Chapter 1

Exponential Function and Logarithm

[read `examples/solow/sections/6inverse.ftl.tex`]

Signature 1. $\mathbb{R}_{>0}$ is the collection of positive real numbers.

Lemma 2. $\mathbb{R}_{>0}$ is an open set.

Proof. Let $x \in \mathbb{R}_{>0}$. Then $B_x(x) \subseteq \mathbb{R}_{>0}$. Indeed $B_x(x) = (x-x, x+x)_{OI} = (0, x+x)_{OI} \subseteq \mathbb{R}_{>0}$. □

Lemma 3 (positive reals open neighbourhood). Let a be a positive real number. $\mathbb{R}_{>0}$ is an open neighbourhood of a.

1.1 e and Logarithm

Let a, b, x, y denote real numbers.

Signature 4 (natural exponential function). exp is a real map such that $\mathrm{D}\exp = \exp$.

Axiom 5 (exp of 0). $\exp(x) = 1$ iff $x = 0$.

Axiom 6 (exp calculation axiom). $\exp(a + b) = \exp(a) \cdot \exp(b)$.

Corollary 7. exp is continuously differentiable.

Lemma 8 (lemma exp nonzero). $\exp(x) \neq 0$.

Proof by contradiction. Assume the contrary. Take a real number x such that $\exp(x) = 0$. $0 = 0 \cdot \exp(-x) = \exp(x) \cdot \exp(-x) = \exp(x - x) = \exp(0) = 1$. □

Lemma 9 (lemma exp positive). $\exp(x)$ is a positive real number.

Proof. Assume $\exp(x) < 0$.

(1) Case $x = 0$. Trivial.

(2) Case $x < 0$. Then $\exp(x) < 0 < \exp(0)$ and exp is continuous. [timelimit 10]

There exists $c \in (x, 0)_{OI}$ such that $\exp(c) = 0$ (by Bolzanos Theorem).

Contradiction (by lemma exp nonzero). [timelimit 3] End.

(3) Case $x > 0$. Then $(-\exp)(0) < 0 < (-\exp)(x)$ and $(-\exp)$ is continuous. [timelimit 10] There exists $c \in (0, x)_{OI}$ such that $(-\exp)(c) = 0$ (by Bolzanos Theorem). [timelimit 3] Contradiction (by lemma exp nonzero). End. □

Proposition 10 (exp inequality). $1 + x \leq \exp(x)$.

Proof. Take $f = (\mathrm{f}_{1\cdot_+1} + (-\exp))$.

0 is a maximum of f (by sufficient condition for maximum on interval). Proof. $\mathbb{R}$ is an interval. (A) f is differentiable and $\mathrm{D}f = (\mathrm{f}_{0\cdot_+1} + (-\exp))$

(by minus derivative 2, derivative addition rule 2, natural exponential function, derivative of linear function).

Let us show that for every real number y $\partial\mathrm{D}f(y) < 0$. Let y be a real number.

$\partial\mathrm{D}f(y). = \mathrm{D}(\mathrm{f}_{0\cdot_+1} + (-\exp))(y)$ (by deriv and derivin, A)

$. = (\mathrm{Df}_{0\cdot_+1} + \mathrm{D}(-\exp))(y)$ (by derivative addition rule 2)

$. = (\mathrm{f}_{0\cdot_+0} + (-\mathrm{D}\exp))(y)$ (by minus derivative 2, derivative of linear function)

$. = (\mathrm{f}_{0\cdot_+0} + (-\exp))(y)$ (by natural exponential function)

$. = ((0 \cdot y) + 0) - \exp(y)$ (by def add func, linear function, def func minus)

$. = -\exp(y) + 0$ (by eq lemma 9, existence zero, add comm)

$. = -\exp(y)$ (by existence zero).

$-\exp(y) < 0$ (by ineq lemma 2, lemma exp positive).

Therefore $\partial\mathrm{D}f(y) < 0$. End.

0 is a local maximum of f (by sufficient condition local maximum). Proof.

(B) $\exp(0) = 1$.

$\partial f(0). = (\mathrm{f}_{0\cdot_+1} + (-\exp))(0)$ (by A, deriv and derivin)

$. = \mathrm{f}_{0\cdot_+1}(0) - \exp(0)$ (by def add func, def func minus)

$. = 1 - 1$ (by linear lem 1, B)

$. = 0$ (by additive inverse). $\partial\mathrm{D}f(0) < 0$. End.

f is strictly convex on $\mathbb{R}$. End.

