Chapter 1

Locality

[read `examples/solow/sections/3differentiation.ftl.tex`]

1.1 Open Sets and Intervals

1.1.1 Balls and Open Sets

As a small proposition we show that alternatively we could have introduced continuity not via domain restrictors but rather with balls.

Let f denote a real map. Let a, b, x, y, z denote real number. Let r denote a positive real number.

Definition 1. $B_r(a) = \{x \in \mathbb{R} | d_{\mathbb{R}}(x, a) < r\}$.

Definition 2. An open set is a subset S of $\mathbb{R}$ such that for every $a \in S$ there exists a positive real number r such that $B_r(a) \subseteq S$.

Lemma 3 (R is open). $\mathbb{R}$ is an open set.

Lemma 4 (ball nonempty). $a \in B_r(a)$.

[timelimit 10]

Proposition 5 (equivalence domain restrictor and ball). f is continuous in a iff for every positive real number ε there exists a positive real number δ such that $f[B_\delta(a)] \subseteq B_\varepsilon(f(a))$.

Proof. Case f is continuous in a.

Let ε be a positive real number. Take a positive real number δ such that δ is a domain restrictor for ε and f around a and $f(a)$. Let us show that $f[B_\delta(a)] \subseteq B_\varepsilon(f(a))$.

Let $y \in f[B_\delta(a)]$. Take $x \in B_\delta(a)$ such that $f(x) = y$. $d_{\mathbb{R}}(x, a) < \delta$. $|f(x) - f(a)| < \varepsilon$ (by domain restrictor lemma 3). Indeed $|x - a| < \delta$. Therefore $y = f(x) \in B_\varepsilon(f(a))$. End. End.

Case for every positive real number ε there exists a positive real number δ such that $f[B_\delta(a)] \subseteq B_\varepsilon(f(a))$.

Let ε be a positive real number. Take a positive real number δ such that $f[B_\delta(a)] \subseteq B_\varepsilon(f(a))$.

δ is a domain restrictor for ε and f around a and $f(a)$. Indeed we can show that for all $x \in \mathbb{R} \setminus \{a\}$ if $d_{\mathbb{R}}(x,a) < \delta$ then $d_{\mathbb{R}}(f(x),f(a)) < \varepsilon$. Let $x \in \mathbb{R} \setminus \{a\}$. Assume $d_{\mathbb{R}}(x,a) < \delta$. Thus $x \in B_\delta(a)$. Hence $f(x) \in f[B_\delta(a)]$ and $f(x) \in B_\varepsilon(f(a))$. Therefore $d_{\mathbb{R}}(f(x),f(a)) < \varepsilon$. End. End. □

Lemma 6 (lemma 01). $|x-y| < r$ iff $x \in B_r(y)$.

Lemma 7 (ball subsets). Let r, ρ be positive real numbers. If $r < \rho$ then $B_r(a) \subseteq B_\rho(a)$.

Proof. Assume $r < \rho$. Let $x \in B_r(a)$. Then $|x-a| < \rho$. Indeed $|x-a| < r < \rho$. Therefore $x \in B_\rho(a)$. □

Lemma 8 (intersection of open sets). Let A, B be open sets. Then $A \cap B$ is an open set.

Proof. Let x be an element of $A \cap B$. Take a positive real number τ such that $B_\tau(x) \subseteq A$. Take a positive real number θ such that $B_\theta(x) \subseteq B$. Take a positive real number δ such that $\delta < \tau$ and $\delta < \theta$ (by ineq lemma 19). Then $B_\delta(x) \subseteq B_\tau(x) \subseteq A$ and $B_\delta(x) \subseteq B_\theta(x) \subseteq B$ (by ball subsets). Therefore $B_\delta(x)$ is a subset of $A \cap B$. □

1.1.2 Intervals

Definition 9. $(x,y)_{OI} = \{z \in \mathbb{R} | x < z < y\}$.

Lemma 10. $(x,y)_{OI}$ is nonempty iff $x < y$.

Proof. Case $x < y$ (by ineq lemma 20). End. □

[timelimit 30]

Lemma 11 (ball and open interval). $B_r(x)$ is equal to $(x-r, x+r)_{OI}$.

Proof. Take $I = (x-r, x+r)_{OI}$. $B_r(x) \subseteq I$. Proof.

Let $y \in B_r(x)$. $(y,x) \in \mathrm{dom}(d_{\mathbb{R}})$. Then $d_{\mathbb{R}}(y,x) < r$. $d_{\mathbb{R}}(y,x) = |y-x|$.

Case $y - x \geq 0$. $y \geq x$ and $y > x - r$. $|y-x| = y - x < r$. Thus $y < r + x$ (by ineq lemma 15). Therefore $y \in I$. End.

