Chapter 1

Inverse Function Theorem

[read `examples/solow/sections/5extrema.ftl.tex`]

Now we are finally able to devote ourselves explicitly to the inverse function theorem. The inverse functon theorem says that under some conditions $(f^{-1})'(y) = 1/f'(f^{-1}(y))$. One such condition, for which we want to prove the theorem, is that f is continuously differentiable and $f'(x)$ is nonzero for $y = f(x)$. In lemma lemma mon 4 we show that in that case there exists an open neighbourhood E of x such that f is strictly monotone on E. Then we know that $(f \restriction E)^{-1}$ exists. In lemma lemma mon 3 we show that the inverse is continuous in y. This we eventually use in our actually proof of the inverse function theorem to show that a function that we define there is continuous in y and therefore a difference quotient map.

Let f denote a real map. Let a denote a real number. Let U denote an interval. Let E denote a subset of $\mathbb{R}$.

Lemma 1 (lemma mon 1). Let f be strictly monotone on E. $f \restriction E$ is injective.

Proof. We can show that for all distinct elements x, y in E $f(x) \neq f(y)$. Let x, y be distinct elements in E. $x, y \in E$.

Case f is strictly increasing on E. If $x < y$ then $f(x) < f(y)$. If $x > y$ then $f(x) > f(y)$. $f(x) \neq f(y)$. End.

Case f is strictly decreasing on E. If $x < y$ then $f(x) > f(y)$. If $x > y$ then $f(x) < f(y)$. $f(x) \neq f(y)$. End. End. □

We want to show that if E is an open neighbourhood of a and f continuous and strictly monotone on E, then $f[E]$ is an open neighbourhood of $f(a)$. Let us assume f is strictly increasing as the case for f strictly decreasing works analogously. We need to show that $f[E]$ is open. Therefore let ξ be an element of $f[E]$. Take $x \in E$ $f(x) = \xi$ and, because E is open, there exists $\delta > 0$ such that $B_\delta(x) \subseteq E$. One corollary of the intermediate value theorem that we proved in the monotonicity subsection of the section about extrema, that for continuous strictly increasing functions $f[(x - \delta, x + \delta)] = (f(x - \delta), f(x + \delta)) \ni \xi$. Thus

$$\exists d > 0 : \ B_d(\xi) \subseteq (f(x - \delta), f(x + \delta)) \subseteq f[E].$$

Lemma 2 (lemma mon 2). Let E be an open neighbourhood of a. Let f be continuous and strictly monotone on E. Then $f[E]$ is an open neighbourhood of $f(a)$.

Proof. $f(a)$ is an element of $f[E]$. $f[E]$ is a subset of $\mathbb{R}$.

$f[E]$ is an open set. Indeed we can show that for every $\xi \in f[E]$ there exists a positive real number r such that $B_r(\xi) \subseteq f[E]$.

Let $\xi \in f[E]$. Take $x \in E$ such that $f(x) = \xi$. Take a positive real number δ such that $B_\delta(x) \subseteq E$. Take a positive real number d such that $d < \delta$. Take $y = x - d$, $z = x + d$. Then $y < x < z$.

(1) Then $x, y, z \in B_\delta(x)$. Proof. $x - \delta < x - d$ (by ineq lemma 22). Indeed $x - d \neq x - \delta$. [timelimit 30] $x - d, x + d \in (x - \delta, x + \delta)_{OI}$. Indeed $x + d$ is a real number and $x + d < x + \delta$. End.

(2) $B_\delta(x)$ is an interval (by ball and open interval). Indeed $(x - \delta, x + \delta)_{OI}$ is an interval (by interval lemma).