Thus $1 + x \leq \exp(x)$. Proof. $f(x) \leq f(0)$.

$(1 + x) - \exp(x) = ((1 \cdot x) + 1) - \exp(x) = \mathrm{f}_{1\cdot_+1}(x) - \exp(x) = \mathrm{f}_{1\cdot_+1}(x) + (-\exp)(x) = (\mathrm{f}_{1\cdot_+1} + (-\exp))(x) = f(x)$

$\leq f(0) = (\mathrm{f}_{1\cdot_+1} + (-\exp))(0) = ((1 \cdot 0) + 1) - \exp(0) = 1 - \exp(0) = 0.$

Thus $(1 + x) - \exp(x) \leq 0$. Therefore $(1 + x) \leq \exp(x)$ (by ineq lemma 27). End. □

Proposition 11 (range exp). $\mathrm{Range}(\exp) = \mathbb{R}_{>0}$.

Proof. $\mathrm{Range}(\exp) \subseteq \mathbb{R}_{>0}$. Proof. Let $y \in \mathrm{Range}(\exp)$. Then y is nonzero (by lemma exp nonzero). Take a real number x such that $\exp(x) = y$. Then $\exp(x)$ is positive (by lemma exp positive). qed.

$\mathbb{R}_{>0} \subseteq$ Range(exp). Proof. Let $y \in \mathbb{R}_{>0}$.

(1) Case $y = 1$. Trivial.

(2) Case $y > 1$. Then $y < 1 + y \leq \exp(y)$. $y \in (\exp(0), \exp(y))_{OI}$. [timelimit 20] Thus $y \in \exp[(0, y)_{OI}]$. Indeed $(\exp(0), \exp(y))_{OI} \subseteq \exp[(0, y)_{OI}]$ (by intermediate value theorem I). Indeed $0 < y$ and exp is continuous. [timelimit 3] Therefore there exists $x \in (0, y)_{OI}$ such that $\exp(x) = y$. End.

(3) Case $y < 1$. Take $\upsilon = 1/y$. Then $\upsilon = 1/y > 1/1 = 1$. Then $\upsilon < 1 + \upsilon \leq \exp(\upsilon)$. $\upsilon \in (\exp(0), \exp(\upsilon))_{OI}$. [timelimit 20] Thus $\upsilon \in \exp[(0, \upsilon)_{OI}]$. Indeed $(\exp(0), \exp(\upsilon))_{OI} \subseteq \exp[(0, \upsilon)_{OI}]$ (by intermediate value theorem I). Indeed $0 < \upsilon$ and exp is continuous. [timelimit 3] Take $x \in (0, \upsilon)_{OI}$ such that $\exp(x) = \upsilon$. Then $-x \in \text{dom}(\exp)$. $\exp(-x) = 1/\exp(x) = 1/\upsilon = y$. End. qed. □

Lemma 12 (lemma 1). $\exp(-a) = 1/\exp(a)$.

Proof. $\exp(a) \cdot \exp(-a) = \exp(a - a) = \exp(0) = 1$. □

Proposition 13 (exp is injective). exp is injective.

Proof by contradiction. Assume exp is not injective.

Take distinct real numbers x, y such that $\exp(x) = \exp(y)$. $1 = \exp(x) \cdot \exp(-x) = \exp(y) \cdot \exp(-x) = \exp(y - x)$. Thus $y - x = 0$. $x = y$. Contradiction. □

Definition 14 (log). $\ln = \exp^{-1}$.

Lemma 15 (domain log). $\text{dom}(\ln) = \mathbb{R}_{>0}$.

Corollary 16. For every positive real number y there exists a real number x such that $\exp(x) = y$.

Proof. Let y be a positive real number. Take $x = \ln(y)$. x is a real number and $\exp(x) = y$. □

Lemma 17. Let x, y be positive. Then $x \cdot y$ is a positive real number and $\ln(x) + \ln(y) = \ln(x \cdot y)$.

