Definition 1.

$$\omega := \left\{ n \in \mathbb{O} \;\middle|\; \begin{array}{l} n \in X \text{ for every } X \subset \mathbb{O} \text{ such that } 0 \in X \text{ and for all } x \in X \\ \text{we have } \mathrm{succ}(x) \in X \end{array} \right\}.$$

Let a *natural number* stand for an element of ω.

Proposition 2. $0 \in \omega$.

Proposition 3. Let $n \in \omega$. Then $\mathrm{succ}(n) \in \omega$.

Proposition 4. Let $\Phi \subset \omega$. Assume that $0 \in \Phi$ and for every $x \in \Phi$ we have $\mathrm{succ}(x) \in \Phi$. Then $\Phi = \omega$.

Proof. Suppose $\Phi \neq \omega$. Consider an element n of ω that is not contained in Φ. Take $\Phi' = \Phi \setminus \{n\}$.

(1) $0 \in \Phi'$. Indeed $0 \in \Phi$ and $0 \neq n$.

(2) For each $x \in \Phi'$ we have $\mathrm{succ}(x) \in \Phi'$.

Proof. Let $x \in \Phi'$. Then $\mathrm{succ}(x) \in \Phi$.

Let us show that $\mathrm{succ}(x) \neq n$. Assume $\mathrm{succ}(x) = n$. Then $x \notin \Phi$. Indeed $n \notin \Phi$ and if $x \in \Phi$ then $n = \mathrm{succ}(x) \in \Phi$. Contradiction. End.

Thus $\mathrm{succ}(x) \in \Phi'$. □

Therefore every element of ω lies in Φ'. Indeed $\Phi' \subset \mathbb{O}$. Consequently $n \in \Phi'$. Contradiction. ■

Corollary 5. ω is a set.

Proof. Define $f(n) := \mathrm{succ}(n)$ for $n \in \omega$. Take a subset X of ω that is inductive regarding 0 and f. Indeed f is a map from ω to ω. Then we have $0 \in X$ and for each $n \in X$ we have $\mathrm{succ}(n) \in X$. Thus $X = \omega$. Therefore ω is a set. ■

Proposition 6. Let $n \in \omega$. Then $n = 0$ or $n = \mathrm{succ}(m)$ for some $m \in \omega$.

Proof. Assume the contrary. Consider a $k \in \omega$ such that neither $k = 0$ nor $k = \mathrm{succ}(m)$ for some $m \in \omega$. Take a class ω' such that $\omega' = \omega \setminus \{k\}$. Then ω' is a set.

(1) $0 \in \omega'$. Indeed $k \neq 0$.

(2) For all $m \in \omega'$ we have $\mathrm{succ}(m) \in \omega'$.

Proof. Let $m \in \omega'$. Then $\mathrm{succ}(m) \neq k$. Hence $\mathrm{succ}(m) \in \omega'$. □

Thus every element of ω is contained in ω'. Therefore $k \in \omega'$. Contradiction. ■

Proposition 7. Every element of ω is an ordinal.