(3) Case f is strictly increasing on $B_\delta(x)$. [prover vampire] $f[(y, z)_{OI}] = (f(y), f(z))_{OI}$ (by cor I intermediate value theorem, 1, 2). Indeed f is a continuous real map such that f is strictly increasing on $B_\delta(x)$. ξ is an element of $(f(y), f(z))_{OI}$. Proof. $f(y) < f(x)$ (by 1,3). Indeed $y < x$. $f(x) < f(z)$ (by 1,3). Indeed $x < z$. Therefore $f(y) < \xi < f(z)$. End. Take a positive real number r such that $B_r(\xi) \subseteq (f(y), f(z))_{OI}$ (by open interval is open). Indeed $(f(y), f(z))_{OI}$ is an open set (by open interval is open). $B_r(\xi)$ is a subset of $f[E]$. Proof. $(y, z)_{OI} \subseteq E$. Indeed $(y, z)_{OI} \subseteq B_\delta(x) \subseteq E$. Therefore $f[(y, z)_{OI}]$ is a subset of $f[E]$. Thus $B_r(\xi) \subseteq f[E]$. Indeed $B_r(\xi) \subseteq (f(y), f(z))_{OI} = f[(y, z)_{OI}] \subseteq f[E]$. End. End.

Case f is strictly decreasing on $B_\delta(x)$. [timelimit 20] $f[(y, z)_{OI}] = (f(z), f(y))_{OI}$ (by cor II intermediate value theorem, 1, 2). Indeed f is a continuous real map such that f is strictly decreasing on $B_\delta(x)$. ξ is an element of $(f(z), f(y))_{OI}$. Indeed $f(z) < \xi < f(y)$. Indeed $y < x < z$. [timelimit 3] Take a positive real number r such that $B_r(\xi) \subseteq (f(z), f(y))_{OI}$ (by open interval is open). $B_r(\xi)$ is a subset of $f[E]$. Indeed $f[(y, z)_{OI}]$ is a subset of $f[E]$. Indeed $(y, z)_{OI} \subseteq B_\delta(x) \subseteq E$. End. [prover eprover]

f is strictly increasing on $B_\delta(x)$ or f is strictly decreasing on $B_\delta(x)$. Indeed f is strictly monotone on $B_\delta(x)$ (by subset monotone lemma). qed. □

In lemma mon 3 we want to show that $(f \restriction E)^{-1}$ is locally continuous in $f(a)$. From lemma mon 1 we already know that $(f \restriction E)^{-1}$ exists. Thus we can take a real map extension thereof and want to show that $\lim_{x \to f(a)} h(x) = h(f(a)) = a$. We assume f to be strictly increasing as the strictly decreasing case is again analogous. Since E is open, we can take a positive real number d such that $[a - d, a + d] \subseteq E$ and $d < \varepsilon$ for any positive ε. Because $f(a - d) < f(a) < f(a + d)$ we can take $\delta = \min\{f(a) - f(a - d), f(a + d) - f(a)\}$ (or rather $\delta < f(a) - f(a - d)$ and $\delta < f(a + d) - f(a)$, as we did not define any minimum function). Then δ is a domain restrictor for ε and h around $f(a)$ and a. Since f is invertible,

$$\begin{aligned} &\text{if } f[B_d(a)] = f[(a - d, a + d)] = (f(a - d), f(a + d)) \supseteq B_\delta(f(a)), \\ &\text{then } (f \restriction E)^{-1}[B_\delta(f(a))] \subseteq B_d(a) \subseteq B_\varepsilon(a). \end{aligned}$$

Formally it takes some time to convince the ATP to verify this, but this is how the proof works.

Lemma 3 (lemma mon 3). Let E be an open neighbourhood of a. Let f be strictly monotone on E and continuous. $(f \restriction E)^{-1}$ is a real map on $f[E]$ and $(f \restriction E)^{-1}$ is locally continuous in $f(a)$.

Proof. Take $b = f(a)$. (0) $f \restriction E$ is an injective map. $f[E]$ is a subset of $\mathbb{R}$.

Take $g = (f \restriction E)^{-1}$.

(A) g is a real map on $f[E]$. Indeed $\mathrm{Range}((f \restriction E)^{-1}) \subseteq \mathbb{R}$ and $\mathrm{dom}(g) = f[E]$.

(B) Let h be a real map extension of g. [prover vampire] $\lim_{x \to b} h(x) = h(b)$. Indeed we can show that for every positive real number ε there exists a domain restrictor for ε and h around b and $h(b)$. Let ε be a positive real number.

