Signature 1. Let L, L' be lists. $L +\!\!+ L'$ is a list.

Axiom 2. Let L be a list. Then $[] +\!\!+ L = L$.

Axiom 3. Let a be an object and L, L' be lists. Then $(a :: L) +\!\!+ L' = a :: (L +\!\!+ L')$.

Proposition 4. $L +\!\!+ [] = L$ for every list L.

Proof by induction on L. Let L be a list.

Case $L = []$. $\square$

Case $L = a :: L'$ for some object a and some list L'. Consider an object a and a list L' such that $L = a :: L'$. Then $L' \prec L$. Hence $L' +\!\!+ [] = L'$. Thus

$$\begin{aligned} L +\!\!+ [] &= (a :: L') +\!\!+ [] \\ &= a :: (L' +\!\!+ []) \\ &= a :: L' \\ &= L \end{aligned}$$

. $\square$

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Proposition 5. $L +\!\!+ (L' +\!\!+ L'') = (L +\!\!+ L') +\!\!+ L''$ for any lists L, L', L''.

Proof by induction on L. Let L be a list.

Case $L = []$. $\square$

Case $L = a :: L'''$ for some object a and some list L'''. Consider an object a and a list L''' such that $L = a :: L'''$. Then $L''' \prec L$. Let L', L'' be lists.

Then $L''' +\!\!+ (L' +\!\!+ L'') = (L''' +\!\!+ L') +\!\!+ L''$. Thus

$$\begin{aligned} L +\!\!+ (L' +\!\!+ L'') &= (a :: L''') +\!\!+ (L' +\!\!+ L'') \\ &= a :: (L''' +\!\!+ (L' +\!\!+ L'')) \\ &= a :: ((L''' +\!\!+ L') +\!\!+ L'') \\ &= (a :: (L''' +\!\!+ L')) +\!\!+ L'' \\ &= ((a :: L''') +\!\!+ L') +\!\!+ L'' \\ &= (L +\!\!+ L') +\!\!+ L'' \end{aligned}$$

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