Proof. Take a real number a such that $x = \exp(a)$. Take a real number b such that $y = \exp(b)$. $\exp(a) \cdot \exp(b)$ is a positive real number. $\ln(x) + \ln(y) = \ln(\exp(a)) + \ln(\exp(b)) = a + b = \ln(\exp(a + b)) = \ln(\exp(a) \cdot \exp(b)) = \ln(x \cdot y)$. □

Lemma 18. Let x, y be positive. Then $\frac{x}{y}$ is a positive real number and $\ln(x) - \ln(y) = \ln\left(\frac{x}{y}\right)$.

Proof. $\frac{x}{y}$ is a positive real number. Take a real number a such that $x = \exp(a)$. Take a real number b such that $y = \exp(b)$. $\ln(x) - \ln(y) = \ln(\exp(a)) - \ln(\exp(b)) = a - b = \ln(\exp(a - b)) = \ln(\exp(a) \cdot \exp(-b)) = \ln(\exp(a) \cdot (1/\exp(b))) = \ln(x \cdot (1/y)) = \ln\left(\frac{x}{y}\right)$. □

1.2 Exponential and Power Function

Let a, x, y denote real numbers.

Definition 19 (exponential function). Let a be positive. $\exp_a = \exp \circ \mathrm{f}_{\ln(a)\cdot_+0}$.

Lemma 20 (evaluated exponential function). Let a be positive. $\exp_a(x) = \exp(\ln(a) \cdot x)$.

Proof. $\exp_a(x). = (\exp \circ \mathrm{f}_{\ln(a)\cdot_+0})(x)$

$. = \exp(\mathrm{f}_{\ln(a)\cdot_+0}(x))$ (by composition)

$. = \exp((\ln(a) \cdot x) + 0)$ (by linear function)

$. = \exp(\ln(a) \cdot x)$ (by existence zero). □

Lemma 21 (lemma exponential function). $\exp_{\exp(x)}(y) = \exp(x \cdot y)$.

Proof. $\exp_{\exp(x)}(y). = \exp(\ln(\exp(x)) \cdot y)$ (by evaluated exponential function)

$. = \exp(x \cdot y)$ (by log, lemma inverse). □

Lemma 22 (lemma 2 exponential function). Let a be positive. $\exp_{\exp_a(x)}(y) = \exp_a(x \cdot y)$.

Proof. $\exp_{\exp_a(x)}(y). = \exp_{\exp(\ln(a)\cdot x)}(y)$ (by evaluated exponential function)

$. = \exp((\ln(a) \cdot x) \cdot y)$ (by lemma exponential function)

$. = \exp_a(x \cdot y)$ (by mult ass, evaluated exponential function). □

Lemma 23 (lemma 3). Let a be a positive real number. $\exp_a(x) \cdot \exp_a(y) = \exp_a(x + y)$.

Proof. $\exp_a(x) \cdot \exp_a(y)$

$. = \exp(\ln(a) \cdot x) \cdot \exp(\ln(a) \cdot y)$ (by evaluated exponential function)

$. = \exp((\ln(a) \cdot x) + (\ln(a) \cdot y))$ (by exp calculation axiom)

$. = \exp(\ln(a) \cdot (x + y))$ (by distributive law)

$. = \exp_a(x + y)$ (by evaluated exponential function). □

Lemma 24 (lemma 4). Let x be a positive real number. $1/x = \exp_x(-1)$.

Proof. $\ln(x) \cdot (-1), \exp_x(-1)$ are a real numbers.

$x \cdot \exp_x(-1)$

$. = x \cdot \exp(\ln(x) \cdot (-1))$ (by evaluated exponential function)

$. = x \cdot \exp(-\ln(x))$ (by mult comm, eq lemma 12)

$. = x \cdot 1/\exp(\ln(x))$ (by lemma 1)

$. = x \cdot (1/x)$ (by log, inverse)

$. = 1.$ □

Definition 25 (power function). $\mathrm{pow}_a = ((\exp \circ \mathrm{f}_{a\cdot_+0}) \circ \ln) \upharpoonright \mathrm{dom}(\ln)$.

Lemma 26. pow_a is a real map on $\mathbb{R}_{>0}$.

Proof. $\mathrm{dom}(\mathrm{pow}_a) = \mathrm{dom}(\ln) = \mathbb{R}_{>0}$.

