Formalization of the Solow Growth Model

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ATP proof searches are restricted by wall clock timeouts, which depend significantly on a system's speed and state. Changing the timeout from the standard of 3 seconds to 10, 20, or more seconds can be done by inserting `[timelimit X]` commands. The timeouts in this formalization are chosen such that the proof runs through on a mid-range consumer laptop (ASUS Zenbook 15, Linux). Making the code run on a different laptop may require increasing the timeout or, in very stubborn cases, adding additional proof steps. Sometimes, changing the ATP by inserting `[prover vampire]`, for example may prove helpful as well.

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Chapter 1

Preparations

[read `examples/100_theorems.ftl.tex`]

In the first section we want to name all those small lemmata that we will need for the following formalization. Especially for Frerix' rewriting algorithm any rewriting step needs to be explicitly mentioned. Therefore, many small lemmata are mentioned here, although automated theorem provers do not have any difficulty proving those theorems even without any provided proof. In the subsection 1.1 some lemmata, axioms and definitions are repeated as lemmata that are already given in `100_theorems.ftl.tex`, just in order to give them names so that they can be referenced later.
One term we already define here: as a "real map" we understand a function from and onto the real numbers.

1.1 Old Lemmata

Lemma 1 (composition). Let F, G be maps such that $\text{Range}(G) \subseteq \text{dom}(F)$. $F \circ G$ is a map H such that $\text{dom}(H) = \text{dom}(G)$ and $H(x) = F(G(x))$ for all $x \in \text{dom}(H)$.

Lemma 2 (inverse). Let F be an injective map. F^{-1} is the map G such that $\text{dom}(G) = \text{Range}(F)$ and for all $v \in \text{dom}(G)$ $G(v) \in \text{dom}(F)$ and $F(G(v)) = v$.

Let x, y, z denote real numbers.

Lemma 3 (add comm). $x + y = y + x$.

Lemma 4 (add ass). $(x + y) + z = x + (y + z)$.

Lemma 5 (existence zero). 0 is a real number such that for every real number x $x+0 = x$.

Lemma 6 (additive inverse). $-x$ is a real number such that $x + (-x) = 0$.

Lemma 7 (mult comm). $x \cdot y = y \cdot x$.

Lemma 8 (mult ass). $((x \cdot y)) \cdot z = x \cdot (y \cdot z)$.

Lemma 9 (existence one). 1 is a real number such that $1 \neq 0$ and for every real number x $1 \cdot x = x$.

Lemma 10 (multiplicative inverse). Assume $x \neq 0$. $1/x$ is a real number such that

$x \cdot (1/x) = 1$.

Lemma 11 (distributive law). $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

Proposition 12 (eq lemma 1). $((y \cdot x)) + (z \cdot x) = (y + z) \cdot x$.

Proposition 13 (eq lemma 2). Let $x + y = x + z$. Then $y = z$.

Proposition 14 (eq lemma 3). If $x + y = x$ then $y = 0$.

Proposition 15 (eq lemma 4). If $x + y = 0$ then $y = -x$.

Proposition 16 (eq lemma 45). $-(-x) = x$.

Proposition 17 (eq lemma 5). Let x be nonzero and $x \cdot y = x \cdot z$. Then $y = z$.

Proposition 18 (eq lemma 6). If x is nonzero and $x \cdot y = x$ then $y = 1$.

Proposition 19 (eq lemma 7). If x is nonzero and $x \cdot y = 1$ then $y = 1/x$.

Proposition 20 (eq lemma 8). If x is nonzero then $1/(1/x) = x$.

Proposition 21 (eq lemma 9). $0 \cdot x = 0$.

Proposition 22 (eq lemma 10). If x is nonzero and $y \neq 0$ then $x \cdot y \neq 0$.

Proposition 23 (eq lemma 11). $(-x) \cdot y = -(x \cdot y)$.

Proposition 24 (eq lemma 12). $-x = -1 \cdot x$.

Proposition 25 (eq lemma 13). $(-x) \cdot (-y) = x \cdot y$.

Lemma 26 (eq lemma 14). $(1 - x) \cdot y = y - (x \cdot y)$.

Lemma 27 (eq lemma 15). $-(x + y) = -x - y$.

Lemma 28 (eq lemma 16). Let x be a real number. $x + x = 2 \cdot x$.

Proof. $x + x = (1 \cdot x) + (1 \cdot x) = (1 + 1) \cdot x = 2 \cdot x$. □

Lemma 29 (eq lemma 17). Let $z \neq 0$. Then $x = \frac{z \cdot x}{z}$.

Lemma 30 (eq lemma 18). Let w be a nonzero real number. Then $\frac{x-y}{w} + \frac{y-z}{w} = \frac{x-z}{w}$.

Lemma 31 (reflexivity). $x \leq x$.

Lemma 32 (antisymmetry). If $x \leq y$ and $x \geq y$ then $x = y$.

Lemma 33 (transitivity). If $x \leq y$ and $y \leq z$ then $x \leq z$.

Lemma 34 (completeness). $x \leq y$ or $x \geq y$.

Lemma 35 (ineq axiom addition). If $y \leq z$ then $x + y \leq x + z$ and $y + x \leq z + x$.

Lemma 36 (ineq axiom multiplication). If $x \geq 0$ and $y \leq z$ then $x \cdot y \leq x \cdot z$.

Lemma 37 (ineq lemma 1). $x \leq y$ iff $y \not< x$.

Proposition 38 (ineq lemma 2). $x > 0$ iff $-x < 0$.

Proposition 39 (ineq lemma 3). If $x \neq 0$ then $x \cdot x > 0$.

Proposition 40 (ineq lemma 4). $1 > 0$.

Proposition 41 (ineq lemma 5). $x \leq y$ iff $-x \geq -y$.

Proposition 42 (ineq lemma 6). If $x < 0$ and $y < z$ then $x \cdot y > x \cdot z$.

Proposition 43 (ineq lemma 7). $-1 < 0$.

Proposition 44 (ineq lemma 8). $x - 1 < x$.

Proposition 45 (ineq lemma 9). $x < y < x + 1$ iff $0 < y - x < 1$.

Proposition 46 (ineq lemma 10). If $0 < y$ then $0 < 1/y$.

Lemma 47 (ineq lemma 11). If $\neg x \leq y$ then $x \geq y$.

Proposition 48 (ineq lemma 12). Assume $0 < x \leq y$. Then $1/y \leq 1/x$.

Let E denote a subset of $\mathbb{R}$.

Lemma 49 (completeness axiom). Assume that E is nonempty and bounded above. Then E has a supremum.

Lemma 50 (potence lemma 1). $x^1 = x$.

Lemma 51 (potence lemma 2). $x^2 = x \cdot x$.

Lemma 52 (potence lemma 3). $x \cdot x^i = x^{i+1}$ for any natural number i.

1.2 New Lemmata

Let a real map stand for a map from $\mathbb{R}$ to $\mathbb{R}$.

Let w, x, y, z denote real numbers. [timelimit 10]

Lemma 53 (fraction lemma). Assume $y \cdot z \neq 0$. Then $y \neq 0$ and $z \neq 0$ and $\frac{x}{y \cdot z} = \frac{x}{y} \cdot 1/z$.

Proof. If $y = 0$ or $z = 0$ then $y \cdot z = 0$. Therefore y, z are nonzero. $1/(y \cdot z) = 1/y \cdot 1/z$. Indeed then $(y \cdot z) \cdot (1/y \cdot 1/z) = 1$. □

Lemma 54 (fraction lemma 2). Assume y is nonzero. Then $y \cdot \frac{x}{y} = x$.

Lemma 55 (frac lemma 1). Let $x \geq 0$ and $y > 0$. Then $\frac{x}{y} \geq 0$.

Lemma 56 (frac lemma 2). Let $x \geq 0$ and $y < 0$. Then $\frac{x}{y} \leq 0$.

Proof. $\frac{x}{-y} \geq 0$ (by frac lemma 1). Indeed $-y$ is positive. $-\frac{x}{y} = \frac{x}{-y}$. □

Lemma 57 (frac lemma 3). Let $x \leq 0$ and $y > 0$. Then $\frac{x}{y} \leq 0$.

Proof. $\frac{-x}{y} \geq 0$ (by frac lemma 1). Indeed $-x \geq 0$. $-\frac{x}{y} = \frac{-x}{y}$. □

Lemma 58 (frac lemma 4). Let $x \leq 0$ and $y < 0$. Then $\frac{x}{y} \geq 0$.

Proof. $\frac{-x}{y} \leq 0$ (by frac lemma 2). Indeed $-x \geq 0$. □

Lemma 59 (frac lemma 5). Let x, y be nonzero. $\frac{x}{y} \neq 0$.

Lemma 60 (eq lemma 0). $0 = -0$.

Lemma 61 (eq lemma 20). Assume z is nonzero. If $x = y \cdot z$ then $y = \frac{x}{z}$.

Lemma 62 (eq lemma 21). $(-x) \cdot y = x \cdot (-y)$.

Lemma 63 (eq lemma 26). Let x, y be nonzero. $\frac{x}{y} \cdot \frac{y}{x} = 1$.

Lemma 64 (eq lemma 27). $x + y = z$ iff $x = z - y$.

Lemma 65 (eq lemma 28). Let x be nonzero. $-1/x = 1/(-x)$.

Proof. $(-x) \cdot (-1/x) = 1$. □

Lemma 66 (eq lemma 29). Let x, y be nonzero. $\frac{x}{x \cdot y} = 1/y$.

Lemma 67 (eq lemma 30). Let x, y be nonzero. $1/x \cdot 1/y = 1/(x \cdot y)$.

Proof. $(x \cdot y) \cdot (1/x \cdot 1/y) . = ((x \cdot y) \cdot 1/x) \cdot 1/y$ (by mult ass, mult comm)

$. = 1$ (by mult ass, mult comm, multiplicative inverse, existence one). □

Lemma 68 (eq lemma 22). Let x, y be nonzero. $1/\frac{x}{y} = \frac{y}{x}$.

Proof. $1/\frac{x}{y} = \frac{y}{x}$ (by eq lemma 7, eq lemma 26). Indeed $\frac{x}{y}$ is a nonzero real number. □

Lemma 69 (eq lemma 23). $\frac{x}{2} = x - \frac{x}{2}$.

Proof. $\frac{x}{2} + \frac{x}{2} . = 2 \cdot \frac{x}{2}$ (by eq lemma 16)

$. = x$ (by fraction lemma 2). $\frac{x}{2} = x - \frac{x}{2}$ (by eq lemma 27). □

Lemma 70 (eq lemma 24). Let x, y be nonzero. $1/x + 1/y = \frac{x+y}{x \cdot y}$.

Proof. $1/x + 1/y . = \frac{y}{y \cdot x} + \frac{x}{x \cdot y}$ (by eq lemma 29) $. = \frac{x+y}{x \cdot y}$ (by mult comm, eq lemma 1, add comm). □

Lemma 71 (eq lemma 25). Let x, y be nonzero. $1/x - 1/y = \frac{y-x}{x \cdot y}$.

Proof. $1/x - 1/y . = \frac{x-y}{x \cdot (-y)}$ (by eq lemma 28, eq lemma 24). $\frac{x-y}{x \cdot (-y)} . = \frac{-(y-x)}{x \cdot (-y)}$ (by eq lemma 15, eq lemma 45, add comm) $. = \frac{-(y-x)}{-(x \cdot y)}$ (by eq lemma 21, eq lemma 11) $. = \frac{y-x}{x \cdot y}$ (by eq lemma 28, eq lemma 13). □

Lemma 72. Let y be positive. $x - y < x + y$.

Lemma 73 (ineq lemma 13). If $w < x$ and $y < z$ then $w + y < x + z$.

Lemma 74 (ineq lemma 14). If $y < z$ then $y + x < z + x$.

Lemma 75 (ineq lemma 145). If $y \leq z$ then $y + x \leq z + x$.

Lemma 76 (ineq lemma 15). $x - y < z$ iff $x < z + y$.

Proof. Case $x - y < z$. $x = (x - y) + y < z + y$. End.

Case $x < z + y$. $x - y < (z + y) - y = z$. End. □

Lemma 77 (ineq lemma 16). Let w, x, y, z be nonnegative real numbers such that $w < x$ and $y < z$. Then $w \cdot y < x \cdot z$.

Proof. $w \cdot y \leq w \cdot z = z \cdot w \leq z \cdot x$ (by ineq axiom multiplication, mult comm). $w \cdot y \neq x \cdot z$. □

Lemma 78 (ineq lemma 17). Let x, y, z be nonnegative. If $y < z$ then $x \cdot y \leq x \cdot z$.

Lemma 79 (ineq lemma 18). $x - y < z$ iff $x - z < y$.

Proof. $x - y < z$ iff $x < z + y$ (by ineq lemma 15).

$x < y + z$ iff $x - z < y$ (by ineq lemma 15). □

Lemma 80 (ineq lemma x1). Let y be positive. Then there exists a positive real number

x such that $x < y$.

Lemma 81 (ineq lemma 19). Let x, y be positive. Then there exists a positive real number z such that $z < x$ and $z < y$.

Proof. Case $x \leq y$. There exists a positive real number z such that $z < x$. End.

Case $x > y$. There exists a positive real number z such that $z < y$. End. □

Lemma 82 (ineq lemma 195). Let x, y, z be positive. Then there exists a positive real number w such that $w < x$ and ($w < y$ and $w < z$).

Proof. Take a positive real number v such that $v < x$ and $v < y$ (by ineq lemma 19). Take a positive real number w such that $w < v$ and $w < z$ (by ineq lemma 19). Then $w < x$ and $w < y$ and $w < z$. □

Lemma 83 (ineq lemma 20). Assume $x < y$. There exists a real number z such that $x < z < y$.

Proof. $y - x$ is a positive real number. Take a positive real number z such that $z < y - x$ (by ineq lemma x1). $x + z$ is a real number and $x < x + z < y$. □

Lemma 84 (ineq lemma 21). Let x, y be nonnegative. $x + y$ is a nonnegative real number.

Lemma 85 (ineq lemma 215). Let x, y be positive. $x \cdot y$ is a positive real number.

Lemma 86 (ineq lemma 22). Assume $y \geq z$. $x - y \leq x - z$.

Proof. $-y \leq -z$. Therefore $x - y \leq x - z$. □

Lemma 87 (ineq lemma 23). Let y be nonnegative. $x + y \geq x$.

Lemma 88 (ineq lemma 24). Assume y is positive. $x - y < x$.

Lemma 89 (ineq lemma 25). Let z be positive. If $x < y$ then $x \cdot z < y \cdot z$.

Lemma 90 (ineq lemma 26). If $x \geq 0$ and $y \leq z$ then $y \cdot x \leq z \cdot x$.

Lemma 91 (ineq lemma 27). $x \leq y$ iff $x - y \leq 0$.

Lemma 92 (ineq lemma 28). $x < y - z$ iff $x + z < y$.

Proof. Case $x < y - z$. $x + z < (y - z) + z = y$. End.

Case $x + z < y$. $x = (x + z) - z.(x + z) - z < y - z$ (by ineq lemma 14). End. □

Lemma 93 (ineq lemma 285). $x + z < y$ iff $z < y - x$.

Lemma 94 (ineq lemma 29). If $x \leq y < z$ then $x < z$.

Lemma 95 (ineq lemma 30). If $x - y \geq z$ then $x - z \geq y$.

Lemma 96 (inverse lemma). Let f be an injective map. For all $x \in \mathrm{dom}(f)$ and all $y \in \mathrm{Range}(f)$ $f(x) = y$ iff $f^{-1}(y) = x$.

Lemma 97 (function lemma 2). Let f, g be maps such that $\mathrm{dom}(f) = \mathrm{dom}(g)$. If for every $x \in \mathrm{dom}(g)$ $f(x) = g(x)$ then $f = g$.

Lemma 98. Let f be a real map and x be a real number. Then $f(x)$ is a real number.

Proof. $f(x) \in \mathbb{R}$. □

Lemma 99. Let f, g be real maps. $g \circ f$ is a real map.

Proof. $\mathrm{dom}(g \circ f) = \mathrm{dom}(f) = \mathbb{R}$. □

Lemma 100 (1 is positive). 1 is a positive real number.

Lemma 101 (lemma x01). Let S be a nonempty subset of $\mathbb{R}$. Let s be the supremum of S. Let ε be a positive real number. Then there exists $x \in S$ such that $s - \varepsilon < x$.

Proof. Take $K = s - \varepsilon$. Assume the contrary. Then $s - \varepsilon$ is an upper bound of S. Indeed we can show that for all $x \in S$ $x \leq K$. Assume the contrary. Take $x \in S$ such that $x > K$. Case $x > s$. $x \notin S$. End. Case $x \leq s$. Then there exists $\xi \in S$ such that $s - \varepsilon < \xi$. End. End. □

Lemma 102 (lemma inverse). Let f be an injective map. For all $v \in \mathrm{dom}(f)$ $f(v) \in \mathrm{dom}(f^{-1})$ and $f^{-1}(f(v)) = v$.

Proof. Let $v \in \mathrm{dom}(f)$. Then $f(v) \in \mathrm{dom}(f^{-1})$. Indeed $\mathrm{Range}(f) = \mathrm{dom}(f^{-1})$. Take $w = f^{-1}(f(v))$. $f(w). = f(f^{-1}(f(v))). = f(v)$ (by inverse). $w = v$. Indeed f is injective. □

[timelimit 3]

Chapter 2

Metric

First we need to import the previous file, which we do by the following command.

[read `examples/solow/sections/0preparation.ftl.tex`]

The aim of the first chapter is to introduce absolute values, which are necessary for the following definitions of continuity and differentiability. We do so in the context of metrics as this is commonly done in standard textbooks (see [**?**]).

2.1 Definition of a Metric

Let X, Y denote sets.

Definition 103. Let d be a map from $(X \times X)$ to $\mathbb{R}$. d is positive definit on X iff for all $x, y \in X$ $d(x, y) = 0$ iff $x = y$.

Definition 104. Let d be a map from $X \times X$ to $\mathbb{R}$. d is reflexive on X iff for all $x, y \in X$ $d(x, y) = d(y, x)$.

Definition 105. Let d be a map from $X \times X$ to $\mathbb{R}$. d holds triangle inequality on X iff For all $x, y, z \in X$ $d(x, z), d(x, y), d(y, z)$ are real numbers and $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 106. A metric on X is a map d from $X \times X$ to $\mathbb{R}$ such that d is positive definit on X and d is reflexive on X and d holds triangle inequality on X.

[prover vampire][timelimit 20]

Lemma 107. Let $x, y \in X$ and d be a metric on X. Then $d(x, y)$ is a real number and $d(x, y) \geq 0$.

Proof. d holds triangle inequality on X. Thus $d(x, x) \leq d(x, y) + d(y, x)$. $0 = d(x, x) \leq d(x, y) + d(y, x) = d(x, y) + d(x, y) = 2 \cdot d(x, y)$. Thus $0 \leq 2 \cdot d(x, y)$. $d(x, y) \geq 0$. Indeed $d(x, y)$ is a real number. □

Lemma 108. Let Y be a subset of X. Then $Y \times Y$ is a subset of $X \times X$.

Lemma 109. Let d be a map from X to $\mathbb{R}$ and Y be a subset of X. Then $d \restriction Y$ is a map from Y to $\mathbb{R}$.

Proposition 110. Let d be a map from $X \times X$ to $\mathbb{R}$. Let Y be a subset of X. If d is a metric

on X then $d \restriction (Y \times Y)$ is a metric on Y.

Proof. Take $\delta = d \restriction (Y \times Y)$. [timelimit 20] δ is a map from $Y \times Y$ to $\mathbb{R}$. Thus for all $x, y \in Y$ $(x, y) \in (Y \times Y)$ and $\delta(x, y)$ is a real number. [timelimit 3]

Assume d is a metric on X. δ is positive definit on Y. δ is reflexive on Y. δ holds triangle inequality on Y. Proof. Let $x, y, z \in Y$. Then $d(x, y), d(y, z)$ are real numbers and $\delta(x, z) = d(x, z) \leq d(x, y) + d(y, z) = \delta(x, y) + \delta(y, z)$. Therefore $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$. End. □

2.2 Absolute Value

Let a, x, y denote real numbers.

Definition 111 (def absolute value). $|a|$ is a nonnegative real number such that (if $a \geq 0$ then $|a| = a$) and if $a < 0$ then $|a| = -a$.

Lemma 112. $|a| = a$ or $|a| = -a$.

Lemma 113 (absolute value nonnegative). $|a|$ is nonnegative.

Lemma 114 (absolute value positive). Let a be nonzero. $|a|$ is positive.

Lemma 115 (absolute lemma 1). $|a| = |-a|$.

[timelimit 30]

Lemma 116 (absolute lemma 2). $|x \cdot y| = |x| \cdot |y|$.

Proof. $x \cdot y \geq 0$ or $x \cdot y < 0$. If $x \cdot y \geq 0$ then $|x \cdot y| = x \cdot y = |x| \cdot |y|$. [timelimit 3] If $x \cdot y < 0$ then $|x \cdot y| = -x \cdot y = |x| \cdot |y|$. □

Lemma 117 (absolute lemma 3). $|x - y| = |y - x|$.

Proof. $|x - y| \,. = |-(x - y)|$ (by absolute lemma 1)

$. = |-x - (-y)|$ (by eq lemma 15)

$. = |y - x|$ (by eq lemma 45, add comm). □

Lemma 118 (absolute lemma 4). $|x + y| \leq |x| + |y|$.

Proof. If $x \geq 0$ and $y \geq 0$ then $|x + y| = |x| + |y|$. If $x \leq 0$ and $y \leq 0$ then $|x + y| = |x| + |y|$. Case $x > 0$ and $y < 0$. If $|x| \geq |y|$ then $|x + y| = x + y < x = |x| < |x| + |y|$.

If $|x| < |y|$ then $|x + y| = -(x + y) = -y - x < -y = |y| < |y| + |x|$. End.

Case $x < 0$ and $y > 0$. If $|y| \geq |x|$ then $|x + y| = x + y < y = |y| < |x| + |y|$.

If $|y| < |x|$ then $|x + y| = -(x + y) = -x - y < -x = |x| < |y| + |x|$. End. □

Lemma 119. $|x| + |y| = ||x| + |y||$.

Lemma 120 (absolute lemma 5). Let a be nonzero. $|1/a| = 1/|a|$.

2.3 The Canonical Metric on the Real Numbers

Definition 121 (Manhattan metric). $d_{\mathbb{R}}$ is a map from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ such that for all $x, y \in \mathbb{R}$ $d_{\mathbb{R}}(x, y) = |x - y|$.

Theorem 122. $d_{\mathbb{R}}$ is a metric on $\mathbb{R}$.

Proof. $\mathbb{R}$ is a set and $d_{\mathbb{R}}$ is a map from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$.

(1) $d_{\mathbb{R}}$ is positive definit on $\mathbb{R}$. Indeed we can show that for all real numbers x, z $d_{\mathbb{R}}(x, z) = 0$ iff $x = z$. Let x, z be real numbers. If $x = z$ then $d_{\mathbb{R}}(x, z) = 0$. Case $x \neq z$. Then $x - z \neq 0$ and $-(x - z) \neq 0$. Thus $|x - z| \neq 0$. Thus $d_{\mathbb{R}}(x, z) \neq 0$. End. End.

(2) $d_{\mathbb{R}}$ is reflexive on $\mathbb{R}$. Indeed $d_{\mathbb{R}}(x, z) = |x - z| = |z - x| = d_{\mathbb{R}}(z, x)$ for all real numbers x, z.

(3) $d_{\mathbb{R}}$ holds triangle inequality on $\mathbb{R}$. Indeed we can show that for all real numbers x, y, z $d_{\mathbb{R}}(x, z) \leq d_{\mathbb{R}}(x, y) + d_{\mathbb{R}}(y, z)$. Let x, y, z be a real number.

$d_{\mathbb{R}}(x, z) = |x - z| = |(x + (-y + y)) - z| = |(x - y) + (y - z)| \leq ||x - y| + |y - z|| = |x - y| + |y - z| = d_{\mathbb{R}}(x, y) + d_{\mathbb{R}}(y, z)$. End. □

Lemma 123 (absolute lemma 6). Let x, y be real numbers. $|x| - |y| \leq |x - y|$.

Proof. $d_{\mathbb{R}}$ holds triangle inequality on $\mathbb{R}$. Thus $d_{\mathbb{R}}(x, 0) \leq d_{\mathbb{R}}(x, y) + d_{\mathbb{R}}(y, 0)$. $|x| - |y| = d_{\mathbb{R}}(x, 0) - |y| \leq (d_{\mathbb{R}}(x, y) + d_{\mathbb{R}}(y, 0)) - |y| = (|x - y| + |y|) - |y| = |x - y|$. □

Chapter 3

Continuity

[read `examples/solow/sections/1metrics.ftl.tex`][prover eprover]

In this chapter we introduce the concept of continuity and prove some common notions such as that linear functions are continuous and if f and g are continuous, then so are $(f+g)$, $(f \cdot g)$, and $(f \circ g)$.

3.1 Definition of Continuity

The typical definition of continuity is the famous ε-δ criterion. $f : \mathbb{R} \to \mathbb{R}$ is continuous in a iff

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in \mathbb{R} : \ |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

This definition is overwhelms, however, the type verification system. Instead we will use Skolem-functions to break this definitions apart.
Thus we introduce the term "domain restrictor" referring to the $\delta = \delta(\varepsilon)$ such that

$$\forall x \in \mathbb{R} \setminus \{a\} : \ |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

We exclude a from the set of real numbers, for which it should hold, because we want to reuse the concept of domain restrictor in the following chapter for our definition of differentiation. There we will have that

$$\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \varepsilon,$$

where the option of x being equal to a would result in a division by zero error.
We define f being continuous in a as $\lim_{x \to a} f(x) = f(a)$, where $\lim_{x \to a} f(x)$ is a real number, for which we can find a domain restrictor for every positive real number ε.
Furthermore, we introduce a more general notion of domain restrictor, which would allow to define continuity more generally for any real-valued function and not just those with a domain of $\mathbb{R}$. We do show that our specific definition for $\mathbb{R}$ is equivalent to the more general one when it is applied to $\mathbb{R}$. The choice to define continuity only on $\mathbb{R}$ is, however, a very deliberate one, as this is the case that we need for almost all the following theorems that we are interested in. In order to make our definition more focussed, we reduce the complexity for the theorem prover.

Let a, x, y denote real numbers. Let δ, ε denote positive real numbers. Let f denote a real maps.

Definition 124 (def domain restrictor). A domain restrictor for ε and f around a and y is a positive real number δ such that for all $x \in \mathbb{R} \setminus \{a\}$ if $d_{\mathbb{R}}(x, a) < \delta$ then $d_{\mathbb{R}}(f(x), y) < \varepsilon$.

Lemma 125 (domain restrictor lemma). Let δ be a domain restrictor for ε and f around a and y and d be a positive real number such that $d < \delta$. Then d is a domain restrictor for ε and f around a and y.

Proof. Let us show that for all $x \in \mathbb{R} \setminus \{a\}$ if $d_{\mathbb{R}}(x, a) < d$ then $d_{\mathbb{R}}(f(x), y) < \varepsilon$. Let $x \in \mathbb{R} \setminus \{a\}$. Assume $d_{\mathbb{R}}(x, a) < d$. $d < \delta$. Thus $d_{\mathbb{R}}(x, a) < \delta$. Therefore $d_{\mathbb{R}}(f(x), y) < \varepsilon$. End. □

[timelimit 100]

Lemma 126 (domain restrictor lemma 2). If there exists $x \in \mathbb{R} \setminus \{a\}$ such that $|x - a| < \delta$ and $|f(x) - y| \geq \varepsilon$ then δ is not a domain restrictor for ε and f around a and y.

[timelimit 10]

Lemma 127 (domain restrictor lemma 3). Let ε be a positive real number. Let δ be a domain restrictor for ε and f around a and $f(a)$. If $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

Proof. Case $x = a$. Trivial.

Case $x \neq a$. Then $x \in \mathbb{R} \setminus \{a\}$. Assume $|x - a| < \delta$. $(x, a), (f(x), f(a)) \in \mathrm{dom}(d_{\mathbb{R}})$. Then $d_{\mathbb{R}}(x, a) < \delta$. $|f(x) - f(a)| = d_{\mathbb{R}}(f(x), f(a)) < \varepsilon$. End. □

Definition 128. $\lim_{x \to a} f(x)$ is a real number y such that for every positive real number ε there exists a domain restrictor for ε and f around a and y.

Definition 129. f is continuous in a iff $\lim_{x \to a} f(x) = f(a)$.

Definition 130. f is continuous iff for every real number a f is continuous in a.

3.2 Continuity in the Real Numbers

3.2.1 Linear Function

Let a, b denote real numbers.

Definition 131 (linear function). $\mathrm{f}_{a \cdot _ + b}$ is a real map such that $\mathrm{f}_{a \cdot _ + b}(x) = (a \cdot x) + b$ for all real numbers x.

Lemma 132 (linear lem 1). $\mathrm{f}_{0 \cdot _ + a}(b) = a$.

Lemma 133 (linear lem 2). $\mathrm{f}_{a \cdot _ + 0}(b) = a \cdot b$.

Lemma 134 (linear lem 3). $\mathrm{f}_{a \cdot _ + b}(0) = b$.

To prove that a function $f(x) = a \cdot x + b$ is continuous, we let ε be a positive real number and take $\delta = \frac{\varepsilon}{|a|}$. Then for all x with a distance smaller δ to a we know

$$|f(x) - f(a)| = |(a \cdot x + b) - (a \cdot a + b)| = |a| \cdot |x - a| < |a| \cdot \frac{\varepsilon}{|a|} = \varepsilon.$$

Theorem 135 (continuity linear function). $\mathrm{f}_{a\cdot_+b}$ is continuous.

Proof. Take $f = \mathrm{f}_{a\cdot_+b}$. Let ξ be a real number.

Case $a = 0$. Let ε be a positive real number. For all $x \in \mathbb{R} \setminus \{\xi\}$ $d_{\mathbb{R}}(f(x), f(\xi)) = d_{\mathbb{R}}(b, b) = 0 < \varepsilon$. Therefore 1 is a domain restrictor for ε and f around ξ and $f(\xi)$. End.

Case $a \neq 0$. Let ε be a positive real number. Take $\delta = \left|\frac{\varepsilon}{a}\right|$. Then δ is a positive real number.

δ is a domain restrictor for ε and f around ξ and $f(\xi)$. Proof.

Let $x \in \mathbb{R} \setminus \{\xi\}$. Assume $|x - \xi| < \delta$.

$|f(x) - f(\xi)| = |((a \cdot x) + b) - ((a \cdot \xi) + b)|$.

$|((a \cdot x) + b) - ((a \cdot \xi) + b)| \,. = |((a \cdot x) + (b - b)) - (a \cdot \xi)|$

(by eq lemma 15, add comm, add ass)

$\,. = |(a \cdot x) + (a \cdot -\xi)|$

(by additive inverse, existence zero, mult comm, eq lemma 11)

$\,. = |a| \cdot |x - \xi|$ (by distributive law, absolute lemma 2).

Then $|a| \cdot |x - \xi| < |a| \cdot \delta = |a| \cdot \left|\frac{\varepsilon}{a}\right| = \varepsilon$. End. QED. □

3.2.2 Continuity Under Addition

Let f, g denote real maps.

Definition 136 (def add func). $(f + g)$ is a real map such that $(f + g)(x) = f(x) + g(x)$ for all real numbers x.

For the proof of the continuity of $(f + g)$ we take an δ so small that it serves as a domain restrictor for both f and g for $\varepsilon/2$. Then

$$\begin{aligned} |(f + g)(x) - (f + g)(a)| &= |(f(x) - f(a)) + (g(x) - g(a))| \\ &\leq |f(x) - f(a)| + |g(x) - g(a)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

[timelimit 10]

Theorem 137 (continuity of plus). Let f, g be continuous in a. Then $(f+g)$ is continuous in a.

Proof. Let ε be a positive real number. Take $\epsilon = \frac{\varepsilon}{2}$. Then ϵ is a positive real number.

Take a positive real number τ such that τ is a domain restrictor for ϵ and f around a and $f(a)$. Take a positive real number θ such that θ is a domain restrictor for ϵ and g around a and $g(a)$. Take a positive real number δ such that $\delta < \tau$ and $\delta < \theta$ (by ineq lemma 19).

We can show that δ is a domain restrictor for ε and $(f + g)$ around a and $(f + g)(a)$.

Let $x \in \mathbb{R} \setminus \{a\}$. Assume $|x - a| < \delta$.

Let us show that $|(f + g)(x) - (f + g)(a)| \leq |f(x) - f(a)| + |g(x) - g(a)|$.

$|(f+g)(x)-(f+g)(a)| . = |(f(x)+g(x))-(f(a)+g(a))|$ (by def add func)

$. = |(f(x)-f(a))+(g(x)-g(a))|$

(by eq lemma 15, add ass, add comm).

$|(f(x)-f(a))+(g(x)-g(a))| \leq |f(x)-f(a)|+|g(x)-g(a)|$ (by absolute lemma 4). End.

$|f(x)-f(a)|+|g(x)-g(a)| < \varepsilon$. Proof. $|f(x)-f(a)|$, $|g(x)-g(a)|$ and ϵ are real numbers.

(A) $|f(x)-f(a)| < \epsilon$ (by domain restrictor lemma 3). Indeed δ is a domain restrictor for ϵ and f around a and $f(a)$ (by domain restrictor lemma).

(B) $|g(x)-g(a)| < \epsilon$ (by domain restrictor lemma 3). Indeed δ is a domain restrictor for ϵ and g around a and $g(a)$ (by domain restrictor lemma).

Therefore $|f(x)-f(a)|+|g(x)-g(a)| < \epsilon+\epsilon$ (by ineq lemma 13, A, B). $\epsilon+\epsilon = 2\cdot\epsilon = \varepsilon$. End.

Thus $|(f+g)(x)-(f+g)(a)| \leq \varepsilon$ (by transitivity). QED. □

Corollary 138. Let f be continuous. Let g be continuous. Then $(f+g)$ is continuous.

[timelimit 3]

3.2.3 Continuity Under Multiplication

Definition 139 (def mult func). $(f \cdot g)$ is a real map such that $(f \cdot g)(x) = f(x) \cdot g(x)$ for all real numbers x.

Lemma 140 (lemma 2). $(f \cdot g) = (g \cdot f)$.

Proof. We can show that for every real number x $(f \cdot g)(x) = (g \cdot f)(x)$. Let x be a real number.