Take a positive real number θ such that $B_\theta(a) \subseteq E$. Take a positive real number d such that $d < \varepsilon$ and $d < \theta$.

(C) $[a-d, a+d]_{CI} \subseteq E$. Proof. $[a-d, a+d]_{CI} \subseteq (a-\theta, a+\theta)_{OI}$. Proof. Let $x \in [a-d, a+d]_{CI}$. Then $x = a - d$ or $x = a + d$ or $x \in (a-d, a+d)_{OI}$. $(a-d, a+d)_{OI} \subseteq (a-\theta, a+\theta)_{OI}$ (by open interval lemma). Indeed $a - \theta \leq a - d$ and $a + d \leq a + \theta$. $a - d \in (a-\theta, a+\theta)_{OI}$ and $a + d \in (a-\theta, a+\theta)_{OI}$. Indeed $a - \theta < a - d$ and $a + d < a + \theta$. Indeed $-\theta < -d$. End. $(a-\theta, a+\theta)_{OI} = B_\theta(a)$ (by ball and open interval). $B_\theta(a) \subseteq E$. End. Take $\beta = f(a-d)$, $B = f(a+d)$.

(1) Case f is strictly increasing on E. $b - \beta, B - b$ are positive real numbers. Proof. $a - d, a, a + d \in E$ (by A). Indeed $a - d, a, a + d$ are elements in $[a-d, a+d]_{CI}$. $a - d < a < a + d$. Therefore $f(a-d) < f(a) < f(a+d)$ (by 1). Thus $\beta < b < B$. End. Take a positive real number δ such that $\delta < b - \beta$ and $\delta < B - b$ (by ineq lemma 19).

[timelimit 10] Let us show that δ is a domain restrictor for ε and h around b and $h(b)$ (by domain restrictor lemma 2). Let $\upsilon \in \mathbb{R} \setminus \{b\}$. Assume $|\upsilon - b| < \delta$.

$f[(a-d, a+d)_{OI}] = (f(a-d), f(a+d))_{OI} = (\beta, B)_{OI}$ (by cor I intermediate value theorem). Proof. f is a continuous real map such that f is strictly increasing on $[a-d, a+d]_{CI}$. $a-d, a+d \in [a-d, a+d]_{CI}$. $[a-d, a+d]_{CI}$ is an interval (by interval lemma). End.

$\upsilon \in (\beta, B)_{OI}$. Proof. $\upsilon \in B_\delta(b)$ (by lemma 01). $B_\delta(b) = (b-\delta, b+\delta)_{OI}$ (by ball and open interval). $(b-\delta, b+\delta)_{OI} \subseteq (\beta, B)_{OI}$ (by open interval lemma). Indeed $b - \delta \geq \beta$ and $b + \delta \leq B$. End. Take $y \in (a-d, a+d)_{OI}$ such that $f(y) = \upsilon$.

$|y - a| < d$ (by lemma 01). Indeed $y \in B_d(a)$ (by ball and open interval).

$h(\upsilon) = y$. Proof. $|y - a| < d < \theta$. $y \in E$. $\upsilon \in f[E] = \mathrm{dom}(g)$. $h(\upsilon) = g(\upsilon)$. $(f \restriction E)^{-1}(\upsilon) = y$ (by inverse lemma, 0). Indeed $y \in \mathrm{dom}(f \restriction E)$ and $(f \restriction E)(y) = \upsilon$. End.

$h(b) = a$. Indeed $b \in \mathrm{dom}(g)$ and $g(b) = a$ (by inverse lemma, 0). Indeed $a \in \mathrm{dom}(f \restriction E)$ and $(f \restriction E)(a) = b$. Thus $|h(\upsilon) - h(b)| = |y - a| < d < \varepsilon$. Therefore $|h(\upsilon) - h(b)| < \varepsilon$. End. qed. [timelimit 10]

(2) Case f is strictly decreasing on E. $\beta - b, b - B$ are positive real numbers. Proof. $a - d, a, a + d \in E$ (by A). Indeed $a - d, a, a + d$ are elements in $[a-d, a+d]_{CI}$. $a - d < a < a + d$. Therefore $f(a+d) < f(a) < f(a-d)$ (by 2). Thus $B < b < \beta$. End. Take a positive real number δ such that $\delta < \beta - b$ and $\delta < b - B$ (by ineq lemma 19).

