Proposition 1. Let n be a natural number. Assume $n \geq 2$. Then $n^n > n!$.

Proof. (a) Define $\Phi := \left\{ n' \in \mathbb{N}_{\geq 2} \,\middle|\, {n'}^{n'} > n'! \right\}$.

(1) Φ contains 2.

(2) For all $n' \in \Phi$ we have $n'+1 \in \Phi$.

Proof. Let $n' \in \Phi$. Then $n' \geq 2$.

Take $X = n'+1^{n'} \cdot (n'+1)$ and $Y = {n'}^{n'} \cdot (n'+1)$. Then $X > Y$.

Proof. We have $n'+1 > n'$ and $n' \neq 0$. Thus $n'+1^{n'} > {n'}^{n'}$. $n'+1$ is nonzero. Hence $n'+1^{n'} \cdot (n'+1) > {n'}^{n'} \cdot (n'+1)$. □

Take $Z = n'! \cdot (n'+1)$. Then $Y > Z$. Indeed ${n'}^{n'} > n'!$ and $n'+1 \neq 0$.

Hence $n'+1^{n'+1} = X > Y > Z = n'+1!$. Thus $n'+1^{n'+1} > n'+1!$. □

Therefore Φ contains every element of $\mathbb{N}_{\geq 2}$ (by induction). Consequently $n^n > n!$ (by a). ■

Proposition 2. Let n be a natural number. Assume $n \geq 4$. Then $n! > 2^n$.

Proof. (a) Define $\Phi := \left\{ n' \in \mathbb{N}_{\geq 4} \,\middle|\, n'! > 2^{n'} \right\}$.

(1) Φ contains 4.

Proof.

$$\begin{aligned} & 4! \\ &= 4 \cdot (3 \cdot 2) \\ &= 2 \cdot (2 \cdot (3 \cdot 2)) \\ &= 3 \cdot (2 \cdot (2 \cdot 2)) \\ &> 2 \cdot (2 \cdot (2 \cdot 2)) \\ &= 2^4. \end{aligned}$$

□

(2) For all $n' \in \Phi$ we have $n'+1 \in \Phi$.

Proof. Let $n' \in \Phi$. Then $n' \geq 4$ and $n'! > 2^{n'}$.

Take $X = n'! \cdot (n'+1)$ and $Y = 2^{n'} \cdot (n'+1)$. Then $X > Y$.

Take $Z = 2^{n'} \cdot 2$. Then $Y > Z$ (by preservation of ordering under left-multiplication). Indeed $n' + 1 > 2$ and $2^{n'} \neq 0$.

Hence $n' + 1! = X > Y > Z = 2^{n'+1}$. Thus $n' + 1! > 2^{n'+1}$. $\square$

Therefore Φ contains every element of $\mathbb{N}_{\geq 4}$ (by induction). Consequently $n! > 2^n$ (by a). ■