$(f \cdot g)(x). = f(x) \cdot g(x). = g(x) \cdot f(x)$ (by mult comm) $. = (g \cdot f)(x)$. End. □

The proof of the continuity of $(f \cdot g)$ is certainly not winning any beauty contests, but it is working. We Take a positive real number δ such that δ is a domain restrictor that is so small that $|f(x)-f(a)| < \frac{\varepsilon}{2\cdot(|g(a)|+\varepsilon)} =: \nu$, $|g(x)-g(a)| < \frac{\varepsilon}{2\cdot(|f(a)|+1)} =: \pi$, and $|g(x)-g(a)| < \varepsilon$.
In (1) we first show that $|(f \cdot g)(x)-(f \cdot g)(a)| \leq |f(x)-f(a)| \cdot |g(x)| + |f(a)| \cdot |g(x)-g(a)|$ by proving in particular (11).
In (2) we show that $|f(x)-f(a)| \cdot |g(x)| < \varepsilon/2$ using (21),(22), and (23).
In (3) we show that also $|f(a)| \cdot |g(x)-g(a)| < \varepsilon/2$ using (31) and (32).
Altogether we get then that $|(f \cdot g)(x)-(f \cdot g)(a)| < \varepsilon$.

Theorem 141 (continuity under multiplication). Let f, g be continuous in a. Then $(f \cdot g)$ is continuous in a.

Proof. Let ε be a positive real number.

$2 \cdot (|g(a)|+\varepsilon)$ is a positive real number. Proof. $|g(a)| \geq 0$ and $\varepsilon > 0$. Thus $|g(a)|+\varepsilon > 0$. [timelimit 20] Therefore $2 \cdot (|g(a)|+\varepsilon) > 0$. [timelimit 3] End.

(01) Take $\nu = \frac{\varepsilon}{2\cdot(|g(a)|+\varepsilon)}$. ν is a positive real number.

$2\cdot(|f(a)|+1)$ is a positive real number. Proof. $|f(a)| \geq 0$. [timelimit 20] Therefore $|f(a)|+1 > 0$. Thus $2 \cdot (|f(a)|+1) > 0$. End.

(02) Take $\pi = \frac{\varepsilon}{2 \cdot (|f(a)|+1)}$. π is a positive real number.

Take a positive real number τ such that τ is a domain restrictor for ν and f around a and $f(a)$. Take a positive real number θ such that θ is a domain restrictor for π and g around a and $g(a)$. Take a positive real number ϑ such that ϑ is a domain restrictor for ε and g around a and $g(a)$. Take a positive real number Θ such that $\Theta < \theta$ and $\Theta < \vartheta$. Take a positive real number δ such that $\delta < \tau$ and $\delta < \Theta$. [timelimit 3]

Let us show that δ is a domain restrictor for for ε and $(f \cdot g)$ around a and $(f \cdot g)(a)$. Let $x \in \mathbb{R} \setminus \{a\}$. Then x is a real number.

Assume $|x - a| < \delta$.

(1) $d_{\mathbb{R}}((f \cdot g)(x), (f \cdot g)(a)) \leq (|f(x) - f(a)| \cdot |g(x)|) + (|f(a)| \cdot |g(x) - g(a)|)$. Proof.

(11) $(f(x) \cdot g(x)) - (f(a) \cdot g(a)) = ((f(x) - f(a)) \cdot g(x)) + (f(a) \cdot (g(x) - g(a)))$. Indeed

$(f(x) \cdot g(x)) - (f(a) \cdot g(a))$

$. = ((f(x) \cdot g(x)) - (f(a) \cdot g(a))) + ((f(a) \cdot g(x)) - (f(a) \cdot g(x)))$

(by existence zero, additive inverse)

$. = ((f(x) \cdot g(x)) - (f(a) \cdot g(x))) + ((f(a) \cdot g(x)) - (g(a) \cdot f(a)))$

(by add ass, add comm, mult comm)

$. = ((f(x) - f(a)) \cdot g(x)) + (f(a) \cdot (g(x) - g(a)))$

(by distributive law, eq lemma 1,mult comm, eq lemma 11).

$d_{\mathbb{R}}((f \cdot g)(x), (f \cdot g)(a)). = |(f \cdot g)(x) - (f \cdot g)(a)|$ (by Manhattan metric)

$. = |(f(x) \cdot g(x)) - (f(a) \cdot g(a))|$ (by def mult func)

$. = |((f(x) - f(a)) \cdot g(x)) + (f(a) \cdot (g(x) - g(a)))|$ (by 11).

$|((f(x) - f(a)) \cdot g(x)) + (f(a) \cdot (g(x) - g(a)))| \leq |(f(x) - f(a)) \cdot g(x)| + |f(a) \cdot (g(x) - g(a))|$ (by absolute lemma 4).

$|(f(x) - f(a)) \cdot g(x)| + |f(a) \cdot (g(x) - g(a))|$

$. = (|f(x) - f(a)| \cdot |g(x)|) + (|f(a)| \cdot |g(x) - g(a)|)$ (by absolute lemma 2). qed.

(2) $|f(x) - f(a)| \cdot |g(x)| < \frac{\varepsilon}{2}$. Proof. [timelimit 20]

(21) $|f(x) - f(a)| < \nu$ (by domain restrictor lemma 3). Indeed δ is a domain restrictor for ν and f around a and $f(a)$ (by domain restrictor lemma).

(22) $|g(x)| < |g(a)| + \varepsilon$.

Proof. δ is a domain restrictor for ε and g around a and $g(a)$ (by domain restrictor lemma). Indeed $\delta < \vartheta$.

$|g(x)| - |g(a)| \leq |g(x) - g(a)| = d_{\mathbb{R}}(g(x), g(a)) < \varepsilon$.

Therefore $|g(x)| < \varepsilon + |g(a)|$ (by ineq lemma 15). Indeed $|g(x)| - |g(a)| < \varepsilon$. End.[timelimit 3]

$|f(x) - f(a)| \cdot |g(x)| < \nu \cdot (|g(a)| + \varepsilon)$. Proof. Take $r = |f(x) - f(a)|, \rho = |g(x)|, p = (|g(a)| + \varepsilon)$. [timelimit 10] r, ρ, p, ν are nonnegative real numbers. $r \cdot \rho < \nu \cdot p$ (by ineq lemma 16). Indeed $r < \nu$ and $\rho < p$ (by 21, 22). [timelimit 3] End.

(23) $\nu \cdot (|g(a)| + \varepsilon) = \frac{\varepsilon}{2}$.

Proof. (231) Take $c = |g(a)| + \varepsilon$. $\nu \cdot (|g(a)| + \varepsilon)$. $= \frac{\varepsilon}{2 \cdot (|g(a)| + \varepsilon)} \cdot (|g(a)| + \varepsilon)$ (by 01)

$. = \frac{\varepsilon}{2 \cdot c} \cdot c$ (by 231)

$. = (\frac{\varepsilon}{2} \cdot 1/c) \cdot c$ (by fraction lemma)

$. = \frac{\varepsilon}{2} \cdot (c \cdot 1/c)$ (by mult ass, mult comm)

$. = \frac{\varepsilon}{2}$ (by multiplicative inverse, mult comm, existence one). End. qed.

(3) $|f(a)| \cdot |g(x) - g(a)| < \frac{\varepsilon}{2}$. Proof. [timelimit 20]

(31) $|g(x) - g(a)| < \pi$ (by domain restrictor lemma 3). Indeed Thus δ is a domain restrictor for π and g around a and $g(a)$ (by domain restrictor lemma). Indeed $\delta < \theta$.[timelimit 3]

$|f(a)| \cdot |g(x) - g(a)| < (|f(a)| + 1) \cdot \pi$. Proof. Take $q = |f(a)|$, $R = |g(x) - g(a)|$, $P = |f(a)| + 1$. [timelimit 10] q, R, P, π are nonnegative real numbers. $q \cdot R < P \cdot \pi$ (by ineq lemma 16). Indeed $q < P$ and $R < \pi$. [timelimit 3] End.

(32) $(|f(a)| + 1) \cdot \pi = \frac{\varepsilon}{2}$.

Proof. (321) Take $c = |f(a)| + 1$.

$(|f(a)| + 1) \cdot \pi. = (|f(a)| + 1) \cdot \frac{\varepsilon}{2 \cdot (|f(a)| + 1)}$ (by 02)

$. = c \cdot \frac{\varepsilon}{2 \cdot c}$ (by 321)

$. = c \cdot (\frac{\varepsilon}{2} \cdot 1/c)$ (by fraction lemma)

$. = (c \cdot 1/c) \cdot \frac{\varepsilon}{2}$ (by mult comm, mult ass)

$. = \frac{\varepsilon}{2}$ (by multiplicative inverse, mult comm, existence one). End.

qed.

(4) $d_{\mathbb{R}}((f \cdot g)(x), (f \cdot g)(a)) < \varepsilon$. Proof.

Take $Q = (|f(x) - f(a)| \cdot |g(x)|)$, $S = (|f(a)| \cdot |g(x) - g(a)|)$, $\epsilon = \frac{\varepsilon}{2}$.

$Q + S < \epsilon + \epsilon$ (by ineq lemma 13). Indeed $Q < \epsilon$ and $S < \epsilon$ (by 2,3). $\epsilon + \epsilon = 2 \cdot \frac{\varepsilon}{2} = \varepsilon$.

$d_{\mathbb{R}}((f \cdot g)(x), (f \cdot g)(a)) \leq \varepsilon$ (by transitivity).

Indeed $d_{\mathbb{R}}((f \cdot g)(x), (f \cdot g)(a)) \leq Q + S < \varepsilon$ (by 1). End. End. □

Corollary 142. Let f be continuous. Let g be continuous. $(f \cdot g)$ is continuous.

3.2.4 Continuity Under composition

The proof of continuity of $g \circ f$ is straightforward. Let d be a domain restrictor for f and ε. Then Take a positive real number δ such that δ is a domain restrictor for g and d. Then δ also serves as a domain restrictor for $g \circ f$ and ε.

[timelimit 50]

Theorem 143 (continuity of composition). Let f be continuous in a and g be continuous in $f(a)$. Then $g \circ f$ is continuous in a.

Proof. $g \circ f$ is a real map. Let ε be a positive real number.

(1) Take a positive real number d such that d is a domain restrictor for ε and g around $f(a)$ and $g(f(a))$.

(2) Take a positive real number δ such that δ is a domain restrictor for d and f around a and $f(a)$.

Let us show that δ is a domain restrictor for ε and $(g \circ f)$ around a and $g(f(a))$. Let $x \in \mathbb{R} \setminus \{a\}$. Assume $|x - a| < \delta$. Then $|f(x) - f(a)| < d$. $|g(f(x)) - g(f(a))| < \varepsilon$ (by domain restrictor lemma 3, 1). Indeed $f(x), f(a)$ are real numbers. End. □

[timelimit 3]

Corollary 144. Let f be continuous. Let g be continuous. $g \circ f$ is continuous.

Chapter 4

Differentiation

[read `examples/solow/sections/2continuity.ftl.tex`]

In this chapter we want to define differentiability and prove some common derivation rules: The derivative of a linear function is its slope, $(f+g)'(x) = f'(x) + g'(x)$, $(f \cdot g)'(x) = (f' \cdot g)(x) + (f \cdot g')(x)$, and the chain rule $(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$.
In regards of notation $\frac{\Delta f}{\Delta a}(x)$ means $\frac{f(x)-f(a)}{x-a}$, $\partial f(x)$ is a real number that is the slope of f in x, and $\mathrm{D}f$ is a function, names f's derivative.

4.1 Definition of Differentiability

Let a denote a real number. Let f denote a real map.

Definition 145 (def diffquot). $\frac{\Delta f}{\Delta a}$ is a real map such that $\frac{\Delta f}{\Delta a}(x) = \frac{f(x)-f(a)}{x-a}$ for all $x \in \mathbb{R} \setminus \{a\}$.

Definition 146. f is differentiable in a iff there exists a real number y such that $\lim_{x \to a} \frac{\Delta f}{\Delta a}(x) = y$.

Definition 147. f is differentiable iff for every real number a f is differentiable in a.

Definition 148. Assume f is differentiable in a. The derivative of f in a is a real number y such that $\lim_{x \to a} \frac{\Delta f}{\Delta a}(x) = y$.

Let $\partial f(x)$ stand for the derivative of f in x.

Definition 149. Let f be differentiable. The derivative of f is a real map d such that for every real number x $d(x)$ is equal to the derivative of f in x.

Let $\mathrm{D}f$ stand for the derivative of f.

Lemma 150 (deriv and derivin). Let f be differentiable. $\mathrm{D}f(a) = \partial f(a)$.

4.2 Equivalent Definition of Differentiability

Although the above definition of differentiability is standard, we give here an equivalent definition and also prove the equivalence, because this one will be more useful to prove the following theorems. We define a function as the difference quotient of a function f; i.e. $\varphi(x) = \frac{f(x)-f(a)}{x-a}$ for all $x \neq a$. If we can find such a function φ that is in addition continuous in a, then f is differentiable in a. This map we call difference quotient map.

Let a denote a real number. Let f denote a real map. [timelimit 50]

Definition 151. The difference quotient map of f in a is a real map φ such that φ is continuous in a and for all real numbers x $f(x) = ((x-a)\cdot\varphi(x)) + f(a)$.

Proposition 152 (prop difference quotient map). Let φ be the difference quotient map of f in a. f is differentiable in a and $\partial f(a) = \varphi(a)$ and for every $x \in \mathbb{R} \setminus \{a\}$ $\varphi(x) = \frac{\Delta f}{\Delta a}(x)$.

Proof. Let us show that $\varphi(x) = \frac{\Delta f}{\Delta a}(x)$ for every $x \in \mathbb{R} \setminus \{a\}$. Let $x \in \mathbb{R} \setminus \{a\}$. Then $x - a \neq 0$. $f(x) - f(a) = (((x-a)\cdot\varphi(x)) + f(a)) - f(a) = ((x-a)\cdot\varphi(x)) + (f(a) - f(a)) = (x-a)\cdot\varphi(x)$. $\frac{\Delta f}{\Delta a}(x) = \frac{f(x)-f(a)}{x-a} = \frac{(x-a)\cdot\varphi(x)}{x-a} = \varphi(x)$. End.

$\varphi(a)$ is a real number.

$\lim_{x\to a} \frac{\Delta f}{\Delta a}(x) = \varphi(a)$. Proof. Let ε be a positive real number.

Take a positive real number δ such that δ is a domain restrictor for ε and φ around a and $\varphi(a)$.

Then δ is a domain restrictor for ε and $\frac{\Delta f}{\Delta a}$ around a and $\varphi(a)$. Indeed for every $x \in \mathbb{R} \setminus \{a\}$ if $d_{\mathbb{R}}(x,a) < \delta$ then $\frac{\Delta f}{\Delta a}(x) = \varphi(x)$ and $d_{\mathbb{R}}(\frac{\Delta f}{\Delta a}(x), \varphi(a)) = d_{\mathbb{R}}(\varphi(x), \varphi(a)) < \varepsilon$. End.

$\partial f(a) = \lim_{x\to a} \frac{\Delta f}{\Delta a}(x) = \varphi(a)$. □

Theorem 153. f is differentiable in a iff there exists a difference quotient map of f in a.

Proof. Case f is differentiable in a. Define

$$\varphi(x) = \begin{cases} \frac{\Delta f}{\Delta a}(x) & : x \neq a \\ \partial f(a) & : x = a \end{cases}$$

for $x \in \mathbb{R}$. φ is a real map. Indeed $\mathrm{Range}(\varphi) \subseteq \mathbb{R}$.

Let us show that φ is continuous in a. Let ε be a positive real number. Take a positive real number δ such that δ is a domain restrictor for ε and $\frac{\Delta f}{\Delta a}$ around a and $\partial f(a)$. Indeed for every positive real number ϵ there exists a domain restrictor for ϵ and $\frac{\Delta f}{\Delta a}$ around a and $\partial f(a)$.

Then δ is a domain restrictor for ε and φ around a and $\partial f(a)$. Indeed for every $x \in \mathbb{R} \setminus \{a\}$ if $d_{\mathbb{R}}(x,a) < \delta$ then $\varphi(x) = \frac{\Delta f}{\Delta a}(x)$ and $d_{\mathbb{R}}(\varphi(x), \partial f(a)) = d_{\mathbb{R}}\left(\frac{\Delta f}{\Delta a}(x), \partial f(a)\right) < \varepsilon$. End.

Let us show that for all real numbers x $f(x) = ((x-a)\cdot\varphi(x)) + f(a)$. Let x be a real number.

Case $x = a$. Trivial.

Case $x \neq a$. $((x-a)\cdot\varphi(x)) + f(a) = \left((x-a)\cdot\frac{\Delta f}{\Delta a}(x)\right) + f(a) = \left((x-a)\cdot\frac{f(x)-f(a)}{x-a}\right) + f(a) = \left(\frac{x-a}{x-a}\cdot(f(x)-f(a))\right) + f(a) = (f(x)-f(a)) + f(a) = f(x)$. End. End.

Therefore φ is a difference quotient map of f in a. End.

Case there exists a difference quotient map of f in a. Trivial. □

[timelimit 10]

Theorem 154. Let f be differentiable in a. Then f is continuous in a.

Proof. Take the difference quotient map φ of f in a. For all $x \in \mathrm{dom}(f)$

$f(x) = ((x-a)\cdot\varphi(x)) + f(a) = (\mathrm{f}_{1\cdot_+-a}(x)\cdot\varphi(x)) + f(a) = (\mathrm{f}_{1\cdot_+-a}(x)\cdot\varphi(x)) + \mathrm{f}_{0\cdot_+f(a)}(x) = ((\mathrm{f}_{1\cdot_+-a}\cdot\varphi) + \mathrm{f}_{0\cdot_+f(a)})(x)$.

[prover vampire] Therefore $f = ((\mathrm{f}_{1\cdot_+-a}\cdot\varphi) + \mathrm{f}_{0\cdot_+f(a)})$. [prover eprover]

$((\mathrm{f}_{1\cdot_+-a}\cdot\varphi) + \mathrm{f}_{0\cdot_+f(a)})$ is continuous in a. Indeed $(\mathrm{f}_{1\cdot_+-a}\cdot\varphi)$ is continuous in a. □

Corollary 155. Let f be a differentiable real map. Then f is continuous.

[timelimit 3]

4.3 Differentiation Rules

4.3.1 Derivative of a Linear Function

Let a, b denote real numbers.

Theorem 156 (derivative of linear function). $\mathrm{f}_{a\cdot_+b}$ is differentiable and $\mathrm{D}\mathrm{f}_{a\cdot_+b} = \mathrm{f}_{0\cdot_+a}$.

Proof. Take $f = \mathrm{f}_{a\cdot_+b}$. Let ξ be a real number. [timelimit 30]

Let us show that $\partial f(\xi) = a$. Let ε be a positive real number.

We can show that 1 is a domain restrictor for ε and $\frac{\Delta f}{\Delta \xi}$ around ξ and a. Let $x \in \mathbb{R}\setminus\{\xi\}$. Thus $x - \xi \neq 0$.

Then $f(x) - f(\xi) = a\cdot(x-\xi)$. Indeed $((a\cdot x)+b) - ((a\cdot\xi)+b). = ((a\cdot x)+b) + (-b - (a\cdot\xi))(byeqlemma15, addcomm). = ((a\cdot x) + (b-b)) - (a\cdot\xi)(byaddass). = (a\cdot x) - (a\cdot\xi)(byadditiveinverse, existencezero). = (a\cdot x) + (a\cdot(-\xi))(bymultcomm, eqlemma11). = a\cdot(x-\xi)$ (by distributive law).

$\frac{\Delta f}{\Delta\xi}(x) = \frac{f(x)-f(\xi)}{x-\xi} = \frac{a\cdot(x-\xi)}{x-\xi} = a$. [timelimit 3] $d_{\mathbb{R}}(\frac{\Delta f}{\Delta\xi}(x), a) = d_{\mathbb{R}}(a,a) = 0 < \varepsilon$. End. End.

Therefore $\mathrm{D}f(\xi) = a = \mathrm{f}_{0\cdot_+a}(\xi)$ (by linear lem 1). □

4.3.2 Derivative Under Addition

Let f, g denote real maps.

Theorem 157 (derivative addition rule). Let f, g be differentiable in a. $(f+g)$ is differentiable in a and $\partial(f+g)(a) = \partial f(a) + \partial g(a)$.

Proof. Take difference quotient map φ of f in a. Take difference quotient map γ of g in a. Take $\psi = (\varphi + \gamma)$.

Let us show that ψ is the difference quotient map of $(f+g)$ in a. φ, γ are continuous in a. Thus ψ is continuous in a.

We can show that for all real numbers x $(f+g)(x) = ((x-a)\cdot\psi(x)) + (f+g)(a)$. [timelimit 10] Let x be a real number.

$(f+g)(x) = f(x) + g(x)$

$= (((x-a)\cdot\varphi(x)) + f(a)) + (((x-a)\cdot\gamma(x)) + g(a))$

$= (((x-a)\cdot\varphi(x)) + (((x-a)\cdot\gamma(x)) + f(a))) + g(a)$

$= (((x-a)\cdot\varphi(x)) + ((x-a)\cdot\gamma(x))) + (f(a)+g(a))$

$= ((x-a)\cdot(\varphi(x)+\gamma(x))) + (f+g)(a)$

$= ((x-a)\cdot\psi(x)) + (f+g)(a)$. End. End.

$\partial(f+g)(a) = \psi(a) = \varphi(a) + \gamma(a) = \partial f(a) + \partial g(a)$. □

Corollary 158 (derivative addition rule 2). Let f, g be differentiable real maps. $(f+g)$ is differentiable and $\mathrm{D}(f+g) = (\mathrm{D}f + \mathrm{D}g)$.

Proof. For all real numbers x

$\mathrm{D}(f+g)(x) = \partial(f+g)(x) = \partial f(x) + \partial g(x) = \mathrm{D}f(x) + \mathrm{D}g(x) = (\mathrm{D}f + \mathrm{D}g)(x)$. □

4.3.3 Derivative Under Multiplication

Theorem 159 (derivative multiplication rule). Let f, g be differentiable in a. $(f\cdot g)$ is differentiable in a and $\partial(f\cdot g)(a) = (\partial f(a)\cdot g(a)) + (f(a)\cdot\partial g(a))$.

Proof. Take difference quotient map φ of f in a. Take difference quotient map γ of g in a. Take $\psi = ((\varphi\cdot g) + (\mathrm{f}_{0\cdot_+f(a)}\cdot\gamma))$. ψ is continuous in a. Indeed $(\varphi\cdot g), (\gamma\cdot\mathrm{f}_{0\cdot_+f(a)})$ are continuous in a. [timelimit 30]

(0) $\psi(x) = (\varphi(x)\cdot g(x)) + (f(a)\cdot\gamma(x))$ for any real number x. Indeed for every real number x $(((\varphi\cdot g) + (\mathrm{f}_{0\cdot_+f(a)}\cdot\gamma)))(x) = (\varphi(x)\cdot g(x)) + (f(a)\cdot\gamma(x))$ (by def mult func, def add func, linear lem 1).

Let us show that ψ is the difference quotient map of $(f\cdot g)$ in a. Indeed we can show that for all real numbers x $(f\cdot g)(x) = ((x-a)\cdot\psi(x)) + (f\cdot g)(a)$. Let x be a real number.

(1) $f(x) = ((x-a)\cdot\varphi(x)) + f(a)$.

(2) $g(x) = ((x-a)\cdot\gamma(x)) + g(a)$.

$(f\cdot g)(x). = f(x)\cdot g(x)$ (by def mult func)

$. = (((x-a)\cdot\varphi(x)) + f(a))\cdot g(x)$ (by 1)

$. = (((x-a)\cdot\varphi(x))\cdot g(x)) + (f(a)\cdot g(x))$ (by eq lemma 1)

$. = ((x-a)\cdot(\varphi(x)\cdot g(x))) + (f(a)\cdot(((x-a)\cdot\gamma(x)) + g(a)))$ (by mult ass, 2)

$. = (((x-a)\cdot(\varphi(x)\cdot g(x))) + (f(a)\cdot((x-a)\cdot\gamma(x)))) + (f(a)\cdot g(a))$ (by distributive law, add ass)

$. = (((x-a)\cdot(\varphi(x)\cdot g(x))) + ((x-a)\cdot(f(a)\cdot\gamma(x)))) + (f(a)\cdot g(a))$ (by mult comm, mult ass)

$. = ((x-a)\cdot((\varphi(x)\cdot g(x)) + (f(a)\cdot\gamma(x)))) + (f(a)\cdot g(a))$ (by distributive law)

$. = ((x-a)\cdot\psi(x)) + (f\cdot g)(a)$ (by 0, def mult func). End.

$\partial(f \cdot g)(a) = \psi(a) = (\varphi \cdot g)(a) + (\mathrm{f}_{0\cdot -+f(a)} \cdot \gamma)(a) = (\varphi(a) \cdot g(a)) + (f(a) \cdot \gamma(a))$

$= (\partial f(a) \cdot g(a)) + (f(a) \cdot \partial g(a))$. □

[prover vampire]

Corollary 160 (derivative multiplication rule 2). Let f, g be differentiable real maps. $(f \cdot g)$ is differentiable and $\mathrm{D}(f \cdot g) = ((\mathrm{D}f \cdot g) + (f \cdot \mathrm{D}g))$.

Proof. Let x be a real number.

$\mathrm{D}(f \cdot g)(x) = \partial(f \cdot g)(x) = (\partial f(x) \cdot g(x)) + (f(x) \cdot \partial g(x)) = (\mathrm{D}f(x) \cdot g(x)) + (f(x) \cdot \mathrm{D}g(x)) = ((\mathrm{D}f \cdot g) + (f \cdot \mathrm{D}g))(x)$. □

[prover eprover]

4.3.4 Derivative Chain Rule

Theorem 161 (derivative chain rule). Let f be differentiable in a and g be differentiable in $f(a)$. $g \circ f$ is differentiable in a and $\partial(g \circ f)(a) = \partial g(f(a)) \cdot \partial f(a)$.

Proof. $g \circ f$ is a real map. Take difference quotient map φ of f in a. Take difference quotient map γ of g in $f(a)$. Take $\psi = ((\gamma \circ f) \cdot \varphi)$.

(1) Then ψ is continuous in a and for every $x \in \mathbb{R}$ $\psi(x) = \gamma(f(x)) \cdot \varphi(x)$.

ψ is the difference quotient map of $g \circ f$ in a.

Indeed we can show that for all real numbers x $(g \circ f)(x) = ((x - a) \cdot \psi(x)) + (g \circ f)(a)$. Let x be a real number.

(2)$f(x) - f(a) = (x - a) \cdot \varphi(x)$. (3) For all real numbers y $g(y) = ((y - f(a)) \cdot \gamma(y)) + g(f(a))$.

$g(f(x))$. $= ((f(x) - f(a)) \cdot \gamma(f(x))) + g(f(a))$ (by 3)

$. = (((x - a) \cdot \varphi(x)) \cdot \gamma(f(x))) + g(f(a))$ (by 2)

$. = ((x - a) \cdot (\gamma(f(x)) \cdot \varphi(x))) + g(f(a))$ (by mult ass, mult comm)

$. = ((x - a) \cdot \psi(x)) + g(f(a))$ (by 1). End.

$\partial(g \circ f)(a) = \psi(a) = \gamma(f(a)) \cdot \varphi(a) = \partial g(f(a)) \cdot \partial f(a)$. □

Corollary 162 (derivative chain rule 2). Let f, g be differentiable real maps. $g \circ f$ is differentiable and $\mathrm{D}(g \circ f) = ((\mathrm{D}g \circ f) \cdot \mathrm{D}f)$.

Proof. Let x be a real number. (1) $\mathrm{D}g(f(x)) = (\mathrm{D}g \circ f)(x)$ (by composition).

$\mathrm{D}(g \circ f)(x)$. $= \partial(g \circ f)(x)$ (by deriv and derivin)

$. = \partial g(f(x)) \cdot \partial f(x)$ (by derivative chain rule)

$. = \mathrm{D}g(f(x)) \cdot \mathrm{D}f(x)$ (by deriv and derivin)

$. = (\mathrm{D}g \circ f)(x) \cdot \mathrm{D}f(x)$ (by 1)

$. = ((\mathrm{D}g \circ f) \cdot \mathrm{D}f)(x)$ (by def mult func). □

4.3.5 Derivative With a Minus

In contrast to the previous proofs of differentiability, in the proof about the differentiability of $(-f)$ we use Frerix' rewriting algorithm. Sometimes this rewriting mechanism solves equations that the normal mechanism was not able to do. On the other hand, even the smallest rewriting step must be precisely named in the rewriting algorithm for it to solve the equations, which is unhelpful with regard to the goal of making the formalization appear more natural. Perhaps the biggest advantage of Frerix' algorithm is its speed. It makes the verification process significantly faster than it would otherwise be.

Definition 163 (def func minus). $(-f)$ is a real map such that $(-f)(x) = -f(x)$ for any real number x.

Lemma 164 (minus function lemma). $(-f) = \mathrm{f}_{-1\cdot_+0} \circ f$.

Proof. Let x be a real number.

$(\mathrm{f}_{-1\cdot_+0} \circ f)(x). = -f(x)$ (by composition, linear lem 2, eq lemma 12) $. = (-f)(x)$. □

Lemma 165 (minus derivative). Let f be differentiable in a. Then $(-f)$ is differentiable in a and $\partial(-f)(a) = -\partial f(a)$.

Proof. $\partial(-f)(a). = \partial(\mathrm{f}_{-1\cdot_+0} \circ f)(a)$ (by minus function lemma)

$. = \partial\mathrm{f}_{-1\cdot_+0}(f(a)) \cdot \partial f(a)$ (by derivative chain rule)

$. = \mathrm{f}_{0\cdot_+-1}(f(a)) \cdot \partial f(a)$ (by deriv and derivin, derivative of linear function)

$. = (-1) \cdot \partial f(a)$ (by linear lem 1)

$. = -\partial f(a)$ (by eq lemma 12). □

Corollary 166 (minus derivative 2). Let f be differentiable. Then $(-f)$ is differentiable and $\mathrm{D}(-f) = (-\mathrm{D}f)$.

Chapter 5

Locality

[read `examples/solow/sections/3differentiation.ftl.tex`]

5.1 Open Sets and Intervals

5.1.1 Balls and Open Sets

As a small proposition we show that alternatively we could have introduced continuity not via domain restrictors but rather with balls.

Let f denote a real map. Let a, b, x, y, z denote real number. Let r denote a positive real number.

Definition 167. $B_r(a) = \{x \in \mathbb{R} | d_{\mathbb{R}}(x, a) < r\}$.

Definition 168. An open set is a subset S of $\mathbb{R}$ such that for every $a \in S$ there exists a positive real number r such that $B_r(a) \subseteq S$.

Lemma 169 (R is open). $\mathbb{R}$ is an open set.

Lemma 170 (ball nonempty). $a \in B_r(a)$.

[timelimit 10]

Proposition 171 (equivalence domain restrictor and ball). f is continuous in a iff for every positive real number ε there exists a positive real number δ such that $f[B_\delta(a)] \subseteq B_\varepsilon(f(a))$.

Proof. Case f is continuous in a.

Let ε be a positive real number. Take a positive real number δ such that δ is a domain restrictor for ε and f around a and $f(a)$. Let us show that $f[B_\delta(a)] \subseteq B_\varepsilon(f(a))$.

Let $y \in f[B_\delta(a)]$. Take $x \in B_\delta(a)$ such that $f(x) = y$. $d_{\mathbb{R}}(x, a) < \delta$. $|f(x) - f(a)| < \varepsilon$ (by domain restrictor lemma 3). Indeed $|x - a| < \delta$. Therefore $y = f(x) \in B_\varepsilon(f(a))$. End. End.

Case for every positive real number ε there exists a positive real number δ such that $f[B_\delta(a)] \subseteq B_\varepsilon(f(a))$.

Let ε be a positive real number. Take a positive real number δ such that $f[B_\delta(a)] \subseteq B_\varepsilon(f(a))$.

δ is a domain restrictor for ε and f around a and $f(a)$. Indeed we can show that for all $x \in \mathbb{R} \setminus \{a\}$ if $d_{\mathbb{R}}(x,a) < \delta$ then $d_{\mathbb{R}}(f(x),f(a)) < \varepsilon$. Let $x \in \mathbb{R} \setminus \{a\}$. Assume $d_{\mathbb{R}}(x,a) < \delta$. Thus $x \in B_\delta(a)$. Hence $f(x) \in f[B_\delta(a)]$ and $f(x) \in B_\varepsilon(f(a))$. Therefore $d_{\mathbb{R}}(f(x),f(a)) < \varepsilon$. End. End. □

Lemma 172 (lemma 01). $|x-y| < r$ iff $x \in B_r(y)$.

Lemma 173 (ball subsets). Let r, ρ be positive real numbers. If $r < \rho$ then $B_r(a) \subseteq B_\rho(a)$.

Proof. Assume $r < \rho$. Let $x \in B_r(a)$. Then $|x-a| < \rho$. Indeed $|x-a| < r < \rho$. Therefore $x \in B_\rho(a)$. □

Lemma 174 (intersection of open sets). Let A, B be open sets. Then $A \cap B$ is an open set.

Proof. Let x be an element of $A \cap B$. Take a positive real number τ such that $B_\tau(x) \subseteq A$. Take a positive real number θ such that $B_\theta(x) \subseteq B$. Take a positive real number δ such that $\delta < \tau$ and $\delta < \theta$ (by ineq lemma 19). Then $B_\delta(x) \subseteq B_\tau(x) \subseteq A$ and $B_\delta(x) \subseteq B_\theta(x) \subseteq B$ (by ball subsets). Therefore $B_\delta(x)$ is a subset of $A \cap B$. □

5.1.2 Intervals

Definition 175. $(x,y)_{OI} = \{z \in \mathbb{R} | x < z < y\}$.

Lemma 176. $(x,y)_{OI}$ is nonempty iff $x < y$.

Proof. Case $x < y$ (by ineq lemma 20). End. □

[timelimit 30]

Lemma 177 (ball and open interval). $B_r(x)$ is equal to $(x-r, x+r)_{OI}$.

Proof. Take $I = (x-r, x+r)_{OI}$. $B_r(x) \subseteq I$. Proof.

Let $y \in B_r(x)$. $(y,x) \in \mathrm{dom}(d_{\mathbb{R}})$. Then $d_{\mathbb{R}}(y,x) < r$. $d_{\mathbb{R}}(y,x) = |y-x|$.

Case $y - x \geq 0$. $y \geq x$ and $y > x - r$. $|y-x| = y - x < r$. Thus $y < r + x$ (by ineq lemma 15). Therefore $y \in I$. End.

Case $y - x < 0$. Then $y < x$ and $y < x + r$. $|y-x| = -(y-x) = x - y < r$. Thus $x - r < y$ (by ineq lemma 18). Therefore $y \in I$. End.

End.

$I \subseteq B_r(x)$. Proof.

Let $y \in I$. $x - y < r$ (by ineq lemma 18). Indeed $x - r < y$. $y - x < r$ (by ineq lemma 15). Indeed $y < r + x$. $|y-x| < r$. Indeed $y - x < r$ and $-(y-x) < r$. $(x,y) \in \mathbb{R} \times \mathbb{R}$. Therefore $d_{\mathbb{R}}(y,x) = |y-x| < r$. Hence $y \in B_r(x)$. End. □

Lemma 178 (open interval lemma). If $a \geq x$ and $b \leq y$ then $(a,b)_{OI} \subseteq (x,y)_{OI}$.