We can show that $\mathrm{pow}_a(x)$ is a real number for all $x \in \mathbb{R}_{>0}$. Let $x \in \mathbb{R}_{>0}$. $\mathrm{pow}_a = ((\exp \circ \mathrm{f}_{a\cdot_+0}) \circ \ln) \restriction \mathrm{dom}(\ln)$. $(((\exp \circ \mathrm{f}_{a\cdot_+0}) \circ \ln) \restriction \mathrm{dom}(\ln))(x) = ((\exp \circ \mathrm{f}_{a\cdot_+0}) \circ \ln)(x)$. $((\exp \circ \mathrm{f}_{a\cdot_+0}) \circ \ln)(x) = \exp(\mathrm{f}_{a\cdot_+0}(\ln(x)))$ (by composition). $\ln(x)$, $\mathrm{f}_{a\cdot_+0}(\ln(x))$, $\exp(\mathrm{f}_{a\cdot_+0}(\ln(x)))$ are real numbers. End.

Thus $\mathrm{Range}(\mathrm{pow}_a) \subseteq \mathbb{R}$. □

Lemma 27 (evaluated power function). Let b be a positive real number. $\mathrm{pow}_a(b) = \exp(a \cdot \ln(b))$.

Proof. $\mathrm{dom}(\mathrm{pow}_a) = \mathrm{dom}(\ln)$ and $b \in \mathrm{dom}(\ln)$.

$\mathrm{pow}_a(b) = ((\exp \circ \mathrm{f}_{a\cdot_+0}) \circ \ln)(b) = \exp(\mathrm{f}_{a\cdot_+0}(\ln(b))) = \exp(a \cdot \ln(b))$. □

Proposition 28 (equivalence of exponential and power function). Let b be a positive real number. $\mathrm{pow}_a(b) = \exp_b(a)$.

Proof. $\mathrm{pow}_a(b). = \exp(a \cdot \ln(b))$ (by evaluated power function)

$. = \exp(\ln(b) \cdot a)$ (by mult comm)

$. = \exp_b(a)$ (by evaluated exponential function). □

In `100_theorems.ftl.tex` x^n was already defined recursively as $x^{n-1} \cdot x$ with $x^0 = 1$ for any natural number n. Here we just want to show that our concept of exponentiation is merely an extension thereof.

Theorem 29 (equivalence exponential function and exponent). Let x be a positive real number. For every natural numbers n $\exp_x(n) = x^n$.

Proof by induction on n. Let n be a natural number.

Case $n = 0$. $\exp_x(0) = \exp(\ln(x) \cdot 0) = \exp(0) = 1 = x^0$. End.

Case $n = 1$. $\exp_x(1) = \exp(\ln(x) \cdot 1) = \exp(\ln(x)) = x = x^1$. End.

Case $n > 1$. Take a natural number m such that $m = n - 1$.m is inductively smaller than n. Thus $\exp_x(m) = x^m$ and $\exp_x(1) = x^1$. $\exp_x(m)$, $\exp_x(1)$ are real numbers. $\exp_x(m+1). = \exp_x(m) \cdot \exp_x(1)$ (by lemma 3).

$\exp_x(n) = \exp_x(m+1)$

$= \exp_x(m) \cdot \exp_x(1)$

$= x^m \cdot \exp_x(1)$

$= x^m \cdot x^1 = x^{m+1} = x^n$. End.[timelimit 3] □

Corollary 30 (equivalence exponential function and exponent 2). Let x be a positive real number and n be a natural number. x^n is positive and $1/x^n = \exp_x(-n)$.

Proof. x^n is positive. Indeed $x^n = \exp_x(n)$ and $\exp_x(n)$ is positive. $\exp_x(-n). = \exp_{\exp_x(n)}(-1)$ (by eq lemma 12, mult comm, lemma 2 exponential function) $. = 1/x^n$ (by equivalence exponential function and exponent, lemma 4). □

Corollary 31. Let n be a natural number and x be a positive real number. $\mathrm{pow}_n(x) = x^n$.

Proof. $\mathrm{pow}_n(x). = \exp_x(n)$ (by equivalence of exponential and power function)

$. = x^n$ (by equivalence exponential function and exponent). □

Lemma 32. Let x be a real number and a be a positive real number. $\exp_a(x)$ is a positive real number and $x \cdot \ln(a) = \ln(\exp_a(x))$.

