Chapter 1

Preparations

[read `examples/100_theorems.ftl.tex`]

In the first section we want to name all those small lemmata that we will need for the following formalization. Especially for Frerix' rewriting algorithm any rewriting step needs to be explicitly mentioned. Therefore, many small lemmata are mentioned here, although automated theorem provers do not have any difficulty proving those theorems even without any provided proof. In the subsection 1.1 some lemmata, axioms and definitions are repeated as lemmata that are already given in `100_theorems.ftl.tex`, just in order to give them names so that they can be referenced later.
One term we already define here: as a "real map" we understand a function from and onto the real numbers.

1.1 Old Lemmata

Lemma 1 (composition). Let F, G be maps such that $\text{Range}(G) \subseteq \text{dom}(F)$. $F \circ G$ is a map H such that $\text{dom}(H) = \text{dom}(G)$ and $H(x) = F(G(x))$ for all $x \in \text{dom}(H)$.

Lemma 2 (inverse). Let F be an injective map. F^{-1} is the map G such that $\text{dom}(G) = \text{Range}(F)$ and for all $v \in \text{dom}(G)$ $G(v) \in \text{dom}(F)$ and $F(G(v)) = v$.

Let x, y, z denote real numbers.

Lemma 3 (add comm). $x + y = y + x$.

Lemma 4 (add ass). $(x + y) + z = x + (y + z)$.

Lemma 5 (existence zero). 0 is a real number such that for every real number x $x+0 = x$.

Lemma 6 (additive inverse). $-x$ is a real number such that $x + (-x) = 0$.

Lemma 7 (mult comm). $x \cdot y = y \cdot x$.

Lemma 8 (mult ass). $((x \cdot y)) \cdot z = x \cdot (y \cdot z)$.

Lemma 9 (existence one). 1 is a real number such that $1 \neq 0$ and for every real number x $1 \cdot x = x$.

Lemma 10 (multiplicative inverse). Assume $x \neq 0$. $1/x$ is a real number such that

$x \cdot (1/x) = 1$.

Lemma 11 (distributive law). $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

Proposition 12 (eq lemma 1). $((y \cdot x)) + (z \cdot x) = (y + z) \cdot x$.

Proposition 13 (eq lemma 2). Let $x + y = x + z$. Then $y = z$.

Proposition 14 (eq lemma 3). If $x + y = x$ then $y = 0$.

Proposition 15 (eq lemma 4). If $x + y = 0$ then $y = -x$.

Proposition 16 (eq lemma 45). $-(-x) = x$.

Proposition 17 (eq lemma 5). Let x be nonzero and $x \cdot y = x \cdot z$. Then $y = z$.

Proposition 18 (eq lemma 6). If x is nonzero and $x \cdot y = x$ then $y = 1$.

Proposition 19 (eq lemma 7). If x is nonzero and $x \cdot y = 1$ then $y = 1/x$.

Proposition 20 (eq lemma 8). If x is nonzero then $1/(1/x) = x$.

Proposition 21 (eq lemma 9). $0 \cdot x = 0$.

Proposition 22 (eq lemma 10). If x is nonzero and $y \neq 0$ then $x \cdot y \neq 0$.

Proposition 23 (eq lemma 11). $(-x) \cdot y = -(x \cdot y)$.

Proposition 24 (eq lemma 12). $-x = -1 \cdot x$.

Proposition 25 (eq lemma 13). $(-x) \cdot (-y) = x \cdot y$.

Lemma 26 (eq lemma 14). $(1 - x) \cdot y = y - (x \cdot y)$.

Lemma 27 (eq lemma 15). $-(x + y) = -x - y$.

Lemma 28 (eq lemma 16). Let x be a real number. $x + x = 2 \cdot x$.

Proof. $x + x = (1 \cdot x) + (1 \cdot x) = (1 + 1) \cdot x = 2 \cdot x$. □

Lemma 29 (eq lemma 17). Let $z \neq 0$. Then $x = \frac{z \cdot x}{z}$.

Lemma 30 (eq lemma 18). Let w be a nonzero real number. Then $\frac{x-y}{w} + \frac{y-z}{w} = \frac{x-z}{w}$.

Lemma 31 (reflexivity). $x \leq x$.

Lemma 32 (antisymmetry). If $x \leq y$ and $x \geq y$ then $x = y$.