Case $y - x < 0$. Then $y < x$ and $y < x + r$. $|y-x| = -(y-x) = x - y < r$. Thus $x - r < y$ (by ineq lemma 18). Therefore $y \in I$. End.

End.

$I \subseteq B_r(x)$. Proof.

Let $y \in I$. $x - y < r$ (by ineq lemma 18). Indeed $x - r < y$. $y - x < r$ (by ineq lemma 15). Indeed $y < r + x$. $|y-x| < r$. Indeed $y - x < r$ and $-(y-x) < r$. $(x,y) \in \mathbb{R} \times \mathbb{R}$. Therefore $d_{\mathbb{R}}(y,x) = |y-x| < r$. Hence $y \in B_r(x)$. End. □

Lemma 12 (open interval lemma). If $a \geq x$ and $b \leq y$ then $(a,b)_{OI} \subseteq (x,y)_{OI}$.

[prover vampire]

Lemma 13 (open interval is open). $(a,b)_{OI}$ is an open set.

Proof. $(a,b)_{OI}$ is a subset of $\mathbb{R}$. Let $x \in (a,b)_{OI}$. Take a positive real number δ such that $\delta < x - a$ and $\delta < b - x$ (by ineq lemma 19). Indeed $x - a$, $b - x$ are positive real

numbers. Then $a < x - \delta$ and $x + \delta < b$ (by ineq lemma 28, ineq lemma 285). Therefore $B_\delta(x) = (x - \delta, x + \delta)_{OI} \subseteq (a, b)_{OI}$ (by ball and open interval, open interval lemma). □

Lemma 14 (ball is open). $B_r(a)$ is an open set.

Proof. $B_r(a)$ is an open set (by ball and open interval, open interval is open). □

Definition 15. An interval is a subset I of $\mathbb{R}$ such that $(x, z)_{OI} \subseteq I$ for every $x, z \in I$.

Lemma 16. Let I be an interval. Let $x, y \in I$. Then $(x, y)_{OI} \subseteq I$ and $(y, x)_{OI} \subseteq I$.

Definition 17. $[a, b]_{CI} = \{x \in \mathbb{R} | a \leq x \leq b\}$.

Lemma 18 (interval lemma). $(a, b)_{OI}$ is an interval and $[a, b]_{CI}$ is an interval.

1.2 Local Functions

Let a denote a real number. Let f denote a real map. Let S denote a subset of $\mathbb{R}$. Let a real map on S stand for a map from S to $\mathbb{R}$.

Definition 19. Let g be a real map on S for some subset S of $\mathbb{R}$. A real map extension of g is a real map f such that $f(x) = g(x)$ for all $x \in \mathrm{dom}(g)$.

Definition 20. Let g be a real map on S for some subset S of $\mathbb{R}$. g is locally continuous in a iff f is continuous in a for every real map extension f of g.

Definition 21. An open neighbourhood of a is an open set S such that $a \in S$.

Theorem 22 (open set continuity extension). Let S be an open neighbourhood of a. Let g be a real map on S. Let f be a real map extension of g such that f is continuous in a.

Then g is locally continuous in a.

Proof. Let h be a real map extension of g. Let ε be a positive real number. Take a positive real number τ such that $B_\tau(a) \subseteq S$. Indeed S is an open set and $a \in S$. Take a positive real number θ such that θ is a domain restrictor for ε and f around a and $f(a)$. Take a positive real number δ such that $\delta < \tau$ and $\delta < \theta$ (by ineq lemma 19). Indeed τ, θ are positive real numbers.

δ is a domain restrictor for ε and h around a and $h(a)$. Proof. Let $x \in \mathbb{R} \setminus \{a\}$. Assume $|x - a| < \delta$. $x \in S$ and $h(x) = g(x)$. Proof. $|x - a| < \tau$. $x \in B_\tau(a)$ (by lemma 01). $B_\tau(a) \subseteq S$. End. [prover eprover]

$(h(x), h(a)), (g(x), g(a)), (f(x), f(a)), (x, a) \in \mathrm{dom}(d_{\mathbb{R}})$. $d_{\mathbb{R}}(h(x), h(a)) = d_{\mathbb{R}}(g(x), g(a)) = d_{\mathbb{R}}(f(x), f(a)) < \varepsilon$. Indeed $d_{\mathbb{R}}(x, a) < \theta$. End. □

Corollary 23 (continuity under domain restriction). Let E be an open neighbourhood of a and f be continuous in a. Then $f \restriction E$ is locally continuous in a.

