**Proposition 1.** $\emptyset$ is a set.

*Proof.* Take a set $x$ (by Set Existence Axiom). Define $A := \{y \in x \mid y \neq y\}$. Then $A$ is a set (by Separation Axiom). We have $A = \emptyset$. Hence $\emptyset$ is a set. ■

**Proposition 2.** Let $x, y$ be sets. Then $x \cup y$ is a set.

*Proof.* Take $X = \{x,y\}$. Then $X$ is a set. Hence $\bigcup X$ is a set (by Union Axiom). Indeed $X$ is a system of sets. We have $x \cup y = \bigcup X$. Thus $x \cup y$ is a set. ■

**Proposition 3.** Let $x, y$ be sets. Then $x \cap y$ is a set.

*Proof.* We have $x \cap y \subset x$. Hence $x \cap y$ is a set (by Separation Axiom). ■

**Proposition 4.** Let $x, y$ be sets. Then $x \setminus y$ is a set.

*Proof.* We have $x \setminus y \subset x$. Hence $x \setminus y$ is a set (by Separation Axiom). ■

**Proposition 5.** Let $x, y$ be sets. Then $x \times y$ is a set.

*Proof.* $\{a\}$ and $\{a,b\}$ are sets for each $a \in x$ and each $b \in y$. Define $P := \{\{\{a\},\{a,b\}\} \mid a \in x \text{ and } b \in y\}$.

(1) $P$ is a set.

*Proof.* Let us show that $P \subset \mathcal{P}(\mathcal{P}(x \cup y))$. Let $p \in P$. Consider $a \in x$ and $b \in y$ such that $p = \{\{a\},\{a,b\}\}$. Then $a, b \in x \cup y$. Hence $\{a\}, \{a,b\} \in \mathcal{P}(x \cup y)$. Thus $\{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(x \cup y))$. End.

$x \cup y$ is a set. Consequently $\mathcal{P}(\mathcal{P}(x \cup y))$ is a set (by Powerset Axiom).

Therefore $P$ is a set (by Separation Axiom). $\square$

Define $l(p) :=$ "choose $a \in x$, choose $b \in y$ such that $p = \{\{a\},\{a,b\}\}$ in $a$" for $p \in P$. Define $r(p) :=$ "choose $a \in x$, choose $b \in y$ such that $p = \{\{a\},\{a,b\}\}$ in $b$" for $p \in P$.

Define $f(p) := (l(p), r(p))$ for $p \in P$.

Let us show that for any objects $u, u', v, v'$ if $\{\{u\},\{u,v\}\} = \{\{u'\},\{u',v'\}\}$ then $u = u'$ and $v = v'$. Let $u, u', v, v'$ be objects. Assume $\{\{u\},\{u,v\}\} = \{\{u'\},\{u',v'\}\}$. Then ($\{u\} = \{u'\}$ or $\{u\} = \{u',v'\}$) and ($\{u,v\} = \{u'\}$ or $\{u,v\} = \{u',v'\}$). Thus ($\{u\} = \{u'\}$ and ($\{u,v\} = \{u'\}$ or $\{u,v\} = \{u',v'\}$)) or ($\{u\} = \{u',v'\}$ and ($\{u,v\} = \{u'\}$ or $\{u,v\} = \{u',v'\}$)).

*Case* $\{u\} = \{u'\}$ *and* ($\{u,v\} = \{u'\}$ *or* $\{u,v\} = \{u',v'\}$). We have $\{u\} = \{u'\}$. Hence $u = u'$.

*Case* $\{u,v\} = \{u'\}$. Then $u = u' = v$. Hence $\{\{u\},\{u,u\}\} = \{\{u\},\{u,v'\}\}$ (by 1). Thus $\{\{u\}\} = \{\{u\},\{u,v'\}\}$. Therefore $\{u\} = \{u,v'\}$. Consequently $v' = u = v$. $\square$

*Case* $\{u,v\} = \{u',v'\}$. Then $\{u,v\} = \{u,v'\}$. Hence $v = v'$. $\square$

$\square$

*Case* $\{u\} = \{u',v'\}$ *and* ($\{u,v\} = \{u'\}$ *or* $\{u,v\} = \{u',v'\}$). We have $\{u\} = \{u',v'\}$. Hence $u = u'$.

*Case* $\{u,v\} = \{u'\}$. Then $u = v = u'$. Hence $v = v'$. $\square$

*Case* $\{u,v\} = \{u',v'\}$. Then $\{u,v\} = \{u,v'\}$. Hence $v = v'$. $\square$

$\square$

End.

$\{\{a\},\{a,b\}\} \in \mathrm{dom}(f)$ for any $a \in x$ and any $b \in y$.

*Proof.* Let $a \in x$ and $b \in y$. Then $\{\{a\},\{a,b\}\} \in P$. $\square$

Let us show that for any $a \in x$ and any $b \in y$ we have $f(\{\{a\},\{a,b\}\}) = (a, b)$. Let $a \in x$ and $b \in y$. Take $p = \{\{a\},\{a,b\}\}$. Then $p$ is a set. Then we can choose $a' \in x$ and $b' \in y$ such that $p = \{\{a'\},\{a',b'\}\}$ and $l(p) = a'$. Then $a = a'$ and $b = b'$. Hence $l(p) = a$. Choose $a'' \in x$ and $b'' \in y$ such that $p = \{\{a''\},\{a'',b''\}\}$ and $r(p) = b''$. Then $a = a''$ and $b = b''$. Thus $r(p) = b$. Therefore $f(p) = (a, b)$. End.