Lemma 33 (transitivity). If $x \leq y$ and $y \leq z$ then $x \leq z$.

Lemma 34 (completeness). $x \leq y$ or $x \geq y$.

Lemma 35 (ineq axiom addition). If $y \leq z$ then $x + y \leq x + z$ and $y + x \leq z + x$.

Lemma 36 (ineq axiom multiplication). If $x \geq 0$ and $y \leq z$ then $x \cdot y \leq x \cdot z$.

Lemma 37 (ineq lemma 1). $x \leq y$ iff $y \not< x$.

Proposition 38 (ineq lemma 2). $x > 0$ iff $-x < 0$.

Proposition 39 (ineq lemma 3). If $x \neq 0$ then $x \cdot x > 0$.

Proposition 40 (ineq lemma 4). $1 > 0$.

Proposition 41 (ineq lemma 5). $x \leq y$ iff $-x \geq -y$.

Proposition 42 (ineq lemma 6). If $x < 0$ and $y < z$ then $x \cdot y > x \cdot z$.

Proposition 43 (ineq lemma 7). $-1 < 0$.

Proposition 44 (ineq lemma 8). $x - 1 < x$.

Proposition 45 (ineq lemma 9). $x < y < x + 1$ iff $0 < y - x < 1$.

Proposition 46 (ineq lemma 10). If $0 < y$ then $0 < 1/y$.

Lemma 47 (ineq lemma 11). If $\neg x \leq y$ then $x \geq y$.

Proposition 48 (ineq lemma 12). Assume $0 < x \leq y$. Then $1/y \leq 1/x$.

Let E denote a subset of $\mathbb{R}$.

Lemma 49 (completeness axiom). Assume that E is nonempty and bounded above. Then E has a supremum.

Lemma 50 (potence lemma 1). $x^1 = x$.

Lemma 51 (potence lemma 2). $x^2 = x \cdot x$.

Lemma 52 (potence lemma 3). $x \cdot x^i = x^{i+1}$ for any natural number i.

1.2 New Lemmata

Let a real map stand for a map from $\mathbb{R}$ to $\mathbb{R}$.

Let w, x, y, z denote real numbers. [timelimit 10]

Lemma 53 (fraction lemma). Assume $y \cdot z \neq 0$. Then $y \neq 0$ and $z \neq 0$ and $\frac{x}{y \cdot z} = \frac{x}{y} \cdot 1/z$.

Proof. If $y = 0$ or $z = 0$ then $y \cdot z = 0$. Therefore y, z are nonzero. $1/(y \cdot z) = 1/y \cdot 1/z$. Indeed then $(y \cdot z) \cdot (1/y \cdot 1/z) = 1$. □

Lemma 54 (fraction lemma 2). Assume y is nonzero. Then $y \cdot \frac{x}{y} = x$.

Lemma 55 (frac lemma 1). Let $x \geq 0$ and $y > 0$. Then $\frac{x}{y} \geq 0$.

Lemma 56 (frac lemma 2). Let $x \geq 0$ and $y < 0$. Then $\frac{x}{y} \leq 0$.

Proof. $\frac{x}{-y} \geq 0$ (by frac lemma 1). Indeed $-y$ is positive. $-\frac{x}{y} = \frac{x}{-y}$. □

Lemma 57 (frac lemma 3). Let $x \leq 0$ and $y > 0$. Then $\frac{x}{y} \leq 0$.

Proof. $\frac{-x}{y} \geq 0$ (by frac lemma 1). Indeed $-x \geq 0$. $-\frac{x}{y} = \frac{-x}{y}$. □

Lemma 58 (frac lemma 4). Let $x \leq 0$ and $y < 0$. Then $\frac{x}{y} \geq 0$.

Proof. $\frac{-x}{y} \leq 0$ (by frac lemma 2). Indeed $-x \geq 0$. □

Lemma 59 (frac lemma 5). Let x, y be nonzero. $\frac{x}{y} \neq 0$.

Lemma 60 (eq lemma 0). $0 = -0$.

Lemma 61 (eq lemma 20). Assume z is nonzero. If $x = y \cdot z$ then $y = \frac{x}{z}$.

Lemma 62 (eq lemma 21). $(-x) \cdot y = x \cdot (-y)$.

Lemma 63 (eq lemma 26). Let x, y be nonzero. $\frac{x}{y} \cdot \frac{y}{x} = 1$.

