Proposition 1. Let A, B be classes and $f : A \to B$ and $X \subset A$. Then $f[X] = \{f(x) \mid x \in X\}$.

Corollary 2. Let A, B be classes and $f : A \to B$. Then $f[A] = \text{range}(f)$.

Corollary 3. Let A, B be classes and $f : A \to B$ and $X \subset A$. Then $f[X] = \text{range}(f \restriction X)$.

Proposition 4. Let A be a class and $X \subset A$. Then $\text{id}_A[X] = X$.

Proposition 5. Let B be a class and $Y \subset B$. Then $\text{id}_B^{-1}[Y] = Y$.

Proposition 6. Let A, B be classes and $f : A \to B$ and $Y \subset B$ and $a \in A$. Then $f(a) \in Y$ iff $a \in f^{-1}[Y]$.

Proof. We have $f^{-1}[Y] = \{x \in A \mid f(x) \in Y\}$. Hence $a \in f^{-1}[Y]$ iff $a \in A$ and $f(a) \in Y$. Then we have the thesis. ■

Proposition 7. Let A, B be classes and $f : A \to B$. Then $f[A] \subset B$.

Proof. $f[A] = f[\text{dom}(f)] = \text{range}(f) \subset B$. ■

Proposition 8. Let A, B be classes and $f : A \to B$. Then $f^{-1}[B] = A$.

Proof. We have $f^{-1}[B] = \{a \in A \mid f(a) \in B\}$. $f(a) \in B$ for all $a \in A$. Hence the thesis. ■

Proposition 9. Let A, B be classes and $f : A \to B$. Then $f[f^{-1}[B]] = f[A]$.

Proof. Let us show that $f[f^{-1}[B]] \subset f[A]$. Let $b \in f[f^{-1}[B]]$. Take $a \in f^{-1}[B] \cap A$ such that $b = f(a)$. Then $a \in A$. Hence $b \in f[A]$. End.

Let us show that $f[A] \subset f[f^{-1}[B]]$. Let $b \in f[A]$. Take $a \in A$ such that $b = f(a)$. We have $b \in B$. Hence $a \in f^{-1}[B]$. Thus $f(a) \in f[f^{-1}[B]]$. Indeed $f^{-1}[B] \subset A$. Therefore $b \in f[f^{-1}[B]]$. End. ■

Proposition 10. Let A, B be classes and $f : A \to B$. Then $f^{-1}[f[A]] = A$.

Proof. $f^{-1}[f[A]] = \{a \in A \mid f(a) \in f[A]\}$. For all $a \in A$ we have $f(a) \in f[A]$. Hence every element of $f^{-1}[f[A]]$ is contained in A and every element of A is contained in $f^{-1}[f[A]]$. Thus $f^{-1}[f[A]] = A$. ■

Proposition 11. Let A, B be classes and $f : A \to B$ and $Y \subset B$. Then $f[f^{-1}[Y]] = Y \cap f[A]$.

Proof. Let us show that $f[f^{-1}[Y]] \subset Y \cap f[A]$. Let $b \in f[f^{-1}[Y]]$. Take $a \in f^{-1}[Y]$ such that $b = f(a)$. Then $f(a) \in Y \cap f[A]$. Hence we have $b \in Y \cap f[A]$. End.

Let us show that $Y \cap f[A] \subset f[f^{-1}[Y]]$. Let $b \in Y \cap f[A]$. Take $a \in A$ such that $b = f(a)$. Then $a \in f^{-1}[Y]$. Hence $f(a) \in f[f^{-1}[Y]]$. End. ■

Corollary 12. Let A, B be classes and $f : A \to B$ and $Y \subset B$. Then $f[f^{-1}[Y]] \subset Y$.

Proposition 13. Let A, B be classes and $f : A \to B$ and $X \subset A$. Then $f^{-1}[f[X]] \supset X$.

Proof. Let $a \in X$. Then $f(a) \in f[X]$. Hence $a \in f^{-1}[f[X]]$. Indeed $f[X] \subset B$. ■

Proposition 14. Let A, B be classes and $f : A \to B$ and $X \subset A$. Then $f[X] = \emptyset$ iff $X = \emptyset$.

Proof.

Case $f[X]$ is empty. Then there is no $a \in X$ such that $f(a) \in f[X]$. For all $a \in X$ we have $f(a) \in f[X]$. Hence X is empty. □

Case X is empty. For all $b \in f[X]$ we have $b = f(a)$ for some $a \in X$. There is no $a \in X$. Hence $f[X]$ is empty. □

■

Proposition 15. Let A, B be classes and $f : A \to B$ and $Y \subset B$. Then $f^{-1}[Y] = \emptyset$ iff $Y \subset B \setminus f[A]$.

Proof.