Let us show that δ is a domain restrictor for ε and h around b and $h(b)$ (by domain restrictor

lemma 2). Let $\upsilon \in \mathbb{R} \setminus \{b\}$. Assume $|\upsilon - b| < \delta$.

$f[(a-d, a+d)_{OI}] = (f(a+d), f(a-d))_{OI} = (B, \beta)_{OI}$ (by cor II intermediate value theorem). Proof. f is a continuous real map such that f is strictly decreasing on $[a-d, a+d]_{CI}$ (by C, 2). $a-d, a+d$ are elements in $[a-d, a+d]_{CI}$. $[a-d, a+d]_{CI}$ is an interval (by interval lemma). End.

$\upsilon \in (B, \beta)_{OI}$. Proof. $\upsilon \in B_\delta(b)$ (by lemma 01). $B_\delta(b) = (b-\delta, b+\delta)_{OI}$ (by ball and open interval). $(b-\delta, b+\delta)_{OI} \subseteq (B, \beta)_{OI}$ (by open interval lemma). Indeed $b - \delta \geq B$ and $b + \delta \leq \beta$. End. Take $y \in (a-d, a+d)_{OI}$ such that $f(y) = \upsilon$.

$|y - a| < d$ (by lemma 01). Indeed $y \in B_d(a)$ (by ball and open interval).

$h(\upsilon) = y$. Proof. $|y-a| < d < \theta$. $y \in E$. $\upsilon \in f[E] = \mathrm{dom}(g)$. $h(\upsilon) = g(\upsilon)$. $(f \restriction E)^{-1}(\upsilon) = y$ (by inverse lemma, 0). Indeed $y \in \mathrm{dom}(f \restriction E)$ and $(f \restriction E)(y) = \upsilon$. End.

$b \in \mathrm{dom}(g)$. $h(b) = a$. Indeed $g(b) = a$ (by inverse lemma, 0). Indeed $a \in \mathrm{dom}(f \restriction E)$ and $(f \restriction E)(a) = b$. $|h(\upsilon) - h(b)| = |y - a| < d < \varepsilon$. Therefore $|h(\upsilon) - h(b)| < \varepsilon$. End. qed. End. [timelimit 3] □

In the next lemma we want to show if f is continuously differentiable and $f'(a) \neq 0$ then there also exists such an open neighborhood E of a, where f is strictly monotone. Let us just look at the case of $f'(a)$ being positive. Then we can Take a positive real number δ such that δ is a domain restrictor such that

$$\forall x \in B_\delta(a) : \ \left|f'(x) - f'(a)\right| < f'(a).$$

Then for all those x $f'(x) > 0$, because if not, then $|f'(x) - f'(a)| = f'(a) - f'(x) \geq f'(a)$. Therefore f is strictly increasing on the neighborhood $B_\delta(a)$ of a.

Let f is continuously differentiable stand for f is differentiable and $\mathrm{D}f$ is continuous. [prover eprover] [timelimit 10]

Lemma 4 (lemma mon 4). Let f be continuously differentiable and $\partial f(a) \neq 0$. Then there exists an open neighbourhood E of a such that f is strictly monotone on E.

Proof. Take $\varepsilon = |\partial f(a)|$. [prover vampire][timelimit 50] Take a positive real number δ such that δ is a domain restrictor for ε and $\mathrm{D}f$ around a and $\mathrm{D}f(a)$. Indeed $\mathrm{D}f$ is a real map and $\mathrm{D}f$ is continuous in a and ε is a positive real number. [prover eprover][timelimit 10] $B_\delta(a)$ is an open neighbourhood of a (by ball is open). Indeed $a \in B_\delta(a)$. Take $B = B_\delta(a)$.

Let us show that f is strictly monotone on B.