[prover vampire]

Lemma 179 (open interval is open). $(a,b)_{OI}$ is an open set.

Proof. $(a,b)_{OI}$ is a subset of $\mathbb{R}$. Let $x \in (a,b)_{OI}$. Take a positive real number δ such

that $\delta < x - a$ and $\delta < b - x$ (by ineq lemma 19). Indeed $x - a$, $b - x$ are positive real numbers. Then $a < x - \delta$ and $x + \delta < b$ (by ineq lemma 28, ineq lemma 285). Therefore $B_\delta(x) = (x - \delta, x + \delta)_{OI} \subseteq (a, b)_{OI}$ (by ball and open interval, open interval lemma). □

Lemma 180 (ball is open). $B_r(a)$ is an open set.

Proof. $B_r(a)$ is an open set (by ball and open interval, open interval is open). □

Definition 181. An interval is a subset I of $\mathbb{R}$ such that $(x, z)_{OI} \subseteq I$ for every $x, z \in I$.

Lemma 182. Let I be an interval. Let $x, y \in I$. Then $(x, y)_{OI} \subseteq I$ and $(y, x)_{OI} \subseteq I$.

Definition 183. $[a, b]_{CI} = \{x \in \mathbb{R} | a \leq x \leq b\}$.

Lemma 184 (interval lemma). $(a, b)_{OI}$ is an interval and $[a, b]_{CI}$ is an interval.

5.2 Local Functions

Let a denote a real number. Let f denote a real map. Let S denote a subset of $\mathbb{R}$. Let a real map on S stand for a map from S to $\mathbb{R}$.

Definition 185. Let g be a real map on S for some subset S of $\mathbb{R}$. A real map extension of g is a real map f such that $f(x) = g(x)$ for all $x \in \mathrm{dom}(g)$.

Definition 186. Let g be a real map on S for some subset S of $\mathbb{R}$. g is locally continuous in a iff f is continuous in a for every real map extension f of g.

Definition 187. An open neighbourhood of a is an open set S such that $a \in S$.

Theorem 188 (open set continuity extension). Let S be an open neighbourhood of a. Let g be a real map on S. Let f be a real map extension of g such that f is continuous in a.

Then g is locally continuous in a.

Proof. Let h be a real map extension of g. Let ε be a positive real number. Take a positive real number τ such that $B_\tau(a) \subseteq S$. Indeed S is an open set and $a \in S$. Take a positive real number θ such that θ is a domain restrictor for ε and f around a and $f(a)$. Take a positive real number δ such that $\delta < \tau$ and $\delta < \theta$ (by ineq lemma 19). Indeed τ, θ are positive real numbers.

δ is a domain restrictor for ε and h around a and $h(a)$. Proof. Let $x \in \mathbb{R} \setminus \{a\}$. Assume $|x - a| < \delta$. $x \in S$ and $h(x) = g(x)$. Proof. $|x - a| < \tau$. $x \in B_\tau(a)$ (by lemma 01). $B_\tau(a) \subseteq S$. End. [prover eprover]

$(h(x), h(a)), (g(x), g(a)), (f(x), f(a)), (x, a) \in \mathrm{dom}(d_{\mathbb{R}})$. $d_{\mathbb{R}}(h(x), h(a)) = d_{\mathbb{R}}(g(x), g(a)) = d_{\mathbb{R}}(f(x), f(a)) < \varepsilon$. Indeed $d_{\mathbb{R}}(x, a) < \theta$. End. □

Corollary 189 (continuity under domain restriction). Let E be an open neighbourhood of a and f be continuous in a. Then $f \restriction E$ is locally continuous in a.

Proof. f is real map extension of $f \restriction E$. $f \restriction E$ is locally continuous in a (by open set continuity extension). Indeed $f \restriction E$ is a real map on E. □

Definition 190. Let g be a real map on S for some subset S of $\mathbb{R}$. g is locally differentiable in a iff for every real map extension f of g f is differentiable in a.

Definition 191. Let g be a real map on S for some subset S of $\mathbb{R}$. The canonical real map extension of g is a real map extension ext of g such that $ext(x) = 0$ for all $x \in \mathbb{R} \setminus \mathrm{dom}(g)$.

Let $ext|_g$ stand for the canonical real map extension of g.

Definition 192. Let g be a real map on S for some subset S of $\mathbb{R}$. Assume g is locally differentiable in a. $\partial_\ell g(a)$ is a real number such that $\partial_\ell g(a) = \partial ext|_g(a)$.

The following theorem aims to show that if a function is locally differentiable in some point a, then every real map extension thereof has the same derivative in that point. We prove that by defining a function h, which is according to some real map extension on one side of a and according to another on the other side. Since h is thus also a real map extension, we know it is differentiable in a. Then we can show that the derivative of h must be equal to the derivative of both real map extensions, which yields a contradiction when they are unequal.

Theorem 193 (lemma differentiation extension). Let S be a subset of $\mathbb{R}$ and g be a real map on S. Assume a is an element of S and g is locally differentiable in a. Let f be a real map extension of g. Then f is differentiable in a and $\partial f(a) = \partial_\ell g(a)$.

Proof. f is differentiable in a. Assume $\partial f(a) \neq \partial_\ell g(a)$. Take a difference quotient map φ of f in a. Take difference quotient map γ of $ext|_g$ in a.

Define

$$h(x) = \begin{cases} ext|_g(x) & : x \leq a \\ f(x) & : x > a \end{cases}$$

for $x \in \mathbb{R}$. h is a real map. Indeed $h(x)$ is a real number for any $x \in \mathbb{R}$.

$h(a) = g(a) = f(a)$. h is a real map extension of g. Indeed we can show that for every $x \in \mathrm{dom}(g)$ $h(x) = g(x)$. Let $x \in \mathrm{dom}(g)$. If $x \leq a$ then $h(x) = ext|_g(x) = g(x)$. If $x > a$ then $h(x) = f(x) = g(x)$. End. h is differentiable in a. Take a difference quotient map ψ of h in a.

(A) For any real number x such that $x < a$ $\psi(x) = \gamma(x)$. Proof. Let x be a real number such that $x < a$. Thus $x \in \mathbb{R} \setminus \{a\}$. $\psi(x) = \frac{\Delta h}{\Delta a}(x)$ (by prop difference quotient map). $\frac{\Delta h}{\Delta a}(x) = \frac{h(x)-h(a)}{x-a} = \frac{ext|_g(x)-ext|_g(a)}{x-a} = \frac{\Delta ext|_g}{\Delta a}(x)$. $\frac{\Delta ext|_g}{\Delta a}(x) = \gamma(x)$ (by prop difference quotient map). End.

(B) For any real number x such that $x > a$ $\psi(x) = \varphi(x)$. Proof. Let x be a real number such that $x > a$. Thus $x \in \mathbb{R} \setminus \{a\}$. $\psi(x) = \frac{\Delta h}{\Delta a}(x)$ (by prop difference quotient map). $\frac{\Delta h}{\Delta a}(x) = \frac{h(x)-h(a)}{x-a} = \frac{f(x)-f(a)}{x-a} = \frac{\Delta f}{\Delta a}(x)$. $\frac{\Delta f}{\Delta a}(x) = \varphi(x)$ (by prop difference quotient map). End. [prover eprover]

(C) $\psi(a) = \partial_\ell g(a)$. Proof. Assume the contrary.

(C1) Take $\kappa = |\psi(a) - \gamma(a)|$. κ is a positive real number (by absolute value positive). Indeed $\psi(a) - \gamma(a)$ is nonzero.

Take a positive real number ε such that $\varepsilon < \kappa$. Take a positive real number d such that d is a domain restrictor for $\kappa - \varepsilon$ and γ around a and $\gamma(a)$. Indeed $\kappa - \varepsilon$ is a positive real number and γ is continuous in a. Take a positive real number δ such that δ is a domain restrictor for ε and ψ around a and $\psi(a)$. Take a positive real number ι such that $\iota < d$ and $\iota < \delta$ (by ineq lemma 19). Take $x = a - \iota$.

(C2) $\psi(x) = \gamma(x)$ (by A). Indeed $x < a$. [timelimit 20]

(C3) $|\gamma(x) - \gamma(a)| = d_{\mathbb{R}}(\gamma(x), \gamma(a)) < \kappa - \varepsilon$. Indeed $d_{\mathbb{R}}(x, a) < d$.

(C4) $|\psi(x) - \psi(a)| < \varepsilon$ (by domain restrictor lemma 3). Indeed $|x - a| < \delta$. [timelimit 3]

We can show that $|\psi(x) - \psi(a)| \geq \varepsilon$.

$|\psi(x) - \psi(a)| . = |\gamma(x) - \psi(a)|$ (by C2)

$. = |\psi(a) - \gamma(x)|$ (by absolute lemma 3)

$. = |(\psi(a) - \gamma(x)) + (\gamma(a) - \gamma(a))|$ (by existence zero, additive inverse)

$. = |(\psi(a) + (\gamma(a) - \gamma(x))) - \gamma(a)|$ (by add comm, add ass)

$. = |(\psi(a) - \gamma(a)) + (\gamma(a) - \gamma(x))|$ (by add comm, add ass)

$. = |(\psi(a) - \gamma(a)) - (\gamma(x) - \gamma(a))|$ (by eq lemma 15, add comm, eq lemma 45).

$|(\psi(a) - \gamma(a)) - (\gamma(x) - \gamma(a))| \geq |\psi(a) - \gamma(a)| - |\gamma(x) - \gamma(a)|$ (by absolute lemma 6).

$|\psi(a) - \gamma(a)| - |\gamma(x) - \gamma(a)| . = \kappa - |\gamma(x) - \gamma(a)|$ (by C1).

$\kappa - |\gamma(x) - \gamma(a)| \geq \kappa - (\kappa - \varepsilon)$ (by ineq lemma 22, C3).

$\kappa - (\kappa - \varepsilon). = (\kappa - \kappa) + \varepsilon$ (by eq lemma 15, eq lemma 45, add ass)

$. = \varepsilon + 0$ (by additive inverse, add comm)

$. = \varepsilon$ (by existence zero). qed.

Contradiction. End.

(D) ψ is not continuous in a. Proof.

(D1) Take $\kappa = |\partial_\ell g(a) - \varphi(a)|$. κ is a positive real number. Indeed $\varphi(a) = \partial f(a)$.

Take a positive real number ε such that $\varepsilon < \kappa$. Take a positive real number d such that d is a domain restrictor for $\kappa - \varepsilon$ and φ around a and $\varphi(a)$. Indeed $\kappa - \varepsilon$ is a positive real number and φ is continuous in a. Take a positive real number δ such that δ is a domain restrictor for ε and ψ around a and $\psi(a)$. Take a positive real number ι such that $\iota < d$ and $\iota < \delta$ (by ineq lemma 19). Take $x = a + \iota$. [timelimit 10]

(D2) $|\varphi(x) - \varphi(a)| < \kappa - \varepsilon$. Indeed $|x - a| < d$.

(D3) $|\psi(x) - \psi(a)| < \varepsilon$. Indeed $|x - a| < \delta$. [timelimit 3]

We can show that $|\psi(x) - \psi(a)| \geq \varepsilon$.

$|\psi(x) - \psi(a)| . = |\varphi(x) - \partial_\ell g(a)|$ (by B, C)

$. = |\partial_\ell g(a) - \varphi(x)|$ (by absolute lemma 3)

$. = |(\partial_\ell g(a) - \varphi(x)) + (\varphi(a) - \varphi(a))|$ (by existence zero, additive inverse)

$. = |(\partial_\ell g(a) + (\varphi(a) - \varphi(x))) - \varphi(a)|$ (by add comm, add ass)

$. = |(\partial_\ell g(a) - \varphi(a)) + (\varphi(a) - \varphi(x))|$ (by add comm, add ass)

$. = |(\partial_\ell g(a) - \varphi(a)) - (\varphi(x) - \varphi(a))|$ (by eq lemma 15, add comm, eq lemma 45).

$|(\partial_\ell g(a) - \varphi(a)) - (\varphi(x) - \varphi(a))| \geq |\partial_\ell g(a) - \varphi(a)| - |\varphi(x) - \varphi(a)|$ (by absolute lemma 6).

$|\partial_\ell g(a) - \varphi(a)| - |\varphi(x) - \varphi(a)| . = \kappa - |\varphi(x) - \varphi(a)|$ (by D1).

$\kappa - |\varphi(x) - \varphi(a)| \geq \kappa - (\kappa - \varepsilon)$ (by ineq lemma 22, D2).

$\kappa - (\kappa - \varepsilon). = (\kappa - \kappa) + \varepsilon$ (by eq lemma 15, eq lemma 45, add ass)

$. = \varepsilon + 0$ (by additive inverse, add comm)

$. = \varepsilon$ (by existence zero). qed. End. □

This theorem is basically the other side of the coin of the previous theorem. While above we assume a local function to be locally differentiable to say that a real map extension thereof is then also differentiable, this theorem assumes differentiability of a real map extension to say that the local function is locally differentiable.

Theorem 194 (open set differentiation extension). Let E be an open neighbourhood of a. Let g be a real map on E. Let f be a real map extension of g such that f is differentiable in a. Then g is locally differentiable in a and $\partial_\ell g(a) = \partial f(a)$.

Proof. Let h be a real map extension of g. Then $\frac{\Delta h}{\Delta a}$ is a real map. Take a difference quotient map ψ of f in a. Define

$$\varphi(x) = \begin{cases} \psi(x) & : x \in E \\ \frac{\Delta h}{\Delta a}(x) & : x \notin E \end{cases}$$

for $x \in \mathbb{R}$. φ is a real map. Indeed $\varphi(x)$ is a real number for any real number x.

Let us show that φ is a difference quotient map of h in a.

(1) φ is continuous in a. Proof.

ψ is continuous in a. $\psi \restriction E$ is locally continuous in a (by continuity under domain restriction). $\psi \restriction E = \varphi \restriction E$. Thus $\varphi \restriction E$ is locally continuous in a. Therefore φ is continuous in a. Indeed φ is a real map extension of $\varphi \restriction E$. End.

(2) $h(x) = ((x - a) \cdot \varphi(x)) + h(a)$ for any real number x. Proof. Let x be a real number.

Case $x \notin E$. $((x - a) \cdot \varphi(x)) + h(a) = ((x - a) \cdot \frac{\Delta h}{\Delta a}(x)) + h(a)$.

$\left((x - a) \cdot \frac{\Delta h}{\Delta a}(x)\right) + h(a). = \left((x - a) \cdot \frac{h(x) - h(a)}{x - a}\right) + h(a)$ (by def diffquot)

$. = (h(x) - h(a)) + h(a)$ (by fraction lemma 2)

$. = h(x)$ (by add ass, add comm, additive inverse, existence zero). End.

Case $x \in E$. $((x - a) \cdot \varphi(x)) + h(a) = ((x - a) \cdot \psi(x)) + f(a) = f(x) = g(x) = h(x)$. End. End. End.

$\partial h(a) = \varphi(a) = \psi(a) = \partial f(a)$. □

5.3 Intermediate Value Theorem

For Bolzano's theorem we define a set H as those values with an image smaller zero. We know that it has a supremum. When we have shown that the image of the supremum can neither be positive nor negative, it needs to be zero and we are done.

Let f denote a real map. Let a, b, x, z denote real numbers.

[prover vampire][timelimit 50]

Theorem 195 (Bolzanos Theorem). Let f be continuous. Assume $a < b$ and $f(a) < 0 < f(b)$. Then there exists $c \in (a,b)_{OI}$ such that $f(c) = 0$.

Proof. Take $I = [a,b]_{CI}$. Define $H = \{x \in I | f(x) < 0\}$. Take a supremum s of H (by completeness axiom). Indeed H is bounded above and H is a nonempty subset of $\mathbb{R}$. $s \in I$.

Let us show that $f(s) = 0$. If $\neg f(s) < 0$ and $\neg f(s) > 0$ then $f(s) = 0$.

Case $f(s) < 0$. Take a positive real number δ such that δ is a domain restrictor for $|f(s)|$ and f around s and $f(s)$.

Take a positive real number ι such that $\iota < \delta$ and $\iota < b - s$ (by ineq lemma 19). Indeed $b - s$ is a positive real number. Take $c = s + \iota$.

We can show that $c \in H$.

$c = s + \iota < s + (b - s) = b$ and $a \leq s \leq c$. Thus $c \in I$.

Let us show that $f(c) < 0$. Assume $f(c) \geq 0$. $|f(c) - f(s)| = f(c) - f(s)$. Indeed $-f(s) \geq 0$ and $f(c) \geq 0$ and $f(c) - f(s) \geq 0$ (by ineq lemma 21).

$|f(c) - f(s)| < |f(s)|$ (by domain restrictor lemma 3). Indeed $|c - s| = \iota < \delta$.

Then $|f(s)| > f(c) - f(s)$ and $f(c) < |f(s)| + f(s) = -f(s) + f(s) = 0$. Contradiction. End. End.

Hence s is not an upper bound of H. Indeed $s < s + \iota = c$. End.

Case $f(s) > 0$. Take a positive real number δ such that δ is a domain restrictor for $f(s)$ and f around s and $f(s)$.

$s - a$ is a positive real number. Indeed $a \leq s$ and $a \neq s$. Indeed $f(a) \neq f(s)$. Take a positive real number ι such that $\iota < \delta$ and $\iota < s - a$ (by ineq lemma 19). Take $c = s - \iota$.

We can show that for all $x \in [c,s]_{CI}$ $f(x) > 0$.

Assume the contrary. Take $x \in [c,s]_{CI}$ such that $f(x) \leq 0$. Then $f(s) - f(x) \geq 0$ (by ineq lemma 21). Indeed $-f(x) \geq 0$ and $f(s) \geq 0$.

$|f(x) - f(s)| < f(s)$ (by domain restrictor lemma 3). Proof. $s - x \leq s - c$ (by ineq lemma 22). Indeed $c \leq x$. Thus $|x - s| = |s - x| = s - x \leq s - c = s - (s - \iota) = \iota < \delta$. Therefore $|x - s| < \delta$ and x is a real number. End.

Then $f(s) > |f(x) - f(s)| = |f(s) - f(x)| = f(s) - f(x)$. Thus $-f(x) < 0$. End.

Thus c is an upper bound of H. Proof. Assume the contrary. Take $x \in H$ such that $c < x$. Then $x \leq s$. Therefore $x \in [c,s]_{CI}$ and $f(x) > 0$. Contradiction. End.

$c = s - \iota < s$. Contradiction. End. qed. □

Corollary 196 (intermediate value theorem I). Let f be continuous. Assume $x < z$. Then $(f(x), f(z))_{OI} \subseteq f[(x,z)_{OI}]$.

Proof. Case $x = z$. Trivial.

Assume $x \neq z$. Let $\upsilon \in (f(x), f(z))_{OI}$.

Take $g = \mathrm{f}_{1 \cdot _ + - \upsilon} \circ f$. [timelimit 50] $g(x) = f(x) - \upsilon < 0$ and $g(z) = f(z) - \upsilon > 0$. Indeed For any real number r $g(r) = f(r) - \upsilon$ and $f(x) < \upsilon < f(z)$. Take $y \in (x,z)_{OI}$ such that $g(y) = 0$ (by Bolzanos Theorem). Indeed g is a continuous real map and $x < z$ and $g(x) < 0 < g(z)$.

Then $f(y) = (f(y) - \upsilon) + \upsilon = g(y) + \upsilon = \upsilon$. [timelimit 3] Therefore $\upsilon \in f[(x,z)_{OI}]$. □

[prover eprover][timelimit 3]

Corollary 197 (intermediate value theorem II). Let f be continuous. Assume $x < z$. Then $(f(z), f(x))_{OI} \subseteq f[(x,z)_{OI}]$.

Proof. Case $x = z$. Trivial. Assume $x \neq z$.

Let $\upsilon \in (f(z), f(x))_{OI}$. Take $g = \mathrm{f}_{-1\cdot_+\upsilon} \circ f$.

(A) For any real number r $g(r) = ((-1) \cdot f(r)) + \upsilon = -f(r) + \upsilon$.

$g(x) = -f(x) + \upsilon < 0$ and $g(z) = -f(z) + \upsilon > 0$. Indeed $f(z) < \upsilon < f(x)$.

[timelimit 10] (B) Take $y \in (x,z)_{OI}$ such that $g(y) = 0$ (by Bolzanos Theorem). Indeed g is a continuous real map and $x < z$ and $g(x) < 0 < g(z)$. [timelimit 3]

$f(y) = \upsilon$. Proof.

$\upsilon. = \upsilon - (-f(y) + \upsilon)$ (by existence zero, eq lemma 0, B,A)

$. = f(y) + (\upsilon - \upsilon)$ (by eq lemma 15, eq lemma 45,add comm, add ass)

$. = f(y)$ (by additive inverse, existence zero). End.

Therefore $\upsilon \in f[(x,z)_{OI}]$. □

5.4 Division Map

The continuity of a division map we will later need for the inverse function theorem. Having proved the inverse function theorem we can show the differentiability of a division map using the exponential function.

Let f denote a real map. Let a denote a real number.

Definition 198 (def division map). A division map of f in a is a real map g such that (for all $x \in E$ $f(x)$ is nonzero and $g(x) = 1/f(x)$) for some open neighbourhood E of a.

[prover vampire]

Proposition 199 (existence division map). Let f be continuous in a and $f(a)$ is nonzero. Then there exists a division map of f in a.

Proof. Take a positive real number δ such that δ is a domain restrictor for $|f(a)|$ and f around a and $f(a)$. Indeed $|f(a)|$ is a positive real number. [prover eprover] Take $E = B_\delta(a)$.

(1) E is an open neighbourhood of a (by ball nonempty, ball is open).

(2) For all $x \in E$ $f(x)$ is nonzero. Proof. Let x be an element in $E \setminus \{a\}$. Then $d_{\mathbb{R}}(x,a) < \delta$ (by lemma 01). Assume $f(x) = 0$. Then $d_{\mathbb{R}}(f(x), f(a)) = |0 - f(a)| = |f(a)|$. Thus δ is not a domain restrictor for $|f(a)|$ and f around a and $f(a)$ (by domain restrictor lemma 2). Indeed $x \in \mathbb{R} \setminus \{a\}$. Contradiction. End.

Define

$$g(x) = \begin{cases} 1/f(x) & : x \in E \\ 0 & : x \notin E \end{cases}$$

for $x \in \mathbb{R}$. g is a real map. Indeed $g(x)$ is a real number for any real number x.

Therefore g is a division map of f in a. □

[prover eprover]

The proof of the continuity of the division map is very unbeautiful. A particular nuisance is the introduction of Θ, because we cannot define δ as a positive real value $\delta < \tau$ and $\delta < \theta$ and $\delta < \vartheta$ immediately, although we have already defined in ineq lemma 195 that such a value exists. The reasoner seems to be overwhelmed by the number of conditions necessary to be fulfilled for this lemma.

By choosing δ so small that if $|x - a| < \delta$ then $|f(x) - f(a)| < \frac{|f(a)|}{2}$, we can show that $1/|f(x)| < 2/|f(a)|$. Choosing δ such that $|f(x) - f(a)| < \frac{|f(a)|^2}{2}\varepsilon$ as well, we can derive the following:

$$|1/f(x) - 1/f(a)| = \left|\frac{f(a) - f(x)}{f(x)f(a)}\right| = \frac{1}{|f(x)|}\frac{1}{|f(a)|}|f(x) - f(a)| < \frac{2}{|f(a)|}\frac{1}{|f(a)|}\frac{|f(a)|^2}{2}\varepsilon = \varepsilon.$$

Theorem 200 (continuity division map). Let f be continuous in a. Let g be a division map of f in a. Then g is continuous in a.

Proof. Let ε be a positive real number.

Take $\nu = \frac{|f(a)|}{2}$ and $\upsilon = f(a)$. ν is a positive real number. Take a positive real number τ such that τ is a domain restrictor for ν and f around a and $f(a)$. Indeed $\frac{|f(a)|}{2}$ is a positive real number.

Take $c = \frac{|f(a)|^2}{2} \cdot \varepsilon$. c is a positive real number (by ineq lemma 215). Indeed $|f(a)|^2, 1/2$ are positive real numbers. Take a positive real number θ such that θ is a domain restrictor for c and f around a and $f(a)$.

Take an open neighbourhood E of a such that for every $x \in E$ $f(x)$ is nonzero and $g(x) = 1/f(x)$. Take a positive real number ϑ such that $B_\vartheta(a) \subseteq E$.

Take a positive real number Θ such that $\Theta < \tau$ and $\Theta < \vartheta$ (by ineq lemma 19). Take a positive real number δ such that $\delta < \theta$ and $\delta < \Theta$ (by ineq lemma 19). Then $\delta < \tau$ and $\delta < \vartheta$.

Let us show that δ is a domain restrictor for ε and g around a and $g(a)$. Let $x \in \mathbb{R} \setminus \{a\}$. Assume $|x - a| < \delta$. $x, f(x), f(a)$ are real numbers. [timelimit 10]

(A) $|f(x) - f(a)| < \nu$ (by domain restrictor lemma 3). Indeed $|x - a| < \tau$.

(B) $|f(x) - f(a)| < c$ (by domain restrictor lemma 3). Indeed $|x - a| < \theta$.

(C) $g(a) = 1/f(a)$ and $f(x)$ is nonzero and $g(x) = 1/f(x)$. Indeed we can show that x, a are elements in E. $|x - a| < \delta < \vartheta$. Thus $x \in B_\vartheta(a)$. End. [timelimit 3]

(0) $\frac{|f(a)|}{2} < |f(x)|$. Proof.

(01) $|f(a)| < \frac{|f(a)|}{2} + |f(x)|$. Proof. $|f(a)| . = (|f(a)| - |f(x)|) + |f(x)|$ (by existence zero, additive inverse, add comm, add ass).

$(|f(a)| - |f(x)|) + |f(x)| \leq |f(a) - f(x)| + |f(x)|$ (by ineq lemma 145, absolute lemma 6).

$|f(a) - f(x)| + |f(x)| . = |f(x) - f(a)| + |f(x)|$ (by absolute lemma 3).

$|f(x) - f(a)| + |f(x)| < \nu + |f(x)|$ (by ineq lemma 14, A).

Thus $|f(a)| < \nu + |f(x)|$. Indeed $|f(a)| \leq \nu + |f(x)|$ (by transitivity).

Indeed $|f(a)| \leq |f(x) - f(a)| + |f(x)| \leq \nu + |f(x)|$. End.

$\frac{|f(a)|}{2} . = |f(a)| - \frac{|f(a)|}{2}$ (by eq lemma 23).

$|f(a)| - \frac{|f(a)|}{2} < (\frac{|f(a)|}{2} + |f(x)|) - \frac{|f(a)|}{2}$ (by ineq lemma 14, 01).

$(\frac{|f(a)|}{2} + |f(x)|) - \frac{|f(a)|}{2} . = |f(x)| + (\frac{|f(a)|}{2} - \frac{|f(a)|}{2})$ (by add comm, add ass)

$. = |f(x)|$ (by additive inverse, existence zero). End.

(1) $1/|f(x)| < \frac{2}{|f(a)|}$. Proof. $1/|f(x)| \leq 1/\frac{|f(a)|}{2}$ (by ineq lemma 12). Indeed $0 < \frac{|f(a)|}{2} \leq |f(x)|$.

$1/\frac{|f(a)|}{2} . = \frac{2}{|f(a)|}$ (by eq lemma 22). Therefore $1/|f(x)| \leq \frac{2}{|f(a)|}$. $1/f(x) \neq \frac{2}{|f(a)|}$. End.

$|g(x) - g(a)| . = |1/f(x) - 1/f(a)|$ (by C) $. = \left|\frac{f(a)-f(x)}{f(x)\cdot f(a)}\right|$ (by eq lemma 25)

$. = \left|1/f(x) \cdot \frac{f(a)-f(x)}{f(a)}\right|$ (by mult comm, fraction lemma)

$. = |1/f(x)| \cdot \frac{|f(a)-f(x)|}{|f(a)|}$ (by absolute lemma 2)

$. = 1/|f(x)| \cdot \frac{|f(x)-f(a)|}{|f(a)|}$ (by absolute lemma 3, absolute lemma 5).[timelimit 10]

$1/|f(x)| \cdot \frac{|f(x)-f(a)|}{|f(a)|} \leq \frac{2}{|f(a)|} \cdot \frac{|f(x)-f(a)|}{|f(a)|}$ (by ineq lemma 26). Proof.

$1/|f(x)|, \frac{|f(x)-f(a)|}{|f(a)|}, \frac{2}{|f(a)|}$ are real numbers. $1/|f(x)| \leq \frac{2}{|f(a)|}$ (by 1). $\frac{|f(x)-f(a)|}{|f(a)|} \geq 0$ (by frac lemma 1). Indeed $|f(x) - f(a)| \geq 0$ and $|f(a)| > 0$. End.

$\frac{2}{|f(a)|} \cdot \frac{|f(x)-f(a)|}{|f(a)|} . = (\frac{2}{|f(a)|} \cdot 1/|f(a)|) \cdot |f(x) - f(a)|$ (by mult comm, mult ass)

$. = |f(x) - f(a)| \cdot \frac{2}{|f(a)|^2}$ (by fraction lemma, potence lemma 2, mult comm).

$|f(x) - f(a)| \cdot \frac{2}{|f(a)|^2} < c \cdot \frac{2}{|f(a)|^2}$ (by ineq lemma 25). Proof.

$|f(x) - f(a)|, \frac{2}{|f(a)|^2}, c$ are real numbers. $\frac{2}{|f(a)|^2}$ is positive (by ineq lemma 215). Indeed $2, |f(a)|^2, 1/(|f(a)|^2)$ are positive real numbers. $|f(x) - f(a)| < c$ (by B). End.

$\frac{2}{|f(a)|^2} \cdot \left(\frac{|f(a)|^2}{2} \cdot \varepsilon\right) . = \left(\frac{2}{|f(a)|^2} \cdot \frac{|f(a)|^2}{2}\right) \cdot \varepsilon$ (by mult ass) $. = \varepsilon$ (by eq lemma 26, existence one).

Therefore $|g(x) - g(a)| < \varepsilon$ (by ineq lemma 29). Indeed $|g(x) - g(a)| \leq \frac{2}{|f(a)|} \cdot \frac{|f(x)-f(a)|}{|f(a)|} < \varepsilon$ and $|g(x) - g(a)|, \frac{2}{|f(a)|} \cdot \frac{|f(x)-f(a)|}{|f(a)|}, \varepsilon$ are real numbers. [timelimit 3] qed. □

Chapter 6

Extrema and Mean Value Theorem

In the last chapter before we are able to take on the inverse function theorem, we have to prove the mean value theorem and some propositions about the relation of derivatives and the slope of the function, i.e. a function is strictly increasing if its derivative is larger than zero and strictly decreasing if smaller than zero. At the end of the chapter we also provide a proof that $f'(\xi) = 0$ and $f''(\xi) < 0$ is a sufficient condition for ξ being a maximum. Using this condition for calculating some optimal state is absolutely ubiquitous in economics.
For the proof of the mean value theorem, we use Rolle's theorem, which says that when you got a differentiable function f with $f(a) = f(b)$ for some $a < b$ then there needs to be some value ξ in between such $f'(\xi) = 0$. For the proof of Rolle's theorem we need that such a function is bounded on $[a, b]$, for which we need the extreme value theorem, and that the derivative of a maximum and minimum is zero, which we proof in section 6.2.

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Before we can devote ourselves to the intermediate value theorem, we need to state the extreme value theorem.

6.1 Extreme Value Theorem

Let a, b denote real numbers. Let f denote a real map. Let S denote a subset of $\mathbb{R}$.

Let S is bounded stand for S is bounded above and S is bounded below. Let a bounds S stand for a is an upper bound of S and $-a$ is a lower bound of S.

The following theorem is the boundedness theorem, which says that the image of a continuous function on a closed interval is bounded. This theorem we need in order to prove the extreme value theorem, namely when we have a continuous function f on $[a, b]$ that we can find an $x \in [a, b]$ such that $f(x)$ is the supremum of the image of f on $[a, b]$.
In the proof of the boundedness theorem we basically follow along the lines of mathcenter.oxford.emory.edu. Here we define a function h such that

$$h(x) = \begin{cases} f[[a, \xi]_{CI}] & : \xi \leq b \\ \emptyset & : \xi > b \end{cases}$$

for any real number x. Furthermore we define $B = \{x \in \mathbb{R} | a \leq x \leq b \text{ and } h(x) \text{ is bounded}\}$. We then can take the supremum s of B. In (1) we show that $s = b$ and in (2) that $b \in B$. Then we

are done. To prove (1) and (2) we choose a value l very close to s (b), for which we still, however, know that f is bounded on $[a, l]$. Then we utilize that f is continuous in s and b and therefore we can construct an upper bound for both.

Lemma 201 (bounded lemma 1). a bounds S iff $|s| \leq a$ for every $s \in S$.

Lemma 202 (bounded lemma 2). S is bounded iff there exists a real number K such that K bounds S.

Proof. Case S is bounded. Take an upper bound u of S. Take a lower bound l of S. Take a real number K such that $K \geq |u|$ and $K \geq |l|$. Then K bounds S. End.

Case there exists a real number K such that K bounds S. Trivial. □

Theorem 203 (boundedness theorem). Let f be continuous. $f[[a, b]_{CI}] \subseteq \mathbb{R}$ and $f[[a, b]_{CI}]$ is bounded.

Proof. Case $a \geq b$. Trivial. Assume $a < b$.

For any real number ξ $[a, \xi]_{CI}$ is a subset of $\mathbb{R}$. Define

$$h(\xi) = \begin{cases} f[[a, \xi]_{CI}] & : \xi \leq b \\ \emptyset & : \xi > b \end{cases}$$

for $\xi \in \mathbb{R}$.

For every real number x $h(x)$ is a subset of $\mathbb{R}$. Proof. Let x be a real number. Case $x > b$. Trivial. Case $x \leq b$. $f[M] \subseteq \mathbb{R}$ for any $M \subseteq \mathbb{R}$. Therefore $f[[a, x]_{CI}]$ is a subset of $\mathbb{R}$. End. End.

(A) For all real number x, y if $x \leq y \leq b$ then $h(x) \subseteq h(y)$.

Proof. Let x, y be real numbers. Assume $x \leq y \leq b$. Let $xi \in h(x)$. Then $xi \in f[[a, x]_{CI}]$. Take $xd \in [a, x]_{CI}$ such that $f(xd) = xi$. Then $xd \in [a, y]_{CI}$ and $y \leq b$. $f(xd) \in f[[a, y]_{CI}]$. Therefore $xi = f(xd) \in h(y)$. End.

Define $B = \{x \in \mathbb{R} | a \leq x \leq b \text{ and } h(x) \text{ is bounded}\}$.

Take supremum s of B (by completeness axiom). Indeed we can show that B is a nonempty subset of $\mathbb{R}$ and bounded above.

$h(a)$ is bounded. Indeed ($f(a) \geq y$ for all $y \in h(a)$) and ($f(a) \leq y$ for all $y \in h(a)$). Therefore B is nonempty.

