# Part I
# Sub- and Superclasses

**Definition 1.** Let $A$ be a class. A *subclass of $A$* is a class $B$ such that every element of $B$ is an element of $A$. Let $B \subset A$ stand for $B$ is a subclass of $A$. Let a *superclass of $B$* stand for a class $A$ such that $B \subset A$. Let $B \supset A$ stand for $B$ is a superclass of $A$. Let a *proper subclass of $A$* stand for a subclass $B$ of $A$ such that $B \neq A$. Let $B \subsetneq A$ stand for $B$ is a proper subclass of $A$. Let a *proper superclass of $B$* stand for a superclass $A$ of $B$ such that $A \neq B$. Let $B \supsetneq A$ stand for $B$ is a proper superclass of $A$. Let $A$ *includes* $B$ stand for $B \subset A$. Let $B$ is *included in* $A$ stand for $B \subset A$.

**Proposition 2.** Let $A$ be a class. Then $A \subset A$.

*Proof.* Every element of $A$ is contained in $A$. Therefore $A \subset A$. ■

**Proposition 3.** Let $A, B, C$ be classes. If $A \subset B$ and $B \subset C$ then $A \subset C$.

*Proof.* Assume $A \subset B$ and $B \subset C$. Then every element of $A$ is contained in $B$ and every element of $B$ is contained in $C$. Hence every element of $A$ is contained in $C$. Thus $A \subset C$. ■

**Proposition 4.** Let $A, B$ be classes. If $A \subset B$ and $B \subset A$ then $A = B$.

*Proof.* Assume $A \subset B$ and $B \subset A$. Then every element of $A$ is contained in $B$ and every element of $B$ is contained in $A$. Hence $A = B$. ■

# Part II
# The Empty Class

**Definition 5.** Let $A$ be a class. $A$ is *empty* iff $A$ has no elements. Let $A$ is *nonempty* stand for $A$ is not empty.

**Definition 6.** $\emptyset := \{x \mid x \neq x\}$.

**Proposition 7.** Let $A$ be a class. $A$ is empty iff $A = \emptyset$.

*Proof.* We can show that $\emptyset$ is empty. Indeed any element $x$ of $\emptyset$ is nonequal to $x$. Hence if $A = \emptyset$ then $A$ is empty. If $A$ is empty then $A$ and $\emptyset$ have no elements. Hence if $A$ is empty then $A \subset \emptyset$ and $\emptyset \subset A$. Thus if $A$ is empty then $A = \emptyset$. ■

**Corollary 8.** $\emptyset$ is empty.

**Corollary 9.** Let $A$ be a class. Then $\emptyset \subset A$.

*Proof.* $\emptyset$ has no elements. Hence every element of $\emptyset$ is contained in $A$. ■

# Part III
# Unordered Pairs

**Definition 10.** Let $a, b$ be objects. $\{a,b\} := \{x \mid x = a \text{ or } x = b\}$. Let the *unordered pair of $a$ and $b$* stand for $\{a,b\}$.

**Definition 11.** An *unordered pair* is a class $A$ such that $A = \{a,b\}$ for some distinct objects $a, b$.

**Definition 12.** Let $a$ be an object. $\{a\} := \{x \mid x = a\}$. Let the *singleton class of $a$* stand for $\{a\}$.

**Definition 13.** A *singleton class* is a class $A$ such that $A = \{a\}$ for some object $a$.

**Proposition 14.** Let $a, a', b, b'$ be objects. Assume $\{a,b\} = \{a',b'\}$. Then ($a = a'$ and $b = b'$) or ($a = b'$ and $b = a'$).

*Proof.* We have $a = a'$ or $a = b'$. If $a = a'$ then $b = b'$. If $a = b'$ then $b = a'$. Hence ($a = a'$ and $b = b'$) or ($a = b'$ and $b = a'$). ■

**Corollary 15.** Let $a, a'$ be objects. If $\{a\} = \{a'\}$ then $a = a'$.

**Definition 16.** Let $A$ be a class. A *unique element of $A$* is an element $a$ of $A$ such that for each $x \in A$ we have $x = a$.

**Proposition 17.** Let $A$ be a class. Then $A$ has a unique element iff $A = \{a\}$ for some object $a$.

# Part IV
# Unions, Intersections, Complements

**Definition 18.** Let $A, B$ be classes. $A \cup B := \{x \mid x \in A \text{ or } x \in B\}$. Let the *union of $A$ and $B$* stand for $A \cup B$.

**Definition 19.** Let $A, B$ be classes. $A \cap B := \{x \mid x \in A \text{ and } x \in B\}$. Let the *intersection of $A$ and $B$* stand for $A \cap B$.

**Definition 20.** Let $A, B$ be classes. $A \setminus B := \{x \mid x \in A \text{ and } x \notin B\}$. Let the *complement of $B$ in $A$* stand for $A \setminus B$.

# Part V
# Disjoint Classes

**Definition 21.** Let $A, B$ be classes. $A$ and $B$ are *disjoint* iff $A$ and $B$ have no common elements.

**Proposition 22.** Let $A, B$ be classes. Then $A$ and $B$ are disjoint iff $A \cap B$ is empty.