Proof. f is real map extension of $f \restriction E$. $f \restriction E$ is locally continuous in a (by open set continuity extension). Indeed $f \restriction E$ is a real map on E. □

Definition 24. Let g be a real map on S for some subset S of $\mathbb{R}$. g is locally differentiable in a iff for every real map extension f of g f is differentiable in a.

Definition 25. Let g be a real map on S for some subset S of $\mathbb{R}$. The canonical real map

extension of g is a real map extension ext of g such that $ext(x) = 0$ for all $x \in \mathbb{R} \setminus \mathrm{dom}(g)$.

Let $ext|_g$ stand for the canonical real map extension of g.

Definition 26. Let g be a real map on S for some subset S of $\mathbb{R}$. Assume g is locally differentiable in a. $\partial_\ell g(a)$ is a real number such that $\partial_\ell g(a) = \partial ext|_g(a)$.

The following theorem aims to show that if a function is locally differentiable in some point a, then every real map extension thereof has the same derivative in that point. We prove that by defining a function h, which is according to some real map extension on one side of a and according to another on the other side. Since h is thus also a real map extension, we know it is differentiable in a. Then we can show that the derivative of h must be equal to the derivative of both real map extensions, which yields a contradiction when they are unequal.

Theorem 27 (lemma differentiation extension). Let S be a subset of $\mathbb{R}$ and g be a real map on S. Assume a is an element of S and g is locally differentiable in a. Let f be a real map extension of g. Then f is differentiable in a and $\partial f(a) = \partial_\ell g(a)$.

Proof. f is differentiable in a. Assume $\partial f(a) \neq \partial_\ell g(a)$. Take a difference quotient map φ of f in a. Take difference quotient map γ of $ext|_g$ in a.

Define

$$h(x) = \begin{cases} ext|_g(x) & : x \leq a \\ f(x) & : x > a \end{cases}$$

for $x \in \mathbb{R}$. h is a real map. Indeed $h(x)$ is a real number for any $x \in \mathbb{R}$.

$h(a) = g(a) = f(a)$. h is a real map extension of g. Indeed we can show that for every $x \in \mathrm{dom}(g)$ $h(x) = g(x)$. Let $x \in \mathrm{dom}(g)$. If $x \leq a$ then $h(x) = ext|_g(x) = g(x)$. If $x > a$ then $h(x) = f(x) = g(x)$. End. h is differentiable in a. Take a difference quotient map ψ of h in a.

(A) For any real number x such that $x < a$ $\psi(x) = \gamma(x)$. Proof. Let x be a real number such that $x < a$. Thus $x \in \mathbb{R} \setminus \{a\}$. $\psi(x) = \frac{\Delta h}{\Delta a}(x)$ (by prop difference quotient map). $\frac{\Delta h}{\Delta a}(x) = \frac{h(x)-h(a)}{x-a} = \frac{ext|_g(x)-ext|_g(a)}{x-a} = \frac{\Delta ext|_g}{\Delta a}(x)$. $\frac{\Delta ext|_g}{\Delta a}(x) = \gamma(x)$ (by prop difference quotient map). End.

(B) For any real number x such that $x > a$ $\psi(x) = \varphi(x)$. Proof. Let x be a real number such that $x > a$. Thus $x \in \mathbb{R} \setminus \{a\}$. $\psi(x) = \frac{\Delta h}{\Delta a}(x)$ (by prop difference quotient map). $\frac{\Delta h}{\Delta a}(x) = \frac{h(x)-h(a)}{x-a} = \frac{f(x)-f(a)}{x-a} = \frac{\Delta f}{\Delta a}(x)$. $\frac{\Delta f}{\Delta a}(x) = \varphi(x)$ (by prop difference quotient map). End. [prover eprover]

(C) $\psi(a) = \partial_\ell g(a)$. Proof. Assume the contrary.

(C1) Take $\kappa = |\psi(a) - \gamma(a)|$. κ is a positive real number (by absolute value positive). Indeed $\psi(a) - \gamma(a)$ is nonzero.

Take a positive real number ε such that $\varepsilon < \kappa$. Take a positive real number d such that d is a domain restrictor for $\kappa - \varepsilon$ and γ around a and $\gamma(a)$. Indeed $\kappa - \varepsilon$ is a positive real number and γ is continuous in a. Take a positive real number δ such that δ is a domain restrictor for ε and ψ around a and $\psi(a)$. Take a positive real number ι such that $\iota < d$ and $\iota < \delta$ (by ineq lemma 19). Take $x = a - \iota$.

(C2) $\psi(x) = \gamma(x)$ (by A). Indeed $x < a$. [timelimit 20]

(C3) $|\gamma(x) - \gamma(a)| = d_{\mathbb{R}}(\gamma(x), \gamma(a)) < \kappa - \varepsilon$. Indeed $d_{\mathbb{R}}(x, a) < d$.