B is bounded above. Indeed for every $x \in B$ $x \leq b$. End.

(1) $s = b$. Proof by contradiction. b is an upper bound of B. Therefore $s \leq b$. Assume $s < b$.

(10) Take a positive real number δ such that δ is a domain restrictor for 1 and f around s and $f(s)$. Indeed 1 is a positive real number and f is continuous in s.

Take $l \in B$ such that $s - \delta < l$ (by lemma x01). Indeed B is a nonempty subset of $\mathbb{R}$ and B is bounded above.

Take real number K such that K bounds $h(l)$ (by bounded lemma 2). Indeed $h(l)$ is bounded. [prover vampire] Take a real number u such that $s < u < s + \delta$ and $u \leq b$. Proof. $b - s$ is a positive real number. Take positive real number υ such that $\upsilon < \delta$ and $\upsilon < b - s$ (by ineq

lemma 19). Take $u = s + \upsilon$. $s < u < s + \delta$. Indeed $s < s + \upsilon < s + \delta$. Indeed $0 < \upsilon < \delta$. $u < b$. End.

$h(u) = f[[a, u]_{CI}]$.

$h(u)$ is bounded (by bounded lemma 1, bounded lemma 2). Indeed for every $y \in h(u)$ $|y| \leq K + 2$. Indeed we can show that for every $x \in [a, u]_{CI}$ $|f(x)| \leq K + 2$. Let $x \in [a, u]_{CI}$. Then x is a real number such that $a \leq x \leq u$. [timelimit 20]

Case $x \leq l$. $f(x) \in f[[a, l]_{CI}] = h(l)$. Therefore $|f(x)| \leq K \leq K + 2$. Indeed K bounds $h(l)$. End.

Case $x > l$.

(1A) $|f(x) - f(s)| < 1$ (by domain restrictor lemma 3, 10, 1 is positive). Proof. $|x - s| < \delta$. Indeed $s - \delta < x < s + \delta$. $f(x), f(s)$ are real numbers. End.

(1B) $|f(l) - f(s)| < 1$ (by domain restrictor lemma 3, 10, 1 is positive). Proof. $|l - s| < \delta$. $f(l), f(s)$ are real numbers. End. [timelimit 3][prover eprover]

(1C) $|f(x)| < 2 + |f(l)|$ (by ineq lemma 15). Proof.

$|f(x)| - |f(l)| \leq |f(x) - f(l)|$ (by absolute lemma 6).

$|f(x) - f(l)| \,. = |(f(x) - f(l)) + (f(s) - f(s))|$ (by existence zero, additive inverse)

$\,. = |(f(x) - f(s)) + (f(s) - f(l))|$ (by add ass, add comm).

$|(f(x) - f(s)) + (f(s) - f(l))| \leq |f(x) - f(s)| + |f(s) - f(l)|$ (by absolute lemma 4).

$|f(x) - f(s)| + |f(s) - f(l)| < 1 + |f(s) - f(l)|$ (by ineq lemma 14, 1A).

$1 + |f(s) - f(l)| \,. = |f(l) - f(s)| + 1$ (by add comm, absolute lemma 3).

$|f(l) - f(s)| + 1 < 1 + 1 = 2$ (by ineq lemma 14, 1B).

Thus $|f(x)| - |f(l)| < 2$. End. [timelimit 10]

(1D) $|f(l)| \leq K$. Proof. $l \in B$. $h(l)$ is bounded and K bounds $h(l)$. $f(l) \in f[[a, l]_{CI}] = h(l)$. Indeed $l \in [a, l]_{CI}$. End. [timelimit 3]

Therefore $|f(x)| \leq 2 + |f(l)| = |f(l)| + 2 \leq K + 2$ (by 1C, add comm, ineq lemma 145, 1D). Then $|f(x)| \leq K + 2$ (by transitivity). End. qed. [timelimit 10]

Then $u \in B$ and $s < u$. Contradiction. [timelimit 3] End.

(2) $b \in B$. Proof.

(20) Take a positive real number δ such that δ is a domain restrictor for 1 and f around b and $f(b)$. Indeed 1 is a positive real number and f is continuous in b.

Take $l \in B$ such that $b - \delta < l$ (by lemma x01). Indeed B is a nonempty subset of $\mathbb{R}$ and B is bounded above and b is the supremum of B.

Take a real number K such that K bounds $h(l)$ (by bounded lemma 2). Indeed $h(l)$ is bounded.

$h(b) = f[[a, b]_{CI}]$. [timelimit 10]

(21) For every $y \in h(b)$ $|y| \leq K + 2$. Indeed we can show that for every $x \in [a, b]_{CI}$ $|f(x)| \leq K + 2$. Let $x \in [a, b]_{CI}$. Then x is a real number such that $a \leq x \leq b$.

Case $x \leq l$. $f(x) \in f[[a, l]_{CI}] = h(l)$. Therefore $|f(x)| \leq K \leq K + 2$. Indeed K bounds $h(l)$.

End.

Case $x > l$.

(2A) $|f(x) - f(b)| < 1$ (by domain restrictor lemma 3, 20, 1 is positive). Proof. $|x - b| < \delta$. Indeed $b - \delta < x \leq b$. $f(x), f(b)$ are real numbers. End.

(2B) $|f(l) - f(b)| < 1$ (by domain restrictor lemma 3, 20, 1 is positive). Proof. $|l - b| < \delta$. $f(l), f(b)$ are real numbers. End.

(2C) $|f(x)| < 2 + |f(l)|$ (by ineq lemma 15). Proof.

$|f(x)| - |f(l)| \leq |f(x) - f(l)|$ (by absolute lemma 6).

$|f(x) - f(l)| . = |(f(x) - f(l)) + (f(b) - f(b))|$ (by existence zero, additive inverse)

$. = |((f(x) - f(l)) - f(b)) + f(b)|$ (by add comm, add ass)

$. = |(f(x) - f(b)) + (f(b) - f(l))|$ (by add comm, add ass).

$|(f(x) - f(b)) + (f(b) - f(l))| \leq |f(x) - f(b)| + |f(b) - f(l)|$ (by absolute lemma 4).

$|f(x) - f(b)| + |f(b) - f(l)| < 1 + |f(b) - f(l)|$ (by ineq lemma 14, 2A).

$1 + |f(b) - f(l)| . = |f(l) - f(b)| + 1$ (by add comm, absolute lemma 3).

$|f(l) - f(b)| + 1 < 1 + 1 = 2$ (by ineq lemma 14, 2B).

Thus $|f(x)| - |f(l)| < 2$. End.

(2D) $|f(l)| \leq K$. Proof. $l \in B$ and $h(l)$ is bounded. $f(l) \in f[[a, l]_{CI}] = h(l)$. Indeed $l \in [a, l]_{CI}$. End.

Therefore $|f(x)| \leq 2 + |f(l)| = |f(l)| + 2 \leq K + 2$ (by 2C, add comm, ineq lemma 145, 2D).

Therefore $|f(x)| \leq K + 2$ (by transitivity). End. qed. [prover vampire][timelimit 50]

$h(b)$ is bounded (by bounded lemma 1, bounded lemma 2, 21). End. [prover eprover][timelimit 3]

$h(b) = f[[a, b]_{CI}]$. Thus $f[[a, b]_{CI}]$ is bounded. □

From the boundedness theorem we know that $f[[a, b]]$ has a supremum that we call s. Now we assume, for purpose of contradiction, that no such a c as in the theorem exists and therefore is $f(x) < s$ for all $x \in [a, b]$. Then we define

$$g(x) = \begin{cases} \frac{1}{s - f(a)} & : x \leq a \\ \frac{1}{s - f(x)} & : x \in (a, b) \\ \frac{1}{s - f(b)} & : x \geq b \end{cases}$$

for all real numbers x and show that g is continuous. The continuity seems obvious to us, but formally it takes up very significant time and most of the space of the proof to show the continuity in any point x for all cases when $x < a$, $x = a$, $x \in (a, b)$, $x = b$, and $x > b$. Now that g is continuous, we know from the boundedness theorem that $g[[a, b]]$ is bounded. Therefore we can take a positive real number K that bounds $g[[a, b]]$ and then we can show that $s - 1/K$ is an upper bound of $f[[a, b]]$, which serves us as a contradiction to s being the lowest upper bound of $f[[a, b]]$, i.e. its supremum.

[timelimit 100]

Theorem 204 (extreme value theorem). Let f be continuous and $a < b$. $f[[a,b]_{CI}]$ is a subset of $\mathbb{R}$ and there exists a $c \in [a,b]_{CI}$ such that $f(c)$ is an upper bound of $f[[a,b]_{CI}]$.

Proof. Take $S = f[[a,b]_{CI}]$. S is a subset of $\mathbb{R}$. [prover vampire]

Take a supremum s of S (by completeness axiom). Proof. S is bounded above. Indeed S is bounded (by boundedness theorem). S is nonempty. Indeed $f(a)$ is an element of S. End.

Case there exists $c \in [a,b]_{CI}$ such that $f(c) = s$. Trivial.

Case for every $c \in [a,b]_{CI}$ $f(c) \neq s$. Take $\varphi = \mathrm{f}_{-1\cdot_+s} \circ f$.

(A) $\varphi(x) = s - f(x)$ for any $x \in [a,b]_{CI}$. Proof. Let $x \in [a,b]_{CI}$.

$(\mathrm{f}_{-1\cdot_+s} \circ f)(x). = ((-1) \cdot f(x)) + s$ (by composition, linear function)

$. = s - f(x)$ (by eq lemma 12, add comm). End.

(B) φ is a continuous real map.

(C) $\varphi(x)$ is a positive real number for any $x \in [a,b]_{CI}$. Proof. Let $x \in [a,b]_{CI}$. Then $s - f(x)$ is a positive real number. Indeed $f(x) < s$. $\varphi(x)$ is a positive real number (by A). End.

For any real number x $x \leq a$ or $x \in (a,b)_{OI}$ or $x \geq b$. $\varphi(x)$ is nonzero for any $x \in [a,b]_{CI}$.

Define

$$g(x) = \begin{cases} 1/\varphi(a) & : x \leq a \\ 1/\varphi(x) & : x \in (a,b)_{OI} \\ 1/\varphi(b) & : x \geq b \end{cases}$$

for $x \in \mathbb{R}$. [prover vampire] g is a real map. Indeed $g(x)$ is a real number for any real number x. Proof. Let x be a real number. Case $x \leq a$ or $x \geq b$. $\varphi(a), \varphi(b)$ are positive real numbers (by C). Indeed $a, b \in [a,b]_{CI}$. $1/\varphi(a), 1/\varphi(b)$ are real numbers (by multiplicative inverse). Therefore $g(x)$ is a real number. End. Case $x \in (a,b)_{OI}$. Trivial. End. [timelimit 10][prover eprover]

g is continuous. Indeed we can show that for any real number ξ g is continuous in ξ. Let ξ be a real number. $\xi < a$ or $\xi > b$ or $\xi \in (a,b)_{OI}$ or $\xi = a$ or $\xi = b$.

Case $\xi < a$. $a - \xi$ is a positive real number. We can show that for any positive real number ε $a - \xi$ is a domain restrictor for ε and g around ξ and $g(\xi)$. Let ε be a positive real number.

Let $x \in \mathbb{R} \setminus \{\xi\}$. Assume $|x - \xi| < a - \xi$.

Then $x < a$. Proof. If $x < \xi$ then $x < \xi < a$. If $x \geq \xi$ then $|x - \xi| = x - \xi < a - \xi$ and $x < a$. End.

$\varphi(a)$ is nonzero. Thus $|g(x) - g(\xi)| = |1/\varphi(a) - 1/\varphi(a)| = 0 < \varepsilon$. End. qed.

Case $\xi > b$. $\xi - b$ is a positive real number. We can show that for any positive real number ε $\xi - b$ is a domain restrictor for ε and g around ξ and $g(\xi)$. Let ε be a positive real number.

Let $x \in \mathbb{R} \setminus \{\xi\}$. Assume $|x - \xi| < \xi - b$.

Then $x > b$. Proof. If $x \geq \xi$ then $b < \xi \leq x$. If $x < \xi$ then $|x - \xi| = |\xi - x| = \xi - x < \xi - b$ and $b < x$. End.

$\varphi(b)$ is nonzero. Thus $|g(x) - g(\xi)| = |1/\varphi(b) - 1/\varphi(b)| = 0 < \varepsilon$. End. qed.

Case $\xi \in (a, b)_{OI}$. g is a division map of φ in ξ. Proof. Then $(a, b)_{OI}$ is an open neighbourhood of ξ. Then ($\varphi(x)$ is nonzero and $g(x) = 1/\varphi(x)$) for all $x \in (a, b)_{OI}$. End.

g is continuous in ξ (by continuity division map). Indeed φ is continuous in ξ. End.[timelimit 50]

Case $\xi = a$. Take a division map h of φ in a (by existence division map, B). Indeed $\varphi(a) \neq 0$ (by C). h is continuous in a (by continuity division map, B).

Let ε be a positive real number. Take an open neighbourhood E of a such that for every $x \in E$ $\varphi(x)$ is nonzero and $h(x) = 1/\varphi(x)$. Take a positive real number τ such that $B_\tau(a) \subseteq E$. [prover vampire] Take a positive real number θ such that $\theta < \tau$ and $\theta < b - a$ (by ineq lemma 19). Indeed $\tau, b - a$ are positive real numbers. Take a positive real number ϑ such that ϑ is a domain restrictor for ε and h around a and $h(a)$. Take a positive real number δ such that $\delta < \theta$ and $\delta < \vartheta$ (by ineq lemma 19). Then δ is a domain restrictor for ε and h around a and $h(a)$ (by domain restrictor lemma). For all real numbers x if $|x - a| < \delta$ then $|x - a| < \theta < \tau$ and $|x - a| < \theta < b - a$. For all real numbers x if $|x - a| < \delta$ then $|x - a| < \tau$ and $x \in E$ and $\varphi(x)$ is nonzero and $h(x) = 1/\varphi(x)$.

Let us show that δ is a domain restrictor for ε and g around a and $g(a)$. Let $x \in \mathbb{R} \setminus \{a\}$. Assume $|x - a| < \delta$.

Case $x \leq a$. $|g(x) - g(a)| = |1/\varphi(a) - 1/\varphi(a)| = 0 < \varepsilon$. End.

Case $x > a$. $x - a = |x - a| < \delta < b - a$. $x - a < b - a$ and $x < b$. Then $x \in [a, b]_{CI}$. [prover eprover] $g(x) = 1/\varphi(x) = h(x)$. $g(a) = 1/\varphi(a) = h(a)$. Therefore $|g(x) - g(a)| = |h(x) - h(a)| < \varepsilon$. End. qed. End.

Case $\xi = b$. Take a division map h of φ in b (by existence division map, B). Indeed $\varphi(b) \neq 0$ (by C). h is continuous in b (by continuity division map, B).

Let ε be a positive real number. Take an open neighbourhood E of b such that for every $x \in E$ $\varphi(x)$ is nonzero and $h(x) = 1/\varphi(x)$. Take a positive real number τ such that $B_\tau(b) \subseteq E$. [prover vampire] Take a positive real number θ such that $\theta < \tau$ and $\theta < b - a$ (by ineq lemma 19). Indeed $\tau, b - a$ are positive real numbers. Take a positive real number ϑ such that ϑ is a domain restrictor for ε and h around b and $h(b)$. Take a positive real number δ such that $\delta < \theta$ and $\delta < \vartheta$ (by ineq lemma 19). Then δ is a domain restrictor for ε and h around b and $h(b)$ (by domain restrictor lemma). For all real numbers x if $|x - b| < \delta$ then $|x - b| < \theta < \tau$ and $|x - b| < \theta < b - a$. For all real numbers x if $|x - b| < \delta$ then $|x - b| < \tau$ and $x \in E$ and $h(x) = 1/\varphi(x)$.

Let us show that δ is a domain restrictor for ε and g around b and $g(b)$. Let $x \in \mathbb{R} \setminus \{b\}$. Assume $|x - b| < \delta$.

Case $x \geq b$. $|g(x) - g(b)| = |1/\varphi(b) - 1/\varphi(b)| = 0 < \varepsilon$. End.

Case $x < b$. $x \in [a, b]_{CI}$. Indeed $b - x < \delta < b - a$ and $a < x$. Then $g(x) = 1/\varphi(x) = h(x)$. $g(b) = 1/\varphi(b) = h(b)$. Therefore $|g(x) - g(b)| = |h(x) - h(b)| < \varepsilon$ (by domain restrictor lemma 3). End. qed. End. qed. [timelimit 50][prover eprover]

$g[[a, b]_{CI}]$ is a subset of $\mathbb{R}$. Take a real number κ such that κ bounds $g[[a, b]_{CI}]$ (by bounded lemma 2). Indeed $g[[a, b]_{CI}]$ is bounded (by boundedness theorem). Take a positive real number

K such that $K \geq \kappa$. Then K bounds $g[[a,b]_{CI}]$.

Let us show that $s - 1/K$ is an upper bound of $f[[a,b]_{CI}]$. Let $y \in f[[a,b]_{CI}]$. Take $x \in [a,b]_{CI}$ such that $f(x) = y$. Then $g(x) \leq K$. Therefore $1/\varphi(x) \leq K$. $1/(1/\varphi(x)) \geq 1/K$ (by ineq lemma 12). Indeed $0 < 1/\varphi(x) \leq K$. $\varphi(x) \geq 1/K$ (by eq lemma 8). $s - f(x) \geq 1/K$ (by A). $f(x) \leq s - 1/K$ (by ineq lemma 30). End.

$s - 1/K < s$. Contradiction. End. □

Lemma 205 (extreme value theorem 2). Let f be continuous and $a < b$. There exists an $i \in [a,b]_{CI}$ such that $f(i)$ is a lower bound of $f[[a,b]_{CI}]$.

Proof. $(-f)$ is continuous. $(-f)[[a,b]_{CI}]$ is a subset of $\mathbb{R}$. Take $i \in [a,b]_{CI}$ such that $(-f)(i)$ is an upper bound of $(-f)[[a,b]_{CI}]$ (by extreme value theorem).

Then $f(i)$ is a lower bound of $f[[a,b]_{CI}]$. Proof. [prover vampire]

Let $y \in f[[a,b]_{CI}]$. Then y is a real number and $-y \in (-f)[[a,b]_{CI}]$. [prover eprover] Thus $-f(i) = (-f)(i) \geq -y$ and $y \geq f(i)$. End. □

6.2 Local Extrema

Let f denote a real map. Let ξ denote a real number.

Definition 206. A maximum domain restrictor for f around ξ is a positive real number δ such that for all $x \in B_\delta(\xi)$ $f(x) \leq f(\xi)$.

Definition 207. A local maximum of f is a real number ξ such that there exists a positive real number δ such that δ is a maximum domain restrictor for f around ξ.

The proof that the derivative of a local maximum is zero, is a proof by contradiction. Assume $f'(\xi) = y$ for some $y \neq 0$. If $y > 0$ then however small you choose δ you can always find an x such that $\frac{f(x)-f(\xi)}{x-\xi} < 0$ and therefore having at least a distance y to y. In turn if $y < 0$ then again however small you choose δ you can find an x such that $\frac{f(x)-f(\xi)}{x-\xi} > 0$ and therefore having at least a distance y to y.

[prover vampire][timelimit 3]

Theorem 208 (derivative of maximum). Let ξ be a local maximum of f. Let f be differentiable in ξ. Then $\partial f(\xi) = 0$.

Proof. Take a maximum domain restrictor τ for f around ξ. Take $y = \partial f(\xi)$.

We can show that $y = 0$. Assume $y \neq 0$. Take $\varepsilon = |y|$. Take a positive real number θ such that θ is a domain restrictor for ε and $\frac{\Delta f}{\Delta \xi}$ around ξ and y. Indeed $\lim_{x \to \xi} \frac{\Delta f}{\Delta \xi}(x) = y$. Take a positive real number δ such that $\delta < \tau$ and $\delta < \theta$ (by ineq lemma 19).

Case $y > 0$. Take a real number x such that $\xi < x < \xi + \delta$ (by ineq lemma 20). Indeed $\xi < \xi + \delta$. Then $|x - \xi| = x - \xi < \delta < \theta$.

Let us show that $\left|\frac{\Delta f}{\Delta \xi}(x) - y\right| \geq \varepsilon$. $f(x) - f(\xi) \leq 0$ (by ineq lemma 27). Proof. $f(x), f(\xi)$ are real numbers. ξ is a local maximum of f. $x \in B_\tau(\xi)$. $f(x) \leq f(\xi)$. End. $x - \xi > 0$. Therefore $\frac{\Delta f}{\Delta \xi}(x) = \frac{f(x)-f(\xi)}{x-\xi} \leq 0$ (by frac lemma 3). $\frac{\Delta f}{\Delta \xi}(x) - y < 0$. Indeed $-y < 0$ and $\frac{\Delta f}{\Delta \xi}(x) \leq 0$. Thus

$\left|\frac{\Delta f}{\Delta \xi}(x) - y\right| = -(\frac{\Delta f}{\Delta \xi}(x) - y).\ -(\frac{\Delta f}{\Delta \xi}(x) - y). = y - \frac{\Delta f}{\Delta \xi}(x)$ (by eq lemma 15, eq lemma 45, add comm). $y - \frac{\Delta f}{\Delta \xi}(x) \geq y = \varepsilon$ (by ineq lemma 23). Indeed $-\frac{\Delta f}{\Delta \xi}(x) \geq 0$. End.

[timelimit 10] Thus θ is not a domain restrictor for ε and $\frac{\Delta f}{\Delta \xi}$ around ξ and y (by domain restrictor lemma 2). Indeed $x \in \mathbb{R} \setminus \{\xi\}$ and $|x - \xi| < \theta$. [timelimit 3] Contradiction. qed.

Case $y < 0$. Take a real number x such that $\xi - \delta < x < \xi$ (by ineq lemma 20). Indeed $\xi - \delta < \xi$ (by ineq lemma 24). [timelimit 30] Then $|x - \xi| = |\xi - x| = \xi - x < \delta < \theta$.

Let us show that $\left|\frac{\Delta f}{\Delta \xi}(x) - y\right| \geq \varepsilon$. $f(x) - f(\xi) \leq 0$ (by ineq lemma 27). Proof. $f(x), f(\xi)$ are real numbers. ξ is a local maximum of f. $x \in B_\tau(\xi)$. $f(x) \leq f(\xi)$. End. $x - \xi < 0$. Therefore $\frac{\Delta f}{\Delta \xi}(x) = \frac{f(x) - f(\xi)}{x - \xi} \geq 0$ (by frac lemma 4). $\frac{\Delta f}{\Delta \xi}(x) - y \geq 0$ (by ineq lemma 21). Indeed $-y \geq 0$. $\left|\frac{\Delta f}{\Delta \xi}(x) - y\right| = \frac{\Delta f}{\Delta \xi}(x) - y$.

$-y + \frac{\Delta f}{\Delta \xi}(x) \geq -y = \varepsilon$ (by ineq lemma 23). End.

Take $h = \frac{\Delta f}{\Delta \xi}$. Thus θ is not a domain restrictor for ε and h around ξ and y (by domain restrictor lemma 2). Indeed $x \in \mathbb{R} \setminus \{\xi\}$ and $|x - \xi| < \theta$ and h is a real map and $|h(x) - y| \geq \varepsilon$. [timelimit 3] Contradiction. qed. QED. □

Definition 209. A local minimum of f is a real number ξ such that ξ is a local maximum of $(-f)$.

Corollary 210 (derivative of minimum). Let ξ be a local minimum of f. Let f be differentiable in ξ. Then $\partial f(\xi) = 0$.

Proof. (A) ξ is a local maximum of $(-f)$.

$-\partial f(\xi) = \partial(-f)(\xi) = 0$ (by minus derivative, A, derivative of maximum). □

6.3 Mean Value Theorem

In the proof of Rolle's theorem we basically have three cases. If we have an inner local maximum, then the derivative of that local maximum is zero. If we have a local minimum, then the derivative of that local minimum is zero. And if we have neither, the function must be constant on the interval and therefore the derivative of any value therein must be zero.

Let f denote a real map. Let a, b denote real numbers.

Theorem 211 (Rolles theorem). Let f be differentiable. Let $a < b$ and $f(a) = f(b)$. Then there exists a $\xi \in (a, b)_{OI}$ such that $\partial f(\xi) = 0$.

Proof. Take $M = f[[a, b]_{CI}]$. M is a subset of $\mathbb{R}$. Take $s \in [a, b]_{CI}$ such that $f(s)$ is an upper bound of M (by extreme value theorem). Indeed f is continuous and $a < b$.

Case $a \neq s \neq b$. Then s is a local maximum of f. Proof. Take a positive real number δ such that $\delta < s - a$ and $\delta < b - s$ (by ineq lemma 19). Indeed $s - a, b - s$ are positive real numbers. Then δ is a maximum domain restrictor for f around s. Proof. Let $x \in B_\delta(s)$. (A) Then $x \in [a, b]_{CI}$. Proof. $x \in (s - \delta, s + \delta)_{OI}$ (by ball and open interval). $a \leq x \leq b$ (by transitivity). Indeed $a < s - \delta < x < s + \delta < b$. End. Therefore $f(s) \geq f(x)$. Indeed $f(x) \in f[[a, b]_{CI}]$ (by A). End. End.

$\partial f(s) = 0$ (by derivative of maximum). End.

Case $s = a$ or $s = b$. Take $i \in [a,b]_{CI}$ such that $f(i)$ is a lower bound of M (by extreme value theorem 2). Indeed f is continuous and $a < b$.

Case $a \neq i \neq b$. i is a local minimum of f. Proof. Take a positive real number δ such that $\delta < i - a$ and $\delta < b - i$ (by ineq lemma 19). Indeed $i - a, b - i$ are positive real numbers. Then δ is a maximum domain restrictor for $(-f)$ around i. Proof. Let $x \in B_\delta(i)$. Then $x \in [a,b]_{CI}$. Proof. $x \in (i-\delta, i+\delta)_{OI}$ (by ball and open interval). $a \leq x \leq b$ (by transitivity). Indeed $a < i - \delta < x < i + \delta < b$. End. [timelimit 10] Therefore $(-f)(i) \geq (-f)(x)$. Indeed $f(i) \leq f(x)$ and $(-f)(i) = -f(i) \geq -f(x) = (-f)(x)$. [timelimit 3] End. End.

Thus $\partial f(i) = 0$ (by derivative of minimum). End.

Case $i = a$ or $i = b$. [timelimit 10] (A) Then for all $x \in (a,b)_{OI}$ $f(x) = f(a)$. Proof. $f(s)$ is an upper bound of M and $f(s) = f(a)$. $f(i)$ is a lower bound of M and $f(i) = f(a)$. For all $y \in M$ $y = f(a)$. Therefore for all $x \in (a,b)_{OI}$ $f(x) = f(a)$. End. Take $e \in (a,b)_{OI}$. Take a positive real number δ such that $\delta < b - e$ and $\delta < e - a$ (by ineq lemma 19). Indeed $a < e < b$ and $b - e, e - a$ are positive real numbers.

Let us show that for any positive real number ε δ is a domain restrictor for ε and $\frac{\Delta f}{\Delta e}$ around e and 0. Let ε be a positive real number. We can show that δ is a domain restrictor for ε and $\frac{\Delta f}{\Delta e}$ around e and 0. Let $x \in \mathbb{R} \setminus \{e\}$. Assume $|x - e| < \delta$. $x \in (e-\delta, e+\delta)_{OI}$. $(e-\delta, e+\delta)_{OI} \subseteq (a,b)_{OI}$ (by open interval lemma). Indeed $a < e - \delta < e + \delta < b$. $x - e$ is nonzero. $\frac{\Delta f}{\Delta e}(x). = \frac{f(x)-f(e)}{x-e}$ (by def diffquot) $. = 0$ (by A, additive inverse, eq lemma 9). Thus $\left|\frac{\Delta f}{\Delta e}(x) - 0\right| < \varepsilon$. End. End. End. End.[timelimit 3] □

Following the standard proof of the mean value theorem based upon Rolle's theorem we define a function $h(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$. Then $h(a) = f(a) = h(b)$ and using Rolle's theorem we find an ξ such that $h'(\xi) = 0$. Then we know that $f'(\xi) = \frac{f(b)-f(a)}{b-a}$.

[prover eprover]

Corollary 212 (mean value theorem). Let f be differentiable and $a < b$. Then there exists $\xi \in (a,b)_{OI}$ such that $\partial f(\xi) = \frac{\Delta f}{\Delta a}(b)$.

Proof. (A) Take $h = (f + \mathrm{f}_{-\frac{\Delta f}{\Delta a}(b)\cdot - + \frac{\Delta f}{\Delta a}(b)\cdot a})$.

h is differentiable.

$h(a) = h(b)$. Proof.

$h(a). = f(a) + \left(\left(-\frac{\Delta f}{\Delta a}(b) \cdot a\right) + \left(\frac{\Delta f}{\Delta a}(b) \cdot a\right)\right)$ (by A, def add func,linear function)

$. = f(a) + \left(\frac{\Delta f}{\Delta a}(b) \cdot (a - a)\right)$ (by eq lemma 21, distributive law, add comm)

$. = f(a)$ (by additive inverse, mult comm, eq lemma 9, existence zero).

$h(b). = f(b) + \left(\left(-\frac{\Delta f}{\Delta a}(b) \cdot b\right) + \left(\frac{\Delta f}{\Delta a}(b) \cdot a\right)\right)$ (by A, def add func,linear function)

$. = f(b) + \left(\frac{\Delta f}{\Delta a}(b) \cdot (a - b)\right)$ (by eq lemma 21, distributive law, add comm)

$. = f(b) + \left((a-b) \cdot \frac{f(b)-f(a)}{b-a}\right)$ (by mult comm, def diffquot)

$. = f(b) - \left((b-a) \cdot \frac{f(b)-f(a)}{b-a}\right)$ (by eq lemma 15, eq lemma 45, add comm, eq lemma 11)

$. = (f(b) - f(b)) + f(a)$ (by fraction lemma 2, eq lemma 15, eq lemma 45, add ass)

$. = f(a)$ (by additive inverse, add comm, existence zero). End. [timelimit 10]

(B) Take $\xi \in (a,b)_{OI}$ such that $\partial h(\xi) = 0$ (by Rolles theorem). Indeed h is a differentiable real map and $a < b$ and $h(a) = h(b)$.[timelimit 3]

$\partial f(\xi) = \frac{\Delta f}{\Delta a}(b)$. Proof.

(C) $\mathrm{D}h. = (\mathrm{D}f + \mathrm{f}_{0\cdot_-+-\frac{\Delta f}{\Delta a}(b)})$ (by A, derivative addition rule 2, derivative of linear function).

$\mathrm{D}h(\xi). = \mathrm{D}f(\xi) + \mathrm{f}_{0\cdot_-+-\frac{\Delta f}{\Delta a}(b)}(\xi)$ (by C, def add func) $. = \mathrm{D}f(\xi) - \frac{\Delta f}{\Delta a}(b)$ (by linear lem 1).

$\mathrm{D}h(\xi) = 0$ (by B). Thus $\mathrm{D}f(\xi) = \frac{\Delta f}{\Delta a}(b)$. End. □

Definition 213. Let S be a subset of $\mathbb{R}$. f is constant on S iff for all $x, y \in S$ $f(x) = f(y)$.

Proposition 214 (derivative zero then constant). Let I be an interval and f be continuous. If $\partial f(x) = 0$ for all $x \in I$ then f is constant on I.

Proof. Assume $\partial f(x) = 0$ for all $x \in I$. Assume f is not constant on I. Take $a, b \in I$ such that $f(a) \neq f(b)$. $a = b$ or $a < b$ or $b < a$.

Case $a = b$. Trivial. [timelimit 10]

Case $a < b$. Then take $\xi \in (a,b)_{OI}$ such that $\partial f(\xi) = \frac{\Delta f}{\Delta a}(b)$ (by mean value theorem). [timelimit 3] $\partial f(\xi) = \frac{\Delta f}{\Delta a}(b) = \frac{f(b)-f(a)}{b-a}$ (by def diffquot). $\frac{f(b)-f(a)}{b-a} \neq 0$ (by frac lemma 5). Indeed $f(b) - f(a), b - a$ are nonzero. $\xi \in I$. Contradiction. End.

Case $b < a$. [timelimit 20] Then take $\xi \in (b,a)_{OI}$ such that $\partial f(\xi) = \frac{\Delta f}{\Delta b}(a)$ (by mean value theorem). [timelimit 3] $\partial f(\xi) = \frac{\Delta f}{\Delta b}(a) = \frac{f(a)-f(b)}{a-b}$ (by def diffquot). $\frac{f(a)-f(b)}{a-b} \neq 0$ (by frac lemma 5). Indeed $f(a) - f(b), a - b$ are nonzero. $\xi \in I$. End. □

6.4 Monotonicity

Let f denote a real map. Let S denote a subset of $\mathbb{R}$.

Definition 215. f is strictly increasing on S iff for all $x, y \in S$ if $x < y$ then $f(x) < f(y)$.

Definition 216. f is strictly decreasing on S iff for all $x, y \in S$ if $x < y$ then $f(y) < f(x)$.

Definition 217. f is strictly monotone on S iff f is strictly increasing on S or f is strictly decreasing on S.

Let f is strictly monotone stand for f is strictly monotone on $\mathrm{dom}(f)$.

Definition 218. f is increasing on S iff for all $x, y \in S$ if $x \leq y$ then $f(x) \leq f(y)$.

Definition 219. f is decreasing on S iff for all $x, y \in S$ if $x \leq y$ then $f(x) \geq f(y)$.

Lemma 220 (subset monotone lemma). Let f be strictly monotone on S. Let U be a subset of S. Then f is strictly monotone on U.

Proof. Case f is strictly increasing on S. Trivial. Case f is strictly decreasing on S. Trivial. □

[prover vampire]

Lemma 221 (lemma minus f increasing). f is strictly decreasing on S iff $(-f)$ is strictly increasing on S.

Proof. Case f is strictly decreasing on S. Let $x, y \in S$. Assume $x < y$. Then $(-f)(x) < (-f)(y)$. Indeed $-(-f)(x) = f(x) > f(y) = -(-f)(y)$. Indeed $f(x) > f(y)$. End.

Case $(-f)$ is strictly increasing on S. Let $x, y \in S$. Assume $x < y$. Then $f(x) > f(y)$. Indeed $-f(x) = (-f)(x) < (-f)(y) = -f(y)$. End. □

Let E denote an interval. [timelimit 10]

Corollary 222 (cor I intermediate value theorem). Let f be a continuous real map such that f be strictly increasing on E. Let $x, z \in E$. Then $(f(x), f(z))_{OI} = f[(x, z)_{OI}]$.

Proof. Let us show that $(f(x), f(z))_{OI} \subseteq f[(x, z)_{OI}]$. Case $x < z$ (by intermediate value theorem I). End. Case $x \geq z$. Then $f(x) \geq f(z)$ and $(f(x), f(z))_{OI} = \emptyset$. End. End.