Case $\partial f(a) > 0$. Then f is strictly increasing on B (by derivative strictly increasing, ball and open interval, interval lemma). Indeed we can show that $\partial f(x) > 0$ for any $x \in B$.

Let $x \in B \setminus \{a\}$. Assume $\partial f(x) \leq 0$. Then $|\partial f(x) - \partial f(a)| = |\partial f(a) - \partial f(x)| = \partial f(a) - \partial f(x) \geq \partial f(a) = \varepsilon$. [prover vampire][timelimit 30]

δ is not a domain restrictor for ε and $\mathrm{D}f$ around a and $\mathrm{D}f(a)$ (by domain restrictor lemma 2). Indeed $x \in \mathbb{R} \setminus \{a\}$ and $|x - a| < \delta$ and $|\mathrm{D}f(x) - \mathrm{D}f(a)| \geq \varepsilon$. Contradiction. [timelimit 3] End. End.

Case $\partial f(a) < 0$. Then f is strictly decreasing on B (by derivative strictly decreasing, ball and open interval, interval lemma). Indeed we can show that $\partial f(x) < 0$ for any $x \in B$.

Let $x \in B \setminus \{a\}$. Assume $\partial f(x) \geq 0$. Then $|\partial f(x) - \partial f(a)| = \partial f(x) - \partial f(a) \geq -\partial f(a) = \varepsilon$. [timelimit 30][prover vampire]

δ is not a domain restrictor for ε and $\mathrm{D}f$ around a and $\mathrm{D}f(a)$ (by domain restrictor lemma 2). Indeed $x \in \mathbb{R} \setminus \{a\}$ and $|x - a| < \delta$ and $|\mathrm{D}f(x) - \mathrm{D}f(a)| \geq \varepsilon$. Contradiction. End. End. qed. [prover eprover] □

So we want to show that $(f \restriction E)^{-1}$ is locally differentiable in $f(a)$ for some neighborhood E of a. By lemma mon 4 we can find a neighborhood $E = B_d(a)$ of a, where f is strictly monotone on. By lemma mon 1 we know then that $f \restriction E$ is injective and therefore $(f \restriction E)^{-1}$ exists.
Now we define

$$ext(y) = \begin{cases} (f \restriction E)^{-1}(y) & : y \in f[E] \\ a & : y \notin f[E] \end{cases}$$

for all real numbers y, which makes ext a real map extension of $(f \restriction E)^{-1}$.
By lemma mon 2 we know that $f[E]$ is actually an open environment of $f(a)$. Thus the idea is to utilize the open set differentiation extension theorem from the section Locality, which says that if we have a differentiable real map extension of a partial function defined on an open set, the partial function is locally differentiable. So what is left to show is that ext is differentiable, which we do by showing the existence of a difference quotient map. For that we define

$$\psi(y) = \begin{cases} 1/\varphi((f \restriction E)^{-1}(y)) & : y \in f[E] \\ 0 & : y \notin f[E] \end{cases}$$

for any real number y, where φ is the difference quotient map of f in a; φ exists, because f is differentiable in a. ψ is a difference quotient map for ext in $f(a)$ if it satisfies

(1) $ext(y) = (y - f(a))\psi(y) + ext(f(a))$ for all y.

(2) ψ is continuous in $f(a)$

(1) we can show by case distinction with $y \in f[E]$ and $y \notin f[E]$ and then some simple calculation. For (2) we need lemma mon 3 though, which says that $(f \restriction E)^{-1}$ is locally continuous in $f(a)$. Furthermore φ is continuous in a and therefore by the continuity of composition and the continuity of a division map is ψ also continuous in $f(a)$.
Thus ext is differentiable in $f(a)$ and since $\mathrm{dom}((f \restriction E)^{-1}) = f[E]$ is an open set, $(f \restriction E)^{-1}$ is locally differentiable in $f(a)$ by open set differentiation extension and

$$\partial_\ell (f \restriction E)^{-1}(f(a)) = \psi(f(a)) = 1/\varphi((f \restriction E)^{-1}(f(a))) = 1/f'(a).$$

Theorem 5 (inverse function theorem). Let f be continuously differentiable and $\partial f(a) \neq 0$. Then there exists an open neighbourhood E of a such that f is strictly monotone on E and $(f \restriction E)^{-1}$ is a real map on $f[E]$ and $(f \restriction E)^{-1}$ is locally differentiable in $f(a)$ and $\partial_\ell (f \restriction E)^{-1}(f(a)) = 1/\partial f(a)$.

