# Chapter 1

# Metric

First we need to import the previous file, which we do by the following command.

[read `examples/solow/sections/0preparation.ftl.tex`]

The aim of the first chapter is to introduce absolute values, which are necessary for the following definitions of continuity and differentiability. We do so in the context of metrics as this is commonly done in standard textbooks (see [**?**]).

## 1.1 Definition of a Metric

Let $X, Y$ denote sets.

**Definition 1.** Let $d$ be a map from $(X \times X)$ to $\mathbb{R}$. $d$ is positive definit on $X$ iff for all $x, y \in X$ $d(x, y) = 0$ iff $x = y$.

**Definition 2.** Let $d$ be a map from $X \times X$ to $\mathbb{R}$. $d$ is reflexive on $X$ iff for all $x, y \in X$ $d(x, y) = d(y, x)$.

**Definition 3.** Let $d$ be a map from $X \times X$ to $\mathbb{R}$. $d$ holds triangle inequality on $X$ iff For all $x, y, z \in X$ $d(x, z), d(x, y), d(y, z)$ are real numbers and $d(x, z) \leq d(x, y) + d(y, z)$.

**Definition 4.** A metric on $X$ is a map $d$ from $X \times X$ to $\mathbb{R}$ such that $d$ is positive definit on $X$ and $d$ is reflexive on $X$ and $d$ holds triangle inequality on $X$.

[prover vampire][timelimit 20]

**Lemma 5.** Let $x, y \in X$ and $d$ be a metric on $X$. Then $d(x, y)$ is a real number and $d(x, y) \geq 0$.

*Proof.* $d$ holds triangle inequality on $X$. Thus $d(x, x) \leq d(x, y) + d(y, x)$. $0 = d(x, x) \leq d(x, y) + d(y, x) = d(x, y) + d(x, y) = 2 \cdot d(x, y)$. Thus $0 \leq 2 \cdot d(x, y)$. $d(x, y) \geq 0$. Indeed $d(x, y)$ is a real number. □

**Lemma 6.** Let $Y$ be a subset of $X$. Then $Y \times Y$ is a subset of $X \times X$.

**Lemma 7.** Let $d$ be a map from $X$ to $\mathbb{R}$ and $Y$ be a subset of $X$. Then $d \restriction Y$ is a map from $Y$ to $\mathbb{R}$.

**Proposition 8.** Let $d$ be a map from $X \times X$ to $\mathbb{R}$. Let $Y$ be a subset of $X$. If $d$ is a metric

on $X$ then $d \restriction (Y \times Y)$ is a metric on $Y$.

*Proof.* Take $\delta = d \restriction (Y \times Y)$. [timelimit 20] $\delta$ is a map from $Y \times Y$ to $\mathbb{R}$. Thus for all $x, y \in Y$ $(x, y) \in (Y \times Y)$ and $\delta(x, y)$ is a real number. [timelimit 3]

Assume $d$ is a metric on $X$. $\delta$ is positive definit on $Y$. $\delta$ is reflexive on $Y$. $\delta$ holds triangle inequality on $Y$. Proof. Let $x, y, z \in Y$. Then $d(x, y), d(y, z)$ are real numbers and $\delta(x, z) = d(x, z) \leq d(x, y) + d(y, z) = \delta(x, y) + \delta(y, z)$. Therefore $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$. End. □

## 1.2 Absolute Value

Let $a, x, y$ denote real numbers.

**Definition 9 (def absolute value).** $|a|$ is a nonnegative real number such that (if $a \geq 0$ then $|a| = a$) and if $a < 0$ then $|a| = -a$.