Let us show that $f[(x, z)_{OI}] \subseteq (f(x), f(z))_{OI}$. Let $\upsilon \in f[(x, z)_{OI}]$. Take $y \in (x, z)_{OI}$ such that $f(y) = \upsilon$. Then $f(x) < f(y) < f(z)$. Indeed $x, y, z \in E$ and $x < y < z$ and f is strictly increasing on E. Therefore $\upsilon \in (f(x), f(z))_{OI}$. End. □

Corollary 223 (cor II intermediate value theorem). Let f be a continuous real map such that f is strictly decreasing on E. Let $x, z \in E$. Then $(f(z), f(x))_{OI} = f[(x, z)_{OI}]$.

Proof. Let us show that $(f(z), f(x))_{OI} \subseteq f[(x, z)_{OI}]$. Case $x \geq z$. $f(x) \leq f(z)$. End. Case $x < z$ (by intermediate value theorem II). End. End.

Let us show that $f[(x, z)_{OI}] \subseteq (f(z), f(x))_{OI}$. Let $\upsilon \in f[(x, z)_{OI}]$. Take $y \in (x, z)_{OI}$ such that $f(y) = \upsilon$. Then $f(x) > f(y) > f(z)$. Indeed $x, y, z \in E$ and $x < y < z$ and f is strictly decreasing on E. Therefore $\upsilon \in (f(z), f(x))_{OI}$. End. □

[timelimit 3]

Lemma 224 (derivative strictly increasing). Let f be differentiable. Let $\partial f(x) > 0$ for every $x \in E$. Then f is strictly increasing on E.

Proof by contradiction. Assume the contrary. Take $x, y \in E$ such that $x < y$ and $f(x) \geq f(y)$. Indeed f is not strictly increasing on E. Take $\xi \in (x, y)_{OI}$ such that $\partial f(\xi) = \frac{\Delta f}{\Delta x}(y)$ (by mean value theorem). $\xi \in E$.

$\partial f(\xi) \leq 0$. Proof. $f(y) - f(x) \leq f(x) - f(x) = 0$. $y - x > 0$.

Therefore $\partial f(\xi) = \frac{\Delta f}{\Delta x}(y) = \frac{f(y)-f(x)}{y-x} \leq 0$ (by frac lemma 3). End.

Contradiction. □

Corollary 225 (derivative strictly decreasing). Let f be differentiable and $\partial f(x) < 0$ for every $x \in E$. Then f is strictly decreasing on E.

Proof. E is a subset of $\mathbb{R}$. $(-f)$ is strictly increasing on E (by derivative strictly increasing). Indeed for every $x \in E$ $\partial(-f)(x) = -\partial f(x) > 0$.

Thus f is strictly decreasing on E. Proof. Let $x, y \in E$. Assume $x < y$. $(-f)(x) < (-f)(y)$

(by lemma minus f increasing). Thus $-f(x) < -f(y)$ and $f(y) < f(x)$. End. □

Lemma 226 (derivative increasing). Let f be differentiable and $\partial f(x) \geq 0$ for every $x \in E$. Then f is increasing on E.

Proof. Let $x, y \in E$. Assume that $x < y$ and $f(x) > f(y)$. Take $\xi \in (x, y)_{OI}$ such that $\partial f(\xi) = \frac{\Delta f}{\Delta x}(y)$ (by mean value theorem). $\xi \in E$.

$\partial f(\xi) < 0$. Proof. [prover eprover] $f(y) - f(x) < f(x) - f(x) = 0$. Indeed $f(y) < f(x)$. $y - x > 0$.

Therefore $\partial f(\xi) = \frac{\Delta f}{\Delta x}(y) = \frac{f(y)-f(x)}{y-x} < 0$ (by frac lemma 3, frac lemma 5). End.

Contradiction. □

Corollary 227 (derivative decreasing). Let f be differentiable and $\partial f(x) \leq 0$ for every $x \in E$. Then f is decreasing on E.

Proof. $(-f)$ is increasing on E (by derivative increasing). Indeed for every $x \in E$ $\partial(-f)(x) = -\partial f(x) \geq 0$.

Thus f is decreasing on E. Proof. Let $x, y \in E$. Assume $x < y$. Then

$-f(x) = (-f)(x) \leq (-f)(y) = -f(y)$. Thus $-f(x) \leq -f(y)$ and $f(y) \leq f(x)$. End. □

6.5 Sufficient Condition for a Local Maximum

If we know that $f'(\xi) = 0$ and $f''(\xi) < 0$, we want to show in (A) that f is increasing on some interval $[\xi - \delta, \xi]$ and in (B) that f is decreasing on $[\xi, \xi + \delta]$. Knowing (A) and (B) we can show that ξ is a local maximum.

Let f denote a real map. Let ξ denote a real number. [prover eprover]

Theorem 228 (sufficient condition local maximum). Let f be differentiable and $\partial f(\xi) = 0$. Let $\mathrm{D}f$ be differentiable in ξ and $\partial \mathrm{D}f(\xi) < 0$. Then ξ is a local maximum of f.

Proof. Take $\upsilon = \partial \mathrm{D}f(\xi)$. Take $\varepsilon = -\upsilon$. ε is a positive real number. [prover vampire] Take a positive real number d such that d is a domain restrictor for ε and $\frac{\Delta \mathrm{D}f}{\Delta \xi}$ around ξ and υ. Indeed $\lim_{x \to \xi} \frac{\Delta \mathrm{D}f}{\Delta \xi}(x) = \upsilon$. [prover eprover] Take a positive real number δ such that $\delta < d$.

Take $I = [\xi - \delta, \xi]_{CI}$. Take $J = [\xi, \xi + \delta]_{CI}$. I is an interval and J is an interval (by interval lemma).

(A) f is increasing on I (by derivative increasing). Indeed we can show that for every $x \in I$ $\partial f(x) \geq 0$.

Assume the contrary. Take $x \in I$ such that $\partial f(x) < 0$.

$\left| \frac{\Delta \mathrm{D}f}{\Delta \xi}(x) - \upsilon \right| \geq \varepsilon$. Proof.

(1) $\mathrm{D}f(\xi) = 0$ and $x - \xi$ is nonzero. $\frac{\Delta \mathrm{D}f}{\Delta \xi}(x) = \frac{\mathrm{D}f(x) - \mathrm{D}f(\xi)}{x - \xi}$ (by def diffquot).

$\frac{\mathrm{D}f(x) - \mathrm{D}f(\xi)}{x - \xi} . = \frac{\mathrm{D}f(x)}{x - \xi}$ (by 1, eq lemma 0, existence zero).

$\frac{\mathrm{D}f(x)}{x - \xi} > 0$ (by frac lemma 4, frac lemma 5). Indeed $x - \xi < 0$ and $\mathrm{D}f(x) < 0$.

(AA) Thus $\frac{\Delta \mathrm{D}f}{\Delta \xi}(x) > 0$.

Therefore $\left|\frac{\Delta \mathrm{D}f}{\Delta \xi}(x) - \upsilon\right| = \frac{\Delta \mathrm{D}f}{\Delta \xi}(x) - \upsilon$ (by ineq lemma 21).

$-\upsilon + \frac{\Delta \mathrm{D}f}{\Delta \xi}(x) \geq -\upsilon + 0 = \varepsilon$ (by ineq axiom addition, AA). End.

$|x - \xi| < d$. Proof. $|x - \xi| = |\xi - x| = \xi - x \leq \delta < d$. Indeed $\xi - x \leq \xi - (\xi - \delta) = \delta$. End.

[timelimit 20] d is not a domain restrictor for ε and $\frac{\Delta \mathrm{D}f}{\Delta \xi}$ around ξ and υ (by domain restrictor lemma 2). Indeed $x \in \mathbb{R} \setminus \{\xi\}$ and $\frac{\Delta \mathrm{D}f}{\Delta \xi}$ is a real map. [timelimit 3] Contradiction. qed.

(B) f is decreasing on J (by derivative decreasing). Indeed we can show that for every $x \in J$ $\partial f(x) \leq 0$. Assume the contrary. Take $x \in J$ such that $\partial f(x) > 0$.

$\left|\frac{\Delta \mathrm{D}f}{\Delta \xi}(x) - \upsilon\right| \geq \varepsilon$. Proof. $\frac{\Delta \mathrm{D}f}{\Delta \xi}(x) = \frac{\mathrm{D}f(x) - \mathrm{D}f(\xi)}{x - \xi} = \frac{\mathrm{D}f(x)}{x - \xi}$.

$\frac{\mathrm{D}f(x)}{x - \xi} > 0$ (by frac lemma 1, frac lemma 5). Indeed $x - \xi > 0$ and $\mathrm{D}f(x) > 0$.

(BB) Thus $\frac{\Delta \mathrm{D}f}{\Delta \xi}(x) > 0$.

Therefore $\left|\frac{\Delta \mathrm{D}f}{\Delta \xi}(x) - \upsilon\right| = \frac{\Delta \mathrm{D}f}{\Delta \xi}(x) - \upsilon$ (by ineq lemma 21).

$-\upsilon + \frac{\Delta \mathrm{D}f}{\Delta \xi}(x) \geq -\upsilon + 0 = \varepsilon$ (by ineq axiom addition, BB). End.

$|x - \xi| < d$. Proof. $|x - \xi| = x - \xi \leq \delta < d$. Indeed $x \leq \xi + \delta$. End.

[timelimit 20] d is not a domain restrictor for ε and $\frac{\Delta \mathrm{D}f}{\Delta \xi}$ around ξ and υ (by domain restrictor lemma 2). Indeed $x \in \mathbb{R} \setminus \{\xi\}$ and $\frac{\Delta \mathrm{D}f}{\Delta \xi}$ is a real map. [timelimit 3] Contradiction. qed.

δ is a maximum domain restrictor for f around ξ. Proof. $f(x) \leq f(\xi)$ for all $x \in I$. $f(\xi) \geq f(x)$ for all $x \in J$. $B_\delta(\xi) = (\xi - \delta, \xi + \delta)_{OI} \subseteq [\xi - \delta, \xi + \delta]_{CI} \subseteq I \cup J$. End.

Therefore ξ is a local maximum of f. □

Corollary 229 (sufficient condition local minimum). Let f be differentiable and $\partial f(\xi) = 0$. Let $\mathrm{D}f$ be differentiable in ξ and $\partial \mathrm{D}f(\xi) > 0$. Then ξ is a local minimum of f.

Proof. $(-f)$ is differentiable and $\partial(-f)(\xi) = 0$. $\mathrm{D}(-f)$ is differentiable in ξ and $\partial \mathrm{D}(-f)(\xi) < 0$. Then ξ is a local maximum of $(-f)$ (by sufficient condition local maximum). Therefore ξ is a local minimum of f. □

Let U denote an interval.

Definition 230. f is strictly convex on U iff for every $x \in U$ f is differentiable in x and $\mathrm{D}f$ is differentiable in x and $\partial \mathrm{D}f(x) < 0$.

Let f is strictly convex stand for f is strictly convex on $\mathbb{R}$.

Definition 231. A maximum of f on U is an element ξ of U such that for every $x \in U$ $f(\xi) \geq f(x)$.

Let a maximum of f stand for a maximum of f on $\mathrm{dom}(f)$. [prover vampire]

Corollary 232 (sufficient condition for maximum on interval). Let f be strictly convex on U. Let ξ be an element of U. If ξ is a local maximum of f then ξ is a maximum of f on U.

Proof. Let ξ be a local maximum of f. (A) Then $\partial f(\xi) = 0$ (by derivative of maximum). Define $I = \{x \in U | x \leq \xi\}$. Define $J = \{x \in U | x \geq \xi\}$. [timelimit 10] $I \cup J = U$. I is an interval and J is an interval.

Let us show that $f(\xi) \geq f(x)$ for any $x \in I$. $\mathrm{D}f$ is decreasing on I (by derivative decreasing). Indeed for every $x \in I$ $\partial \mathrm{D}f(x) \leq 0$. For every $x \in I$ $\partial f(x) = \mathrm{D}f(x) \geq \mathrm{D}f(\xi) = 0$. [timelimit 20] Therefore f is increasing on I (by derivative increasing). [timelimit 3] End.

Let us show that $f(\xi) \geq f(x)$ for any $x \in J$. $\mathrm{D}f$ is decreasing on J (by derivative decreasing). Indeed for every $x \in J$ $\partial \mathrm{D}f(x) \leq 0$. For every $x \in J$ $\partial f(x) = \mathrm{D}f(x) \leq \mathrm{D}f(\xi) = 0$. [timelimit 20] Therefore f is decreasing on J (by derivative decreasing). [timelimit 3] End. □

Lemma 233 (maximum is local maximum). Let E be an open neighbourhood of ξ such that E is an interval. Let ξ is a maximum of f on E. Then ξ is a local maximum of f.

Proof. Take a positive real number δ such that $B_\delta(\xi) \subseteq E$. Then δ is a maximum domain restrictor for f around ξ. □

[prover eprover]

Chapter 7

Inverse Function Theorem

[read `examples/solow/sections/5extrema.ftl.tex`]

Now we are finally able to devote ourselves explicitly to the inverse function theorem. The inverse functon theorem says that under some conditions $(f^{-1})'(y) = 1/f'(f^{-1}(y))$. One such condition, for which we want to prove the theorem, is that f is continuously differentiable and $f'(x)$ is nonzero for $y = f(x)$. In lemma lemma mon 4 we show that in that case there exists an open neighbourhood E of x such that f is strictly monotone on E. Then we know that $(f \restriction E)^{-1}$ exists. In lemma lemma mon 3 we show that the inverse is continuous in y. This we eventually use in our actually proof of the inverse function theorem to show that a function that we define there is continuous in y and therefore a difference quotient map.

Let f denote a real map. Let a denote a real number. Let U denote an interval. Let E denote a subset of $\mathbb{R}$.

Lemma 234 (lemma mon 1). Let f be strictly monotone on E. $f \restriction E$ is injective.

Proof. We can show that for all distinct elements x, y in E $f(x) \neq f(y)$. Let x, y be distinct elements in E. $x, y \in E$.

Case f is strictly increasing on E. If $x < y$ then $f(x) < f(y)$. If $x > y$ then $f(x) > f(y)$. $f(x) \neq f(y)$. End.

Case f is strictly decreasing on E. If $x < y$ then $f(x) > f(y)$. If $x > y$ then $f(x) < f(y)$. $f(x) \neq f(y)$. End. End. □

We want to show that if E is an open neighbourhood of a and f continuous and strictly monotone on E, then $f[E]$ is an open neighbourhood of $f(a)$. Let us assume f is strictly increasing as the case for f strictly decreasing works analogously. We need to show that $f[E]$ is open. Therefore let ξ be an element of $f[E]$. Take $x \in E$ $f(x) = \xi$ and, because E is open, there exists $\delta > 0$ such that $B_\delta(x) \subseteq E$. One corollary of the intermediate value theorem that we proved in the monotonicity subsection of the section about extrema, that for continuous strictly increasing functions $f[(x - \delta, x + \delta)] = (f(x - \delta), f(x + \delta)) \ni \xi$. Thus

$$\exists d > 0 : \ B_d(\xi) \subseteq (f(x - \delta), f(x + \delta)) \subseteq f[E].$$

Lemma 235 (lemma mon 2). Let E be an open neighbourhood of a. Let f be continuous and strictly monotone on E. Then $f[E]$ is an open neighbourhood of $f(a)$.

Proof. $f(a)$ is an element of $f[E]$. $f[E]$ is a subset of $\mathbb{R}$.

$f[E]$ is an open set. Indeed we can show that for every $\xi \in f[E]$ there exists a positive real number r such that $B_r(\xi) \subseteq f[E]$.

Let $\xi \in f[E]$. Take $x \in E$ such that $f(x) = \xi$. Take a positive real number δ such that $B_\delta(x) \subseteq E$. Take a positive real number d such that $d < \delta$. Take $y = x - d$, $z = x + d$. Then $y < x < z$.

(1) Then $x, y, z \in B_\delta(x)$. Proof. $x - \delta < x - d$ (by ineq lemma 22). Indeed $x - d \neq x - \delta$. [timelimit 30] $x - d, x + d \in (x - \delta, x + \delta)_{OI}$. Indeed $x + d$ is a real number and $x + d < x + \delta$. End.

(2) $B_\delta(x)$ is an interval (by ball and open interval). Indeed $(x - \delta, x + \delta)_{OI}$ is an interval (by interval lemma).

(3) Case f is strictly increasing on $B_\delta(x)$. [prover vampire] $f[(y, z)_{OI}] = (f(y), f(z))_{OI}$ (by cor I intermediate value theorem, 1, 2). Indeed f is a continuous real map such that f is strictly increasing on $B_\delta(x)$. ξ is an element of $(f(y), f(z))_{OI}$. Proof. $f(y) < f(x)$ (by 1,3). Indeed $y < x$. $f(x) < f(z)$ (by 1,3). Indeed $x < z$. Therefore $f(y) < \xi < f(z)$. End. Take a positive real number r such that $B_r(\xi) \subseteq (f(y), f(z))_{OI}$ (by open interval is open). Indeed $(f(y), f(z))_{OI}$ is an open set (by open interval is open). $B_r(\xi)$ is a subset of $f[E]$. Proof. $(y, z)_{OI} \subseteq E$. Indeed $(y, z)_{OI} \subseteq B_\delta(x) \subseteq E$. Therefore $f[(y, z)_{OI}]$ is a subset of $f[E]$. Thus $B_r(\xi) \subseteq f[E]$. Indeed $B_r(\xi) \subseteq (f(y), f(z))_{OI} = f[(y, z)_{OI}] \subseteq f[E]$. End. End.

Case f is strictly decreasing on $B_\delta(x)$. [timelimit 20] $f[(y, z)_{OI}] = (f(z), f(y))_{OI}$ (by cor II intermediate value theorem, 1, 2). Indeed f is a continuous real map such that f is strictly decreasing on $B_\delta(x)$. ξ is an element of $(f(z), f(y))_{OI}$. Indeed $f(z) < \xi < f(y)$. Indeed $y < x < z$. [timelimit 3] Take a positive real number r such that $B_r(\xi) \subseteq (f(z), f(y))_{OI}$ (by open interval is open). $B_r(\xi)$ is a subset of $f[E]$. Indeed $f[(y, z)_{OI}]$ is a subset of $f[E]$. Indeed $(y, z)_{OI} \subseteq B_\delta(x) \subseteq E$. End. [prover eprover]

f is strictly increasing on $B_\delta(x)$ or f is strictly decreasing on $B_\delta(x)$. Indeed f is strictly monotone on $B_\delta(x)$ (by subset monotone lemma). qed. □

In lemma mon 3 we want to show that $(f \restriction E)^{-1}$ is locally continuous in $f(a)$. From lemma mon 1 we already know that $(f \restriction E)^{-1}$ exists. Thus we can take a real map extension thereof and want to show that $\lim_{x \to f(a)} h(x) = h(f(a)) = a$. We assume f to be strictly increasing as the strictly decreasing case is again analogous. Since E is open, we can take a positive real number d such that $[a - d, a + d] \subseteq E$ and $d < \varepsilon$ for any positive ε. Because $f(a - d) < f(a) < f(a + d)$ we can take $\delta = \min\{f(a) - f(a - d), f(a + d) - f(a)\}$ (or rather $\delta < f(a) - f(a - d)$ and $\delta < f(a + d) - f(a)$, as we did not define any minimum function). Then δ is a domain restrictor for ε and h around $f(a)$ and a. Since f is invertible,

$$\begin{aligned} &\text{if } f[B_d(a)] = f[(a - d, a + d)] = (f(a - d), f(a + d)) \supseteq B_\delta(f(a)), \\ &\text{then } (f \restriction E)^{-1}[B_\delta(f(a))] \subseteq B_d(a) \subseteq B_\varepsilon(a). \end{aligned}$$

Formally it takes some time to convince the ATP to verify this, but this is how the proof works.

Lemma 236 (lemma mon 3). Let E be an open neighbourhood of a. Let f be strictly monotone on E and continuous. $(f \restriction E)^{-1}$ is a real map on $f[E]$ and $(f \restriction E)^{-1}$ is locally continuous in $f(a)$.

Proof. Take $b = f(a)$. (0) $f \restriction E$ is an injective map. $f[E]$ is a subset of $\mathbb{R}$.

Take $g = (f \restriction E)^{-1}$.

(A) g is a real map on $f[E]$. Indeed $\mathrm{Range}((f \restriction E)^{-1}) \subseteq \mathbb{R}$ and $\mathrm{dom}(g) = f[E]$.

(B) Let h be a real map extension of g. [prover vampire] $\lim_{x \to b} h(x) = h(b)$. Indeed we can show that for every positive real number ε there exists a domain restrictor for ε and h around b and $h(b)$. Let ε be a positive real number.

Take a positive real number θ such that $B_\theta(a) \subseteq E$. Take a positive real number d such that $d < \varepsilon$ and $d < \theta$.

(C) $[a-d, a+d]_{CI} \subseteq E$. Proof. $[a-d, a+d]_{CI} \subseteq (a-\theta, a+\theta)_{OI}$. Proof. Let $x \in [a-d, a+d]_{CI}$. Then $x = a - d$ or $x = a + d$ or $x \in (a-d, a+d)_{OI}$. $(a-d, a+d)_{OI} \subseteq (a-\theta, a+\theta)_{OI}$ (by open interval lemma). Indeed $a - \theta \leq a - d$ and $a + d \leq a + \theta$. $a - d \in (a-\theta, a+\theta)_{OI}$ and $a + d \in (a-\theta, a+\theta)_{OI}$. Indeed $a - \theta < a - d$ and $a + d < a + \theta$. Indeed $-\theta < -d$. End. $(a-\theta, a+\theta)_{OI} = B_\theta(a)$ (by ball and open interval). $B_\theta(a) \subseteq E$. End. Take $\beta = f(a-d)$, $B = f(a+d)$.

(1) Case f is strictly increasing on E. $b - \beta, B - b$ are positive real numbers. Proof. $a - d, a, a + d \in E$ (by A). Indeed $a - d, a, a + d$ are elements in $[a-d, a+d]_{CI}$. $a - d < a < a + d$. Therefore $f(a-d) < f(a) < f(a+d)$ (by 1). Thus $\beta < b < B$. End. Take a positive real number δ such that $\delta < b - \beta$ and $\delta < B - b$ (by ineq lemma 19).

[timelimit 10] Let us show that δ is a domain restrictor for ε and h around b and $h(b)$ (by domain restrictor lemma 2). Let $\upsilon \in \mathbb{R} \setminus \{b\}$. Assume $|\upsilon - b| < \delta$.

$f[(a-d, a+d)_{OI}] = (f(a-d), f(a+d))_{OI} = (\beta, B)_{OI}$ (by cor I intermediate value theorem). Proof. f is a continuous real map such that f is strictly increasing on $[a-d, a+d]_{CI}$. $a-d, a+d \in [a-d, a+d]_{CI}$. $[a-d, a+d]_{CI}$ is an interval (by interval lemma). End.

$\upsilon \in (\beta, B)_{OI}$. Proof. $\upsilon \in B_\delta(b)$ (by lemma 01). $B_\delta(b) = (b-\delta, b+\delta)_{OI}$ (by ball and open interval). $(b-\delta, b+\delta)_{OI} \subseteq (\beta, B)_{OI}$ (by open interval lemma). Indeed $b - \delta \geq \beta$ and $b + \delta \leq B$. End. Take $y \in (a-d, a+d)_{OI}$ such that $f(y) = \upsilon$.

$|y - a| < d$ (by lemma 01). Indeed $y \in B_d(a)$ (by ball and open interval).

$h(\upsilon) = y$. Proof. $|y - a| < d < \theta$. $y \in E$. $\upsilon \in f[E] = \mathrm{dom}(g)$. $h(\upsilon) = g(\upsilon)$. $(f \restriction E)^{-1}(\upsilon) = y$ (by inverse lemma, 0). Indeed $y \in \mathrm{dom}(f \restriction E)$ and $(f \restriction E)(y) = \upsilon$. End.

$h(b) = a$. Indeed $b \in \mathrm{dom}(g)$ and $g(b) = a$ (by inverse lemma, 0). Indeed $a \in \mathrm{dom}(f \restriction E)$ and $(f \restriction E)(a) = b$. Thus $|h(\upsilon) - h(b)| = |y - a| < d < \varepsilon$. Therefore $|h(\upsilon) - h(b)| < \varepsilon$. End. qed. [timelimit 10]

(2) Case f is strictly decreasing on E. $\beta - b, b - B$ are positive real numbers. Proof. $a - d, a, a + d \in E$ (by A). Indeed $a - d, a, a + d$ are elements in $[a-d, a+d]_{CI}$. $a - d < a < a + d$. Therefore $f(a+d) < f(a) < f(a-d)$ (by 2). Thus $B < b < \beta$. End. Take a positive real number δ such that $\delta < \beta - b$ and $\delta < b - B$ (by ineq lemma 19).

Let us show that δ is a domain restrictor for ε and h around b and $h(b)$ (by domain restrictor

lemma 2). Let $\upsilon \in \mathbb{R} \setminus \{b\}$. Assume $|\upsilon - b| < \delta$.

$f[(a-d, a+d)_{OI}] = (f(a+d), f(a-d))_{OI} = (B, \beta)_{OI}$ (by cor II intermediate value theorem). Proof. f is a continuous real map such that f is strictly decreasing on $[a-d, a+d]_{CI}$ (by C, 2). $a-d, a+d$ are elements in $[a-d, a+d]_{CI}$. $[a-d, a+d]_{CI}$ is an interval (by interval lemma). End.

$\upsilon \in (B, \beta)_{OI}$. Proof. $\upsilon \in B_\delta(b)$ (by lemma 01). $B_\delta(b) = (b-\delta, b+\delta)_{OI}$ (by ball and open interval). $(b-\delta, b+\delta)_{OI} \subseteq (B, \beta)_{OI}$ (by open interval lemma). Indeed $b - \delta \geq B$ and $b + \delta \leq \beta$. End. Take $y \in (a-d, a+d)_{OI}$ such that $f(y) = \upsilon$.

$|y - a| < d$ (by lemma 01). Indeed $y \in B_d(a)$ (by ball and open interval).

$h(\upsilon) = y$. Proof. $|y-a| < d < \theta$. $y \in E$. $\upsilon \in f[E] = \mathrm{dom}(g)$. $h(\upsilon) = g(\upsilon)$. $(f \restriction E)^{-1}(\upsilon) = y$ (by inverse lemma, 0). Indeed $y \in \mathrm{dom}(f \restriction E)$ and $(f \restriction E)(y) = \upsilon$. End.

$b \in \mathrm{dom}(g)$. $h(b) = a$. Indeed $g(b) = a$ (by inverse lemma, 0). Indeed $a \in \mathrm{dom}(f \restriction E)$ and $(f \restriction E)(a) = b$. $|h(\upsilon) - h(b)| = |y - a| < d < \varepsilon$. Therefore $|h(\upsilon) - h(b)| < \varepsilon$. End. qed. End. [timelimit 3] □

In the next lemma we want to show if f is continuously differentiable and $f'(a) \neq 0$ then there also exists such an open neighborhood E of a, where f is strictly monotone. Let us just look at the case of $f'(a)$ being positive. Then we can Take a positive real number δ such that δ is a domain restrictor such that

$$\forall x \in B_\delta(a) : \ \left|f'(x) - f'(a)\right| < f'(a).$$

Then for all those x $f'(x) > 0$, because if not, then $|f'(x) - f'(a)| = f'(a) - f'(x) \geq f'(a)$. Therefore f is strictly increasing on the neighborhood $B_\delta(a)$ of a.

Let f is continuously differentiable stand for f is differentiable and $\mathrm{D}f$ is continuous. [prover eprover] [timelimit 10]

Lemma 237 (lemma mon 4). Let f be continuously differentiable and $\partial f(a) \neq 0$. Then there exists an open neighbourhood E of a such that f is strictly monotone on E.

Proof. Take $\varepsilon = |\partial f(a)|$. [prover vampire][timelimit 50] Take a positive real number δ such that δ is a domain restrictor for ε and $\mathrm{D}f$ around a and $\mathrm{D}f(a)$. Indeed $\mathrm{D}f$ is a real map and $\mathrm{D}f$ is continuous in a and ε is a positive real number. [prover eprover][timelimit 10] $B_\delta(a)$ is an open neighbourhood of a (by ball is open). Indeed $a \in B_\delta(a)$. Take $B = B_\delta(a)$.

Let us show that f is strictly monotone on B.

Case $\partial f(a) > 0$. Then f is strictly increasing on B (by derivative strictly increasing, ball and open interval, interval lemma). Indeed we can show that $\partial f(x) > 0$ for any $x \in B$.

Let $x \in B \setminus \{a\}$. Assume $\partial f(x) \leq 0$. Then $|\partial f(x) - \partial f(a)| = |\partial f(a) - \partial f(x)| = \partial f(a) - \partial f(x) \geq \partial f(a) = \varepsilon$. [prover vampire][timelimit 30]

δ is not a domain restrictor for ε and $\mathrm{D}f$ around a and $\mathrm{D}f(a)$ (by domain restrictor lemma 2). Indeed $x \in \mathbb{R} \setminus \{a\}$ and $|x - a| < \delta$ and $|\mathrm{D}f(x) - \mathrm{D}f(a)| \geq \varepsilon$. Contradiction. [timelimit 3] End. End.

Case $\partial f(a) < 0$. Then f is strictly decreasing on B (by derivative strictly decreasing, ball and open interval, interval lemma). Indeed we can show that $\partial f(x) < 0$ for any $x \in B$.

Let $x \in B \setminus \{a\}$. Assume $\partial f(x) \geq 0$. Then $|\partial f(x) - \partial f(a)| = \partial f(x) - \partial f(a) \geq -\partial f(a) = \varepsilon$. [timelimit 30][prover vampire]

δ is not a domain restrictor for ε and $\mathrm{D}f$ around a and $\mathrm{D}f(a)$ (by domain restrictor lemma 2). Indeed $x \in \mathbb{R} \setminus \{a\}$ and $|x - a| < \delta$ and $|\mathrm{D}f(x) - \mathrm{D}f(a)| \geq \varepsilon$. Contradiction. End. End. qed. [prover eprover] □

So we want to show that $(f \restriction E)^{-1}$ is locally differentiable in $f(a)$ for some neighborhood E of a. By lemma mon 4 we can find a neighborhood $E = B_d(a)$ of a, where f is strictly monotone on. By lemma mon 1 we know then that $f \restriction E$ is injective and therefore $(f \restriction E)^{-1}$ exists.
Now we define

$$ext(y) = \begin{cases} (f \restriction E)^{-1}(y) & : y \in f[E] \\ a & : y \notin f[E] \end{cases}$$

for all real numbers y, which makes ext a real map extension of $(f \restriction E)^{-1}$.
By lemma mon 2 we know that $f[E]$ is actually an open environment of $f(a)$. Thus the idea is to utilize the open set differentiation extension theorem from the section Locality, which says that if we have a differentiable real map extension of a partial function defined on an open set, the partial function is locally differentiable. So what is left to show is that ext is differentiable, which we do by showing the existence of a difference quotient map. For that we define

$$\psi(y) = \begin{cases} 1/\varphi((f \restriction E)^{-1}(y)) & : y \in f[E] \\ 0 & : y \notin f[E] \end{cases}$$

for any real number y, where φ is the difference quotient map of f in a; φ exists, because f is differentiable in a. ψ is a difference quotient map for ext in $f(a)$ if it satisfies

(1) $ext(y) = (y - f(a))\psi(y) + ext(f(a))$ for all y.

(2) ψ is continuous in $f(a)$

(1) we can show by case distinction with $y \in f[E]$ and $y \notin f[E]$ and then some simple calculation. For (2) we need lemma mon 3 though, which says that $(f \restriction E)^{-1}$ is locally continuous in $f(a)$. Furthermore φ is continuous in a and therefore by the continuity of composition and the continuity of a division map is ψ also continuous in $f(a)$.
Thus ext is differentiable in $f(a)$ and since $\mathrm{dom}((f \restriction E)^{-1}) = f[E]$ is an open set, $(f \restriction E)^{-1}$ is locally differentiable in $f(a)$ by open set differentiation extension and

$$\partial_\ell (f \restriction E)^{-1}(f(a)) = \psi(f(a)) = 1/\varphi((f \restriction E)^{-1}(f(a))) = 1/f'(a).$$

Theorem 238 (inverse function theorem). Let f be continuously differentiable and $\partial f(a) \neq 0$. Then there exists an open neighbourhood E of a such that f is strictly monotone on E and $(f \restriction E)^{-1}$ is a real map on $f[E]$ and $(f \restriction E)^{-1}$ is locally differentiable in $f(a)$ and $\partial_\ell (f \restriction E)^{-1}(f(a)) = 1/\partial f(a)$.

Proof. Take an open neighbourhood N of a such that f is strictly monotone on N (by lemma mon 4). Take a positive real number d such that $B_d(a) \subseteq N$. Take a positive real number δ such that $\delta < d$. Take $E = B_\delta(a)$.

(0) f is strictly monotone on E (by subset monotone lemma). Indeed E is a subset of N. Indeed $E = B_\delta(a) \subseteq B_d(a) \subseteq N$ (by ball subsets).

$f \restriction E$ is injective (by lemma mon 1, 0). Indeed E is a subset of $\mathbb{R}$. Thus there exists $(f \restriction E)^{-1}$.

Take $g = f \restriction E$. g is a real map on E. Take $D = \mathrm{Range}(g)$. D is an open neighbourhood of $f(a)$ (by lemma mon 2, 0). Indeed f is continuous and $D = f[E]$ and E is an open neighbourhood of a. g^{-1} is a real map on D.

(3) Take a difference quotient map φ of f in a. For all $y \in D$ there exists an $x \in E$ such that $x = g^{-1}(y)$.

Define

$$ext(y) = \begin{cases} g^{-1}(y) & : y \in D \\ a & : y \notin D \end{cases}$$

for $y \in \mathbb{R}$. ext is a real map extension of g^{-1}. Indeed $ext(y)$ is a real number for any real number y. Indeed a is a real number and $g^{-1}(y)$ is a real number for every $y \in D$.

For all $y \in D$ $\varphi(g^{-1}(y)) \neq 0$. Proof. $\mathrm{Range}(g^{-1}) \subseteq E$. Indeed $E = \mathrm{dom}(g)$. We can show that for all $x \in E$ $\varphi(x) \neq 0$. Let $x \in E$.

Case $x = a$. Trivial.

Case $x \neq a$. $\varphi(x) = \frac{\Delta f}{\Delta a}(x)$ (by prop difference quotient map, 3). Indeed f is differentiable in a and $x \in \mathbb{R} \setminus \{a\}$. Therefore $x \neq a$ and $x - a$ is nonzero. $\frac{\Delta f}{\Delta a}(x) = \frac{f(x)-f(a)}{x-a}$ (by def diffquot). (4) $f(x) - f(a)$ is nonzero. Indeed $x \neq a$ and $x, a \in E$ and $f(x) \neq f(a)$. $\frac{f(x)-f(a)}{x-a} \neq 0$ (by frac lemma 5,4). Indeed $x - a$ is nonzero. End. End. End.