(C4) $|\psi(x) - \psi(a)| < \varepsilon$ (by domain restrictor lemma 3). Indeed $|x - a| < \delta$. [timelimit 3]

We can show that $|\psi(x) - \psi(a)| \geq \varepsilon$.

$|\psi(x) - \psi(a)| . = |\gamma(x) - \psi(a)|$ (by C2)

$. = |\psi(a) - \gamma(x)|$ (by absolute lemma 3)

$. = |(\psi(a) - \gamma(x)) + (\gamma(a) - \gamma(a))|$ (by existence zero, additive inverse)

$. = |(\psi(a) + (\gamma(a) - \gamma(x))) - \gamma(a)|$ (by add comm, add ass)

$. = |(\psi(a) - \gamma(a)) + (\gamma(a) - \gamma(x))|$ (by add comm, add ass)

$. = |(\psi(a) - \gamma(a)) - (\gamma(x) - \gamma(a))|$ (by eq lemma 15, add comm, eq lemma 45).

$|(\psi(a) - \gamma(a)) - (\gamma(x) - \gamma(a))| \geq |\psi(a) - \gamma(a)| - |\gamma(x) - \gamma(a)|$ (by absolute lemma 6).

$|\psi(a) - \gamma(a)| - |\gamma(x) - \gamma(a)| . = \kappa - |\gamma(x) - \gamma(a)|$ (by C1).

$\kappa - |\gamma(x) - \gamma(a)| \geq \kappa - (\kappa - \varepsilon)$ (by ineq lemma 22, C3).

$\kappa - (\kappa - \varepsilon). = (\kappa - \kappa) + \varepsilon$ (by eq lemma 15, eq lemma 45, add ass)

$. = \varepsilon + 0$ (by additive inverse, add comm)

$. = \varepsilon$ (by existence zero). qed.

Contradiction. End.

(D) ψ is not continuous in a. Proof.

(D1) Take $\kappa = |\partial_\ell g(a) - \varphi(a)|$. κ is a positive real number. Indeed $\varphi(a) = \partial f(a)$.

Take a positive real number ε such that $\varepsilon < \kappa$. Take a positive real number d such that d is a domain restrictor for $\kappa - \varepsilon$ and φ around a and $\varphi(a)$. Indeed $\kappa - \varepsilon$ is a positive real number and φ is continuous in a. Take a positive real number δ such that δ is a domain restrictor for ε and ψ around a and $\psi(a)$. Take a positive real number ι such that $\iota < d$ and $\iota < \delta$ (by ineq lemma 19). Take $x = a + \iota$. [timelimit 10]

(D2) $|\varphi(x) - \varphi(a)| < \kappa - \varepsilon$. Indeed $|x - a| < d$.

(D3) $|\psi(x) - \psi(a)| < \varepsilon$. Indeed $|x - a| < \delta$. [timelimit 3]

We can show that $|\psi(x) - \psi(a)| \geq \varepsilon$.

$|\psi(x) - \psi(a)| . = |\varphi(x) - \partial_\ell g(a)|$ (by B, C)

$. = |\partial_\ell g(a) - \varphi(x)|$ (by absolute lemma 3)

$. = |(\partial_\ell g(a) - \varphi(x)) + (\varphi(a) - \varphi(a))|$ (by existence zero, additive inverse)

$. = |(\partial_\ell g(a) + (\varphi(a) - \varphi(x))) - \varphi(a)|$ (by add comm, add ass)

$. = |(\partial_\ell g(a) - \varphi(a)) + (\varphi(a) - \varphi(x))|$ (by add comm, add ass)

$. = |(\partial_\ell g(a) - \varphi(a)) - (\varphi(x) - \varphi(a))|$ (by eq lemma 15, add comm, eq lemma 45).

$|(\partial_\ell g(a) - \varphi(a)) - (\varphi(x) - \varphi(a))| \geq |\partial_\ell g(a) - \varphi(a)| - |\varphi(x) - \varphi(a)|$ (by absolute lemma 6).

$|\partial_\ell g(a) - \varphi(a)| - |\varphi(x) - \varphi(a)| . = \kappa - |\varphi(x) - \varphi(a)|$ (by D1).

$\kappa - |\varphi(x) - \varphi(a)| \geq \kappa - (\kappa - \varepsilon)$ (by ineq lemma 22, D2).