Proof. Take an open neighbourhood N of a such that f is strictly monotone on N (by lemma mon 4). Take a positive real number d such that $B_d(a) \subseteq N$. Take a positive real number δ such that $\delta < d$. Take $E = B_\delta(a)$.

(0) f is strictly monotone on E (by subset monotone lemma). Indeed E is a subset of N. Indeed $E = B_\delta(a) \subseteq B_d(a) \subseteq N$ (by ball subsets).

$f \restriction E$ is injective (by lemma mon 1, 0). Indeed E is a subset of $\mathbb{R}$. Thus there exists $(f \restriction E)^{-1}$.

Take $g = f \restriction E$. g is a real map on E. Take $D = \mathrm{Range}(g)$. D is an open neighbourhood of $f(a)$ (by lemma mon 2, 0). Indeed f is continuous and $D = f[E]$ and E is an open neighbourhood of a. g^{-1} is a real map on D.

(3) Take a difference quotient map φ of f in a. For all $y \in D$ there exists an $x \in E$ such that $x = g^{-1}(y)$.

Define

$$ext(y) = \begin{cases} g^{-1}(y) & : y \in D \\ a & : y \notin D \end{cases}$$

for $y \in \mathbb{R}$. ext is a real map extension of g^{-1}. Indeed $ext(y)$ is a real number for any real number y. Indeed a is a real number and $g^{-1}(y)$ is a real number for every $y \in D$.

For all $y \in D$ $\varphi(g^{-1}(y)) \neq 0$. Proof. $\mathrm{Range}(g^{-1}) \subseteq E$. Indeed $E = \mathrm{dom}(g)$. We can show that for all $x \in E$ $\varphi(x) \neq 0$. Let $x \in E$.

Case $x = a$. Trivial.

Case $x \neq a$. $\varphi(x) = \frac{\Delta f}{\Delta a}(x)$ (by prop difference quotient map, 3). Indeed f is differentiable in a and $x \in \mathbb{R} \setminus \{a\}$. Therefore $x \neq a$ and $x - a$ is nonzero. $\frac{\Delta f}{\Delta a}(x) = \frac{f(x)-f(a)}{x-a}$ (by def diffquot). (4) $f(x) - f(a)$ is nonzero. Indeed $x \neq a$ and $x, a \in E$ and $f(x) \neq f(a)$. $\frac{f(x)-f(a)}{x-a} \neq 0$ (by frac lemma 5,4). Indeed $x - a$ is nonzero. End. End. End.

Define

$$\psi(y) = \begin{cases} 1/\varphi(g^{-1}(y)) & : y \in D \\ 0 & : y \notin D \end{cases}$$

for $y \in \mathbb{R}$. ψ is a real map. Indeed $1/\varphi(g^{-1}(y))$ is a real number for any $y \in D$.

Let us show that ψ is a difference quotient map of ext in $f(a)$.

Let us show that for every real number y $ext(y) = ((y - f(a)) \cdot \psi(y)) + ext(f(a))$. Let y be a real number.

(A) $ext(f(a)) = a$. $((y - f(a)) \cdot \psi(y)) + ext(f(a)). = ((y - f(a)) \cdot \psi(y)) + a$ (by A).

Case $y = f(a)$. $((f(a) - f(a)) \cdot \psi(y)) + a. = a$ (by additive inverse, eq lemma 9, add comm, existence zero) $. = ext(f(a))$ (by A). End.

Case $y \in D$ and $y \neq f(a)$.