Proof. Take a real number α such that $a = \exp(\alpha)$. $x \cdot \ln(a) = x \cdot \ln(\exp(\alpha)) = x \cdot \alpha = \ln(\exp(x \cdot \alpha)) = \ln(\exp(\ln(a) \cdot x)) = \ln(\exp(\mathrm{f}_{\ln(a)\cdot_+0}(x))) = \ln(\exp_a(x))$. □

1.3 Derivative Theorems

Let a denote a real number. Let f denote a real map.

Theorem 33 (derivative log). Let x be a positive real number. ln is locally differentiable in x and $\partial_\ell \ln(x) = 1/x$.

Proof. Let us show that for any real map extension ext of ln ext is differentiable in x and $\partial ext(x) = 1/x$. Let ext be a real map extension of ln. Take $\xi = \ln(x)$.

We can show that ext is a locally inverse function to exp in ξ. $\mathbb{R}$ is an open neighbourhood of x (by R is open). $\exp \upharpoonright \mathbb{R}$ is injective (by exp is injective).

$ext \upharpoonright \exp[\mathbb{R}] = (\exp \upharpoonright \mathbb{R})^{-1}$. Proof.

$ext \upharpoonright \exp[\mathbb{R}] = ext \upharpoonright \mathrm{dom}(\ln)$. Indeed $\exp[\mathbb{R}] = \mathrm{dom}(\ln)$. [timelimit 20] $ext \upharpoonright \mathrm{dom}(\ln) = \ln$ (by function lemma 2). Proof. $ext \upharpoonright \mathrm{dom}(\ln), \ln$ are maps such that $\mathrm{dom}(ext \upharpoonright \mathrm{dom}(\ln)) = \mathrm{dom}(\ln)$. $(ext \upharpoonright (\exp[\mathbb{R}]))(y) = \ln(y)$ for all $y \in \mathrm{dom}(\ln)$. End. [timelimit 3]

$\ln = \exp^{-1} = (\exp \upharpoonright \mathbb{R})^{-1}$. Indeed $\exp = (\exp \upharpoonright \mathbb{R})$. End. End.

exp is continuously differentiable and $\partial \exp(\xi) \neq 0$. $\partial ext(\exp(\xi)) = 1/\partial \exp(\xi)$ (by inverse function theorem II). Therefore $\partial ext(x) = \partial ext(\exp(\xi)) = 1/\partial \exp(\xi) = 1/\exp(\xi) = 1/x$. End.

Thus $\partial_\ell \ln(x) = \partial ext|_{\ln}(x) = 1/x$. □

Theorem 34 (derivative exponential function). Let a be a positive real number. $\exp_a$ is differentiable and $\mathrm{D}\exp_a = (\mathrm{f}_{0\cdot_+\ln(a)} \cdot \exp_a)$.

Proof. $\mathrm{D}\exp_a . = \mathrm{D}\exp \circ \mathrm{f}_{\ln(a)\cdot_+0}$ (by exponential function)

$. = (\mathrm{D}\exp \circ \mathrm{f}_{\ln(a)\cdot_+0} \cdot \mathrm{Df}_{\ln(a)\cdot_+0})$ (by derivative chain rule 2)

$. = (\exp \circ \mathrm{f}_{\ln(a)\cdot_+0} \cdot \mathrm{f}_{0\cdot_+\ln(a)})$ (by natural exponential function, derivative of linear function)

$. = (\exp_a \cdot \mathrm{f}_{0\cdot_+\ln(a)})$ (by exponential function)

$. = (\mathrm{f}_{0\cdot_+\ln(a)} \cdot \exp_a)$ (by lemma 2). □

Theorem 35 (derivative power function). Let x be a positive real number. pow_a is locally differentiable in x and $\partial_\ell \mathrm{pow}_a(x) = a \cdot \mathrm{pow}_{a-1}(x)$.

Proof. (0) Take $h = (\mathrm{f}_{a\cdot_+0} \circ ext|_{\ln})$. Then h is a real map and differentiable in x.[timelimit 10] $\exp \circ h$ is a real map and differentiable in x.

(A) $\exp \circ h$ is a real map extension of pow_a. Proof.