$\kappa - (\kappa - \varepsilon). = (\kappa - \kappa) + \varepsilon$ (by eq lemma 15, eq lemma 45, add ass)

$. = \varepsilon + 0$ (by additive inverse, add comm)

$. = \varepsilon$ (by existence zero). qed. End. □

This theorem is basically the other side of the coin of the previous theorem. While above we assume a local function to be locally differentiable to say that a real map extension thereof is then also differentiable, this theorem assumes differentiability of a real map extension to say that the local function is locally differentiable.

Theorem 28 (open set differentiation extension). Let E be an open neighbourhood of a. Let g be a real map on E. Let f be a real map extension of g such that f is differentiable in a. Then g is locally differentiable in a and $\partial_\ell g(a) = \partial f(a)$.

Proof. Let h be a real map extension of g. Then $\frac{\Delta h}{\Delta a}$ is a real map. Take a difference quotient map ψ of f in a. Define

$$\varphi(x) = \begin{cases} \psi(x) & : x \in E \\ \frac{\Delta h}{\Delta a}(x) & : x \notin E \end{cases}$$

for $x \in \mathbb{R}$. φ is a real map. Indeed $\varphi(x)$ is a real number for any real number x.

Let us show that φ is a difference quotient map of h in a.

(1) φ is continuous in a. Proof.

ψ is continuous in a. $\psi \restriction E$ is locally continuous in a (by continuity under domain restriction). $\psi \restriction E = \varphi \restriction E$. Thus $\varphi \restriction E$ is locally continuous in a. Therefore φ is continuous in a. Indeed φ is a real map extension of $\varphi \restriction E$. End.

(2) $h(x) = ((x - a) \cdot \varphi(x)) + h(a)$ for any real number x. Proof. Let x be a real number.

Case $x \notin E$. $((x - a) \cdot \varphi(x)) + h(a) = ((x - a) \cdot \frac{\Delta h}{\Delta a}(x)) + h(a)$.

$\left((x - a) \cdot \frac{\Delta h}{\Delta a}(x)\right) + h(a). = \left((x - a) \cdot \frac{h(x) - h(a)}{x - a}\right) + h(a)$ (by def diffquot)

$. = (h(x) - h(a)) + h(a)$ (by fraction lemma 2)

$. = h(x)$ (by add ass, add comm, additive inverse, existence zero). End.

Case $x \in E$. $((x - a) \cdot \varphi(x)) + h(a) = ((x - a) \cdot \psi(x)) + f(a) = f(x) = g(x) = h(x)$. End. End. End.

$\partial h(a) = \varphi(a) = \psi(a) = \partial f(a)$. □

1.3 Intermediate Value Theorem

For Bolzano's theorem we define a set H as those values with an image smaller zero. We know that it has a supremum. When we have shown that the image of the supremum can neither be positive nor negative, it needs to be zero and we are done.

Let f denote a real map. Let a, b, x, z denote real numbers.

[prover vampire][timelimit 50]

Theorem 29 (Bolzanos Theorem). Let f be continuous. Assume $a < b$ and $f(a) < 0 < f(b)$. Then there exists $c \in (a, b)_{OI}$ such that $f(c) = 0$.

Proof. Take $I = [a, b]_{CI}$. Define $H = \{x \in I | f(x) < 0\}$. Take a supremum s of H (by completeness axiom). Indeed H is bounded above and H is a nonempty subset of $\mathbb{R}$. $s \in I$.

Let us show that $f(s) = 0$. If $\neg f(s) < 0$ and $\neg f(s) > 0$ then $f(s) = 0$.

Case $f(s) < 0$. Take a positive real number δ such that δ is a domain restrictor for $|f(s)|$ and f around s and $f(s)$.

Take a positive real number ι such that $\iota < \delta$ and $\iota < b - s$ (by ineq lemma 19). Indeed $b - s$ is a positive real number. Take $c = s + \iota$.

We can show that $c \in H$.

$c = s + \iota < s + (b - s) = b$ and $a \leq s \leq c$. Thus $c \in I$.

Let us show that $f(c) < 0$. Assume $f(c) \geq 0$. $|f(c) - f(s)| = f(c) - f(s)$. Indeed $-f(s) \geq 0$ and $f(c) \geq 0$ and $f(c) - f(s) \geq 0$ (by ineq lemma 21).

$|f(c) - f(s)| < |f(s)|$ (by domain restrictor lemma 3). Indeed $|c - s| = \iota < \delta$.

Then $|f(s)| > f(c) - f(s)$ and $f(c) < |f(s)| + f(s) = -f(s) + f(s) = 0$. Contradiction. End. End.

Hence s is not an upper bound of H. Indeed $s < s + \iota = c$. End.

Case $f(s) > 0$. Take a positive real number δ such that δ is a domain restrictor for $f(s)$ and f around s and $f(s)$.