(B) $\psi(y) = \frac{g^{-1}(y)-a}{y-f(a)}$. Proof. $\varphi(g^{-1}(y)) = \frac{\Delta f}{\Delta a}(g^{-1}(y))$ (by prop difference quotient map,3). Indeed f is differentiable in $g^{-1}(y)$ and $g^{-1}(y) \in \mathbb{R} \setminus \{a\}$. $g^{-1}(y) \neq a$ and $g^{-1}(y) - a$ is nonzero. $\frac{\Delta f}{\Delta a}(g^{-1}(y)). = \frac{f(g^{-1}(y))-f(a)}{g^{-1}(y)-a}$ (by def diffquot). $f(g^{-1}(y)) - f(a) \neq 0$. $1/\frac{f(g^{-1}(y))-f(a)}{g^{-1}(y)-a}. = \frac{g^{-1}(y)-a}{f(g^{-1}(y))-f(a)}$ (by eq lemma 22). $\psi(y) = 1/\varphi(g^{-1}(y)) = \frac{g^{-1}(y)-a}{y-f(a)}$. End.

$((y - f(a)) \cdot \psi(y)) + a. = ((y - f(a)) \cdot \frac{g^{-1}(y)-a}{y-f(a)}) + a$ (by B) $. = (g^{-1}(y) - a) + a$ (by fraction lemma 2) $. = g^{-1}(y)$ (by add ass, add comm, additive inverse, existence zero).

$ext(y) = g^{-1}(y)$. End.

Case $y \notin D$. (C) $\psi(y) = 0$. $((y - f(a)) \cdot \psi(y)) + a. = a$ (by mult comm, C, eq lemma 9, add

comm, existence zero). $ext(y) = a$. End. End.

Let us show that ψ is continuous in $f(a)$.

(4) E is an open neighbourhood of a. Proof. a is an element of E. E is an open set (by open interval is open). Indeed $E = (a - \delta, a + \delta)_{OI}$ (by ball and open interval). End.

g^{-1} is locally continuous in $f(a)$ (by lemma mon 3, 4, 0). Indeed f is continuous. ext is continuous in $f(a)$. φ is continuous in a. [timelimit 50] Thus $\varphi \circ ext$ is continuous in $f(a)$ (by continuity of composition). Indeed $g^{-1}(f(a)) = a$. ψ is continuous in $f(a)$ (by continuity division map). Proof. $\varphi \circ ext$ is a real map. ψ is a division map of $\varphi \circ ext$ in $f(a)$ (by 4). End. End. End.

Therefore ext is differentiable in $f(a)$. $\mathrm{dom}(g^{-1}) = D$ and D is an open neighbourhood of $f(a)$. g^{-1} is a real map on D. Indeed $\mathrm{Range}(g) = D$ and for every $y \in D$ $g^{-1}(y)$ is a real number. $\partial_\ell g^{-1}(f(a)) = \partial ext(f(a))$ (by open set differentiation extension). Indeed ext is a real map extension of g^{-1} such that ext is differentiable in $f(a)$. [timelimit 3] $\partial ext(f(a)) = \psi(f(a)) = 1/\varphi(g^{-1}(f(a))) = 1/\varphi(a) = 1/\partial f(a)$. $(f \restriction E)^{-1}$ is locally differentiable in $f(a)$. $\partial_\ell (f \restriction E)^{-1}(f(a)) = 1/\partial f(a)$. $f \restriction E$ is injective. $(f \restriction E)^{-1}$ is a real map on $f[E]$. Indeed $\mathrm{Range}(f \restriction E) = f[E]$. □

[timelimit 3]

Definition 6. A locally inverse function to f in a is a real map g such that for some open neighbourhood E of a $f \restriction E$ is injective and $g \restriction f[E] = (f \restriction E)^{-1}$.

Corollary 7 (inverse function theorem II). Let f be continuously differentiable and $\partial f(a) \neq 0$. Let g be a locally inverse function to f in a. Then g is differentiable in $f(a)$ and $\partial g(f(a)) = 1/\partial f(a)$.

