Part I
Finite Classes

Definition 1. Let X be a class and n be a natural number. X *has n elements* iff X is equinumerous to $\{1,\ldots,n\}$.

Definition 2. Let X be a class. X is *finite* iff there exists a natural number n such that X has n elements.

Proposition 3. Let X, Y be classes. If X is finite and Y is equinumerous to X then Y is finite.

Proof. Assume that X is finite and Y is equinumerous to X. Take a natural number n and a bijection f between $\{1,\ldots,n\}$ and X and a bijection g between X and Y. Then $g \circ f$ is a bijection between $\{1,\ldots,n\}$ and Y (by bijectivity of composition of bijections). Indeed X, Y are classes. Hence Y is finite. ■

Proposition 4. Let X be a class. X has zero elements iff $X = \emptyset$.

Proof.

Case X has zero elements. Then X does not contain any element. □

Case $X = \emptyset$. Then X does not contain any element. Hence $X = \{1,\ldots,0\}$. □

■

Proposition 5. Let X be a class. X has one element iff $X = \{a\}$ for some object a.

Proof.

Case X has one element. Take a bijection f between $\{1,\ldots,1\}$ and X. We have $\{1,\ldots,1\} = \{1\}$. Hence $X = \{f(1)\}$. □

Case $X = \{a\}$ for some object a. Consider an object a such that $X = \{a\}$. Define $f(x) := 1$ for $x \in \{a\}$. Then f is a bijection between $\{a\}$ and $\{1,\ldots,1\}$. □

■

Proposition 6. Let X be a class. X has two elements iff $X = \{a,b\}$ for some distinct objects a, b.

Proof.

Case X has two elements. Take a bijection f between $\{1,\ldots,2\}$ and X. We have $\{1,\ldots,2\} = \{1,2\}$. Hence $X = \{f(1),f(2)\}$. We have $f(1) \neq f(2)$. □

Case $X = \{a,b\}$ for some distinct objects a, b. Consider distinct objects a, b such that $X = \{a,b\}$. Define

$$f(x) := \begin{cases} 1 & : x = a \\ 2 & : x = b \end{cases}$$

for $x \in \{a,b\}$. Then f is a bijection between $\{a,b\}$ and $\{1,\ldots,2\}$. Indeed f is injective and surjective onto $\{1,\ldots,2\}$. □

■

Proposition 7. Let n be a natural number and X be a class that has n elements and a be an object such that $a \notin X$. Then $X \cup \{a\}$ has $n+1$ elements.

Proof. Take a bijection f between X and $\{1,\ldots,n\}$. Define

$$g(x) := \begin{cases} f(x) & : x \in X \\ n+1 & : x = a \end{cases}$$

for $x \in X \cup \{a\}$.

(1) g is a map from $X \cup \{a\}$ to $\{1, \ldots, n+1\}$. Indeed we can show that $g(x) \in \{1, \ldots, n+1\}$ for all $x \in X \cup \{a\}$. Let $x \in X \cup \{a\}$. If $x \in X$ then $g(x) \in \{1, \ldots, n\}$. If $x = a$ then $g(x) = n+1$. End.

(2) g is injective.

Proof. Let $x, y \in \text{dom}(g)$. Assume $x \neq y$.

Case $x, y \in X$. □

Case $x \in X$ and $y = a$. □

Case $x = a$ and $y \in X$. □

□

(3) g is surjective onto $\{1, \ldots, n+1\}$. Indeed we can show that for every $k \in \{1, \ldots, n+1\}$ there exists an $x \in \text{dom}(g)$ such that $k = g(x)$.

Proof. Let $k \in \{1, \ldots, n+1\}$.

Case $k \leq n$. Then $k \in \{1, \ldots, n\}$. Hence we can take a $x \in X$ such that $k = f(x)$. □

Case $k = n+1$. Then $k = g(a)$. □

□

Hence g is a bijection between $X \cup \{a\}$ and $\{1, \ldots, n+1\}$. ■

Part II
Infinite Classes

Definition 8. Let X be a class. X is *infinite* iff X is not finite.

Proposition 9. Let X, Y be classes. If X is infinite and Y is equinumerous to X then Y is infinite.

Proof. Assume that Y is equinumerous to X. If Y is finite then X is finite. Hence if X is infinite then Y is infinite. ■