Lemma 64 (eq lemma 27). $x + y = z$ iff $x = z - y$.

Lemma 65 (eq lemma 28). Let x be nonzero. $-1/x = 1/(-x)$.

Proof. $(-x) \cdot (-1/x) = 1$. □

Lemma 66 (eq lemma 29). Let x, y be nonzero. $\frac{x}{x \cdot y} = 1/y$.

Lemma 67 (eq lemma 30). Let x, y be nonzero. $1/x \cdot 1/y = 1/(x \cdot y)$.

Proof. $(x \cdot y) \cdot (1/x \cdot 1/y). = ((x \cdot y) \cdot 1/x) \cdot 1/y$ (by mult ass, mult comm)

$. = 1$ (by mult ass, mult comm, multiplicative inverse, existence one). □

Lemma 68 (eq lemma 22). Let x, y be nonzero. $1/\frac{x}{y} = \frac{y}{x}$.

Proof. $1/\frac{x}{y} = \frac{y}{x}$ (by eq lemma 7, eq lemma 26). Indeed $\frac{x}{y}$ is a nonzero real number. □

Lemma 69 (eq lemma 23). $\frac{x}{2} = x - \frac{x}{2}$.

Proof. $\frac{x}{2} + \frac{x}{2}. = 2 \cdot \frac{x}{2}$ (by eq lemma 16)

$. = x$ (by fraction lemma 2). $\frac{x}{2} = x - \frac{x}{2}$ (by eq lemma 27). □

Lemma 70 (eq lemma 24). Let x, y be nonzero. $1/x + 1/y = \frac{x+y}{x \cdot y}$.

Proof. $1/x + 1/y. = \frac{y}{y \cdot x} + \frac{x}{x \cdot y}$ (by eq lemma 29) $. = \frac{x+y}{x \cdot y}$ (by mult comm, eq lemma 1, add comm). □

Lemma 71 (eq lemma 25). Let x, y be nonzero. $1/x - 1/y = \frac{y-x}{x \cdot y}$.

Proof. $1/x - 1/y. = \frac{x-y}{x \cdot (-y)}$ (by eq lemma 28, eq lemma 24). $\frac{x-y}{x \cdot (-y)}. = \frac{-(y-x)}{x \cdot (-y)}$ (by eq lemma 15, eq lemma 45, add comm) $. = \frac{-(y-x)}{-(x \cdot y)}$ (by eq lemma 21, eq lemma 11) $. = \frac{y-x}{x \cdot y}$ (by eq lemma 28, eq lemma 13). □

Lemma 72. Let y be positive. $x - y < x + y$.

Lemma 73 (ineq lemma 13). If $w < x$ and $y < z$ then $w + y < x + z$.

Lemma 74 (ineq lemma 14). If $y < z$ then $y + x < z + x$.

Lemma 75 (ineq lemma 145). If $y \leq z$ then $y + x \leq z + x$.

Lemma 76 (ineq lemma 15). $x - y < z$ iff $x < z + y$.

Proof. Case $x - y < z$. $x = (x - y) + y < z + y$. End.

Case $x < z + y$. $x - y < (z + y) - y = z$. End. □

Lemma 77 (ineq lemma 16). Let w, x, y, z be nonnegative real numbers such that $w < x$ and $y < z$. Then $w \cdot y < x \cdot z$.

Proof. $w \cdot y \leq w \cdot z = z \cdot w \leq z \cdot x$ (by ineq axiom multiplication, mult comm). $w \cdot y \neq x \cdot z$. □

Lemma 78 (ineq lemma 17). Let x, y, z be nonnegative. If $y < z$ then $x \cdot y \leq x \cdot z$.

Lemma 79 (ineq lemma 18). $x - y < z$ iff $x - z < y$.

Proof. $x - y < z$ iff $x < z + y$ (by ineq lemma 15).

$x < y + z$ iff $x - z < y$ (by ineq lemma 15). □

Lemma 80 (ineq lemma x1). Let y be positive. Then there exists a positive real number

x such that $x < y$.

Lemma 81 (ineq lemma 19). Let x, y be positive. Then there exists a positive real number z such that $z < x$ and $z < y$.

Proof. Case $x \leq y$. There exists a positive real number z such that $z < x$. End.