$s - a$ is a positive real number. Indeed $a \leq s$ and $a \neq s$. Indeed $f(a) \neq f(s)$. Take a positive real number ι such that $\iota < \delta$ and $\iota < s - a$ (by ineq lemma 19). Take $c = s - \iota$.

We can show that for all $x \in [c, s]_{CI}$ $f(x) > 0$.

Assume the contrary. Take $x \in [c, s]_{CI}$ such that $f(x) \leq 0$. Then $f(s) - f(x) \geq 0$ (by ineq lemma 21). Indeed $-f(x) \geq 0$ and $f(s) \geq 0$.

$|f(x) - f(s)| < f(s)$ (by domain restrictor lemma 3). Proof. $s - x \leq s - c$ (by ineq lemma 22). Indeed $c \leq x$. Thus $|x - s| = |s - x| = s - x \leq s - c = s - (s - \iota) = \iota < \delta$. Therefore $|x - s| < \delta$ and x is a real number. End.

Then $f(s) > |f(x) - f(s)| = |f(s) - f(x)| = f(s) - f(x)$. Thus $-f(x) < 0$. End.

Thus c is an upper bound of H. Proof. Assume the contrary. Take $x \in H$ such that $c < x$. Then $x \leq s$. Therefore $x \in [c, s]_{CI}$ and $f(x) > 0$. Contradiction. End.

$c = s - \iota < s$. Contradiction. End. qed. □

Corollary 30 (intermediate value theorem I). Let f be continuous. Assume $x < z$. Then $(f(x), f(z))_{OI} \subseteq f[(x, z)_{OI}]$.

Proof. Case $x = z$. Trivial.

Assume $x \neq z$. Let $\upsilon \in (f(x), f(z))_{OI}$.

Take $g = \mathrm{f}_{1 \cdot _ + - \upsilon} \circ f$. [timelimit 50] $g(x) = f(x) - \upsilon < 0$ and $g(z) = f(z) - \upsilon > 0$. Indeed For any real number r $g(r) = f(r) - \upsilon$ and $f(x) < \upsilon < f(z)$. Take $y \in (x, z)_{OI}$ such that $g(y) = 0$ (by Bolzanos Theorem). Indeed g is a continuous real map and $x < z$ and $g(x) < 0 < g(z)$.

Then $f(y) = (f(y) - \upsilon) + \upsilon = g(y) + \upsilon = \upsilon$. [timelimit 3] Therefore $\upsilon \in f[(x, z)_{OI}]$. □

[prover eprover][timelimit 3]

Corollary 31 (intermediate value theorem II). Let f be continuous. Assume $x < z$. Then $(f(z), f(x))_{OI} \subseteq f[(x, z)_{OI}]$.

Proof. Case $x = z$. Trivial. Assume $x \neq z$.

Let $\upsilon \in (f(z), f(x))_{OI}$. Take $g = \mathrm{f}_{-1 \cdot _ + \upsilon} \circ f$.

(A) For any real number r $g(r) = ((-1) \cdot f(r)) + \upsilon = -f(r) + \upsilon$.

$g(x) = -f(x) + \upsilon < 0$ and $g(z) = -f(z) + \upsilon > 0$. Indeed $f(z) < \upsilon < f(x)$.

[timelimit 10] (B) Take $y \in (x, z)_{OI}$ such that $g(y) = 0$ (by Bolzanos Theorem). Indeed g is a continuous real map and $x < z$ and $g(x) < 0 < g(z)$. [timelimit 3]

$f(y) = \upsilon$. Proof.

$\upsilon. = \upsilon - (-f(y) + \upsilon)$ (by existence zero, eq lemma 0, B,A)

$. = f(y) + (\upsilon - \upsilon)$ (by eq lemma 15, eq lemma 45,add comm, add ass)

$. = f(y)$ (by additive inverse, existence zero). End.

Therefore $\upsilon \in f[(x, z)_{OI}]$. □

1.4 Division Map

The continuity of a division map we will later need for the inverse function theorem. Having proved the inverse function theorem we can show the differentiability of a division map using the exponential function.

Let f denote a real map. Let a denote a real number.

Definition 32 (def division map). A division map of f in a is a real map g such that (for all $x \in E$ $f(x)$ is nonzero and $g(x) = 1/f(x)$) for some open neighbourhood E of a.

[prover vampire]

Proposition 33 (existence division map). Let f be continuous in a and $f(a)$ is nonzero. Then there exists a division map of f in a.

Proof. Take a positive real number δ such that δ is a domain restrictor for $|f(a)|$ and f around a and $f(a)$. Indeed $|f(a)|$ is a positive real number. [prover eprover] Take $E = B_\delta(a)$.