Let $y \in \mathrm{dom}(\ln)$. (A1) $ext|_{\ln}(y) = \ln(y)$.

(A2) Take $\eta = (\exp \circ \mathrm{f}_{a\cdot_+0})$. η is a real map.

$\eta \circ \ln$ is a real map on $\mathrm{dom}(\ln)$. Indeed $\mathrm{dom}(\eta \circ \ln) = \mathrm{dom}(\ln)$.

$(\exp \circ (\mathrm{f}_{a\cdot_+0} \circ ext|_{\ln}))(y) . = \exp(\mathrm{f}_{a\cdot_+0}(ext|_{\ln}(y)))$ (by composition)

$. = \exp(\mathrm{f}_{a\cdot_+0}(\ln(y)))$ (by A1)

$. = ((\exp \circ \mathrm{f}_{a\cdot_+0}) \circ \ln)(y)$ (by composition)

$. = (\eta \circ \ln)(y)$ (by A2).

Take $\mu = \eta \circ \ln$. Therefore $\mu(y) = (\mu \upharpoonright \mathrm{dom}(\ln))(y)$. Then $(\eta \circ \ln)(y) = ((\eta \circ \ln) \upharpoonright \mathrm{dom}(\ln))(y).((\eta \circ \ln) \upharpoonright \mathrm{dom}(\ln))(y) = \mathrm{pow}_a(y)$. End.

(B) $\partial ext|_{\ln}(x) = \partial_\ell \ln(x) = 1/x$.

(C) $\exp(h(x)) = \mathrm{pow}_a(x)$. Indeed then $(\exp \circ h)(x) = \mathrm{pow}_a(x)$. [timelimit 3]

$\partial \exp \circ h(x) . = \partial \exp(h(x)) \cdot \partial h(x)$ (by derivative chain rule) $. = \partial \exp(h(x)) \cdot \partial(\mathrm{f}_{a\cdot_+0} \circ ext|_{\ln})(x)$ (by 0)

$. = \exp(h(x)) \cdot (\partial \mathrm{f}_{a\cdot_+0}(ext|_{\ln}(x)) \cdot \partial ext|_{\ln}(x))$ (by deriv and derivin, natural exponential function, derivative chain rule)

$. = \mathrm{pow}_a(x) \cdot (\mathrm{f}_{0\cdot_+a}(ext|_{\ln}(x)) \cdot (1/x))$ (by C, deriv and derivin, derivative of linear function, B)

$. = (\exp_x(a) \cdot (1/x)) \cdot a$ (by linear lem 1, equivalence of exponential and power function, mult comm, mult ass)

$. = (\exp_x(a) \cdot \exp_x(-1)) \cdot a$ (by lemma 4)

$. = \exp_x(a - 1) \cdot a$ (by lemma 3)

$. = a \cdot \mathrm{pow}_{(a-1)}(x)$ (by mult comm, equivalence of exponential and power function).

(D) $\mathbb{R}_{>0}$ is an open neighbourhood of x and pow_a is a real map on $\mathbb{R}_{>0}$.

[timelimit 10] Therefore $\partial_\ell \mathrm{pow}_a(x) = \partial(\exp \circ h)(x)$ (by open set differentiation extension, A, D). Indeed h is differentiable in x. [timelimit 3] $\mathrm{pow}_{(a-1)}(x)$ is a real number. Therefore $\partial_\ell \mathrm{pow}_a(x) = \partial(\exp \circ h)(x) = a \cdot \mathrm{pow}_{(a-1)}(x)$. □

Lemma 36 (lemma derivative of division map). Let f be differentiable in a and $f(a)$ be positive. Let g be a division map of f in a. Then g is differentiable in a and $\partial g(a) = -\frac{\partial f(a)}{(f(a))^2}$.

Proof. Take an open neighbourhood N of a such that for any $x \in N$ $f(x)$ is nonzero and $g(x) = 1/f(x)$. Take a positive real number τ such that $B_\tau(a)$ is a subset of N. Take a positive real number θ such that θ is a domain restrictor for $f(a)$ and f around a and $f(a)$. Indeed f is continuous in a. Take a positive real number δ such that $\delta < \tau$ and $\delta < \theta$ (by ineq lemma 19). Take $E = B_\delta(a)$.