Define

$$\psi(y) = \begin{cases} 1/\varphi(g^{-1}(y)) & : y \in D \\ 0 & : y \notin D \end{cases}$$

for $y \in \mathbb{R}$. ψ is a real map. Indeed $1/\varphi(g^{-1}(y))$ is a real number for any $y \in D$.

Let us show that ψ is a difference quotient map of ext in $f(a)$.

Let us show that for every real number y $ext(y) = ((y - f(a)) \cdot \psi(y)) + ext(f(a))$. Let y be a real number.

(A) $ext(f(a)) = a$. $((y - f(a)) \cdot \psi(y)) + ext(f(a)). = ((y - f(a)) \cdot \psi(y)) + a$ (by A).

Case $y = f(a)$. $((f(a) - f(a)) \cdot \psi(y)) + a. = a$ (by additive inverse, eq lemma 9, add comm, existence zero) $. = ext(f(a))$ (by A). End.

Case $y \in D$ and $y \neq f(a)$.

(B) $\psi(y) = \frac{g^{-1}(y)-a}{y-f(a)}$. Proof. $\varphi(g^{-1}(y)) = \frac{\Delta f}{\Delta a}(g^{-1}(y))$ (by prop difference quotient map,3). Indeed f is differentiable in $g^{-1}(y)$ and $g^{-1}(y) \in \mathbb{R} \setminus \{a\}$. $g^{-1}(y) \neq a$ and $g^{-1}(y) - a$ is nonzero. $\frac{\Delta f}{\Delta a}(g^{-1}(y)). = \frac{f(g^{-1}(y))-f(a)}{g^{-1}(y)-a}$ (by def diffquot). $f(g^{-1}(y)) - f(a) \neq 0$. $1/\frac{f(g^{-1}(y))-f(a)}{g^{-1}(y)-a}. = \frac{g^{-1}(y)-a}{f(g^{-1}(y))-f(a)}$ (by eq lemma 22). $\psi(y) = 1/\varphi(g^{-1}(y)) = \frac{g^{-1}(y)-a}{y-f(a)}$. End.

$((y - f(a)) \cdot \psi(y)) + a. = ((y - f(a)) \cdot \frac{g^{-1}(y)-a}{y-f(a)}) + a$ (by B) $. = (g^{-1}(y) - a) + a$ (by fraction lemma 2) $. = g^{-1}(y)$ (by add ass, add comm, additive inverse, existence zero).

$ext(y) = g^{-1}(y)$. End.

Case $y \notin D$. (C) $\psi(y) = 0$. $((y - f(a)) \cdot \psi(y)) + a. = a$ (by mult comm, C, eq lemma 9, add

comm, existence zero). $ext(y) = a$. End. End.

Let us show that ψ is continuous in $f(a)$.

(4) E is an open neighbourhood of a. Proof. a is an element of E. E is an open set (by open interval is open). Indeed $E = (a - \delta, a + \delta)_{OI}$ (by ball and open interval). End.

g^{-1} is locally continuous in $f(a)$ (by lemma mon 3, 4, 0). Indeed f is continuous. ext is continuous in $f(a)$. φ is continuous in a. [timelimit 50] Thus $\varphi \circ ext$ is continuous in $f(a)$ (by continuity of composition). Indeed $g^{-1}(f(a)) = a$. ψ is continuous in $f(a)$ (by continuity division map). Proof. $\varphi \circ ext$ is a real map. ψ is a division map of $\varphi \circ ext$ in $f(a)$ (by 4). End. End. End.

Therefore ext is differentiable in $f(a)$. $\mathrm{dom}(g^{-1}) = D$ and D is an open neighbourhood of $f(a)$. g^{-1} is a real map on D. Indeed $\mathrm{Range}(g) = D$ and for every $y \in D$ $g^{-1}(y)$ is a real number. $\partial_\ell g^{-1}(f(a)) = \partial ext(f(a))$ (by open set differentiation extension). Indeed ext is a real map extension of g^{-1} such that ext is differentiable in $f(a)$. [timelimit 3] $\partial ext(f(a)) = \psi(f(a)) = 1/\varphi(g^{-1}(f(a))) = 1/\varphi(a) = 1/\partial f(a)$. $(f \restriction E)^{-1}$ is locally differentiable in $f(a)$. $\partial_\ell (f \restriction E)^{-1}(f(a)) = 1/\partial f(a)$. $f \restriction E$ is injective. $(f \restriction E)^{-1}$ is a real map on $f[E]$. Indeed $\mathrm{Range}(f \restriction E) = f[E]$. □

[timelimit 3]

Definition 239. A locally inverse function to f in a is a real map g such that for some open neighbourhood E of a $f \restriction E$ is injective and $g \restriction f[E] = (f \restriction E)^{-1}$.

Corollary 240 (inverse function theorem II). Let f be continuously differentiable and $\partial f(a) \neq 0$. Let g be a locally inverse function to f in a. Then g is differentiable in $f(a)$ and $\partial g(f(a)) = 1/\partial f(a)$.

Proof. Take an open neighbourhood M of a such that $f \restriction M$ is injective and $g \restriction f[M] = (f \restriction M)^{-1}$. [timelimit 100] Take an open neighbourhood N of a such that f is strictly monotone on N and $(f \restriction N)^{-1}$ is a real map on $f[N]$ and $(f \restriction N)^{-1}$ is locally differentiable in $f(a)$ and $\partial_\ell (f \restriction N)^{-1}(f(a)) = 1/\partial f(a)$ (by inverse function theorem). Take $E = N \cap M$.

E is an open neighbourhood of a. Indeed a is an element of E and E is an open set (by intersection of open sets).

$f \restriction E$ is injective. $(f \restriction E)^{-1}$ is a real map on $f[E]$. Indeed $\mathrm{dom}((f \restriction E)^{-1}) = \mathrm{Range}(f \restriction E) = f[E]$ and for every $y \in f[E]$ $(f \restriction E)^{-1}(y)$ is a real number. Define

$$ext(y) = \begin{cases} (f \restriction N)^{-1}(y) & : y \in f[N] \\ 0 & : y \notin f[N] \end{cases}$$

for $y \in \mathbb{R}$. ext is a real map. Indeed $\mathrm{Range}(ext)$ is a subset of $\mathbb{R}$. Indeed for any $y \in \mathrm{dom}(ext)$ $ext(y)$ is a real number. Indeed 0 is a real number and $(f \restriction N)^{-1}(y)$ is a real number for any $y \in f[N]$. $(f \restriction N)^{-1}$ is a real map on $f[N]$.

$f[N]$ is a subset of $\mathbb{R}$. Take $\gamma = (f \restriction N)^{-1}$. γ is a real map on $f[N]$. $f(a)$ is an element of $f[N]$. γ is locally differentiable in $f(a)$. [prover vampire] $\partial_\ell \gamma(f(a)) = \partial ext(f(a))$ (by lemma differentiation extension). Indeed ext is a real map extension of γ. [prover eprover] Thus $\partial_\ell (f \restriction N)^{-1}(f(a)) = \partial ext(f(a))$.

Take $h = (f \restriction E)^{-1}$. h is locally differentiable in $f(a)$ and $\partial ext(f(a)) = \partial_\ell h(f(a))$ (by open set

differentiation extension). Proof. h is a real map on $f[E]$. $f[E]$ is an open neighbourhood of $f(a)$ (by lemma mon 2, 1). Proof. f is strictly monotone on E (by subset monotone lemma). Indeed E is a subset of N. f is continuous. End.

ext is a real map extension of h such that ext is differentiable in $f(a)$. Indeed we can show that for every $y \in f[E]$ $(f \restriction E)^{-1}(y) = ext(y)$. Let $y \in f[E]$. Then $y \in f[N]$. $h(y) = (f \restriction N)^{-1}(y)$. $(f \restriction N)^{-1}(y) = ext(y)$. End. End.

$\partial_\ell (f \restriction E)^{-1}(f(a)) = \partial g(f(a))$ (by lemma differentiation extension). Proof. $f[E]$ is a subset of $\mathbb{R}$. $(f \restriction E)^{-1}$ is a real map on $f[E]$. $f(a)$ is an element of $f[E]$. $(f \restriction E)^{-1}$ is locally differentiable in $f(a)$. g is a real map extension of $(f \restriction E)^{-1}$. Proof. $f \restriction M$ is injective. $(f \restriction M)^{-1}$ is a real map on $f[M]$. $f[E] \subseteq f[M]$. For every $y \in f[E]$ $(f \restriction E)^{-1}(y) = g(y)$. Proof. Assume the contrary. Take $y \in f[E]$ such that $(f \restriction E)^{-1}(y) \neq g(y)$. Take $x = (f \restriction E)^{-1}(y)$. Then $(f \restriction E)(x) = y$. Thus $f(x) = y$. $x \in M$. Indeed $E \subseteq M$. Hence $(f \restriction M)(x) = y$. Thus $(f \restriction M)^{-1}(y) = x$. Indeed $y \in f[M]$. $(f \restriction E)^{-1}(y) = (f \restriction M)^{-1}(y)$. $(f \restriction M)^{-1}(y) \neq g(y)$. Contradiction. End. End. End. □

[timelimit 3]

Chapter 8

Exponential Function and Logarithm

[read `examples/solow/sections/6inverse.ftl.tex`]

Signature 241. $\mathbb{R}_{>0}$ is the collection of positive real numbers.

Lemma 242. $\mathbb{R}_{>0}$ is an open set.

Proof. Let $x \in \mathbb{R}_{>0}$. Then $B_x(x) \subseteq \mathbb{R}_{>0}$. Indeed $B_x(x) = (x-x, x+x)_{OI} = (0, x+x)_{OI} \subseteq \mathbb{R}_{>0}$. □

Lemma 243 (positive reals open neighbourhood). Let a be a positive real number. $\mathbb{R}_{>0}$ is an open neighbourhood of a.

8.1 e and Logarithm

Let a, b, x, y denote real numbers.

Signature 244 (natural exponential function). exp is a real map such that $\mathrm{D}\exp = \exp$.

Axiom 245 (exp of 0). $\exp(x) = 1$ iff $x = 0$.

Axiom 246 (exp calculation axiom). $\exp(a + b) = \exp(a) \cdot \exp(b)$.

Corollary 247. exp is continuously differentiable.

Lemma 248 (lemma exp nonzero). $\exp(x) \neq 0$.

Proof by contradiction. Assume the contrary. Take a real number x such that $\exp(x) = 0$. $0 = 0 \cdot \exp(-x) = \exp(x) \cdot \exp(-x) = \exp(x - x) = \exp(0) = 1$. □

Lemma 249 (lemma exp positive). $\exp(x)$ is a positive real number.

Proof. Assume $\exp(x) < 0$.

(1) Case $x = 0$. Trivial.

(2) Case $x < 0$. Then $\exp(x) < 0 < \exp(0)$ and exp is continuous. [timelimit 10]

There exists $c \in (x, 0)_{OI}$ such that $\exp(c) = 0$ (by Bolzanos Theorem).

Contradiction (by lemma exp nonzero). [timelimit 3] End.

(3) Case $x > 0$. Then $(-\exp)(0) < 0 < (-\exp)(x)$ and $(-\exp)$ is continuous. [timelimit 10] There exists $c \in (0, x)_{OI}$ such that $(-\exp)(c) = 0$ (by Bolzanos Theorem). [timelimit 3] Contradiction (by lemma exp nonzero). End. □

Proposition 250 (exp inequality). $1 + x \leq \exp(x)$.

Proof. Take $f = (\mathrm{f}_{1\cdot_+1} + (-\exp))$.

0 is a maximum of f (by sufficient condition for maximum on interval). Proof. $\mathbb{R}$ is an interval. (A) f is differentiable and $\mathrm{D}f = (\mathrm{f}_{0\cdot_+1} + (-\exp))$

(by minus derivative 2, derivative addition rule 2, natural exponential function, derivative of linear function).

Let us show that for every real number y $\partial\mathrm{D}f(y) < 0$. Let y be a real number.

$\partial\mathrm{D}f(y). = \mathrm{D}(\mathrm{f}_{0\cdot_+1} + (-\exp))(y)$ (by deriv and derivin, A)

$. = (\mathrm{Df}_{0\cdot_+1} + \mathrm{D}(-\exp))(y)$ (by derivative addition rule 2)

$. = (\mathrm{f}_{0\cdot_+0} + (-\mathrm{D}\exp))(y)$ (by minus derivative 2, derivative of linear function)

$. = (\mathrm{f}_{0\cdot_+0} + (-\exp))(y)$ (by natural exponential function)

$. = ((0 \cdot y) + 0) - \exp(y)$ (by def add func, linear function, def func minus)

$. = -\exp(y) + 0$ (by eq lemma 9, existence zero, add comm)

$. = -\exp(y)$ (by existence zero).

$-\exp(y) < 0$ (by ineq lemma 2, lemma exp positive).

Therefore $\partial\mathrm{D}f(y) < 0$. End.

0 is a local maximum of f (by sufficient condition local maximum). Proof.

(B) $\exp(0) = 1$.

$\partial f(0). = (\mathrm{f}_{0\cdot_+1} + (-\exp))(0)$ (by A, deriv and derivin)

$. = \mathrm{f}_{0\cdot_+1}(0) - \exp(0)$ (by def add func, def func minus)

$. = 1 - 1$ (by linear lem 1, B)

$. = 0$ (by additive inverse). $\partial\mathrm{D}f(0) < 0$. End.

f is strictly convex on $\mathbb{R}$. End.

Thus $1 + x \leq \exp(x)$. Proof. $f(x) \leq f(0)$.

$(1 + x) - \exp(x) = ((1 \cdot x) + 1) - \exp(x) = \mathrm{f}_{1\cdot_+1}(x) - \exp(x) = \mathrm{f}_{1\cdot_+1}(x) + (-\exp)(x) = (\mathrm{f}_{1\cdot_+1} + (-\exp))(x) = f(x)$

$\leq f(0) = (\mathrm{f}_{1\cdot_+1} + (-\exp))(0) = ((1 \cdot 0) + 1) - \exp(0) = 1 - \exp(0) = 0.$

Thus $(1 + x) - \exp(x) \leq 0$. Therefore $(1 + x) \leq \exp(x)$ (by ineq lemma 27). End. □

Proposition 251 (range exp). $\mathrm{Range}(\exp) = \mathbb{R}_{>0}$.

Proof. $\mathrm{Range}(\exp) \subseteq \mathbb{R}_{>0}$. Proof. Let $y \in \mathrm{Range}(\exp)$. Then y is nonzero (by lemma exp nonzero). Take a real number x such that $\exp(x) = y$. Then $\exp(x)$ is positive (by lemma exp positive). qed.

$\mathbb{R}_{>0} \subseteq$ Range(exp). Proof. Let $y \in \mathbb{R}_{>0}$.

(1) Case $y = 1$. Trivial.

(2) Case $y > 1$. Then $y < 1 + y \leq \exp(y)$. $y \in (\exp(0), \exp(y))_{OI}$. [timelimit 20] Thus $y \in \exp[(0, y)_{OI}]$. Indeed $(\exp(0), \exp(y))_{OI} \subseteq \exp[(0, y)_{OI}]$ (by intermediate value theorem I). Indeed $0 < y$ and exp is continuous. [timelimit 3] Therefore there exists $x \in (0, y)_{OI}$ such that $\exp(x) = y$. End.

(3) Case $y < 1$. Take $\upsilon = 1/y$. Then $\upsilon = 1/y > 1/1 = 1$. Then $\upsilon < 1 + \upsilon \leq \exp(\upsilon)$. $\upsilon \in (\exp(0), \exp(\upsilon))_{OI}$. [timelimit 20] Thus $\upsilon \in \exp[(0, \upsilon)_{OI}]$. Indeed $(\exp(0), \exp(\upsilon))_{OI} \subseteq \exp[(0, \upsilon)_{OI}]$ (by intermediate value theorem I). Indeed $0 < \upsilon$ and exp is continuous. [timelimit 3] Take $x \in (0, \upsilon)_{OI}$ such that $\exp(x) = \upsilon$. Then $-x \in \text{dom}(\exp)$. $\exp(-x) = 1/\exp(x) = 1/\upsilon = y$. End. qed. □

Lemma 252 (lemma 1). $\exp(-a) = 1/\exp(a)$.

Proof. $\exp(a) \cdot \exp(-a) = \exp(a - a) = \exp(0) = 1$. □

Proposition 253 (exp is injective). exp is injective.

Proof by contradiction. Assume exp is not injective.

Take distinct real numbers x, y such that $\exp(x) = \exp(y)$. $1 = \exp(x) \cdot \exp(-x) = \exp(y) \cdot \exp(-x) = \exp(y - x)$. Thus $y - x = 0$. $x = y$. Contradiction. □

Definition 254 (log). $\ln = \exp^{-1}$.

Lemma 255 (domain log). $\text{dom}(\ln) = \mathbb{R}_{>0}$.

Corollary 256. For every positive real number y there exists a real number x such that $\exp(x) = y$.

Proof. Let y be a positive real number. Take $x = \ln(y)$. x is a real number and $\exp(x) = y$. □

Lemma 257. Let x, y be positive. Then $x \cdot y$ is a positive real number and $\ln(x) + \ln(y) = \ln(x \cdot y)$.

Proof. Take a real number a such that $x = \exp(a)$. Take a real number b such that $y = \exp(b)$. $\exp(a) \cdot \exp(b)$ is a positive real number. $\ln(x) + \ln(y) = \ln(\exp(a)) + \ln(\exp(b)) = a + b = \ln(\exp(a + b)) = \ln(\exp(a) \cdot \exp(b)) = \ln(x \cdot y)$. □

Lemma 258. Let x, y be positive. Then $\frac{x}{y}$ is a positive real number and $\ln(x) - \ln(y) = \ln\left(\frac{x}{y}\right)$.

Proof. $\frac{x}{y}$ is a positive real number. Take a real number a such that $x = \exp(a)$. Take a real number b such that $y = \exp(b)$. $\ln(x) - \ln(y) = \ln(\exp(a)) - \ln(\exp(b)) = a - b = \ln(\exp(a - b)) = \ln(\exp(a) \cdot \exp(-b)) = \ln(\exp(a) \cdot (1/\exp(b))) = \ln(x \cdot (1/y)) = \ln\left(\frac{x}{y}\right)$. □

8.2 Exponential and Power Function

Let a, x, y denote real numbers.

Definition 259 (exponential function). Let a be positive. $\exp_a = \exp \circ \mathrm{f}_{\ln(a)\cdot_+0}$.

Lemma 260 (evaluated exponential function). Let a be positive. $\exp_a(x) = \exp(\ln(a) \cdot x)$.

Proof. $\exp_a(x). = (\exp \circ \mathrm{f}_{\ln(a)\cdot_+0})(x)$

$. = \exp(\mathrm{f}_{\ln(a)\cdot_+0}(x))$ (by composition)

$. = \exp((\ln(a) \cdot x) + 0)$ (by linear function)

$. = \exp(\ln(a) \cdot x)$ (by existence zero). □

Lemma 261 (lemma exponential function). $\exp_{\exp(x)}(y) = \exp(x \cdot y)$.

Proof. $\exp_{\exp(x)}(y). = \exp(\ln(\exp(x)) \cdot y)$ (by evaluated exponential function)

$. = \exp(x \cdot y)$ (by log, lemma inverse). □

Lemma 262 (lemma 2 exponential function). Let a be positive. $\exp_{\exp_a(x)}(y) = \exp_a(x \cdot y)$.

Proof. $\exp_{\exp_a(x)}(y). = \exp_{\exp(\ln(a)\cdot x)}(y)$ (by evaluated exponential function)

$. = \exp((\ln(a) \cdot x) \cdot y)$ (by lemma exponential function)

$. = \exp_a(x \cdot y)$ (by mult ass, evaluated exponential function). □

Lemma 263 (lemma 3). Let a be a positive real number. $\exp_a(x) \cdot \exp_a(y) = \exp_a(x+y)$.

Proof. $\exp_a(x) \cdot \exp_a(y)$

$. = \exp(\ln(a) \cdot x) \cdot \exp(\ln(a) \cdot y)$ (by evaluated exponential function)

$. = \exp((\ln(a) \cdot x) + (\ln(a) \cdot y))$ (by exp calculation axiom)

$. = \exp(\ln(a) \cdot (x + y))$ (by distributive law)

$. = \exp_a(x + y)$ (by evaluated exponential function). □

Lemma 264 (lemma 4). Let x be a positive real number. $1/x = \exp_x(-1)$.

Proof. $\ln(x) \cdot (-1), \exp_x(-1)$ are a real numbers.

$x \cdot \exp_x(-1)$

$. = x \cdot \exp(\ln(x) \cdot (-1))$ (by evaluated exponential function)

$. = x \cdot \exp(-\ln(x))$ (by mult comm, eq lemma 12)

$. = x \cdot 1/\exp(\ln(x))$ (by lemma 1)

$. = x \cdot (1/x)$ (by log, inverse)

$. = 1.$ □

Definition 265 (power function). $\mathrm{pow}_a = ((\exp \circ \mathrm{f}_{a\cdot_+0}) \circ \ln) \restriction \mathrm{dom}(\ln)$.

Lemma 266. pow_a is a real map on $\mathbb{R}_{>0}$.

Proof. $\mathrm{dom}(\mathrm{pow}_a) = \mathrm{dom}(\ln) = \mathbb{R}_{>0}$.

We can show that $\mathrm{pow}_a(x)$ is a real number for all $x \in \mathbb{R}_{>0}$. Let $x \in \mathbb{R}_{>0}$. $\mathrm{pow}_a = ((\exp \circ \mathrm{f}_{a\cdot_+0}) \circ \ln) \restriction \mathrm{dom}(\ln)$. $(((\exp \circ \mathrm{f}_{a\cdot_+0}) \circ \ln) \restriction \mathrm{dom}(\ln))(x) = ((\exp \circ \mathrm{f}_{a\cdot_+0}) \circ \ln)(x)$. $((\exp \circ \mathrm{f}_{a\cdot_+0}) \circ \ln)(x) = \exp(\mathrm{f}_{a\cdot_+0}(\ln(x)))$ (by composition). $\ln(x)$, $\mathrm{f}_{a\cdot_+0}(\ln(x))$, $\exp(\mathrm{f}_{a\cdot_+0}(\ln(x)))$ are real numbers. End.

Thus $\mathrm{Range}(\mathrm{pow}_a) \subseteq \mathbb{R}$. □

Lemma 267 (evaluated power function). Let b be a positive real number. $\mathrm{pow}_a(b) = \exp(a \cdot \ln(b))$.

Proof. $\mathrm{dom}(\mathrm{pow}_a) = \mathrm{dom}(\ln)$ and $b \in \mathrm{dom}(\ln)$.

$\mathrm{pow}_a(b) = ((\exp \circ \mathrm{f}_{a\cdot_+0}) \circ \ln)(b) = \exp(\mathrm{f}_{a\cdot_+0}(\ln(b))) = \exp(a \cdot \ln(b))$. □

Proposition 268 (equivalence of exponential and power function). Let b be a positive real number. $\mathrm{pow}_a(b) = \exp_b(a)$.

Proof. $\mathrm{pow}_a(b). = \exp(a \cdot \ln(b))$ (by evaluated power function)

$. = \exp(\ln(b) \cdot a)$ (by mult comm)

$. = \exp_b(a)$ (by evaluated exponential function). □

In `100_theorems.ftl.tex` x^n was already defined recursively as $x^{n-1} \cdot x$ with $x^0 = 1$ for any natural number n. Here we just want to show that our concept of exponentiation is merely an extension thereof.

Theorem 269 (equivalence exponential function and exponent). Let x be a positive real number. For every natural numbers n $\exp_x(n) = x^n$.

Proof by induction on n. Let n be a natural number.

Case $n = 0$. $\exp_x(0) = \exp(\ln(x) \cdot 0) = \exp(0) = 1 = x^0$. End.

Case $n = 1$. $\exp_x(1) = \exp(\ln(x) \cdot 1) = \exp(\ln(x)) = x = x^1$. End.

Case $n > 1$. Take a natural number m such that $m = n - 1$.m is inductively smaller than n. Thus $\exp_x(m) = x^m$ and $\exp_x(1) = x^1$. $\exp_x(m)$, $\exp_x(1)$ are real numbers. $\exp_x(m+1). = \exp_x(m) \cdot \exp_x(1)$ (by lemma 3).

$\exp_x(n) = \exp_x(m+1)$

$= \exp_x(m) \cdot \exp_x(1)$

$= x^m \cdot \exp_x(1)$

$= x^m \cdot x^1 = x^{m+1} = x^n$. End.[timelimit 3] □

Corollary 270 (equivalence exponential function and exponent 2). Let x be a positive real number and n be a natural number. x^n is positive and $1/x^n = \exp_x(-n)$.

Proof. x^n is positive. Indeed $x^n = \exp_x(n)$ and $\exp_x(n)$ is positive. $\exp_x(-n). = \exp_{\exp_x(n)}(-1)$ (by eq lemma 12, mult comm, lemma 2 exponential function) $. = 1/x^n$ (by equivalence exponential function and exponent, lemma 4). □

Corollary 271. Let n be a natural number and x be a positive real number. $\mathrm{pow}_n(x) = x^n$.

Proof. $\mathrm{pow}_n(x). = \exp_x(n)$ (by equivalence of exponential and power function)

$. = x^n$ (by equivalence exponential function and exponent). □

Lemma 272. Let x be a real number and a be a positive real number. $\exp_a(x)$ is a positive real number and $x \cdot \ln(a) = \ln(\exp_a(x))$.

Proof. Take a real number α such that $a = \exp(\alpha)$. $x \cdot \ln(a) = x \cdot \ln(\exp(\alpha)) = x \cdot \alpha = \ln(\exp(x \cdot \alpha)) = \ln(\exp(\ln(a) \cdot x)) = \ln(\exp(\mathrm{f}_{\ln(a)\cdot_+0}(x))) = \ln(\exp_a(x))$. □

8.3 Derivative Theorems

Let a denote a real number. Let f denote a real map.

Theorem 273 (derivative log). Let x be a positive real number. ln is locally differentiable in x and $\partial_\ell \ln(x) = 1/x$.

Proof. Let us show that for any real map extension ext of ln ext is differentiable in x and $\partial ext(x) = 1/x$. Let ext be a real map extension of ln. Take $\xi = \ln(x)$.

We can show that ext is a locally inverse function to exp in ξ. $\mathbb{R}$ is an open neighbourhood of x (by R is open). $\exp \upharpoonright \mathbb{R}$ is injective (by exp is injective).

$ext \upharpoonright \exp[\mathbb{R}] = (\exp \upharpoonright \mathbb{R})^{-1}$. Proof.

$ext \upharpoonright \exp[\mathbb{R}] = ext \upharpoonright \mathrm{dom}(\ln)$. Indeed $\exp[\mathbb{R}] = \mathrm{dom}(\ln)$. [timelimit 20] $ext \upharpoonright \mathrm{dom}(\ln) = \ln$ (by function lemma 2). Proof. $ext \upharpoonright \mathrm{dom}(\ln), \ln$ are maps such that $\mathrm{dom}(ext \upharpoonright \mathrm{dom}(\ln)) = \mathrm{dom}(\ln)$. $(ext \upharpoonright (\exp[\mathbb{R}]))(y) = \ln(y)$ for all $y \in \mathrm{dom}(\ln)$. End. [timelimit 3]

$\ln = \exp^{-1} = (\exp \upharpoonright \mathbb{R})^{-1}$. Indeed $\exp = (\exp \upharpoonright \mathbb{R})$. End. End.

exp is continuously differentiable and $\partial \exp(\xi) \neq 0$. $\partial ext(\exp(\xi)) = 1/\partial \exp(\xi)$ (by inverse function theorem II). Therefore $\partial ext(x) = \partial ext(\exp(\xi)) = 1/\partial \exp(\xi) = 1/\exp(\xi) = 1/x$. End.

Thus $\partial_\ell \ln(x) = \partial ext|_{\ln}(x) = 1/x$. □

Theorem 274 (derivative exponential function). Let a be a positive real number. $\exp_a$ is differentiable and $\mathrm{D}\exp_a = (\mathrm{f}_{0\cdot_+\ln(a)} \cdot \exp_a)$.

Proof. $\mathrm{D}\exp_a . = \mathrm{D}\exp \circ \mathrm{f}_{\ln(a)\cdot_+0}$ (by exponential function)

$. = (\mathrm{D}\exp \circ \mathrm{f}_{\ln(a)\cdot_+0} \cdot \mathrm{Df}_{\ln(a)\cdot_+0})$ (by derivative chain rule 2)

$. = (\exp \circ \mathrm{f}_{\ln(a)\cdot_+0} \cdot \mathrm{f}_{0\cdot_+\ln(a)})$ (by natural exponential function, derivative of linear function)

$. = (\exp_a \cdot \mathrm{f}_{0\cdot_+\ln(a)})$ (by exponential function)

$. = (\mathrm{f}_{0\cdot_+\ln(a)} \cdot \exp_a)$ (by lemma 2). □

Theorem 275 (derivative power function). Let x be a positive real number. pow_a is locally differentiable in x and $\partial_\ell \mathrm{pow}_a(x) = a \cdot \mathrm{pow}_{a-1}(x)$.

Proof. (0) Take $h = (\mathrm{f}_{a\cdot_+0} \circ ext|_{\ln})$. Then h is a real map and differentiable in x.[timelimit 10] $\exp \circ h$ is a real map and differentiable in x.

(A) $\exp \circ h$ is a real map extension of pow_a. Proof.

Let $y \in \mathrm{dom}(\ln)$. (A1) $ext|_{\ln}(y) = \ln(y)$.

(A2) Take $\eta = (\exp \circ \mathrm{f}_{a\cdot_+0})$. η is a real map.

$\eta \circ \ln$ is a real map on $\mathrm{dom}(\ln)$. Indeed $\mathrm{dom}(\eta \circ \ln) = \mathrm{dom}(\ln)$.

$(\exp \circ (\mathrm{f}_{a\cdot_+0} \circ ext|_{\ln}))(y). = \exp(\mathrm{f}_{a\cdot_+0}(ext|_{\ln}(y)))$ (by composition)

$. = \exp(\mathrm{f}_{a\cdot_+0}(\ln(y)))$ (by A1)

$. = ((\exp \circ \mathrm{f}_{a\cdot_+0}) \circ \ln)(y)$ (by composition)

$. = (\eta \circ \ln)(y)$ (by A2).

Take $\mu = \eta \circ \ln$. Therefore $\mu(y) = (\mu \upharpoonright \mathrm{dom}(\ln))(y)$. Then $(\eta \circ \ln)(y) = ((\eta \circ \ln) \upharpoonright \mathrm{dom}(\ln))(y).((\eta \circ \ln) \upharpoonright \mathrm{dom}(\ln))(y) = \mathrm{pow}_a(y)$. End.

(B) $\partial ext|_{\ln}(x) = \partial_\ell \ln(x) = 1/x$.

(C) $\exp(h(x)) = \mathrm{pow}_a(x)$. Indeed then $(\exp \circ h)(x) = \mathrm{pow}_a(x)$. [timelimit 3]

$\partial \exp \circ h(x) . = \partial \exp(h(x)) \cdot \partial h(x)$ (by derivative chain rule) $. = \partial \exp(h(x)) \cdot \partial(\mathrm{f}_{a\cdot_+0} \circ ext|_{\ln})(x)$ (by 0)

$. = \exp(h(x)) \cdot (\partial \mathrm{f}_{a\cdot_+0}(ext|_{\ln}(x)) \cdot \partial ext|_{\ln}(x))$ (by deriv and derivin, natural exponential function, derivative chain rule)

$. = \mathrm{pow}_a(x) \cdot (\mathrm{f}_{0\cdot_+a}(ext|_{\ln}(x)) \cdot (1/x))$ (by C, deriv and derivin, derivative of linear function, B)

$. = (\exp_x(a) \cdot (1/x)) \cdot a$ (by linear lem 1, equivalence of exponential and power function, mult comm, mult ass)

$. = (\exp_x(a) \cdot \exp_x(-1)) \cdot a$ (by lemma 4)

$. = \exp_x(a - 1) \cdot a$ (by lemma 3)

$. = a \cdot \mathrm{pow}_{(a-1)}(x)$ (by mult comm, equivalence of exponential and power function).

(D) $\mathbb{R}_{>0}$ is an open neighbourhood of x and pow_a is a real map on $\mathbb{R}_{>0}$.

[timelimit 10] Therefore $\partial_\ell \mathrm{pow}_a(x) = \partial(\exp \circ h)(x)$ (by open set differentiation extension, A, D). Indeed h is differentiable in x. [timelimit 3] $\mathrm{pow}_{(a-1)}(x)$ is a real number. Therefore $\partial_\ell \mathrm{pow}_a(x) = \partial(\exp \circ h)(x) = a \cdot \mathrm{pow}_{(a-1)}(x)$. □

Lemma 276 (lemma derivative of division map). Let f be differentiable in a and $f(a)$ be positive. Let g be a division map of f in a. Then g is differentiable in a and $\partial g(a) = -\frac{\partial f(a)}{(f(a))^2}$.

Proof. Take an open neighbourhood N of a such that for any $x \in N$ $f(x)$ is nonzero and $g(x) = 1/f(x)$. Take a positive real number τ such that $B_\tau(a)$ is a subset of N. Take a positive real number θ such that θ is a domain restrictor for $f(a)$ and f around a and $f(a)$. Indeed f is continuous in a. Take a positive real number δ such that $\delta < \tau$ and $\delta < \theta$ (by ineq lemma 19). Take $E = B_\delta(a)$.

For every $x \in E$ $f(x)$ is nonzero and $g(x) = 1/f(x)$. Proof. Let $x \in E$. x is an element of N. Indeed $B_\delta(a) \subseteq B_\tau(a) \subseteq N$. End.

(1) For every $x \in E$ $f(x)$ is positive and $g(x) = 1/f(x)$. Proof. [prover vampire] Assume the contrary. Take $x \in E$ such that $f(x)$ is not positive.

(1A) $f(a), -f(x)$ are nonnegative real numbers. Indeed $f(x) = 0$ or $f(x)$ is negative. [prover eprover] Then $|f(x) - f(a)| = |f(a) - f(x)| = f(a) - f(x)$. Indeed $f(a) - f(x)$ is nonnegative

(by ineq lemma 21, 1A). $f(a) - f(x) \geq f(a)$ (by ineq lemma 23, 1A). [timelimit 20] θ is not a domain restrictor for $f(a)$ and f around a and $f(a)$ (by domain restrictor lemma 2). Indeed we can show that there exists $y \in \mathbb{R} \setminus \{a\}$ such that $|y - a| < \theta$ and $|f(y) - f(a)| \geq f(a)$. $|x - a| < \theta$. Indeed $|x - a| < \delta < \theta$. $|f(x) - f(a)| \geq f(a)$. $x \in \mathbb{R} \setminus \{a\}$. End. [timelimit 3] End.

Take $ext = ext|_{\mathrm{pow}_{(-1)}}$. $ext \circ f$ is a real map.

