Ontological Axioms for ℕaproche

Peter Koepke, University of Bonn

2021

1 Basic Ideas

The natural proof assistant ℕaproche is intended to approximate and support ordinary mathematical practices. ℕaproche employs the controlled natural language ForTheL as its input language. Some ForTheL notions are already built into ℕaproche, as well as some basic properties of these notions. There are mathematical objects, and sets and classes of mathematical objects. Sets are classes which are objects themselves and can thus be used in further mathematical constructions. Functions and maps map objects to objects, where functions are those maps which are objects.

Modelling mathematical notions by objects corresponds to the intuition that numbers, points, etc. should not have internal set-theoretical structurings, contrary to purely set-theoretical foundations of mathematics. This is also advantageous for automated proving since it prevents proof searches to dig into non-informative internal structurings.

As in common mathematics, sets and functions are required to satisfy separation and replacement properties known from the set theories of Kelley-Morse or Zermelo-Fraenkel.

The ontology presented here is more hierarchical than ad-hoc first-order axiomatizations in some previous ℕaproche formalizations. Sometimes this results in more complex and longer ontological checking tasks. Controlling the complexitiy of automated proofs will be a major issue for the further development.

2 Importing Some Mathematical Language

We import singular/plural forms of words that will be used in our formalizations (`examples/vocabulary.ftl.tex`). In the long run this should be

replaced by employing a proper English vocabulary. We also import some alternative formulations for useful mathematical phrases (`examples/macros.ftl.tex`).

[read `examples/lang/vocabulary.ftl.tex`] [read `examples/lang/macros.ftl.tex`]

3 Sets and Classes

The notions of *classes* and *sets* are already provided by ℕaproche. Classes are usually defined by abstraction terms $\{\cdots \mid \ldots\}$. Since such terms need to be processed by the parser we cannot introduce them simply by first-order definitions but have to implement them in the software. That sets are classes which are objects, or extensionality can be proved from inbuilt assumptions, but it is important to reprove or restate such facts so that they can directly be accessed by the ATP.

Let S, T denote classes.

Definition 1. The empty set is the set that has no elements.

Definition 2. A subclass of S is a class T such that every $x \in T$ belongs to S.

Let $T \subseteq S$ stand for T is a subclass of S.

Let a proper subclass of S stand for a subclass T of S such that $T \neq S$.

Lemma 3 (Class Extensionality). If S is a subclass of T and T is a subclass of S then $S = T$.

Definition 4. A subset of S is a set X such that $X \subseteq S$.

Let a proper subset of S stand for a subset X of S such that $X \neq S$.

Axiom 5 (Separation Axiom). Assume that X is a set. Let T be a subclass of X. Then T is a set.

Definition 6. The intersection of S and T is $\{x \in S \mid x \in T\}$.

Let $S \cap T$ stand for the intersection of S and T.

Definition 7. The union of S and T is $\{x \mid x \in S \vee x \in T\}$.

Let $S \cup T$ stand for the union of S and T.

Definition 8. The set difference of S and T is $\{x \in S \mid x \notin T\}$.

Let $S \setminus T$ stand for the set difference of S and T.

Definition 9. S is disjoint from T iff there is no element of S that is an element of T.

Definition 10. A family is a set F such that every element of F is a set.

Definition 11. A disjoint family is a family F such that X is disjoint from Y for all nonequal elements X, Y of F.

4 Pairs and Products

Since we prefer objects over sets if possible, we do not work with Kuratowski-style set-theoretical ordered pairs, but axiomatize them as objects.

Let S, T denote classes.

Axiom 12. For any objects a, b, c, d if $(a, b) = (c, d)$ then $a = c$ and $b = d$.

Definition 13. $S \times T = \{(x, y) \mid x \in S \text{ and } y \in T\}$.

Axiom 14. Let X, Y be sets. Then $X \times Y$ is a set.

Lemma 15. Let x, y be objects. If (x, y) is an element of $S \times T$ then x is an element of S and y is an element of T.

5 Functions and Maps

The treatment of functions and maps is similar to that of sets and classes.

Let S, T denote classes. Let f stand for a map.

Axiom 16. Assume that $\mathrm{dom}(f)$ is a set and $f(x)$ is an object for any $x \in \mathrm{dom}(f)$. Then f is a function.

Definition 17. Assume S is a subclass of the domain of f. $f[S] = \{f(x) \mid x \in S\}$.

Let the image of f stand for $f[\mathrm{dom}(f)]$.

Definition 18. f maps elements of S to elements of T iff $\mathrm{dom}(f) = S$

and $f[S] \subseteq T$.

Let $f : S \to T$ stand for f maps elements of S to elements of T.

Axiom 19 (Replacement Axiom). Let X be a set. Assume that X is a subset of the domain of f. Then $f[X]$ is a set.

Let g retracts f stand for $g(f(x)) = x$ for all elements x of $\mathrm{dom}(f)$. Let h sections f stand for $f(h(y)) = y$ for all elements y of $\mathrm{dom}(h)$.

Definition 20. $f : S \leftrightarrow T$ iff $f : S \to T$ and there exists a map g such that $g : T \to S$ and g retracts f and g sections f.