(1) E is an open neighbourhood of a (by ball nonempty, ball is open).

(2) For all $x \in E$ $f(x)$ is nonzero. Proof. Let x be an element in $E \setminus \{a\}$. Then $d_{\mathbb{R}}(x, a) < \delta$ (by lemma 01). Assume $f(x) = 0$. Then $d_{\mathbb{R}}(f(x), f(a)) = |0 - f(a)| = |f(a)|$. Thus δ is not a domain restrictor for $|f(a)|$ and f around a and $f(a)$ (by domain restrictor lemma 2). Indeed $x \in \mathbb{R} \setminus \{a\}$. Contradiction. End.

Define

$$g(x) = \begin{cases} 1/f(x) & : x \in E \\ 0 & : x \notin E \end{cases}$$

for $x \in \mathbb{R}$. g is a real map. Indeed $g(x)$ is a real number for any real number x.

Therefore g is a division map of f in a. □

[prover eprover]

The proof of the continuity of the division map is very unbeautiful. A particular nuisance is the introduction of Θ, because we cannot define δ as a positive real value $\delta < \tau$ and $\delta < \theta$ and $\delta < \vartheta$ immediately, although we have already defined in ineq lemma 195 that such a value exists. The reasoner seems to be overwhelmed by the number of conditions necessary to be fulfilled for this lemma.

By choosing δ so small that if $|x - a| < \delta$ then $|f(x) - f(a)| < \frac{|f(a)|}{2}$, we can show that $1/|f(x)| < 2/|f(a)|$. Choosing δ such that $|f(x) - f(a)| < \frac{|f(a)|^2}{2}\varepsilon$ as well, we can derive the following:

$$|1/f(x) - 1/f(a)| = \left|\frac{f(a) - f(x)}{f(x)f(a)}\right| = \frac{1}{|f(x)|}\frac{1}{|f(a)|}|f(x) - f(a)| < \frac{2}{|f(a)|}\frac{1}{|f(a)|}\frac{|f(a)|^2}{2}\varepsilon = \varepsilon.$$

Theorem 34 (continuity division map). Let f be continuous in a. Let g be a division map of f in a. Then g is continuous in a.

Proof. Let ε be a positive real number.

Take $\nu = \frac{|f(a)|}{2}$ and $\upsilon = f(a)$. ν is a positive real number. Take a positive real number τ such that τ is a domain restrictor for ν and f around a and $f(a)$. Indeed $\frac{|f(a)|}{2}$ is a positive real number.

Take $c = \frac{|f(a)|^2}{2} \cdot \varepsilon$. c is a positive real number (by ineq lemma 215). Indeed $|f(a)|^2, 1/2$ are positive real numbers. Take a positive real number θ such that θ is a domain restrictor for c and f around a and $f(a)$.

Take an open neighbourhood E of a such that for every $x \in E$ $f(x)$ is nonzero and $g(x) = 1/f(x)$. Take a positive real number ϑ such that $B_\vartheta(a) \subseteq E$.

Take a positive real number Θ such that $\Theta < \tau$ and $\Theta < \vartheta$ (by ineq lemma 19). Take a positive real number δ such that $\delta < \theta$ and $\delta < \Theta$ (by ineq lemma 19). Then $\delta < \tau$ and $\delta < \vartheta$.

Let us show that δ is a domain restrictor for ε and g around a and $g(a)$. Let $x \in \mathbb{R} \setminus \{a\}$. Assume $|x - a| < \delta$. $x, f(x), f(a)$ are real numbers. [timelimit 10]

(A) $|f(x) - f(a)| < \nu$ (by domain restrictor lemma 3). Indeed $|x - a| < \tau$.

(B) $|f(x) - f(a)| < c$ (by domain restrictor lemma 3). Indeed $|x - a| < \theta$.

(C) $g(a) = 1/f(a)$ and $f(x)$ is nonzero and $g(x) = 1/f(x)$. Indeed we can show that x, a are elements in E. $|x - a| < \delta < \vartheta$. Thus $x \in B_\vartheta(a)$. End. [timelimit 3]

(0) $\frac{|f(a)|}{2} < |f(x)|$. Proof.

(01) $|f(a)| < \frac{|f(a)|}{2} + |f(x)|$. Proof. $|f(a)| . = (|f(a)| - |f(x)|) + |f(x)|$ (by existence zero, additive inverse, add comm, add ass).

$(|f(a)| - |f(x)|) + |f(x)| \leq |f(a) - f(x)| + |f(x)|$ (by ineq lemma 145, absolute lemma 6).

$|f(a) - f(x)| + |f(x)| . = |f(x) - f(a)| + |f(x)|$ (by absolute lemma 3).