(2) Then $ext \circ f$ is a real map extension of $g \upharpoonright E$. Indeed we can show that $g(x) = (ext \circ f)(x)$ for any $x \in E$. Let $x \in E$. Then $(ext \circ f)(x) = ext(f(x)) = \mathrm{pow}_{(-1)}(f(x)) = 1/f(x) = g(x)$. Indeed $f(x) \in \mathrm{dom}(\mathrm{pow}_{(-1)})$ and $ext(f(x)) = \mathrm{pow}_{(-1)}(f(x))$. End.

(3) $\partial(ext \circ f)(a) = -\frac{\partial f(a)}{(f(a))^2}$. Proof.

(3A) $-1 - 1 = -2$.

(3B) $\mathrm{pow}_{-2}(f(a)). = 1/(f(a))^2$ (by equivalence exponential function and exponent 2). Indeed 2 is a natural number.

(3C) ext is differentiable in $f(a)$ and $\partial ext(f(a)) = \partial_\ell \mathrm{pow}_{(-1)}(f(a))$ (by open set differentiation extension, positive reals open neighbourhood).

$\partial(ext \circ f)(a). = \partial ext(f(a)) \cdot \partial f(a)$ (by derivative chain rule)

$. = \partial_\ell \mathrm{pow}_{(-1)}(f(a)) \cdot \partial f(a)$ (by 3C)

$. = ((-1) \cdot \mathrm{pow}_{-2}(f(a))) \cdot \partial f(a)$ (by derivative power function, 3A)

$. = -\frac{\partial f(a)}{(f(a))^2}$ (by 3B, mult ass, mult comm, eq lemma 12). End.

(4) $g \upharpoonright E$ is a real map on E. [timelimit 30]

(5) $g \upharpoonright E$ is locally differentiable in a and $\partial_\ell(g \upharpoonright E)(a) = \partial(ext \circ f)(a)$ (by open set differentiation extension, 2, 4). Proof. E is an open neighbourhood of a. $ext \circ f$ is a real map extension of $g \upharpoonright E$ such that $ext \circ f$ is differentiable in a. End.

(6) $\partial g(a) = \partial_\ell(g \upharpoonright E)(a)$ (by lemma differentiation extension, 4, 5). Proof. E is a subset of $\mathbb{R}$. a is an element of E. g is a real map extension of $g \upharpoonright E$. End. [timelimit 3]

Therefore $\partial g(a) = -\frac{\partial f(a)}{(f(a))^2}$ (by 3,5,6). □

[timelimit 20]

Theorem 277 (derivative of division map). Let f be differentiable in a. Let g be a division map of f in a. Then g is differentiable in a and $\partial g(a) = -\frac{\partial f(a)}{(f(a))^2}$.

Proof. Case $f(a)$ is positive. Trivial (by lemma derivative of division map). [prover vampire]

Case $f(a)$ is negative. Take $\varphi = (-f)$. Take $\gamma = (-g)$. φ is a real map and φ is differentiable in a and $\varphi(a)$ is positive. Indeed $-f(a)$ is positive. γ is a division map of φ in a. Proof. Take an open neighbourhood E of a such that for any $x \in E$ $f(x)$ is nonzero and $g(x) = 1/f(x)$. For all $x \in E$ x is a real number and $-g(x) = -1/f(x) = 1/(-f(x)) = 1/(-f)(x)$. Then E is an open neighbourhood of a such that for any $x \in E$ $(-f)(x)$ is nonzero and $(-g)(x) = 1/(-f)(x)$. End.[timelimit 3][prover eprover]

$\partial\gamma(a) = -\frac{\partial\varphi(a)}{(\varphi(a))^2}$ (by lemma derivative of division map).

(A) $((-f)(a))^2 . = (-f(a)) \cdot (-f(a))$ (by def func minus, potence lemma 2) $. = (f(a))^2$ (by eq lemma 13, potence lemma 2).

$-\frac{\partial(-f)(a)}{((-f)(a))^2} . = \frac{\partial f(a)}{((-f)(a))^2}$ (by minus derivative, eq lemma 11, eq lemma 45) $. = \frac{\partial f(a)}{(f(a))^2}$ (by A). $\partial g(a) = -\partial(-g)(a) = -\left(-\frac{\partial \varphi(a)}{(\varphi(a))^2}\right) = -\frac{\partial f(a)}{(f(a))^2}$. End.

$f(a)$ is positive or $f(a)$ is negative. Indeed $f(a)$ is nonzero. [prover eprover] □

Definition 278. A real map with positive range is a real map f such that $f(x)$ is positive for any real number x.

Let g denote a real map with positive range.

Definition 279 (def div map). (f/g) is a real map such that $(f/g)(x) = \frac{f(x)}{g(x)}$ for any real number x.

Theorem 280 (derivative of a fraction). Let f, g be differentiable in a.

Then (f/g) is differentiable in a and

$$\partial(f/g)(a) = \frac{(\partial f(a) \cdot g(a)) - (f(a) \cdot \partial g(a))}{(g(a))^2}.$$

Proof. (0) Define $h(x) = 1/g(x)$ for $x \in \mathbb{R}$. h is a real map.

(1) $(f/g) = (f \cdot h)$.

(2) $\partial h(a) = -\frac{\partial g(a)}{(g(a))^2}$ (by derivative of division map). Indeed h is a division map of g in a.

$\partial(f/g)(a). = \partial(f \cdot h)(a)$ (by 1)

$. = (\partial f(a) \cdot h(a)) + (f(a) \cdot \partial h(a))$ (by derivative multiplication rule)

$. = \frac{\partial f(a)}{g(a)} + \left(-f(a) \cdot \frac{\partial g(a)}{(g(a))^2}\right)$ (by 0,2, eq lemma 21)

$. = \frac{\partial f(a) \cdot g(a)}{(g(a))^2} + \frac{-f(a) \cdot \partial g(a)}{(g(a))^2}$ (by fraction lemma, fraction lemma 2, potence lemma 2, mult ass, mult comm)

$. = \frac{(\partial f(a) \cdot g(a)) - (f(a) \cdot \partial g(a))}{(g(a))^2}$ (by eq lemma 1, eq lemma 11). □

Corollary 281 (derivative of a fraction 2). Let f be differentiable and g be differentiable. Then (f/g) is differentiable and $\mathrm{D}(f/g) = (((\mathrm{D}f \cdot g) + (-(f \cdot \mathrm{D}g)))/(g \cdot g))$.

Proof. Let a be a real number.

$\mathrm{D}(f/g)(a). = \partial(f/g)(a)$ (by deriv and derivin)

$. = \frac{(\partial f(a) \cdot g(a)) - (f(a) \cdot \partial g(a))}{(g(a))^2}$ (by derivative of a fraction)

$. = (((\mathrm{D}f \cdot g) + (-(f \cdot \mathrm{D}g)))/(g \cdot g))(a)$ (by deriv and derivin, potence lemma 2, def div map, def add func, def mult func, def func minus). □

Chapter 9

Solow Model

The Solow growth model was introduced by Robert M. Solow in 1956 and sought to explain several consistent patterns in economic data, which seem to hold in the long term and which later in 1961 Nicholas Kaldor summarized in 6 "stylized" facts. These 6 are therefore called Kaldor's facts.

G1 Continued growth in the aggregate volume of production, i.e. in the long term economies seem to grow.

G2 A continued increase in the amount of capital per worker, meaning that each employee manages more and more capital.

G3 A steady rate of profit on capital, i.e. when you invest money, the average return you get on it when discounting inflation, seems to be quite constant over time.

G4 Steady capital-output ratios over long periods, which implies the near-identity in the percentage rates of growth of production and of the capital stock.

G5 A steady share of profits (and of wages) in societies and periods in which the share of investment in output is constant.

G6 There are appreciable differences in the rate of growth of labour productivity and of total output in different societies.

9.1 The Model

The model consists of output (Y_t), capital (K_t), labor (L_t), technology (or efficiency A_t), a savings rate s, and a depreciation rate δ. Later we will also introduce wages w_t and interest rates on capital r_t and a production function f that converts capital, labor, and technology into output.

In economics, time-dependent variables are typically denoted as sequences by x_t, instead of $x(t)$, which is used for functions. This saves some brackets and makes many equations much clearer. Unfortunately, we cannot follow this notational change here because sequences are already introduced in `100_theorems.ftl.tex` with their typical notation. Thus, denoting functions with a domain on $\mathbb{R}$ in the same way as sequences will run into parsing problems. In addition, Naproche does not have any notational rules regarding parentheses implemented. Therefore, $a \cdot b + c$ will return an error and must be rewritten as $(a \cdot b) + c$. This makes the whole formalization a bit heavy on parentheses. Despite that, one aim of this text is to show that formalizations can be done visually quite clearly in Naproche.

[read `examples/solow/sections/7exponential.ftl.tex`]

Let a point in time stand for a real number. Let t denote a point in time.

Let $\hat{f}$ stand for $(\mathrm{D}f/f)$.

Signature 282. An output level is a differentiable real map with positive range.

Let Y denote an output level.

Signature 283. A capital stock is a differentiable real map with positive range.

Let K denote a capital stock.

Signature 284. A labor amount is a differentiable real map with positive range.

Let L denote a labor amount.

Signature 285. A labor growth rate is a nonnegative real number.

Let gL denote a labor growth rate.

Axiom 286 (labor growth rate). $gL = \hat{L}(t)$.

Lemma 287 (lemma gL). $\frac{\partial L(t)}{L(t)} = gL$.

Proof. $gL = \hat{L}(t) = \frac{\partial L(t)}{L(t)}$ (by labor growth rate, def div map, deriv and derivin). □

Signature 288. An efficiency level is a differentiable real map with positive range.

Let A denote an efficiency level.

Signature 289. An efficiency growth rate is a nonnegative real number.

Let gA denote an efficiency growth rate.

Axiom 290 (efficiency growth rate). $gA = \hat{A}(t)$.

Lemma 291 (lemma gA). $\frac{\partial A(t)}{A(t)} = gA$.

Proof. $gA = \hat{A}(t) = \frac{\partial A(t)}{A(t)}$ (by efficiency growth rate, def div map, deriv and derivin). □

Signature 292. A depreciation rate is a positive real number δ such that $\delta < 1$.

Let δ denote a depreciation rate.

Signature 293. A savings rate is a positive real number s such that $s < 1$.

Let s denote a savings rate.

The first relation we want to introduce is the capital accumulation rule. It says that the marginal change in our capital stock depends on how much we save and how much it depreciates. If we save more than it depreciates, our capital stock grows, while if more depreciates than is saved, our capital stock shrinks. Formally, this means:

Axiom 294 (capital accumulation).

$$\partial K(t) = (s \cdot Y(t)) - (\delta \cdot K(t)).$$

Next we introduce capital per efficiency unit labor $k_t = \frac{K_t}{A_t \cdot L_t}$. As we will see, this will be the defining variable in this model. Properly, we would have to define $k(t) := \frac{K(t)}{A(t) \cdot L(t)}$, but because k

then depends on K, A, and L, we would have to write $k_{K,A,L,t}$ as we will later do with firm profits. Because we use k so often, we would like to suppress this extra notation. Thus, we introduce it with a signature and by axiom. If there were two different capital stocks K, this would lead to a contradiction[1], but we will not assume this.

Lemma 295. $(A \cdot L)$ is a real map with positive range and $A(t) \cdot L(t)$ is nonzero.

Signature 296. A capital per efficiency unit labor is a real map with positive range.

Let k denote a capital per efficiency unit labor.

Axiom 297 (def k). $k = (K/(A \cdot L))$.

Lemma 298 (lemma k). $k(t) = \frac{K(t)}{A(t) \cdot L(t)}$.

Proof. $k(t) = (K/(A \cdot L))(t) = \frac{K(t)}{(A \cdot L)(t)} = \frac{K(t)}{A(t) \cdot L(t)}$ (by def k, def div map, def mult func). □

Next we want to introduce the production function and thus the connection between capital, labor, and technology on one side and output on the other. The typical production function is

$$Y_t = F(K_t, A_t L_t),$$

where the assumption of constant returns to scale is typically defined separately as

$$F(\lambda K_t, \lambda A_t L_t) = \lambda Y_t$$

for every positive real number λ. We define a production function in the following way:

Signature 299. A production function is a strictly convex differentiable real map.

Let f denote a production function.

Axiom 300 (constant returns to scale). $Y = ((A \cdot L) \cdot (f \circ k))$.

Lemma 301 (lemma Y). $Y(t) = (A(t) \cdot L(t)) \cdot f(k(t))$.

Proof. $Y(t) = ((A \cdot L) \cdot (f \circ k))(t) = (A \cdot L)(t) \cdot f(k(t)) = (A(t) \cdot L(t)) \cdot f(k(t))$ (by constant returns to scale, def mult func, composition). □

In our definition of the production function, we already assume constant returns to scale implicitly:

$$(\lambda A_t L_t) f\left(\frac{\lambda K_t}{\lambda A_t L_t}\right) = \lambda A_t L_t f(k_t) = \lambda Y_t.$$

In general constant returns to scale seems to be reasonable condition to assume. Two identical bakeries should probably produce basically twice as much as a single of those bakeries would produce. Also, on a country-wide level: if one country is twice as large as another and both are otherwise identical, the larger one should probably have an economy about twice as large. What reasonably holds in a large economy does not necessarily need to be the case on a firm level. When you are a car manufacturer for example the size of your addressed market is crucial. When you are very large, then investing in a huge and highly automated factory makes a lot of sense. However, when you are just producing a couple of hundred cars per year, the upfront cost of the large machinery

[1] Say, there were two different capital stocks K and K'. Then $k(t) = \frac{K(t)}{A(t) \cdot L(t)} \neq \frac{K'(t)}{A(t) \cdot L(t)} = k(t)$.

cannot be justified by the slight efficiency improvements. Therefore, a factory that produces 10,000 cars instead of 100 is probably not needing 100 times the inputs of the latter but fewer, because the processes the larger factory allows are not viable in the smaller one.

Next we discuss the maximization problem of the firm:

Signature 302. A wage is a differentiable real map with positive range.

Let w denote a wage.

Signature 303. An interest rate is a differentiable real map with positive range.

Let r denote an interest rate.

The typical formulation of the maximization problem of a firm is

$$\max_{K_t, L_t} F(K_t, A_t L_t) - r_t K_t - w_t L_t.$$

This is total output minus the costs consisting of the capital ($r_t K_t$) and labor ($w_t L_t$) costs. The firm chooses its capital and labor demand such that it maximizes profits. Here we will just focus on maximizing profits by capital because Kaldor's facts, as we stated them above, only demand that of us. With Π we denote the profits depending upon the capital stock.

Definition 304 (def Pi). $\Pi_{A,L,f,r,w,t} = (\mathrm{f}_{A(t)\cdot L(t)\cdot _ + 0} \circ (f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot _ + 0}) + (-\mathrm{f}_{r(t)\cdot _ + w(t)\cdot L(t)}))$.

Lemma 305. $\Pi_{A,L,f,r,w,t}$ is a differentiable real map.

Later we will add the following axiom, but first we want to show that such a $K(t)$ also exists:

Axiom 306 (profit maximizing K). $K(t)$ is a maximum of $\Pi_{A,L,f,r,w,t}$.

From an economics perspective, we should discuss what this maximization problem of a firm means. We did not introduce any firms, and K is the capital stock of the whole economy. So who is maximizing based on profits? Do we assume that only one firm exists? In a way, we do exactly that. This is obviously not realistic, and monopolies bring about their own set of problems. Actually when we assume constant returns to scale, we implicitly also assume perfect competition - i.e. the firm is making, when it maximizes profits, exactly zero profits[2]. Instead economists would rather speak of a "representative firm". The idea is that we have many small identical firms and therefore perfect competition, preventing profits from realizing, but because every firm faces the very same maximization problem - namely the one above, just with firm specific capital and labor -, total capital and labor demand behave as if they were maximized. This makes it obviously also not so much more realistic, as we certainly do see profits in the real economy. Another implicit assumption we make is that capital demand is equal to capital supply, and - even more problematically - labor demand is equal to labor supply. This means there is no unemployment in this modeled economy and there are no vacancies. On the other hand, we can certainly make the model more complex in order to remedy this, but the model is designed to make statements about long-term growth trends

[2]The proof works as follows: Because $\lambda Y_t = F(\lambda K_t, \lambda A_t L_t)$, taking the derivative with respect to λ on both sides yields $Y_t = \frac{\partial F}{\partial K_t} K_t + \frac{\partial F}{\partial L_t} L_t$ when evaluated for $\lambda = 1$. As we will later see $r_t = \frac{\partial F}{\partial K_t}$ (or rather we show that $r_t = f'(k_t)$, but $f'(k_t) = \frac{\partial F}{\partial K_t}$). Analogously $w_t = \frac{\partial F}{\partial L_t}$ when L_t is chosen optimally. This leaves us, however, with zero profits.

and introducing labor market frictions or imperfect markets will not change the results. Therefore it is probably preferable to keep the model simple and clear, even though we then may have to make some assumption we are not perfectly comfortable with - as long as we know that making the assumptions more realistic does not significantly change the results.

Next we introduce the Inada-conditions. They focus on the concept of diminishing returns on capital and labor that we already introduced in our definition of the production function, when we stated that it was strictly convex, but which did not discuss there. The idea is that when you have no capital, any capital might significantly help. A farmer is already much more productive with a plough than with his bare hands - and even more productive with a tractor. But there is also a limit to the productivity increases more capital brings: Giving a farmer a tractor when he has none makes him much more productive; giving the same tractor to a farmer who already has a hundred, will probably not help him at all. The same argument can be made for labor: If you have a small plot of land, having one worker on it will yield much more crops than leaving it barren. But when you already have a million workers on your small plot of land, adding the one million and first worker will only stand in the way of the others.

In macroeconomics textbooks, you will find the Inada-conditions typically defined like this:

$$\lim_{K_t \to 0} F'(K_t, A_t L_t) = \infty = \lim_{L_t \to 0} F'(K_t, A_t L_t)$$
$$\lim_{K_t \to \infty} F'(K_t, A_t L_t) = 0 = \lim_{L_t \to \infty} F'(K_t, A_t L_t).$$

Infinity is, however, nothing that we have defined so far. For our purposes, we just want to make sure that the production function is such that it is optimal to use at least some capital and labor, while at the same time it is also not optimal to use infinitely much of it.

Axiom 307 (Inada condition I). $\partial f(0) > r(t)$.

Axiom 308 (Inada condition II). There exists a positive real number x such that $\partial f(x) < r(t)$.

Axiom 309 (Inada condition III). There exists a positive real number M such that for all real numbers ξ $f(\xi) < M$.

Axiom 310 (Inada condition IV). $f(0) > 0$.

Utilizing the Inada-conditions, we will show that there exists a profit maximizing capital stock.

Proposition 311 (derivative profit function). For any real number x

$$\mathrm{D}\Pi_{A,L,f,r,w,t}(x) = \partial f\left(\frac{x}{A(t) \cdot L(t)}\right) - r(t).$$

Proof. Let x be a real number. $f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0}$ is a real map.

(A) Take $\xi = (f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0})(x)$. Then ξ is a real number.

(B) $\partial \mathrm{f}_{A(t)\cdot L(t)\cdot_+0}(\xi) = \mathrm{f}_{0\cdot_+A(t)\cdot L(t)}(\xi)$ and $\partial \mathrm{f}_{r(t)\cdot_+w(t)\cdot L(t)}(x) = \mathrm{f}_{0\cdot_+r(t)}(x)$ and $\partial \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0}(x) = \mathrm{f}_{0\cdot_+1/(A(t)\cdot L(t))}(x)$. Proof.

$A(t) \cdot L(t), r(t), w(t) \cdot L(t), x, 1/(A(t) \cdot L(t))$ are real numbers.

$\partial \mathrm{f}_{A(t) \cdot L(t) \cdot_ + 0}(\xi) = \mathrm{Df}_{A(t) \cdot L(t) \cdot_ + 0}(\xi)$

and $\partial \mathrm{f}_{r(t) \cdot_ + w(t) \cdot L(t)}(x) = \mathrm{Df}_{r(t) \cdot_ + w(t) \cdot L(t)}(x)$ and

$\partial \mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0}(x) = \mathrm{Df}_{1/(A(t) \cdot L(t)) \cdot_ + 0}(x)$ (by deriv and derivin).

$\mathrm{Df}_{A(t) \cdot L(t) \cdot_ + 0}(\xi) = \mathrm{f}_{0 \cdot_ + A(t) \cdot L(t)}(\xi)$ and

$\mathrm{Df}_{r(t) \cdot_ + w(t) \cdot L(t)}(x) = \mathrm{f}_{0 \cdot_ + r(t)}(x)$ and

$\mathrm{Df}_{1/(A(t) \cdot L(t)) \cdot_ + 0}(x) = \mathrm{f}_{0 \cdot_ + 1/(A(t) \cdot L(t))}(x)$ (by derivative of linear function). End.

$\mathrm{D}\Pi_{A,L,f,r,w,t}(x). = \partial \Pi_{A,L,f,r,w,t}(x)$ (by deriv and derivin)

$. = \partial(\mathrm{f}_{A(t) \cdot L(t) \cdot_ + 0} \circ (f \circ \mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0}) + (-\mathrm{f}_{r(t) \cdot_ + w(t) \cdot L(t)}))(x)$ (by def Pi)

$. = \partial \mathrm{f}_{A(t) \cdot L(t) \cdot_ + 0} \circ (f \circ \mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0})(x) + \partial(-\mathrm{f}_{r(t) \cdot_ + w(t) \cdot L(t)})(x)$ (by derivative addition rule)

$. = (\partial \mathrm{f}_{A(t) \cdot L(t) \cdot_ + 0}((f \circ \mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0})(x)) \cdot (\partial f(\mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0}(x)) \cdot \partial \mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0}(x))) - \partial \mathrm{f}_{r(t) \cdot_ + w(t) \cdot L(t)}(x)$ (by derivative chain rule, minus derivative)

$. = (\partial \mathrm{f}_{A(t) \cdot L(t) \cdot_ + 0}(\xi) \cdot (\partial f(\mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0}(x)) \cdot \partial \mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0}(x))) - \partial \mathrm{f}_{r(t) \cdot_ + w(t) \cdot L(t)}(x)$ (by A)

$. = (\mathrm{f}_{0 \cdot_ + A(t) \cdot L(t)}(\xi) \cdot (\partial f(\mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0}(x)) \cdot \mathrm{f}_{0 \cdot_ + 1/(A(t) \cdot L(t))}(x))) - \mathrm{f}_{0 \cdot_ + r(t)}(x)$ (by B)

$. = ((A(t) \cdot L(t)) \cdot (\partial f(\mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0}(x)) \cdot (1/(A(t) \cdot L(t))))) - r(t)$ (by A, linear lem 1)

$. = ((A(t) \cdot L(t)) \cdot (\partial f(1/(A(t) \cdot L(t)) \cdot x) \cdot (1/(A(t) \cdot L(t))))) - r(t)$ (by linear lem 2)

$. = \left(((A(t) \cdot L(t)) \cdot 1/(A(t) \cdot L(t))) \cdot \partial f\left(\frac{x}{A(t) \cdot L(t)}\right)\right) - r(t)$ (by mult comm, mult ass)

$. = \partial f\left(\frac{x}{A(t) \cdot L(t)}\right) - r(t)$ (by multiplicative inverse, mult comm, existence one). □

Proposition 312. There exists a maximum of $\Pi_{A,L,f,r,w,t}$.

Proof. Take $\Pi = \Pi_{A,L,f,r,w,t}$. [prover eprover]

(1) For any real number x $\mathrm{D}\Pi(x) = \partial f\left(\frac{x}{A(t) \cdot L(t)}\right) - r(t)$ (by derivative profit function).

(2) $\mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0}$ is a real map.

Let us show that $\mathrm{D}\Pi = \mathrm{f}_{1 \cdot_ + -r(t)} \circ (\mathrm{D}f \circ \mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0})$.

Let x be a real number. $\mathrm{f}_{1 \cdot_ + -r(t)} \circ (\mathrm{D}f \circ \mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0})$ is a real map.

$\mathrm{D}\Pi(x). = \partial f\left(\frac{x}{A(t) \cdot L(t)}\right) - r(t)$ (by 1)

$. = \mathrm{D}f((1/(A(t) \cdot L(t)) \cdot x) + 0) - r(t)$ (by deriv and derivin, mult comm, existence zero)

$. = (1 \cdot (\mathrm{D}f \circ \mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0})(x)) - r(t)$ (by linear function, composition, existence one)

$. = (\mathrm{f}_{1 \cdot_ + -r(t)} \circ (\mathrm{D}f \circ \mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0}))(x)$ (by linear function, composition). End.

Thus $\mathrm{D}\Pi$ is differentiable. $\mathrm{D}f$ is differentiable. Indeed for every real number x $\mathrm{D}f$ is differentiable in x.

Let us show that $\partial \mathrm{D}\Pi(x) < 0$ for every real number x.

$\mathrm{D}(\mathrm{f}_{1 \cdot_ + -r(t)} \circ (\mathrm{D}f \circ \mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0})). = ((\mathrm{D}\mathrm{f}_{1 \cdot_ + -r(t)} \circ (\mathrm{D}f \circ \mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0})) \cdot \mathrm{D}(\mathrm{D}f \circ \mathrm{f}_{1/(A(t) \cdot L(t)) \cdot_ + 0}))$ (by derivative chain rule 2)

$. = ((\mathrm{Df}_{1\cdot_+-r(t)} \circ (\mathrm{D}f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0})) \cdot ((\mathrm{DD}f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0}) \cdot \mathrm{Df}_{1/(A(t)\cdot L(t))\cdot_+0}))$

(by derivative chain rule 2)

$. = ((\mathrm{f}_{0\cdot_+1} \circ (\mathrm{D}f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0})) \cdot ((\mathrm{DD}f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0}) \cdot \mathrm{Df}_{1/(A(t)\cdot L(t))\cdot_+0}))$

(by derivative of linear function)

$. = ((\mathrm{f}_{0\cdot_+1} \circ (\mathrm{D}f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0})) \cdot ((\mathrm{DD}f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0}) \cdot \mathrm{f}_{0\cdot_+1/(A(t)\cdot L(t))}))$

(by derivative of linear function).

Let x be a real number.

$((\mathrm{f}_{0\cdot_+1} \circ (\mathrm{D}f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0})) \cdot ((\mathrm{DD}f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0}) \cdot \mathrm{f}_{0\cdot_+1/(A(t)\cdot L(t))}))(x)$

$. = (\mathrm{f}_{0\cdot_+1} \circ (\mathrm{D}f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0}))(x) \cdot ((\mathrm{DD}f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0}) \cdot \mathrm{f}_{0\cdot_+1/(A(t)\cdot L(t))})(x)$

(by def mult func)

$. = \mathrm{f}_{0\cdot_+1}((\mathrm{D}f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0})(x)) \cdot ((\mathrm{DD}f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0}) \cdot \mathrm{f}_{0\cdot_+1/(A(t)\cdot L(t))})(x)$

(by composition)

$. = 1 \cdot ((\mathrm{DD}f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0}) \cdot \mathrm{f}_{0\cdot_+1/(A(t)\cdot L(t))})(x)$ (by linear lem 1)

$. = (\mathrm{DD}f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0})(x) \cdot \mathrm{f}_{0\cdot_+1/(A(t)\cdot L(t))}(x)$ (by existence one, def mult func)

$. = (\mathrm{DD}f \circ \mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0})(x) \cdot 1/(A(t) \cdot L(t))$ (by linear lem 1)

$. = \mathrm{DD}f(\mathrm{f}_{1/(A(t)\cdot L(t))\cdot_+0}(x)) \cdot 1/(A(t) \cdot L(t))$ (by composition)

$. = \frac{\mathrm{DD}f\left(\frac{x}{A(t)\cdot L(t)}\right)}{A(t)\cdot L(t)}$ (by linear lem 2, mult comm).

We can show that $\frac{\mathrm{DD}f\left(\frac{x}{A(t)\cdot L(t)}\right)}{A(t)\cdot L(t)} < 0$. f is strictly convex. Therefore $\partial \mathrm{D}f\left(\frac{x}{A(t)\cdot L(t)}\right) \leq 0$. Thus $\mathrm{DD}f\left(\frac{x}{A(t)\cdot L(t)}\right) \leq 0$. $A(t) \cdot L(t) > 0$. $\frac{\mathrm{DD}f\left(\frac{x}{A(t)\cdot L(t)}\right)}{A(t)\cdot L(t)} \leq 0$ (by frac lemma 3). $\frac{\mathrm{DD}f\left(\frac{x}{A(t)\cdot L(t)}\right)}{A(t)\cdot L(t)} < 0$. Indeed $\mathrm{DD}f\left(\frac{x}{A(t)\cdot L(t)}\right)$ is nonzero. End. End.

Therefore Π is strictly convex.

Let us show that there exists a positive real number ξ such that $\mathrm{D}\Pi(\xi) = 0$.

$\mathrm{D}\Pi(0). = \partial f\left(\frac{0}{A(t)\cdot L(t)}\right) - r(t)$ (by 1)

$. = \partial f(0) - r(t)$ (by eq lemma 9).

$\partial f(0) - r(t) > 0$. Indeed $\partial f(0) > r(t)$ (by Inada condition I). Thus $\mathrm{D}\Pi(0) > 0$.

Take a positive real number u such that $\partial f(u) < r(t)$.

$\mathrm{D}\Pi((A(t) \cdot L(t)) \cdot u). = \partial f\left(\frac{(A(t)\cdot L(t))\cdot u}{A(t)\cdot L(t)}\right) - r(t)$ (by 1)

$. = \partial f(u) - r(t)$ (by mult ass, fraction lemma 2).

$\partial f(u) - r(t) < 0$. Indeed $\partial f(u) < r(t)$ (by Inada condition II). Therefore $\mathrm{D}\Pi((A(t) \cdot L(t)) \cdot u) = \partial f(u) - r(t) < 0$. [timelimit 10][prover vampire]

Take $\omega = (A(t) \cdot L(t)) \cdot u$. Then $\mathrm{D}\Pi(\omega) < 0 < \mathrm{D}\Pi(0)$ and $0 \in (\mathrm{D}\Pi(\omega), \mathrm{D}\Pi(0))_{OI}$. $\mathrm{D}\Pi$ is a continuous real map and $0 < \omega$. $(\mathrm{D}\Pi(\omega), \mathrm{D}\Pi(0))_{OI} \subseteq \mathrm{D}\Pi[(0, \omega)_{OI}]$ (by intermediate

value theorem II). Therefore $0 \in \mathrm{D}\Pi[(0,\omega)_{OI}]$. [prover eprover] Take $\xi \in (0,\omega)_{OI}$ such that $\mathrm{D}\Pi(\xi) = 0$. ξ is a positive real number. End.

Take a positive real number ξ such that $\partial\Pi(\xi) = 0$. ξ is a local maximum of Π (by sufficient condition local maximum). Indeed $\partial\mathrm{D}\Pi(\xi) < 0$. Then ξ is a maximum of Π (by sufficient condition for maximum on interval). Indeed $\mathbb{R}$ is an interval and ξ is an element in $\mathbb{R}$.[timelimit 3] □

Now that we know that there exists such a profit maximizing capital stock, we assume that firms will choose exactly that:

Axiom 313 (profit maximizing K). $K(t)$ is the maximum of $\Pi_{A,L,f,r,w,t}$.

Next, we can show that there exists a stable-steady state amount of capital per efficiency unit labor. What we will see is that for some function Δ, $\mathrm{D}k(t) = \Delta(k_t)$. Furthermore, there exists a level of capital per efficiency unit labor, κ, such that $\Delta(\kappa) = 0$, i.e. if $k_t = \kappa$ for some t, then it is constant throughout time; meaning, the capital per efficiency unit labor will always be at that level and actually also always has been. This level is called the steady state. In addition, we can show that for any capital levels x below κ, $\Delta(x) > 0$ and for any levels above, $\Delta(x) < 0$; i.e., if the amount of capital is off steady state, it will always converge to κ (although it will never reach that level).

In the proposition derivative k, because we want to use K, L, and A during the proof, although they are otherwise not mentioned in the declaration of the proposition, we have to add them there by

Let K be a capital stock and L be a labor amount and A be an efficiency level.

Proposition 314 (derivative k). Let Y be an output level and K be a capital stock and L be a labor amount and A be an efficiency level. k is differentiable in t and

$$\partial k(t) = (s \cdot f(k(t))) - ((\delta + (gA + gL)) \cdot k(t)).$$

Proof.$((A \cdot L)(t))^2, A(t) \cdot L(t)$ are nonzero.

(A) $\frac{\partial(A\cdot L)(t)}{((A\cdot L)(t))^2} = \frac{gA+gL}{(A\cdot L)(t)}$. Proof. $A(t) \cdot L(t), L(t)$ is nonzero.

(A1) $\frac{\partial A(t)}{A(t)} = gA$ and $\frac{\partial L(t)}{L(t)} = gL$ (by lemma gA, lemma gL).

$\frac{\partial(A\cdot L)(t)}{((A\cdot L)(t))^2} \cdot = \frac{(\partial A(t)\cdot L(t))+(A(t)\cdot \partial L(t))}{((A\cdot L)(t))^2}$ (by derivative multiplication rule)

$. = \left(\frac{\partial A(t)\cdot L(t)}{(A\cdot L)(t)} + \frac{A(t)\cdot \partial L(t)}{(A\cdot L)(t)}\right) \cdot 1/(A \cdot L)(t)$

(by potence lemma 2, eq lemma 30, mult ass, eq lemma 1)

$. = \left(\frac{\partial A(t)\cdot L(t)}{A(t)\cdot L(t)} + \frac{A(t)\cdot \partial L(t)}{A(t)\cdot L(t)}\right) \cdot 1/(A \cdot L)(t)$ (by def mult func)

$. = \left(\frac{\partial A(t)\cdot L(t)}{L(t)\cdot A(t)} + \frac{\partial L(t)\cdot A(t)}{A(t)\cdot L(t)}\right) \cdot 1/(A \cdot L)(t)$(by mult comm)

$. = \left(\left(\partial A(t) \cdot \frac{L(t)}{L(t)\cdot A(t)}\right) + \left(\partial L(t) \cdot \frac{A(t)}{A(t)\cdot L(t)}\right)\right) \cdot 1/(A \cdot L)(t)$ (by mult ass)

$. = \frac{gA+gL}{(A \cdot L)(t)}$ (by eq lemma 29, A1). End. [timelimit 10]

(B) $\partial K(t) - (K(t) \cdot (gA + gL)) = (s \cdot Y(t)) - (K(t) \cdot (\delta + (gA + gL)))$. Proof.

(B1) $\partial K(t) = (s \cdot Y(t)) - (\delta \cdot K(t))$ (by capital accumulation). [timelimit 3]

$\partial K(t) - (K(t) \cdot (gA + gL)) \; . = ((s \cdot Y(t)) - (\delta \cdot K(t))) - (K(t) \cdot (gA + gL))$ (by B1)

$. = (s \cdot Y(t)) - ((K(t) \cdot \delta) + (K(t) \cdot (gA + gL)))$ (by mult comm, add ass, eq lemma 15)

$. = (s \cdot Y(t)) - (K(t) \cdot (\delta + (gA + gL)))$ (by distributive law). End.[prover vampire]

(C) $Y(t) = (A(t) \cdot L(t)) \cdot f(k(t))$ (by lemma Y). [prover eprover]

(D) $k(t) = \frac{K(t)}{A(t) \cdot L(t)}$ (by lemma k).