Case $x > y$. There exists a positive real number z such that $z < y$. End. □

Lemma 82 (ineq lemma 195). Let x, y, z be positive. Then there exists a positive real number w such that $w < x$ and ($w < y$ and $w < z$).

Proof. Take a positive real number v such that $v < x$ and $v < y$ (by ineq lemma 19). Take a positive real number w such that $w < v$ and $w < z$ (by ineq lemma 19). Then $w < x$ and $w < y$ and $w < z$. □

Lemma 83 (ineq lemma 20). Assume $x < y$. There exists a real number z such that $x < z < y$.

Proof. $y - x$ is a positive real number. Take a positive real number z such that $z < y - x$ (by ineq lemma x1). $x + z$ is a real number and $x < x + z < y$. □

Lemma 84 (ineq lemma 21). Let x, y be nonnegative. $x + y$ is a nonnegative real number.

Lemma 85 (ineq lemma 215). Let x, y be positive. $x \cdot y$ is a positive real number.

Lemma 86 (ineq lemma 22). Assume $y \geq z$. $x - y \leq x - z$.

Proof. $-y \leq -z$. Therefore $x - y \leq x - z$. □

Lemma 87 (ineq lemma 23). Let y be nonnegative. $x + y \geq x$.

Lemma 88 (ineq lemma 24). Assume y is positive. $x - y < x$.

Lemma 89 (ineq lemma 25). Let z be positive. If $x < y$ then $x \cdot z < y \cdot z$.

Lemma 90 (ineq lemma 26). If $x \geq 0$ and $y \leq z$ then $y \cdot x \leq z \cdot x$.

Lemma 91 (ineq lemma 27). $x \leq y$ iff $x - y \leq 0$.

Lemma 92 (ineq lemma 28). $x < y - z$ iff $x + z < y$.

Proof. Case $x < y - z$. $x + z < (y - z) + z = y$. End.

Case $x + z < y$. $x = (x + z) - z.(x + z) - z < y - z$ (by ineq lemma 14). End. □

Lemma 93 (ineq lemma 285). $x + z < y$ iff $z < y - x$.

Lemma 94 (ineq lemma 29). If $x \leq y < z$ then $x < z$.

Lemma 95 (ineq lemma 30). If $x - y \geq z$ then $x - z \geq y$.

Lemma 96 (inverse lemma). Let f be an injective map. For all $x \in \mathrm{dom}(f)$ and all $y \in \mathrm{Range}(f)$ $f(x) = y$ iff $f^{-1}(y) = x$.

Lemma 97 (function lemma 2). Let f, g be maps such that $\mathrm{dom}(f) = \mathrm{dom}(g)$. If for every $x \in \mathrm{dom}(g)$ $f(x) = g(x)$ then $f = g$.

Lemma 98. Let f be a real map and x be a real number. Then $f(x)$ is a real number.

Proof. $f(x) \in \mathbb{R}$. □

Lemma 99. Let f, g be real maps. $g \circ f$ is a real map.

Proof. $\mathrm{dom}(g \circ f) = \mathrm{dom}(f) = \mathbb{R}$. □

Lemma 100 (1 is positive). 1 is a positive real number.

Lemma 101 (lemma x01). Let S be a nonempty subset of $\mathbb{R}$. Let s be the supremum of S. Let ε be a positive real number. Then there exists $x \in S$ such that $s - \varepsilon < x$.

Proof. Take $K = s - \varepsilon$. Assume the contrary. Then $s - \varepsilon$ is an upper bound of S. Indeed we can show that for all $x \in S$ $x \leq K$. Assume the contrary. Take $x \in S$ such that $x > K$. Case $x > s$. $x \notin S$. End. Case $x \leq s$. Then there exists $\xi \in S$ such that $s - \varepsilon < \xi$. End. End. □

Lemma 102 (lemma inverse). Let f be an injective map. For all $v \in \mathrm{dom}(f)$ $f(v) \in \mathrm{dom}(f^{-1})$ and $f^{-1}(f(v)) = v$.

Proof. Let $v \in \mathrm{dom}(f)$. Then $f(v) \in \mathrm{dom}(f^{-1})$. Indeed $\mathrm{Range}(f) = \mathrm{dom}(f^{-1})$. Take $w = f^{-1}(f(v))$. $f(w). = f(f^{-1}(f(v))). = f(v)$ (by inverse). $w = v$. Indeed f is injective. □

[timelimit 3]