Case $f^{-1}[Y]$ is empty. Let $b \in Y$. Then $b \in B$.

There is no $a \in A$ such that $b = f(a)$.

Proof. Assume the contrary. Take $a \in A$ such that $b = f(a)$. Then $a \in f^{-1}[Y]$. Contradiction. □

Hence $b \notin f[A]$. Therefore $b \in B \setminus f[A]$. □

Case $Y \subset B \setminus f[A]$. Then no element of Y is an element of $f[A]$. Assume that $f^{-1}[Y]$ is nonempty. Take $a \in f^{-1}[Y]$. Then $f(a) \in Y$ and $f(a) \in f[A]$. Contradiction. □

■

Proposition 16. Let A, B be classes and $f : A \to B$ and $X \subset A$ and $Y \subset B$. Then $f[X] \cap Y = \emptyset$ iff $X \cap f^{-1}[Y] = \emptyset$.

Proof.

Case $f[X] \cap Y$ is empty. Assume that $X \cap f^{-1}[Y]$ is nonempty. Take $a \in X \cap f^{-1}[Y]$. Then $f(a) \in f[X]$ and $f(a) \in Y$. Hence $f(a) \in f[X] \cap Y$. Contradiction. □

Case $X \cap f^{-1}[Y]$ is empty. Assume that $f[X] \cap Y$ is nonempty. Take $b \in f[X] \cap Y$. Consider a $a \in X$ such that $b = f(a)$. Then $a \in f^{-1}[Y]$. Indeed $b \in Y$. Hence $a \in X \cap f^{-1}[Y]$. Contradiction. □

■

Proposition 17. Let A, B, C be classes and $f : A \to B$ and $g : B \to C$ and $X \subset A$. Then $g \circ f[X] = g[f[X]]$.

Proof. $g \circ f[X] = \{g(f(a)) \mid a \in X\}$. $f[X]$ is a subclass of B. We have $g[f[X]] = \{g(b) \mid b \in f[X]\}$ and $f[X] = \{f(a) \mid a \in X\}$. Thus $g[f[X]] = \{g(f(a)) \mid a \in X\}$. Therefore $g \circ f[X] = g[f[X]]$. ■

Proposition 18. Let A, B, C be classes and $f : A \to B$ and $g : B \to C$ and $Z \subset C$. Then $g \circ f^{-1}[Z] = f^{-1}[g^{-1}[Z]]$.

Proof. $g \circ f^{-1}[Z] = \{a \in A \mid g(f(a)) \in Z\}$. We have $g^{-1}[Z] = \{b \in B \mid g(b) \in Z\}$ and $f^{-1}[g^{-1}[Z]] = \left\{a \in A \,\middle|\, f(a) \in g^{-1}[Z]\right\}$. Hence $f^{-1}[g^{-1}[Z]] = \{a \in A \mid g(f(a)) \in Z\}$. Thus $g \circ f^{-1}[Z] = f^{-1}[g^{-1}[Z]]$. ■

Proposition 19. Let A, B be classes and $f : A \to B$ and $X, X' \subset A$. If $X \subset X'$ then $f[X] \subset f[X']$.

Proof. Assume $X \subset X'$. Let $b \in f[X]$. Take $a \in X$ such that $f(a) = b$. Then $a \in X'$. Hence $b = f(a) \in f[X']$. ■

Proposition 20. Let A, B be classes and $f : A \to B$ and $Y, Y' \subset B$. If

$Y \subset Y'$ then $f^{-1}[Y] \subset f^{-1}[Y']$.

Proof. Assume $Y \subset Y'$. Let $a \in f^{-1}[Y]$. Then $f(a) \in Y$. Hence $f(a) \in Y'$. Thus $a \in f^{-1}[Y']$. ■

Proposition 21. Let A, B be classes and $f : A \to B$ and $X, X' \subset A$. Then $f[X \cup X'] = f[X] \cup f[X']$.

Proof. Let us show that $f[X \cup X'] \subset f[X] \cup f[X']$. Let $b \in f[X \cup X']$. Take $a \in X \cup X'$ such that $b = f(a)$. Then $a \in X$ or $a \in X'$. Hence $f(a) \in f[X]$ or $f(a) \in f[X']$. Thus $b = f(a) \in f[X] \cup f[X']$. End.

Let us show that $f[X] \cup f[X'] \subset f[X \cup X']$. Let $b \in f[X] \cup f[X']$.

Case $b \in f[X]$. Take $a \in X$ such that $b = f(a)$. Then $a \in X \cup X'$. Hence $b \in f[X \cup X']$. □

Case $b \in f[X']$. Take $a \in X'$ such that $b = f(a)$. Then $a \in X \cup X'$. Hence $b \in f[X \cup X']$. □

End. ■

Proposition 22. Let A, B be classes and $f : A \to B$ and $Y, Y' \subset B$. Then $f^{-1}[Y \cup Y'] = f^{-1}[Y] \cup f^{-1}[Y']$.