$\partial k(t) = \partial(K/(A \cdot L))(t)$ (by def k).

$\partial(K/(A \cdot L))(t) \; . = \frac{(\partial K(t) \cdot (A \cdot L)(t)) - (K(t) \cdot \partial(A \cdot L)(t))}{((A \cdot L)(t))^2}$ (by derivative of a fraction)

$. = \frac{(\partial K(t) \cdot (A \cdot L)(t))}{((A \cdot L)(t))^2} - \frac{(K(t) \cdot \partial(A \cdot L)(t))}{((A \cdot L)(t))^2}$ (by eq lemma 1, eq lemma 11)

$. = \frac{\partial K(t)}{(A \cdot L)(t)} - \frac{K(t) \cdot (gA+gL)}{(A \cdot L)(t)}$ (by A, potence lemma 2, mult ass, eq lemma 29)

$. = \frac{\partial K(t) - (K(t) \cdot (gA+gL))}{(A \cdot L)(t)}$ (by eq lemma 1, eq lemma 11)

$. = \frac{(s \cdot Y(t)) - (K(t) \cdot (\delta + (gA+gL)))}{(A \cdot L)(t)}$ (by B)

$. = \frac{s \cdot Y(t)}{(A \cdot L)(t)} - \frac{K(t) \cdot (\delta + (gA+gL))}{(A \cdot L)(t)}$ (by eq lemma 1, eq lemma 11)

$. = \frac{s \cdot ((A(t) \cdot L(t)) \cdot f(k(t)))}{(A \cdot L)(t)} - \left((\delta + (gA + gL)) \cdot \frac{K(t)}{A(t) \cdot L(t)}\right)$ (by C, mult comm, mult ass, def mult func)

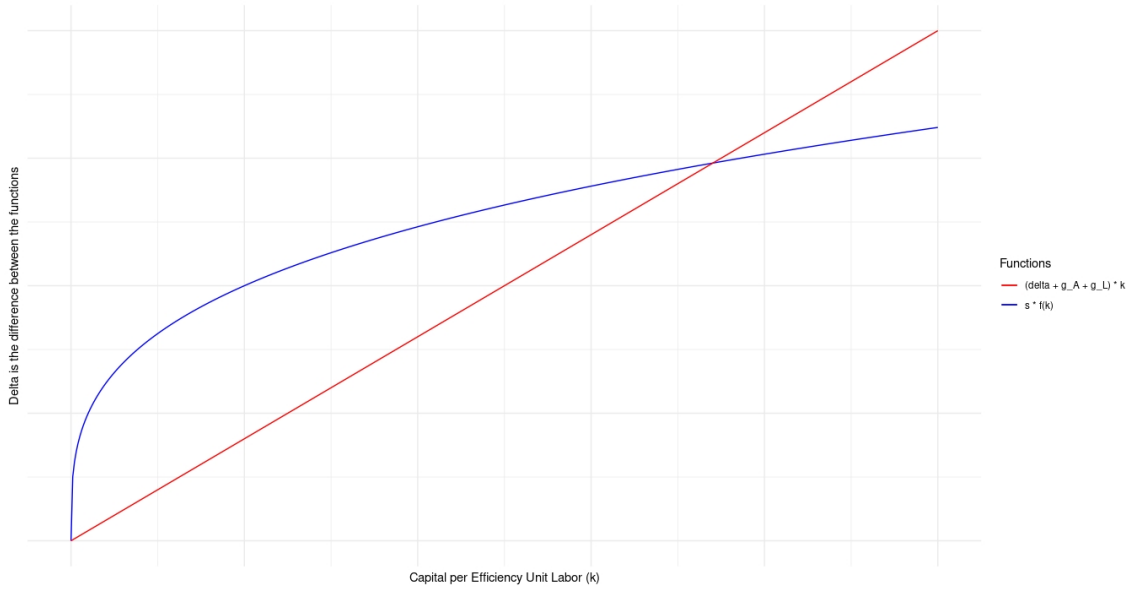
$. = \frac{s \cdot ((A(t) \cdot L(t)) \cdot f(k(t)))}{(A \cdot L)(t)} - ((\delta + (gA + gL)) \cdot k(t))$ (by D)

$. = \frac{s \cdot ((A \cdot L)(t) \cdot f(k(t)))}{(A \cdot L)(t)} - ((\delta + (gA + gL)) \cdot k(t))$ (by def mult func)

$. = \left(s \cdot \left((A \cdot L)(t) \cdot \frac{f(k(t))}{(A \cdot L)(t)}\right)\right) - ((\delta + (gA + gL)) \cdot k(t))$ (by mult ass)

$. = (s \cdot f(k(t))) - ((\delta + (gA + gL)) \cdot k(t))$ (by fraction lemma 2). □

The reason we chose the Inada-conditions the way we did, was to make the production function resemble most closely the following image with the limited concepts, which we have defined so far. This image is always shown for illustrative purposes during any presentation of the Solow model.



The Δ-function that we define right below, is exactly the difference between these to graphs.

Definition 315 (Delta). $\Delta_{s,f,\delta,gA,gL} = (\mathrm{f}_{s\cdot_+0} \circ f + (-\mathrm{f}_{\delta+(gA+gL)\cdot_+0}))$.

Let x denote a real number.

Lemma 316 (lemma Delta). $\Delta_{s,f,\delta,gA,gL}(x) = (s \cdot f(x)) - ((\delta + (gA + gL)) \cdot x)$.

Proof. $\Delta_{s,f,\delta,gA,gL}(x)$. $= (\mathrm{f}_{s\cdot_+0} \circ f + (-\mathrm{f}_{\delta+(gA+gL)\cdot_+0}))(x)$ (by Delta)

$. = (\mathrm{f}_{s\cdot_+0} \circ f)(x) + (-\mathrm{f}_{\delta+(gA+gL)\cdot_+0})(x)$ (by def add func)

$. = (\mathrm{f}_{s\cdot_+0} \circ f)(x) - \mathrm{f}_{\delta+(gA+gL)\cdot_+0}(x)$ (by def func minus)

$. = \mathrm{f}_{s\cdot_+0}(f(x)) - \mathrm{f}_{\delta+(gA+gL)\cdot_+0}(x)$ (by composition)

$. = (s \cdot f(x)) - ((\delta + (gA + gL)) \cdot x)$ (by linear lem 2). □

Proposition 317 (proposition Delta). Let Y be an output level and K be a capital stock and L be a labor amount and A be an efficiency level. $\Delta_{s,f,\delta,gA,gL} \circ k = \mathrm{D}k$.

Proof. Take $\Delta = \Delta_{s,f,\delta,gA,gL}$. Let t be a point in time. Δ, $\Delta \circ k$ are real maps and $k(t)$ is a real number. [timelimit 30]

(1) $\partial k(t) = (s \cdot f(k(t))) - ((\delta + (gA + gL)) \cdot k(t))$ (by derivative k).

(2) $\Delta(k(t)) = (s \cdot f(k(t))) - ((\delta + (gA + gL)) \cdot k(t))$ (by lemma Delta).

(3) $\partial k(t) = (s \cdot f(k(t))) - ((\delta + (gA + gL)) \cdot k(t)) = \Delta(k(t))$ (by 1,2).

Therefore $\mathrm{D}k(t) = \partial k(t) = \Delta(k(t)) = (\Delta \circ k)(t)$ (by deriv and derivin, 3, composition). [timelimit 3] □

Lemma 318 (derivative Delta). $\mathrm{D}\Delta_{s,f,\delta,gA,gL} = \mathrm{f}_{s\cdot_+-(\delta+(gA+gL))} \circ \mathrm{D}f$.

Proof. $\mathrm{f}_{s\cdot_+-(\delta+(gA+gL))} \circ \mathrm{D}f$ is a real map.

Let us show that for all real numbers x $\mathrm{D}\Delta_{s,f,\delta,gA,gL}(x) = (\mathrm{f}_{s\cdot_+-(\delta+(gA+gL))} \circ \mathrm{D}f)(x)$.

$\mathrm{D}\Delta_{s,f,\delta,gA,gL}. = \mathrm{D}(\mathrm{f}_{s\cdot_+0} \circ f + (-\mathrm{f}_{\delta+(gA+gL)\cdot_+0}))$ (by Delta)

$. = (\mathrm{D}(\mathrm{f}_{s\cdot_+0} \circ f) + (-\mathrm{D}\mathrm{f}_{\delta+(gA+gL)\cdot_+0}))$ (by derivative addition rule 2, minus derivative 2)

$. = (((\mathrm{f}_{0\cdot_+s} \circ f) \cdot \mathrm{D}f) + (-\mathrm{f}_{0\cdot_+\delta+(gA+gL)}))$ (by derivative chain rule 2, derivative of linear function).

Let x be a real number.

$(((\mathrm{f}_{0\cdot_+s} \circ f) \cdot \mathrm{D}f) + (-\mathrm{f}_{0\cdot_+\delta+(gA+gL)}))(x). = ((\mathrm{f}_{0\cdot_+s} \circ f) \cdot \mathrm{D}f)(x) - \mathrm{f}_{0\cdot_+\delta+(gA+gL)}(x)$

(by def add func, def func minus)

$. = ((\mathrm{f}_{0\cdot_+s} \circ f)(x) \cdot \mathrm{D}f(x)) - \mathrm{f}_{0\cdot_+\delta+(gA+gL)}(x)$ (by def mult func)

$. = (\mathrm{f}_{0\cdot_+s}(f(x)) \cdot \mathrm{D}f(x)) - \mathrm{f}_{0\cdot_+\delta+(gA+gL)}(x)$ (by composition)

$. = (s \cdot \mathrm{D}f(x)) - (\delta + (gA + gL))$ (by linear lem 1)

$. = \mathrm{f}_{s\cdot_+-(\delta+(gA+gL))}(\mathrm{D}f(x))$ (by linear function)

$. = (\mathrm{f}_{s\cdot_+-(\delta+(gA+gL))} \circ \mathrm{D}f)(x)$ (by composition). End.

Thus $\mathrm{D}\Delta_{s,f,\delta,gA,gL} = \mathrm{f}_{s\cdot_+-(\delta+(gA+gL))} \circ \mathrm{D}f$. □

Lemma 319 (lemma Delta 2). $\Delta_{s,f,\delta,gA,gL}(0) > 0$.

Proof.

$\Delta_{s,f,\delta,gA,gL}(0) = (s \cdot f(0)) - ((\delta + (gA + gL)) \cdot 0)$ (by lemma Delta).

$(s \cdot f(0)) - ((\delta + (gA + gL)) \cdot 0) . = (s \cdot f(0)) - 0$ (by mult comm, eq lemma 9)

$. = s \cdot f(0)$ (by eq lemma 0, existence zero).

$s \cdot f(0) > s \cdot 0 = 0$. Indeed $f(0) > 0$ (by Inada condition IV). □

Lemma 320 (lemma Delta 3). Let ξ be a positive real number such that $\Delta_{s,f,\delta,gA,gL}(\xi) = 0$. Then for all real numbers ζ such that $\zeta > \xi$ $\Delta_{s,f,\delta,gA,gL}(\zeta) < 0$.

Proof. Let ζ be a real number such that $\zeta > \xi$. Take $\Delta = \Delta_{s,f,\delta,gA,gL}$. $\Delta, \mathrm{D}\Delta$ are differentiable real maps and $\mathbb{R}$ is an interval. [prover vampire]

$\mathrm{D}\Delta$ is strictly decreasing on $\mathbb{R}$ (by derivative strictly decreasing). Indeed we can show that $\partial\mathrm{D}\Delta(x) < 0$ for any $x \in \mathbb{R}$. Let x be a real number.

(1) $\mathrm{D}\Delta = \mathrm{f}_{s\cdot_+-(\delta+(gA+gL))} \circ \mathrm{D}f$ (by derivative Delta).

$\partial\mathrm{D}\Delta(x). = \partial(\mathrm{f}_{s\cdot_+-(\delta+(gA+gL))} \circ \mathrm{D}f)(x)$ (by 1)

$. = \partial\mathrm{f}_{s\cdot_+-(\delta+(gA+gL))}(\mathrm{D}f(x)) \cdot \partial\mathrm{D}f(x)$ (by derivative chain rule)

$. = \mathrm{f}_{0\cdot_+s}(\mathrm{D}f(x)) \cdot \partial\mathrm{D}f(x)$ (by deriv and derivin, derivative of linear function)

$. = s \cdot \partial\mathrm{D}f(x)$ (by linear lem 1).

$s \cdot \partial\mathrm{D}f(x) < s \cdot 0 = 0$. Indeed $\partial\mathrm{D}f(x) < 0$. Thus $\partial\mathrm{D}\Delta(x) < 0$. End.

$\Delta(0) > 0$ (by lemma Delta 2).

Take $Z \in (0, \xi)_{OI}$ such that $\partial\Delta(Z) = \frac{\Delta\Delta}{\Delta 0}(\xi)$ (by mean value theorem).

$\mathrm{D}\Delta(Z) < 0$. Proof. $\partial\Delta(Z) = \frac{\Delta\Delta}{\Delta 0}(\xi) = \frac{\Delta(\xi)-\Delta(0)}{\xi-0} = \frac{-\Delta(0)}{\xi} \leq 0$.

$\mathrm{D}\Delta(Z) = \frac{-\Delta(0)}{\xi} < 0$ (by frac lemma 5). End.

Δ is strictly decreasing on $[Z,\zeta]_{CI}$ (by derivative strictly decreasing, interval lemma). Indeed we can show that for all $x \in [Z,\zeta]_{CI}$ $\partial\Delta(x) < 0$. Let $x \in [Z,\zeta]_{CI}$. $\mathrm{D}\Delta(x) \leq \mathrm{D}\Delta(Z) < 0$. Indeed $x \geq Z$ and $\mathrm{D}\Delta$ is strictly decreasing on $\mathbb{R}$. End.[prover eprover]

Therefore $\Delta(\zeta) < \Delta(\xi) = 0$. Indeed $\xi, \zeta \in [Z,\zeta]_{CI}$ and $\xi < \zeta$. □

Proposition 321. There exists a positive real number κ such that $\Delta_{s,f,\delta,gA,gL}(\kappa) = 0$.

Proof. Take $\Delta = \Delta_{s,f,\delta,gA,gL}$. Δ is a continuous real map.

Take a positive real number x such that $\Delta(x) < 0$. Proof.

Take a positive real number M such that $f(\xi) < M$ for all real numbers ξ (by Inada condition III).

(A) Take $c = \frac{s}{\delta+(gA+gL)} \cdot M$. c is a positive real number. Indeed $\delta + (gA + gL), \frac{s}{\delta+(gA+gL)}$ are positive.

$\Delta(c) = (s \cdot f(c)) - ((\delta + (gA + gL)) \cdot c)$ (by lemma Delta).

$(s \cdot f(c)) - ((\delta + (gA + gL)) \cdot c). = (s \cdot f(c)) - \left((\delta + (gA + gL)) \cdot \left(\frac{s}{\delta+(gA+gL)} \cdot M\right)\right)$ (by A)

$. = (s \cdot f(c)) - \left(\left((\delta + (gA + gL)) \cdot \frac{s}{\delta+(gA+gL)}\right) \cdot M\right)$ (by mult ass)

$. = (s \cdot f(c)) - (s \cdot M)$ (by fraction lemma 2).

$(s \cdot f(c)) - (s \cdot M) < (s \cdot M) - (s \cdot M) = 0$. Thus $\Delta(c) < 0$. End.

$\Delta(0) > 0$ (by lemma Delta 2). $0 < x$. $(\Delta(x), \Delta(0))_{OI} \subseteq \Delta[(0,x)_{OI}]$ (by intermediate value theorem II). $0 \in (\Delta(x), \Delta(0))_{OI}$. Therefore take $\kappa \in (0,x)_{OI}$ such that $\Delta(\kappa) = 0$. κ is a positive real number. □

Signature 322. A steady state capital is a positive real number.

Let κ denote a steady state capital.

Axiom 323 (def kappa). $\Delta_{s,f,\delta,gA,gL}(\kappa) = 0$.

Proposition 324 (off steady state). Let x be positive. (If $x < \kappa$ then $\Delta_{s,f,\delta,gA,gL}(x) > 0$) and if $x > \kappa$ then $\Delta_{s,f,\delta,gA,gL}(x) < 0$.

Proof. Take $\Delta = \Delta_{s,f,\delta,gA,gL}$.

Case $x < \kappa$. Assume the contrary. Then $\Delta(x) \leq 0$. [timelimit 20]

Case $\Delta(x) = 0$. Then $\Delta(\kappa) < 0$ (by lemma Delta 3). Contradiction (by def kappa). End.

Case $\Delta(x) < 0$. Δ is a continuous real map and $0 < x$. Therefore $(\Delta(x), \Delta(0))_{OI} \subseteq \Delta[(0,x)_{OI}]$ (by intermediate value theorem II). $0 \in (\Delta(x), \Delta(0))_{OI}$. Indeed $\Delta(x) < 0 < \Delta(0)$ (by lemma Delta 2, lemma Delta 3). Therefore take $\xi \in (0,x)_{OI}$ such that $\Delta(\xi) = 0$. Then $\Delta(\kappa) < 0$ (by lemma Delta 3). Indeed ξ is a positive real number such that $\Delta(\xi) = 0$ and κ is a real number such that $\kappa > \xi$. Contradiction (by def kappa). End. End.

Case $x > \kappa$. [timelimit 50] $\Delta(x) < 0$ (by lemma Delta 3). Indeed κ is a positive real number such that $\Delta(\kappa) = 0$ and x is a real number such that $x > \kappa$ (by def kappa). [timelimit 3] End. □

So κ is our steady-state capital per efficiency unit labor. When our capital per efficiency unit labor is once at a level of κ, it does not change anymore, since $\partial k(t)|_{k=\kappa} = \Delta(\kappa) = 0$. If we are not at the level κ, we know that we converge to this level. For Kaldor's facts, we will assume that we are

in the steady state. However, at G6, we will continue our discussion of what happens off steady state, because that is exactly what is relevant there.

Axiom 325 (steady state). $\partial k(t) = 0$.

Lemma 326 (lemma kappa). Let Y be an output level and K be a capital stock and L be a labor amount and A be an efficiency level. Let s be a savings rate and f be a production function and δ be a depreciation rate and gA be an efficiency growth rate and gL be a labor growth rate. $k(t) = \kappa$.

Proof. Take $\Delta = \Delta_{s,f,\delta,gA,gL}$. k is constant on $\mathbb{R}$ (by derivative zero then constant). Indeed for every point in time τ $\partial k(\tau) = 0$ and $\mathbb{R}$ is an interval and k is a continuous real map.

(1) Take $\iota = k(0)$. ι, κ are positive real numbers. $\Delta \circ k$ is a real map. Then for any point in time τ $k(\tau) = \iota$.

$\Delta(\iota). = (\Delta \circ k)(0)$ (by 1, composition). [timelimit 30][prover vampire] Then $(\Delta \circ k)(0) = \mathrm{D}k(0)$. Indeed $\Delta \circ k = \mathrm{D}k$ (by proposition Delta). $\mathrm{D}k(0) = \partial k(0) = 0$ (by deriv and derivin, steady state). Assume $\iota \neq \kappa$.

Case $\iota < \kappa$. $\Delta(\iota) > 0$ (by off steady state). End.

Case $\iota > \kappa$. $\Delta(\iota) < 0$ (by off steady state). End. [timelimit 3] □

9.2 Kaldor's Facts

Having the model presented, we can now turn to Kaldor's facts.

G1 Continued growth in the aggregate volume of production, i.e. in the long term economies seem to grow.

We will make a stronger claim in this model: Even the aggregate volume of production per capita grows.

Let Y denote an output level. Let K denote a capital stock. Let L denote a labor amount. Let f denote a production function. Let δ denote a depreciation rate. Let gA denote an efficiency growth rate. Let gL denote a labor growth rate. Let t denote a point in time. Let r denote an interest rate. Let κ denote a steady state capital.

Lemma 327. (Y/L) is a real map with positive range.

Proof. Let t be a point in time. Therefore $\frac{Y(t)}{L(t)}$ is a positive real number. Indeed $Y(t), L(t)$ are positive real numbers. □

Proposition 328 (G1). Let A be an efficiency level and f be a production function and k be a capital per efficiency unit labor.

$$(\hat{Y/L})(t) = gA.$$

Proof. (A) $(Y/L) = (A \cdot (f \circ k))$. Proof.

Let τ be a real number. [timelimit 20]

(1) $Y(\tau) = ((A \cdot L) \cdot (f \circ k))(\tau)$ (by constant returns to scale). [timelimit 3]

$(Y/L)(\tau). = \frac{Y(\tau)}{L(\tau)}$ (by def div map)

$. = \frac{((A \cdot L) \cdot (f \circ k))(\tau)}{L(\tau)}$ (by 1)

$. = \frac{(A(\tau) \cdot L(\tau)) \cdot (f \circ k)(\tau)}{L(\tau)}$ (by def mult func)

$. = A(\tau) \cdot (f \circ k)(\tau)$ (by mult ass, fraction lemma 2)

$. = (A \cdot (f \circ k))(\tau)$ (by def mult func). End.

(2) $\partial k(t) = 0$ (by steady state).

(B) $\partial(f \circ k)(t) = 0$. Proof.

$\partial(f \circ k)(t). = \partial f(k(t)) \cdot \partial k(t)$ (by derivative chain rule)

$. = \partial f(k(t)) \cdot 0$ (by 2)

$. = 0$ (by mult comm, eq lemma 9). End.

(2) $\frac{\partial A(t)}{A(t)} = gA$ (by lemma gA).

$L(t), (f \circ k)(t)$ are nonzero. Proof. Assume $(f \circ k)(t) = 0$. Then $Y(t) = (A(t) \cdot L(t)) \cdot (f \circ k)(t) = 0$. End.

$(\hat{Y/L})(t). = (A \cdot (\hat{f} \circ k))(t)$ (by A)

$. = \frac{\mathrm{D}(A \cdot (f \circ k))(t)}{(A \cdot (f \circ k))(t)}$

$. = \frac{\partial(A \cdot (f \circ k))(t)}{(A \cdot (f \circ k))(t)}$ (by deriv and derivin)

$. = \frac{(\partial A(t) \cdot (f \circ k)(t)) + (A(t) \cdot \partial(f \circ k)(t))}{(A \cdot (f \circ k))(t)}$ (by derivative multiplication rule)

$. = \frac{(\partial A(t) \cdot (f \circ k)(t)) + (0 \cdot A(t))}{(A \cdot (f \circ k))(t)}$ (by B, mult comm)

$. = \frac{\partial A(t) \cdot (f \circ k)(t)}{(A \cdot (f \circ k))(t)}$ (by eq lemma 9, existence zero)

$. = \partial A(t) \cdot \frac{(f \circ k)(t)}{(f \circ k)(t) \cdot A(t)}$ (by def mult func, mult comm, mult ass)

$. = \frac{\partial A(t)}{A(t)}$ (by eq lemma 29)

$. = gA$ (by 2). □

We can see that output per capita grows with improvements in technology. So the more efficient we become, the wealthier we will be. Hence, we cannot only show with the model that the first of Kaldor's facts holds within it, but can also point out why this is true in this model and under which conditions this is the case. I think therein lies the biggest advantage of modeling theories: you have to explicitly state every single assumption you make that leads you to a conclusion. This structures one's own thoughts and ideas and makes them transparent to everyone else. By having rigid methods on which you can rely, you can also see where your assumptions lead you. In our case, given our assumptions, only technological advancements can increase our wealth in the steady state.

G2 A continued increase in the amount of capital per worker, meaning that each employee manages more and more capital.

For G2, we see that the capital stock grows in the steady-state with technological advancements and labor force increases. The capital per worker therefore increases with the efficiency gains.

Proposition 329 (G2). Let Y be an output level and δ be a depreciation rate and s be a savings rate and f be a production function. Let A be an efficiency level and L be a labor amount and k be a capital per efficiency unit labor.

$$\hat{K}(t) = gA + gL.$$

Proof. $k(t)$ is a real number.

(A) $\partial K(t) = (s \cdot Y(t)) - (\delta \cdot K(t))$ (by capital accumulation).

(B) $\frac{s \cdot Y(t)}{K(t)} = \delta + (gA + gL)$. Proof.

(B1) $\partial k(t) = (s \cdot f(k(t))) - ((\delta + (gA + gL)) \cdot k(t))$ (by derivative k).

(B2) $\partial k(t) = 0$ (by steady state).

(B3) $s \cdot f(k(t)) = (\delta + (gA + gL)) \cdot k(t)$. Indeed $(s \cdot f(k(t))) - ((\delta + (gA + gL)) \cdot k(t)) = 0$ (by B1, B2). [prover vampire]

(B4) $Y(t) = (A(t) \cdot L(t)) \cdot f(k(t))$ (by lemma Y). [prover eprover]

(B5) $(A(t) \cdot L(t)) \cdot k(t) = K(t)$. Indeed $k(t) = \frac{K(t)}{A(t) \cdot L(t)}$ (by lemma k).

$\frac{s \cdot Y(t)}{K(t)}. = \frac{s \cdot ((A(t) \cdot L(t)) \cdot f(k(t)))}{K(t)}$ (by B4)

$. = \frac{(A(t) \cdot L(t)) \cdot (s \cdot f(k(t)))}{K(t)}$ (by mult comm, mult ass)

$. = \frac{(A(t) \cdot L(t)) \cdot ((\delta + (gA + gL)) \cdot k(t))}{K(t)}$ (by B3)

$. = \frac{(A(t) \cdot L(t)) \cdot (k(t) \cdot (\delta + (gA + gL)))}{K(t)}$ (by mult comm)

$. = ((A(t) \cdot L(t)) \cdot k(t)) \cdot \frac{\delta + (gA + gL)}{K(t)}$ (by mult ass)

$. = K(t) \cdot \frac{\delta + (gA + gL)}{K(t)}$ (by B5)

$. = \delta + (gA + gL)$ (by fraction lemma 2). End.

$\hat{K}(t). = (\mathrm{D}K/K)(t)$

$. = \frac{\partial K(t)}{K(t)}$ (by def div map, deriv and derivin)

$. = \frac{(s \cdot Y(t)) - (\delta \cdot K(t))}{K(t)}$ (by A)

$. = \frac{s \cdot Y(t)}{K(t)} - \frac{\delta \cdot K(t)}{K(t)}$ (by eq lemma 1, eq lemma 11)

$. = \frac{s \cdot Y(t)}{K(t)} - \delta$ (by mult ass, multiplicative inverse, mult comm, existence one)

$. = (\delta + (gA + gL)) - \delta$ (by B)

$. = (gA + gL) + (\delta - \delta)$ (by add comm, add ass)

$. = gA + gL$ (by additive inverse, existence zero). □

G3 A steady rate of profit on capital, i.e. when you invest money, the average return you get on

it when discounting inflation, seems to be quite constant over time.

Profit on capital we interpret as the real interest rate. The real interest rate differs from the nominal interest rate in that the nominal profits are those in euros or dollars, but the real values always account for inflation. Otherwise, we would observe significant changes in the rate of profit on capital depending on whether we are currently in a high or low inflation environment.

[timelimit 10]

Proposition 330 (G3). Let Y be an output level and K be a capital stock and L be a labor amount and A be an efficiency level. Let s be a savings rate and f be a production function and δ be a depreciation rate and gA be an efficiency growth rate and gL be a labor growth rate and w be a wage and k be a capital per efficiency unit labor.

$$r(t) = \partial f(\kappa).$$

Proof. Take $\Pi = \Pi_{A,L,f,r,w,t}$. [prover vampire]

Let us show that $\partial\Pi(K(t)) = 0$. $K(t)$ is the maximum of Π (by profit maximizing K). $\partial\Pi(K(t)) = 0$ (by derivative of maximum). Indeed $K(t)$ is a local maximum of Π (by maximum is local maximum). End.

(1) For any real number x $\mathrm{D}\Pi(x) = \partial f\left(\frac{x}{A(t)\cdot L(t)}\right) - r(t)$ (by derivative profit function).

(2) $\frac{K(t)}{A(t)\cdot L(t)} = k(t)$ (by lemma k).

(3) $k(t) = \kappa$ (by lemma kappa). [prover eprover]

$\mathrm{D}\Pi(K(t)). = \partial f\left(\frac{K(t)}{A(t)\cdot L(t)}\right) - r(t)$ (by 1)

$. = \partial f(k(t)) - r(t)$ (by 2)

$. = \partial f(\kappa) - r(t)$ (by 3).

Thus $r(t) = \partial f(\kappa)$. □

[timelimit 3]

G3 is an interesting observation that holds in many macroeconomic models. The interest on capital that is being paid is equal to the marginal productivity of capital. If the interest rate wer lower than the marginal productivity of capital, then it would increase profits to take out a loan, borrow more capital, but producing significantly more. However, by increasing capital, we reduce its marginal productivity due to diminishing returns on it. Similarly, if the interest rate is higher than the marginal productivity of capital, it would be better to have less capital. Although we would be producing less, we would also be reducing our capital costs, which would save us more than what we lose in output. By decreasing our capital share, we increase the marginal productivity of the remaining capital—the idea being that we essentially discard machines that are not significantly contributing - and in equilibrium, the interest rate aligns with the marginal productivity.

The next logical step that is usually done in extensions of the Solow model is to endogenize the interest rate by introducing households and thus their inclination to either consume or to save and to examine how this relates to the interest rate and the supply of capital.

G4 Steady capital-output ratios over long periods, which implies the near-identity in the percentage rates of growth of production and of the capital stock.

Given that we do steady-state analysis, G4 is trivially the case.

Lemma 331. $\delta + (gA + gL)$ is a positive real number.

Proof. $\delta + (gA + gL)$ is nonzero. [timelimit 20] □

Lemma 332 (lemma G4). Let s be a savings rate and δ be a depreciation rate and gA be an efficiency growth rate and gL be a labor growth rate. Let Y be an output level and K be a capital stock and A be an efficiency level and L be a labor amount and k be a capital per efficiency unit labor and t be a point in time. Then $f(\kappa)$ is a positive real number.

Proof. $Y(t), A(t), L(t)$ are positive real numbers. Therefore $\frac{Y(t)}{A(t) \cdot L(t)}$ is positive. [prover vampire]

(1) $Y(t) = (A(t) \cdot L(t)) \cdot f(k(t))$ (by lemma Y).

(2) $k(t) = \kappa$ (by lemma kappa).[prover eprover]

$\frac{Y(t)}{A(t) \cdot L(t)} \cdot = \frac{(A(t) \cdot L(t)) \cdot f(k(t))}{A(t) \cdot L(t)}$ (by 1)

$. = (A(t) \cdot L(t)) \cdot \frac{f(k(t))}{A(t) \cdot L(t)}$ (by mult ass)

$. = f(k(t))$ (by fraction lemma 2)

$. = f(\kappa)$ (by 2).

Thus $f(\kappa)$ is a positive real number. □

[prover vampire]

Proposition 333 (G4). Let s be a savings rate and δ be a depreciation rate and gA be an efficiency growth rate and gL be a labor growth rate. Let A be an efficiency level and L be a labor amount and k be a capital per efficiency unit labor. $f(\kappa)$ is nonzero and

$$\frac{K(t)}{Y(t)} = \frac{\kappa}{f(\kappa)}.$$

Proof. (0) $Y(t) = (A(t) \cdot L(t)) \cdot f(k(t))$ (by lemma Y).

(1) $\frac{K(t)}{A(t) \cdot L(t)} = k(t)$ (by lemma k).

(2) $k(t) = \kappa$ (by lemma kappa).

$f(\kappa)$ is nonzero (by lemma G4). Thus $f(k(t)), (A(t) \cdot L(t)) \cdot f(k(t))$ are nonzero.[prover eprover]

$\frac{K(t)}{Y(t)} \cdot = \frac{K(t)}{(A(t) \cdot L(t)) \cdot f(k(t))}$ (by 0)

$. = \frac{K(t)}{A(t) \cdot L(t)} \cdot 1/f(k(t))$ (by eq lemma 30, mult ass)

$. = \frac{k(t)}{f(k(t))}$ (by 1)

$. = \frac{\kappa}{f(\kappa)}$ (by 2). □

G5 A steady share of profits (and of wages) in societies and periods in which the share of investment in output is constant.

G5 is a corollary of G3 and G4.

Proposition 334 (G5). Let s be a savings rate and δ be a depreciation rate and gA be an efficiency growth rate and gL be a labor growth rate and w be a wage. Let A be an efficiency level and L be a labor amount and k be a capital per efficiency unit labor. $f(\kappa)$ is nonzero and

$$\frac{r(t) \cdot K(t)}{Y(t)} = \frac{\partial f(\kappa) \cdot \kappa}{f(\kappa)}.$$

Proof. $f(\kappa)$ is nonzero (by lemma G4).

(1) $r(t) = \partial f(\kappa)$ (by G3).

(2) $\frac{K(t)}{Y(t)} = \frac{\kappa}{f(\kappa)}$ (by G4). [prover eprover]

$\frac{r(t) \cdot K(t)}{Y(t)}. = r(t) \cdot \frac{K(t)}{Y(t)}$ (by mult ass)

$. = \partial f(\kappa) \cdot \frac{\kappa}{f(\kappa)}$ (by 1, 2)

$. = \frac{\partial f(\kappa) \cdot \kappa}{f(\kappa)}$ (by mult ass). □

G6 There are appreciable differences in the rate of growth of labour productivity and of total output in different societies.

G6 can be analyzed in the Solow model as being off steady state. A country is in a steady state when the capital per efficiency unit labor is constant, which we can interpret as the long-term state of the economy of an industrialized country like Britain with the appropriate amount of capital for its labor around which it fluctuates with booms and recessions in the short to medium term. If a country has a different growth rate from Britain in the longer term - typically growing much faster for some decades, like China in the last 30 years, or Japan in the 30 years before that, or Germany before the First or after the Second World War -, then this is interpreted in the country having less capital than it should have - so it has much less machinery than an industrialized country. In this catch up phase, where the country builds up its capital until it reaches the point where it is equivalent to Britain, it reaches higher growth rates in the Solow model.

9.3 Cobb-Douglas Example

An alternative to the way we introduced a production function with the Inada conditions, we also could have defined

$$f = ext|_{\mathrm{pow}_a}.$$

This is the Cobb-Douglas production function

$$F(K_t, A_t L_t) = K_t^{\alpha} (A_t L_t)^{1-\alpha},$$

which is equivalent to

$$f(k_t) = k_t^{\alpha}$$

for some $\alpha \in (0, 1)$. Because this is a function that satisfies the standard Inada-conditions stated above, but not the ones that we defined, since we do not have limits against infinity, we cannot define $f = ext|_{\mathrm{pow}_a}$ axiomatically at this time and view it as an example. However in the future, formalizations including examples like this would be a great idea.