For every $x \in E$ $f(x)$ is nonzero and $g(x) = 1/f(x)$. Proof. Let $x \in E$. x is an element of N. Indeed $B_\delta(a) \subseteq B_\tau(a) \subseteq N$. End.

(1) For every $x \in E$ $f(x)$ is positive and $g(x) = 1/f(x)$. Proof. [prover vampire] Assume the contrary. Take $x \in E$ such that $f(x)$ is not positive.

(1A) $f(a), -f(x)$ are nonnegative real numbers. Indeed $f(x) = 0$ or $f(x)$ is negative. [prover eprover] Then $|f(x) - f(a)| = |f(a) - f(x)| = f(a) - f(x)$. Indeed $f(a) - f(x)$ is nonnegative (by ineq lemma 21, 1A). $f(a) - f(x) \geq f(a)$ (by ineq lemma 23, 1A). [timelimit 20] θ is not a

domain restrictor for $f(a)$ and f around a and $f(a)$ (by domain restrictor lemma 2). Indeed we can show that there exists $y \in \mathbb{R} \setminus \{a\}$ such that $|y - a| < \theta$ and $|f(y) - f(a)| \geq f(a)$. $|x - a| < \theta$. Indeed $|x - a| < \delta < \theta$. $|f(x) - f(a)| \geq f(a)$. $x \in \mathbb{R} \setminus \{a\}$. End. [timelimit 3] End.

Take $ext = ext|_{\mathrm{pow}_{(-1)}}$. $ext \circ f$ is a real map.

(2) Then $ext \circ f$ is a real map extension of $g \upharpoonright E$. Indeed we can show that $g(x) = (ext \circ f)(x)$ for any $x \in E$. Let $x \in E$. Then $(ext \circ f)(x) = ext(f(x)) = \mathrm{pow}_{(-1)}(f(x)) = 1/f(x) = g(x)$. Indeed $f(x) \in \mathrm{dom}(\mathrm{pow}_{(-1)})$ and $ext(f(x)) = \mathrm{pow}_{(-1)}(f(x))$. End.

(3) $\partial(ext \circ f)(a) = -\frac{\partial f(a)}{(f(a))^2}$. Proof.

(3A) $-1 - 1 = -2$.

(3B) $\mathrm{pow}_{-2}(f(a)). = 1/(f(a))^2$ (by equivalence exponential function and exponent 2). Indeed 2 is a natural number.

(3C) ext is differentiable in $f(a)$ and $\partial ext(f(a)) = \partial_\ell \mathrm{pow}_{(-1)}(f(a))$ (by open set differentiation extension, positive reals open neighbourhood).

$\partial(ext \circ f)(a). = \partial ext(f(a)) \cdot \partial f(a)$ (by derivative chain rule)

$. = \partial_\ell \mathrm{pow}_{(-1)}(f(a)) \cdot \partial f(a)$ (by 3C)

$. = ((-1) \cdot \mathrm{pow}_{-2}(f(a))) \cdot \partial f(a)$ (by derivative power function, 3A)

$. = -\frac{\partial f(a)}{(f(a))^2}$ (by 3B, mult ass, mult comm, eq lemma 12). End.

(4) $g \upharpoonright E$ is a real map on E. [timelimit 30]

(5) $g \upharpoonright E$ is locally differentiable in a and $\partial_\ell(g \upharpoonright E)(a) = \partial(ext \circ f)(a)$ (by open set differentiation extension, 2, 4). Proof. E is an open neighbourhood of a. $ext \circ f$ is a real map extension of $g \upharpoonright E$ such that $ext \circ f$ is differentiable in a. End.

(6) $\partial g(a) = \partial_\ell(g \upharpoonright E)(a)$ (by lemma differentiation extension, 4, 5). Proof. E is a subset of $\mathbb{R}$. a is an element of E. g is a real map extension of $g \upharpoonright E$. End. [timelimit 3]

Therefore $\partial g(a) = -\frac{\partial f(a)}{(f(a))^2}$ (by 3,5,6). □

[timelimit 20]

Theorem 37 (derivative of division map). Let f be differentiable in a. Let g be a division map of f in a. Then g is differentiable in a and $\partial g(a) = -\frac{\partial f(a)}{(f(a))^2}$.