(2) $x \times y = f[P]$.

*Proof.* For all $p \in P$ we have $l(p) \in x$ and $r(p) \in y$. Hence $f(p) \in x \times y$ for all $p \in P$. Therefore $f[P] \subset x \times y$.

Let us show that $x \times y \subset f[P]$. Let $z \in x \times y$. Take $a \in x$ and $b \in y$ such that $z = (a, b)$. Then $(a, b) = f(\{\{a\},\{a,b\}\})$. Hence there exists a $p \in P$

such that $(a, b) = f(p)$. Thus $(a, b) \in f[P]$. End.

Consequently $x \times y = f[P]$. $\square$

Thus $x \times y$ is the image of some set under some map. Therefore $x \times y$ is a set (by Replacement Axiom). ■

**Proposition 6.** Let $X$ be a nonempty system of sets. Then $\bigcap X$ is a set.

*Proof.* Take an element $x$ of $X$. Then $\bigcap X \subset x$. Hence $\bigcap X$ is a set (by Separation Axiom). ■

**Proposition 7.** Let $f$ be a map such that $\mathrm{dom}(f)$ is a set. Then $\mathrm{range}(f)$ is a set.

*Proof.* $\mathrm{range}(f) = f[\mathrm{dom}(f)]$ and $f[\mathrm{dom}(f)]$ is a set. Hence $\mathrm{range}(f)$ is a set (by Replacement Axiom). ■

**Proposition 8.** Let $A$ be a class and $x$ be a set. Assume that there exists an injective map from $A$ to $x$. Then $A$ is a set.

*Proof.* Consider an injective map $f$ from $A$ to $x$. Then $f^{-1}$ is a bijection between $\mathrm{range}(f)$ and $A$. $\mathrm{range}(f)$ is a set and $A$ is the image of $\mathrm{range}(f)$ under $f^{-1}$. Indeed $\mathrm{range}(f) \subset x$. Thus $A$ is a set (by Replacement Axiom). ■

**Proposition 9.** There exist no sets $x, y$ such that $x \in y$ and $y \in x$.

*Proof.* Assume the contrary. Take sets $x, y$ such that $x \in y$ and $y \in x$. Consider an element $z$ of $\{x,y\}$ such that $\{x,y\}$ and $z$ are disjoint (by Foundation Axiom). Indeed $\{x,y\}$ is a nonempty system of sets. Then we have $z = x$ or $z = y$.

*Case* $z = x$. Then $x$ and $\{x,y\}$ are disjoint. Hence $y \notin x$. Contradiction.

□

*Case* $z = y$. Then $y$ and $\{x,y\}$ are disjoint. Hence $x \notin y$. Contradiction. □

■

**Corollary 10.** Let $x$ be a set. Then $x \notin x$.

**Proposition 11.** Let $f, g$ be functions. Assume that $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $f(a) = g(a)$ for all $a \in \mathrm{dom}(f)$. Then $f = g$.

**Proposition 12.** Let $x, y$ be sets. Then $[x \to y]$ is a set.

*Proof.* Define $R := \{F \in \mathcal{P}(x \times y) \mid$ (for all $a \in x$ there exists a $b \in y$ such that $(a, b) \in F$) and for all $a \in x$

[prover vampire] Every element of $R$ is a set. Define $h(F) := \lambda a \in x.$ "choose $b \in y$ such that $(a, b) \in F$ in $b$" for $F \in R$. [prover eprover]

Let us show that $[x \to y] \subset \mathrm{range}(h)$. Let $f \in [x \to y]$. Define $F := \{(a, f(a)) \mid a \in x\}$.

Then $F \in R$.

*Proof.* Define $g(a) := (a, f(a))$ for $a \in x$. Then $F = \mathrm{range}(g)$. Hence $F$ is a set. Thus $F \in \mathcal{P}(x \times y)$. Indeed $F \subset x \times y$.

(1) For all $a \in x$ there exists a $b \in y$ such that $(a, b) \in F$.

(2) For all $a \in x$ and all $b, b' \in y$ such that $(a, b), (a, b') \in F$ we have $b = b'$.

*Proof.* Let $a \in x$ and $b, b' \in y$. Assume $(a, b), (a, b') \in F$. Take $c, c' \in F$ such that $c = (a, b)$ and $c' = (a, b')$. Take $d, d' \in x$ such that $c = (d, f(d))$ and $c' = (d', f(d'))$. Then $a = d$ and $b = f(d)$ and $a = d'$ and $b' = f(d)$. Hence $b = b'$. □

[prover vampire] Hence the thesis. □

$\mathrm{dom}(f) = \mathrm{dom}(h(F))$ and for each $a \in \mathrm{dom}(f)$ we have $h(F)(a) = f(a)$. Hence $f = h(F)$. Thus $f \in \mathrm{range}(h)$. End.

Therefore $[x \to y]$ is a set. Indeed $R$ is a set. ■