Proof. Let us show that $f^{-1}[Y \cup Y'] \subset f^{-1}[Y] \cup f^{-1}[Y']$. Let $a \in f^{-1}[Y \cup Y']$. Then $f(a) \in Y \cup Y'$. Hence $f(a) \in Y$ or $f(a) \in Y'$. If $f(a) \in Y$ then $a \in f^{-1}[Y]$. If $f(a) \in Y'$ then $a \in f^{-1}[Y']$. Thus $a \in f^{-1}[Y] \cup f^{-1}[Y']$. End.

Let us show that $f^{-1}[Y] \cup f^{-1}[Y'] \subset f^{-1}[Y \cup Y']$. Let $a \in f^{-1}[Y] \cup f^{-1}[Y']$. Then $a \in f^{-1}[Y]$ or $a \in f^{-1}[Y']$. If $a \in f^{-1}[Y]$ then $f(a) \in Y$. If $a \in f^{-1}[Y']$ then $f(a) \in Y'$. Hence $f(a) \in Y \cup Y'$. Thus $a \in f^{-1}[Y \cup Y']$. End. ■

Proposition 23. Let A, B be classes and $f : A \to B$ and $X, X' \subset A$. Then $f[X \cap X'] \subset f[X] \cap f[X']$.

Proof. Let $b \in f[X \cap X']$. Take $a \in X \cap X'$ such that $b = f(a)$. Then $a \in X$ and $a \in X'$. Hence $f(a) \in f[X]$ and $f(a) \in f[X']$. Thus $b \in f[X] \cap f[X]$. ■

Proposition 24. Let A, B be classes and $f : A \to B$ and $Y, Y' \subset B$. Then $f^{-1}[Y \cap Y'] = f^{-1}[Y] \cap f^{-1}[Y']$.

Proof. Let us show that $f^{-1}[Y \cap Y'] \subset f^{-1}[Y] \cap f^{-1}[Y']$. Let $a \in f^{-1}[Y \cap Y']$. Then $f(a) \in Y \cap Y'$. Hence $f(a) \in Y$ and $f(a) \in Y'$. Thus $a \in f^{-1}[Y]$ and $a \in f^{-1}[Y']$. Therefore $a \in f^{-1}[Y] \cap f^{-1}[Y']$. End.

Let us show that $f^{-1}[Y] \cap f^{-1}[Y'] \subset f^{-1}[Y \cap Y']$. Let $a \in f^{-1}[Y] \cap f^{-1}[Y']$. Then $a \in f^{-1}[Y]$ and $a \in f^{-1}[Y']$. Hence $f(a) \in Y$ and $f(a) \in Y'$. Thus $f(a) \in Y \cap Y'$. Therefore $a \in f^{-1}[Y \cap Y']$. End. ■

Proposition 25. Let A, B be classes and $f : A \to B$ and $X, X' \subset A$. Then $f[X \setminus X'] \supset f[X] \setminus f[X']$.

Proof. Let $b \in f[X] \setminus f[X']$. Then $b \in f[X]$ and $b \notin f[X']$. Take $a \in X$ such that $b = f(a)$. If $a \in X'$ then $b \in f[X']$. Hence $a \notin X'$. Thus $a \in X \setminus X'$. Therefore $b = f(a) \in f[X \setminus X']$. ■

Proposition 26. Let A, B be classes and $f : A \to B$ and $Y, Y' \subset B$. Then $f^{-1}[Y \setminus Y'] = f^{-1}[Y] \setminus f^{-1}[Y']$.

Proof. Let us show that $f^{-1}[Y \setminus Y'] \subset f^{-1}[Y] \setminus f^{-1}[Y']$. Let $a \in f^{-1}[Y \setminus Y']$. Then $f(a) \in Y \setminus Y'$. Hence $f(a) \in Y$ and $f(a) \notin Y'$. Thus $a \in f^{-1}[Y]$ and $a \notin f^{-1}[Y']$. Therefore $a \in f^{-1}[Y] \setminus f^{-1}[Y']$. End.

Let us show that $f^{-1}[Y] \setminus f^{-1}[Y'] \subset f^{-1}[Y \setminus Y']$. Let $a \in f^{-1}[Y] \setminus f^{-1}[Y']$. Then $a \in f^{-1}[Y]$ and $a \notin f^{-1}[Y']$. Hence $f(a) \in Y$ and $f(a) \notin Y'$. Thus $f(a) \in Y \setminus Y'$. Therefore $a \in f^{-1}[Y \setminus Y']$. End. ■