Proof. Take an open neighbourhood M of a such that $f \restriction M$ is injective and $g \restriction f[M] = (f \restriction M)^{-1}$. [timelimit 100] Take an open neighbourhood N of a such that f is strictly monotone on N and $(f \restriction N)^{-1}$ is a real map on $f[N]$ and $(f \restriction N)^{-1}$ is locally differentiable in $f(a)$ and $\partial_\ell (f \restriction N)^{-1}(f(a)) = 1/\partial f(a)$ (by inverse function theorem). Take $E = N \cap M$.

E is an open neighbourhood of a. Indeed a is an element of E and E is an open set (by intersection of open sets).

$f \restriction E$ is injective. $(f \restriction E)^{-1}$ is a real map on $f[E]$. Indeed $\mathrm{dom}((f \restriction E)^{-1}) = \mathrm{Range}(f \restriction E) = f[E]$ and for every $y \in f[E]$ $(f \restriction E)^{-1}(y)$ is a real number. Define

$$ext(y) = \begin{cases} (f \restriction N)^{-1}(y) & : y \in f[N] \\ 0 & : y \notin f[N] \end{cases}$$

for $y \in \mathbb{R}$. ext is a real map. Indeed $\mathrm{Range}(ext)$ is a subset of $\mathbb{R}$. Indeed for any $y \in \mathrm{dom}(ext)$ $ext(y)$ is a real number. Indeed 0 is a real number and $(f \restriction N)^{-1}(y)$ is a real number for any $y \in f[N]$. $(f \restriction N)^{-1}$ is a real map on $f[N]$.

$f[N]$ is a subset of $\mathbb{R}$. Take $\gamma = (f \restriction N)^{-1}$. γ is a real map on $f[N]$. $f(a)$ is an element of $f[N]$. γ is locally differentiable in $f(a)$. [prover vampire] $\partial_\ell \gamma(f(a)) = \partial ext(f(a))$ (by lemma differentiation extension). Indeed ext is a real map extension of γ. [prover eprover] Thus $\partial_\ell (f \restriction N)^{-1}(f(a)) = \partial ext(f(a))$.

Take $h = (f \restriction E)^{-1}$. h is locally differentiable in $f(a)$ and $\partial ext(f(a)) = \partial_\ell h(f(a))$ (by open set

differentiation extension). Proof. h is a real map on $f[E]$. $f[E]$ is an open neighbourhood of $f(a)$ (by lemma mon 2, 1). Proof. f is strictly monotone on E (by subset monotone lemma). Indeed E is a subset of N. f is continuous. End.

ext is a real map extension of h such that ext is differentiable in $f(a)$. Indeed we can show that for every $y \in f[E]$ $(f \restriction E)^{-1}(y) = ext(y)$. Let $y \in f[E]$. Then $y \in f[N]$. $h(y) = (f \restriction N)^{-1}(y)$. $(f \restriction N)^{-1}(y) = ext(y)$. End. End.

$\partial_\ell (f \restriction E)^{-1}(f(a)) = \partial g(f(a))$ (by lemma differentiation extension). Proof. $f[E]$ is a subset of $\mathbb{R}$. $(f \restriction E)^{-1}$ is a real map on $f[E]$. $f(a)$ is an element of $f[E]$. $(f \restriction E)^{-1}$ is locally differentiable in $f(a)$. g is a real map extension of $(f \restriction E)^{-1}$. Proof. $f \restriction M$ is injective. $(f \restriction M)^{-1}$ is a real map on $f[M]$. $f[E] \subseteq f[M]$. For every $y \in f[E]$ $(f \restriction E)^{-1}(y) = g(y)$. Proof. Assume the contrary. Take $y \in f[E]$ such that $(f \restriction E)^{-1}(y) \neq g(y)$. Take $x = (f \restriction E)^{-1}(y)$. Then $(f \restriction E)(x) = y$. Thus $f(x) = y$. $x \in M$. Indeed $E \subseteq M$. Hence $(f \restriction M)(x) = y$. Thus $(f \restriction M)^{-1}(y) = x$. Indeed $y \in f[M]$. $(f \restriction E)^{-1}(y) = (f \restriction M)^{-1}(y)$. $(f \restriction M)^{-1}(y) \neq g(y)$. Contradiction. End. End. End. □

[timelimit 3]