$|f(x) - f(a)| + |f(x)| < \nu + |f(x)|$ (by ineq lemma 14, A).

Thus $|f(a)| < \nu + |f(x)|$. Indeed $|f(a)| \leq \nu + |f(x)|$ (by transitivity).

Indeed $|f(a)| \leq |f(x) - f(a)| + |f(x)| \leq \nu + |f(x)|$. End.

$\frac{|f(a)|}{2} . = |f(a)| - \frac{|f(a)|}{2}$ (by eq lemma 23).

$|f(a)| - \frac{|f(a)|}{2} < (\frac{|f(a)|}{2} + |f(x)|) - \frac{|f(a)|}{2}$ (by ineq lemma 14, 01).

$(\frac{|f(a)|}{2} + |f(x)|) - \frac{|f(a)|}{2} . = |f(x)| + (\frac{|f(a)|}{2} - \frac{|f(a)|}{2})$ (by add comm, add ass)

$. = |f(x)|$ (by additive inverse, existence zero). End.

(1) $1/|f(x)| < \frac{2}{|f(a)|}$. Proof. $1/|f(x)| \leq 1/\frac{|f(a)|}{2}$ (by ineq lemma 12). Indeed $0 < \frac{|f(a)|}{2} \leq |f(x)|$.

$1/\frac{|f(a)|}{2} . = \frac{2}{|f(a)|}$ (by eq lemma 22). Therefore $1/|f(x)| \leq \frac{2}{|f(a)|}$. $1/f(x) \neq \frac{2}{|f(a)|}$. End.

$|g(x) - g(a)| . = |1/f(x) - 1/f(a)|$ (by C) $. = \left|\frac{f(a)-f(x)}{f(x)\cdot f(a)}\right|$ (by eq lemma 25)

$. = \left|1/f(x) \cdot \frac{f(a)-f(x)}{f(a)}\right|$ (by mult comm, fraction lemma)

$. = |1/f(x)| \cdot \frac{|f(a)-f(x)|}{|f(a)|}$ (by absolute lemma 2)

$. = 1/|f(x)| \cdot \frac{|f(x)-f(a)|}{|f(a)|}$ (by absolute lemma 3, absolute lemma 5).[timelimit 10]

$1/|f(x)| \cdot \frac{|f(x)-f(a)|}{|f(a)|} \leq \frac{2}{|f(a)|} \cdot \frac{|f(x)-f(a)|}{|f(a)|}$ (by ineq lemma 26). Proof.

$1/|f(x)|, \frac{|f(x)-f(a)|}{|f(a)|}, \frac{2}{|f(a)|}$ are real numbers. $1/|f(x)| \leq \frac{2}{|f(a)|}$ (by 1). $\frac{|f(x)-f(a)|}{|f(a)|} \geq 0$ (by frac lemma 1). Indeed $|f(x) - f(a)| \geq 0$ and $|f(a)| > 0$. End.

$\frac{2}{|f(a)|} \cdot \frac{|f(x)-f(a)|}{|f(a)|} . = (\frac{2}{|f(a)|} \cdot 1/|f(a)|) \cdot |f(x) - f(a)|$ (by mult comm, mult ass)

$. = |f(x) - f(a)| \cdot \frac{2}{|f(a)|^2}$ (by fraction lemma, potence lemma 2, mult comm).

$|f(x) - f(a)| \cdot \frac{2}{|f(a)|^2} < c \cdot \frac{2}{|f(a)|^2}$ (by ineq lemma 25). Proof.

$|f(x) - f(a)|, \frac{2}{|f(a)|^2}, c$ are real numbers. $\frac{2}{|f(a)|^2}$ is positive (by ineq lemma 215). Indeed $2, |f(a)|^2, 1/(|f(a)|^2)$ are positive real numbers. $|f(x) - f(a)| < c$ (by B). End.

$\frac{2}{|f(a)|^2} \cdot \left(\frac{|f(a)|^2}{2} \cdot \varepsilon\right) . = \left(\frac{2}{|f(a)|^2} \cdot \frac{|f(a)|^2}{2}\right) \cdot \varepsilon$ (by mult ass) $. = \varepsilon$ (by eq lemma 26, existence one).

Therefore $|g(x) - g(a)| < \varepsilon$ (by ineq lemma 29). Indeed $|g(x) - g(a)| \leq \frac{2}{|f(a)|} \cdot \frac{|f(x)-f(a)|}{|f(a)|} < \varepsilon$ and $|g(x) - g(a)|, \frac{2}{|f(a)|} \cdot \frac{|f(x)-f(a)|}{|f(a)|}, \varepsilon$ are real numbers. [timelimit 3] qed. □