y \setminus F[c]$. Indeed $v = g(v')$ for some $v' \in y \setminus F[c]$. Thus $G(v) \neq F(u)$. □

Case $u \notin c$ *and* $v \in c$. Then $H(u) = G(u)$ and $H(v) = F(v)$. Hence $u \in g[y \setminus F[c]]$. We have $G(u) = g^{-1}(u) \in y \setminus F[c]$. Indeed $u = g(u')$ for some $u' \in y \setminus F[c]$. Thus $G(u) \neq F(v)$. □

End.

Consequently H is a bijection between x and y (by 4, 5). Therefore x and y are equinumerous. □

■

References

[1] Bernd S. W. Schröder. "The fixed point property for ordered sets". In: *Arabian Journal of Mathematics* 1 (2012), pp. 529–547.

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