Proof. Case $f(a)$ is positive. Trivial (by lemma derivative of division map). [prover vampire]

Case $f(a)$ is negative. Take $\varphi = (-f)$. Take $\gamma = (-g)$. φ is a real map and φ is differentiable in a and $\varphi(a)$ is positive. Indeed $-f(a)$ is positive. γ is a division map of φ in a. Proof. Take an open neighbourhood E of a such that for any $x \in E$ $f(x)$ is nonzero and $g(x) = 1/f(x)$. For all $x \in E$ x is a real number and $-g(x) = -1/f(x) = 1/(-f(x)) = 1/(-f)(x)$. Then E is an open neighbourhood of a such that for any $x \in E$ $(-f)(x)$ is nonzero and $(-g)(x) = 1/(-f)(x)$. End.[timelimit 3][prover eprover]

$\partial\gamma(a) = -\frac{\partial\varphi(a)}{(\varphi(a))^2}$ (by lemma derivative of division map).

(A) $((-f)(a))^2. = (-f(a)) \cdot (-f(a))$ (by def func minus, potence lemma 2) $. = (f(a))^2$ (by eq

lemma 13, potence lemma 2).

$-\frac{\partial(-f)(a)}{((-f)(a))^2} . = \frac{\partial f(a)}{((-f)(a))^2}$ (by minus derivative, eq lemma 11, eq lemma 45) $. = \frac{\partial f(a)}{(f(a))^2}$ (by A).

$\partial g(a) = -\partial(-g)(a) = -\left(-\frac{\partial\varphi(a)}{(\varphi(a))^2}\right) = -\frac{\partial f(a)}{(f(a))^2}$. End.

$f(a)$ is positive or $f(a)$ is negative. Indeed $f(a)$ is nonzero. [prover eprover] □

Definition 38. A real map with positive range is a real map f such that $f(x)$ is positive for any real number x.

Let g denote a real map with positive range.

Definition 39 (def div map). (f/g) is a real map such that $(f/g)(x) = \frac{f(x)}{g(x)}$ for any real number x.

Theorem 40 (derivative of a fraction). Let f, g be differentiable in a.

Then (f/g) is differentiable in a and

$$\partial(f/g)(a) = \frac{(\partial f(a) \cdot g(a)) - (f(a) \cdot \partial g(a))}{(g(a))^2}.$$

Proof. (0) Define $h(x) = 1/g(x)$ for $x \in \mathbb{R}$. h is a real map.

(1) $(f/g) = (f \cdot h)$.

(2) $\partial h(a) = -\frac{\partial g(a)}{(g(a))^2}$ (by derivative of division map). Indeed h is a division map of g in a.

$\partial(f/g)(a). = \partial(f \cdot h)(a)$ (by 1)

$. = (\partial f(a) \cdot h(a)) + (f(a) \cdot \partial h(a))$ (by derivative multiplication rule)

$. = \frac{\partial f(a)}{g(a)} + \left(-f(a) \cdot \frac{\partial g(a)}{(g(a))^2}\right)$ (by 0,2, eq lemma 21)

$. = \frac{\partial f(a) \cdot g(a)}{(g(a))^2} + \frac{-f(a) \cdot \partial g(a)}{(g(a))^2}$ (by fraction lemma, fraction lemma 2, potence lemma 2, mult ass, mult comm)

$. = \frac{(\partial f(a) \cdot g(a)) - (f(a) \cdot \partial g(a))}{(g(a))^2}$ (by eq lemma 1, eq lemma 11). □

Corollary 41 (derivative of a fraction 2). Let f be differentiable and g be differentiable. Then (f/g) is differentiable and $\mathrm{D}(f/g) = (((\mathrm{D}f \cdot g) + (-(f \cdot \mathrm{D}g)))/(g \cdot g))$.

Proof. Let a be a real number.

$\mathrm{D}(f/g)(a). = \partial(f/g)(a)$ (by deriv and derivin)

$. = \frac{(\partial f(a) \cdot g(a)) - (f(a) \cdot \partial g(a))}{(g(a))^2}$ (by derivative of a fraction)

$. = (((\mathrm{D}f \cdot g) + (-(f \cdot \mathrm{D}g)))/(g \cdot g))(a)$ (by deriv and derivin, potence lemma 2, def div map, def add func, def mult func, def func minus). □