**Lemma 10.** $|a| = a$ or $|a| = -a$.

**Lemma 11 (absolute value nonnegative).** $|a|$ is nonnegative.

**Lemma 12 (absolute value positive).** Let $a$ be nonzero. $|a|$ is positive.

**Lemma 13 (absolute lemma 1).** $|a| = |-a|$.

[timelimit 30]

**Lemma 14 (absolute lemma 2).** $|x \cdot y| = |x| \cdot |y|$.

*Proof.* $x \cdot y \geq 0$ or $x \cdot y < 0$. If $x \cdot y \geq 0$ then $|x \cdot y| = x \cdot y = |x| \cdot |y|$. [timelimit 3] If $x \cdot y < 0$ then $|x \cdot y| = -x \cdot y = |x| \cdot |y|$. □

**Lemma 15 (absolute lemma 3).** $|x - y| = |y - x|$.

*Proof.* $|x - y| \,. = |-(x - y)|$ (by absolute lemma 1)

$. = |-x - (-y)|$ (by eq lemma 15)

$. = |y - x|$ (by eq lemma 45, add comm). □

**Lemma 16 (absolute lemma 4).** $|x + y| \leq |x| + |y|$.

*Proof.* If $x \geq 0$ and $y \geq 0$ then $|x + y| = |x| + |y|$. If $x \leq 0$ and $y \leq 0$ then $|x + y| = |x| + |y|$. Case $x > 0$ and $y < 0$. If $|x| \geq |y|$ then $|x + y| = x + y < x = |x| < |x| + |y|$.

If $|x| < |y|$ then $|x + y| = -(x + y) = -y - x < -y = |y| < |y| + |x|$. End.

Case $x < 0$ and $y > 0$. If $|y| \geq |x|$ then $|x + y| = x + y < y = |y| < |x| + |y|$.

If $|y| < |x|$ then $|x + y| = -(x + y) = -x - y < -x = |x| < |y| + |x|$. End. □

**Lemma 17.** $|x| + |y| = ||x| + |y||$.

**Lemma 18 (absolute lemma 5).** Let $a$ be nonzero. $|1/a| = 1/|a|$.

## 1.3 The Canonical Metric on the Real Numbers

**Definition 19 (Manhattan metric).** $d_{\mathbb{R}}$ is a map from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ such that for all $x, y \in \mathbb{R}$ $d_{\mathbb{R}}(x, y) = |x - y|$.

**Theorem 20.** $d_{\mathbb{R}}$ is a metric on $\mathbb{R}$.

*Proof.* $\mathbb{R}$ is a set and $d_{\mathbb{R}}$ is a map from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$.

(1) $d_{\mathbb{R}}$ is positive definit on $\mathbb{R}$. Indeed we can show that for all real numbers $x, z$ $d_{\mathbb{R}}(x, z) = 0$ iff $x = z$. Let $x, z$ be real numbers. If $x = z$ then $d_{\mathbb{R}}(x, z) = 0$. Case $x \neq z$. Then $x - z \neq 0$ and $-(x - z) \neq 0$. Thus $|x - z| \neq 0$. Thus $d_{\mathbb{R}}(x, z) \neq 0$. End. End.

(2) $d_{\mathbb{R}}$ is reflexive on $\mathbb{R}$. Indeed $d_{\mathbb{R}}(x, z) = |x - z| = |z - x| = d_{\mathbb{R}}(z, x)$ for all real numbers $x, z$.

(3) $d_{\mathbb{R}}$ holds triangle inequality on $\mathbb{R}$. Indeed we can show that for all real numbers $x, y, z$ $d_{\mathbb{R}}(x, z) \leq d_{\mathbb{R}}(x, y) + d_{\mathbb{R}}(y, z)$. Let $x, y, z$ be a real number.

$d_{\mathbb{R}}(x, z) = |x - z| = |(x + (-y + y)) - z| = |(x - y) + (y - z)| \leq ||x - y| + |y - z|| = |x - y| + |y - z| = d_{\mathbb{R}}(x, y) + d_{\mathbb{R}}(y, z)$. End. □

**Lemma 21 (absolute lemma 6).** Let $x, y$ be real numbers. $|x| - |y| \leq |x - y|$.

*Proof.* $d_{\mathbb{R}}$ holds triangle inequality on $\mathbb{R}$. Thus $d_{\mathbb{R}}(x, 0) \leq d_{\mathbb{R}}(x, y) + d_{\mathbb{R}}(y, 0)$. $|x| - |y| = d_{\mathbb{R}}(x, 0) - |y| \leq (d_{\mathbb{R}}(x, y) + d_{\mathbb{R}}(y, 0)) - |y| = (|x - y| + |y|) - |y| = |x - y|$. □