Kőnig's Theorem

Steffen Frerix (2018–2019) and Peter Koepke (2018–2021)
University of Bonn

Kőnig's Theorem is an important set-theoretical result about the arithmetic of cardinals. It was proved by Julius Kőnig in [1]. The proof is reminiscent of Cantor's diagonal argument for proving that $\kappa < 2^\kappa$.

The formalization begins with a minimal axiomatic setup on sets, functions and cardinals, sufficient for the proof of the theorem.

On an older laptop with an Intel Pentium N3710 processor ℕaproche takes around 35 seconds to check this text.

1 ForTheL Setup

[read `examples/preliminaries.ftl.tex`]

Let f_i stand for $f(i)$.

2 Cardinals

Let M, N denote sets. Let f denote a function.

Signature 1. A cardinal is a set.

Let A, B, C denote cardinals.

Signature 2. $A < B$ is a relation.

Let A is less than B stand for $A < B$. Let $A \leq B$ stand for $A = B$ or $A < B$.

Axiom 3. $A < B < C$ implies $A < C$.

Axiom 4. $A < B$ or $B < A$ or $B = A$.

Signature 5. The cardinality of M is a cardinal.

Let $\mathrm{card}(M)$ denote the cardinality of M.

Axiom 6. $\mathrm{card}(A) = A$.

Axiom 7. Assume M is a subset of $\mathrm{dom}(f)$. $\mathrm{card}(f[M]) \leq \mathrm{card}(M)$.

Axiom 8. Assume the cardinality of N is less than the cardinality of M. Then $M \setminus N$ has an element.

Axiom 9. Assume $\mathrm{card}(M) \leq \mathrm{card}(N)$. Assume M has an element. There exists a function f such that N is the domain of f and M is the image of f.

3 Sums and Products of cardinals

Let D denote a set.

Definition 10. A sequence of cardinals on D is a function κ such that $\mathrm{dom}(\kappa) = D$ and $\kappa(i)$ is a cardinal for every element i of D.

Definition 11. Let κ be a sequence of cardinals on D.

$$\bigsqcup_{i \in D} \kappa_i = \{(n, i) \mid i \text{ is an element of } D \text{ and } n \text{ is an element of } \kappa_i\}.$$

Axiom 12. Let κ be a sequence of cardinals on D. Then $\bigsqcup_{i \in D} \kappa_i$ is a set.

Definition 13. Let κ be a sequence of cardinals on D. $\sum_{i \in D} \kappa_i = \mathrm{card}(\bigsqcup_{i \in D} \kappa_i)$.

Definition 14. Let κ be a sequence of cardinals on D.

$$\bigtimes_{i \in D} \kappa_i = \left\{ f \;\middle|\; \begin{array}{l} f \text{ is a function and } \mathrm{dom}(f) = D \text{ and } f(i) \text{ is an element of} \\ \kappa_i \text{ for every element } i \text{ of } D \end{array} \right\}.$$

Axiom 15. Let κ be a sequence of cardinals on D. Then $\bigtimes_{i \in D} \kappa_i$ is a set.

Definition 16. Let κ be a sequence of cardinals on D. $\prod_{i \in D} \kappa_i = \mathrm{card}(\bigtimes_{i \in D} \kappa_i)$.

Kőnig's Theorem requires some form of the axiom of choice. Currently choice is built into Naproche by the *choose* construct in function definitions.

The axiom of choice is also required to show that products of non-empty factors are themselves non-empty:

Lemma 17 (Choice). Let λ be a sequence of cardinals on D. Assume that λ_i has an element for every element i of D. Then $\bigtimes_{i \in D} \lambda_i$ has an element.

Proof. Define $f(i) =$ choose an element v of λ_i in v for i in D. Then f is an element of $\bigtimes_{i \in D} \lambda_i$. □

4 Kőnig's theorem

Let D denote a set.

Theorem 18. Let κ, λ be sequences of cardinals on D. Assume that for every element i of D $\kappa_i < \lambda_i$. Then

$$\sum_{i \in D} \kappa_i < \prod_{i \in D} \lambda_i.$$

Proof by contradiction. Assume the contrary. Then

$$\prod_{i \in D} \lambda_i \leq \sum_{i \in D} \kappa_i.$$

[unfoldlow off][prover vampire] Take a function G such that $\bigsqcup_{i \in D} \kappa_i$ is the domain of G and $\bigtimes_{i \in D} \lambda_i$ is the image of G. Indeed $\bigtimes_{i \in D} \lambda_i$ and $\bigtimes_{i \in D} \lambda_i$ are sets and $\bigtimes_{i \in D} \lambda_i$ has an element. [unfoldlow on][prover eprover]

Define $\Delta(i) = \{G(n, i)(i) \mid n \text{ is an element of } \kappa_i\}$ for i in D.

For every element f of $\bigtimes_{i \in D} \lambda_i$ and every element i of D $f(i)$ is an element of λ_i. For every element i of D λ_i is a set. For every element i of D and every element d of $\Delta(i)$ we have $d \in \lambda_i$. For every element i of D $\Delta(i)$ is a set.

For every element i of D $\operatorname{card}(\Delta(i)) < \lambda_i$.
Proof. Let i be an element of D. Define $F(n) = G(n, i)(i)$ for n in κ_i. Then $F[\kappa(i)] = \Delta(i)$. End.

Define $f(i) =$ choose an element v of $\lambda_i \setminus \Delta(i)$ in v for i in D. Then f is an element of $\bigtimes_{i \in D} \lambda_i$. Take an element j of D and an element m of κ_j such that $G(m, j) = f$. $G(m, j)(j)$ is an element of $\Delta(j)$ and $f(j)$ is not an element of $\Delta(j)$. Contradiction. □

References

[1] Gyula Kőnig. "Zum Kontinuumsproblem". In: *Mathematische Annalen* 60 (1905).