Definition 1. Let X be a system of sets. X is *closed under finite unions* iff $\bigcup U \in X$ for every nonempty finite subclass U of X.

Proposition 2. Let X be a system of sets. X is closed under finite unions iff $U \cup V \in X$ for every $U, V \in X$.

Proof.

Case X is closed under finite unions. Let $U, V \in X$. Then $\{U,V\}$ is a nonempty finite subclass of X. Hence $U \cup V = \bigcup \{U,V\} \in X$. $\square$

Case $U \cup V \in X$ for every $U, V \in X$. Define $\Phi := \{n \in \mathbb{N} \mid \bigcup U \in X$ for every nonempty subclass U of X th

(1) Φ contains 0.

(2) For every $n \in \Phi$ we have $n+1 \in \Phi$.

Proof. Let $n \in \Phi$. Then $\bigcup U \in X$ for every nonempty subclass U of X that has n elements.

Let us show that $\bigcup U \in X$ for every nonempty subclass U of X that has $n+1$ elements.

Case $n = 0$. $\square$

Case $n \neq 0$. Let U be a nonempty subclass of X such that U has $n+1$ elements. Take a bijection f between $\{1, \ldots, n+1\}$ and U. We have $\{1, \ldots, n+1\} = \{1, \ldots, n\} \cup \{n+1\}$. Take $V = f[\{1, \ldots, n\}]$. We have $\{1, \ldots, n\} \subset \{1, \ldots, n+1\}$.
Let us show that $V \subset U$. Let $x \in V$. Take $k \in \{1, \ldots, n\}$ such that $x = f(k)$. Indeed we can show that there exists a $k \in \{1, \ldots, n\}$ such that $x = f(k)$. Assume the contrary. Then $x \neq f(k)$ for all $k \in \{1, \ldots, n\}$. Hence $x \notin f[\{1, \ldots, n\}] = V$. Contradiction. End. Hence $x \in U$. Indeed $x \in f[\{1, \ldots, n+1\}]$. End.
V is nonempty. Indeed $f(1) \in f[\{1, \ldots, n\}]$. Indeed $1 \in \{1, \ldots, n\}$. Hence $f \restriction \{1, \ldots, n\}$ is a bijection between $\{1, \ldots, n\}$ and V (by bijectivity of restriction of injection). Thus V has n elements. Consequently $\bigcup V \in X$.
Let us show that $U = V \cup \{f(n+1)\}$.
(1) $f[A \cup B] = f[A] \cup f[B]$ for all $A, B \subset \mathrm{dom}(f)$.
(2) $f[\{a\}] = \{f(a)\}$ for all $a \in \mathrm{dom}(f)$.
Hence $U = f[\mathrm{dom}(f)] = f[\{1, \ldots, n+1\}] = f[\{1, \ldots, n\} \cup \{n+1\}] = f[\{1, \ldots, n\}] \cup f[\{n+1\}] = f[\{1, \ldots$
Indeed $n+1 \in \mathrm{dom}(f)$ and $\{1, \ldots, n\}, \{1+n\} \subset \mathrm{dom}(f)$. End.

Let us show that $\bigcup(A \cup B) = (\bigcup A) \cup (\bigcup B)$ for any nonempty systems of sets A, B. Let A, B be nonempty systems of sets. $\bigcup(A \cup B) \subset (\bigcup A) \cup (\bigcup B)$. $((\bigcup A) \cup (\bigcup B)) \subset \bigcup(A \cup B)$. End.

Let us show that $f(n+1)$ and $f(k)$ are sets for every $k \in \{1, \ldots, n\}$. Let $k \in \{1, \ldots, n\}$. Then $f(k), f(n+1) \in U$. Hence $f(k)$ and $f(n+1)$ are sets. End.

Hence V and $\{f(n+1)\}$ are nonempty systems of sets. Thus $\bigcup U = \bigcup(V \cup \{f(n+1)\}) = (\bigcup V) \cup (\bigcup \{f($ □

End. □

Therefore Φ contains every natural number (by induction). Thus $\bigcup U \in X$ for every nonempty finite subclass U of X. Consequently X is closed under finite unions. □

■